New Results on the Sum of Two Generalized Gaussian Random Variables

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Abstract

We propose in this paper a new method to compute the characteristic function (CF) of generalized Gaussian (GG) random variable in terms of the Fox H function. The CF of the sum of two independent GG random variables is then deduced. Based on this results, the probability density function (PDF) and the cumulative distribution function (CDF) of the sum distribution are obtained. These functions are expressed in terms of the bivariate Fox H function. Next, the statistics of the distribution of the sum, such as the moments, the cumulant, and the kurtosis, are analyzed and computed. Due to the complexity of bivariate Fox H function, a solution to reduce such complexity is to approximate the sum of two independent GG random variables by one GG random variable with suitable shape factor. The approximation method depends on the utility of the system so three methods of estimate the shape factor are studied and presented [1].

Motivation

- Gaussian noise is widely used in communication systems
- In sensor networks and local spectrum sensing, the deployed noise is Generalized Gaussian (GG)
- In UWB, interference and noise can be modeled as GG distribution
- Noise + Interference can be modeled as sum of GG random variables
- The statistics of the SGG distribution (PDF, CDF, moments, cumulant...) should be studied
- An approximation of the SGG by a single GG random can be used to get simplified results

Generalized Gaussian Distribution

Definition

- GG distribution is characterized by a parameter α, called shaping parameter (or exponent parameter)
- The PDF of GGD(μ,σ) is given as

$$f_X(x)=\frac{\alpha\Lambda}{2\Gamma(1/\alpha)}\exp\left(-\Lambda^\alpha|x-\mu|^\alpha\right)\ \forall x\in\mathbb{R},$$

where $\Lambda=\frac{\Lambda_0}{\sigma}=\frac{1}{\sigma}\sqrt{\frac{\Gamma(3/\alpha)}{\Gamma(1/\alpha)}}$, a normalization coefficient.

- Laplacian, Gaussian, and uniform distributions are obtained for $\alpha=1,2,$ and ∞ respectively.

Cumulative Distribution Function

- The complementary CDF is obtained as

$$Q_\alpha(x)=\frac{1}{2\Gamma(1/\alpha)}\Gamma(1/\alpha,\Lambda_0^\alpha x^\alpha),\quad \text{for } x\geq 0,$$

- The CDF is given by

$$F_X(x)=1-Q_\alpha\left(\frac{x-\mu}{\sigma}\right).$$

- Q_α is reduced to the classical Gaussian Q-function for $\alpha=2$.

Characteristic Function

- CHF is the Fourier Transform of the PDF
- PDF is even ($\mu=0$) $\Rightarrow$ CHF becomes cosine transform of the PDF

$$\varphi_\alpha(t)=\int_0^\infty\cos(tx)f_X(x)dx$$

- Using alternative expressions of $\cos(\cdot)$ and $f_X(\cdot)$, CHF appears as an integral of 2 Fox H-functions
- Using integral identity [2, Eq. (2.8.4)], the CHF is expressed as

$$\varphi_\alpha(t)=\frac{\sqrt{\pi}}{\Gamma(1/\alpha)}\mathrm{H}_{1,2}^{1,1}\left[\frac{\sigma^2\Gamma(1/\alpha)}{4\Gamma(3/\alpha)}t^2\,\middle|\,\begin{matrix}(1-\frac{1}{\alpha},\frac{2}{\alpha})\\(0,1),(\frac{1}{2},1)\end{matrix}\right]$$

MGF, Moment, and Cumulant

- The MGF is obtained from the CHF as $M_\alpha(t)=\varphi_\alpha(-it)$
- The odd moments are equal to 0, and the even moments are

$$m_{2n}(X)=\mathbb{E}[X^{2n}]=\sigma^{2n}\frac{\Gamma(1/\alpha)^n}{\Gamma(3/\alpha)^n}\frac{\Gamma(\frac{2n+1}{\alpha})}{\Gamma(1/\alpha)}$$

- Cumulant generating function: $K_\alpha(t)=\log M_\alpha(t)$
- New results: the even cumulants of zero mean GGD

$$k_{2n}(X)=-\sum_{m_1+2m_2+\ldots nm_n=n}\frac{(2n)!(m_1+\cdots+m_n-1)!}{m_1!m_2!\ldots m_n!}\prod_{1\leq j\leq n}\left(-\frac{\sigma^{2j}\Gamma(1/\alpha)^j\Gamma(\frac{2j+1}{\alpha})}{\Gamma(3/\alpha)^j\Gamma(1/\alpha)(2j)!}\right)^{m_j}$$

Kurtosis

- Measures the tailedness of the distribution (i.e. more kurtosis means heavier tail)
- For the GGD, the kurtosis expressed as

$$Kurt(X)=\frac{k_4(X)}{k_2(X)^2}=\frac{\Gamma(1/\alpha)\Gamma(5/\alpha)}{\Gamma(3/\alpha)^2}-3$$

- Examples:
  - $Kurt(Laplace)=3$: heavy tail
  - $Kurt(Gaussian)=0$: normal tail
  - $Kurt(Uniform)=-1.2$: no positive-valued tail
- Applications: estimation of the distribution parameter...

Sum of Generalized Gaussian RV

Characteristic Function of the Sum

- CHF of the sum of 2 independent GGD is the product of their CHFs $\Rightarrow \varphi_Z(t)=\varphi_X(t)\varphi_Y(t)$

PDF and CDF of the Sum

- The PDF of the sum, $f_Z(z)$, is the inverse cosine transform of the CHF

$$f_Z(z)=\frac{1}{2\Gamma(1/\alpha)\Gamma(1/\beta)}\int_0^\infty\cos\left(t(\mu-z)\right)\mathrm{H}_{1,2}^{1,1}\left[A\,t^2\,\middle|\,\begin{matrix}(1-\frac{1}{\alpha},\frac{2}{\alpha})\\(0,1),(\frac{1}{2},1)\end{matrix}\right]\mathrm{H}_{1,2}^{1,1}\left[B\,t^2\,\middle|\,\begin{matrix}(1-\frac{1}{\beta},\frac{2}{\beta})\\(0,1),(\frac{1}{2},1)\end{matrix}\right]dt,$$

where $\mu=\mu_1+\mu_2$, $A=\frac{\sigma_1^2\Gamma(1/\alpha)}{4\Gamma(3/\alpha)}$, and $B=\frac{\sigma_2^2\Gamma(1/\beta)}{4\Gamma(3/\beta)}$

- The CDF is the primitive of the previous formula that vanishes at $-\infty$, so it is the inverse sine transform of $\varphi_Z(t)$.
- PDF of the sum in terms of the BFHF

$$f_Z(z)=\frac{\sqrt{\pi}}{\Gamma(1/\alpha)\Gamma(1/\beta)|z-\mu|}\times\mathrm{H}_{2,0;1,2;1,2}^{0,1;1,1;1,1}\left[\frac{4A}{(z-\mu)^2},\frac{4B}{(z-\mu)^2}\,\middle|\,\begin{matrix}(\frac{1}{2},1,1),(0,1,1)\\ \text{—}\end{matrix}\,\middle|\,\begin{matrix}(1-\frac{1}{\alpha},\frac{2}{\alpha})\\(0,1),(\frac{1}{2},1)\end{matrix}\,\middle|\,\begin{matrix}(1-\frac{1}{\beta},\frac{2}{\beta})\\(0,1),(\frac{1}{2},1)\end{matrix}\right]$$

- CDF of the sum

$$F_Z(z)=\frac{1}{2}+\frac{\sqrt{\pi}\ sign(z-\mu)}{2\Gamma(1/\alpha)\Gamma(1/\beta)}\times\mathrm{H}_{2,0;1,2;1,2}^{0,1;1,1;1,1}\left[\frac{4A}{(z-\mu)^2},\frac{4B}{(z-\mu)^2}\,\middle|\,\begin{matrix}(\frac{1}{2},1,1),(1,1,1)\\ \text{—}\end{matrix}\,\middle|\,\begin{matrix}(1-\frac{1}{\alpha},\frac{2}{\alpha})\\(0,1),(\frac{1}{2},1)\end{matrix}\,\middle|\,\begin{matrix}(1-\frac{1}{\beta},\frac{2}{\beta})\\(0,1),(\frac{1}{2},1)\end{matrix}\right]$$

Statistics of the Sum

- MGF: $M_Z(t)=\varphi_Z(-it)$
- Moment: $m_{2n+1}(Z)=0$ and

$$m_{2n}(Z)=\frac{\sigma_2^{2n}\Gamma(\frac{1}{\beta})^n}{\Gamma(\frac{1}{\alpha})\Gamma(\frac{1}{\beta})\Gamma(\frac{3}{\beta})^n}\sum_{k=0}^{n}\binom{2n}{2k}\left(\frac{\sigma_1^2}{\sigma_2^2}\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{3}{\beta})}{\Gamma(\frac{3}{\alpha})\Gamma(\frac{1}{\beta})}\right)^k\Gamma\left(\frac{2k+1}{\alpha}\right)\Gamma\left(\frac{2n-2k+1}{\beta}\right)$$

- Kurtosis:

$$Kurt(Z)=\frac{\sigma_1^4}{\sigma^4}\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{5}{\alpha})}{\Gamma(\frac{3}{\alpha})^2}+\frac{\sigma_2^4}{\sigma^4}\frac{\Gamma(\frac{1}{\beta})\Gamma(\frac{5}{\beta})}{\Gamma(\frac{3}{\beta})^2}+6\frac{\sigma_1^2\sigma_2^2}{\sigma^4}-3,$$

where $\sigma^2=\sigma_1^2+\sigma_2^2$

Approximation of the PDF of the Sum

Motivation

- Similar distribution properties of the sum of 2 GGD and one GGD [3]
- Approximation of the sum distribution by one GGD
- Estimation of one shape parameter instead of two
- Reducing the complexity of the expression of the PDF of the sum (BFHF)
- Simplification of the PDF/CDF expressions
- Less complexity expressions of system performance metrics (BER, SER...)

Kurtosis Approach

- Equalize the Kurtosis of the sum to the Kurtosis of a new distribution (Z_γ) with shape parameter γ

$$Kurt(Z_\gamma)=Kurt(Z)$$

$$\Longleftrightarrow$$

$$\frac{\Gamma(\frac{1}{\gamma})\Gamma(\frac{5}{\gamma})}{\Gamma(\frac{3}{\gamma})^2}=\frac{1}{(1+\delta)^2}\left(\delta^2\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{5}{\alpha})}{\Gamma(\frac{3}{\alpha})^2}+\frac{\Gamma(\frac{1}{\beta})\Gamma(\frac{5}{\beta})}{\Gamma(\frac{3}{\beta})^2}+6\delta\right),$$

where $\delta=\frac{\sigma_1^2}{\sigma_2^2}$

- Solve the system $h(\gamma)=C$, with unknown γ and C is positive known value

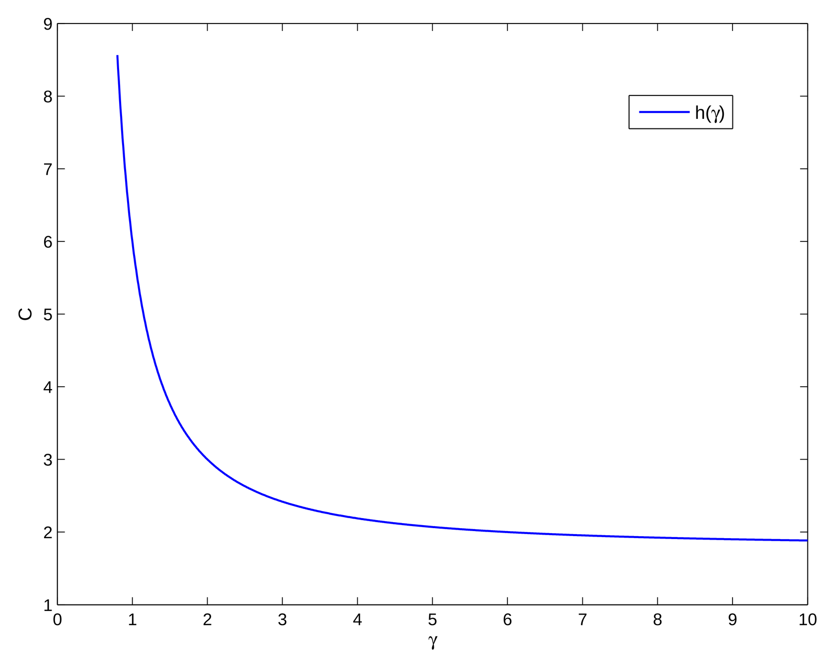

Figure 1: *The curve of $h(\gamma)$ for positive values of γ.*

- $h(\cdot)$ is bijection (Fig.1) $\Rightarrow$ the equation $h(\gamma)=C$ has unique solution γ_{Kurt}

Best Tail Approximation

- Minimizing the square error of the desired tail (defined for $z\geq n\sigma$)

$$\gamma_{Tail}=\arg\min_{\gamma>0}\int_{n\sigma}^{\infty}\left(f_\gamma(z)-f_Z(z)\right)^2dz$$

CDF Approximation

- Minimize the error between the exact and the approximated CDF

$$\gamma_{CDF}=\arg\min_{\gamma>0}\int_{0}^{\infty}\left(F_\gamma(z)-F_Z(z)\right)^2dz$$

Examples of estimated γ

Table 1: *Shape parameter for the approximated distribution with $\sigma_1=1$*

(α,β,δ)	γ_{Kurt}	γ_{CDF}	γ_{Tail}			
			$n=0$	1	2	3
(0.5, 0.5, 1)	0.626	0.467	0.768	0.673	0.624	0.642
(0.5, 0.5, 2)	0.604	0.492	0.762	0.656	0.603	0.584
(0.5, 0.7, 2)	0.633	0.501	0.861	0.741	0.636	0.834
(0.5, 1.2, 1)	0.779	0.602	1.160	1.053	0.757	1.165
(1.5, 1.5, 2)	1.673	1.373	1.738	1.702	1.683	1.664
(1.5, 2.5, 1)	1.908	1.391	1.979	1.959	1.952	1.887
(1.5, 2.5, 2)	1.753	1.443	1.842	1.799	1.771	1.741
(2.5, 3, 3)	2.295	1.941	2.226	2.261	2.267	2.335

- γ_{Kurt} and γ_{Tail} are close to each other (Kurtosis measures the heavy tail)
- γ_{CDF} is little far from other methods $\Rightarrow$ Each method can be used according to the scenario in case.

PDF & CDF Simulations

Impact of Kurtosis and CDF approaches on the PDF

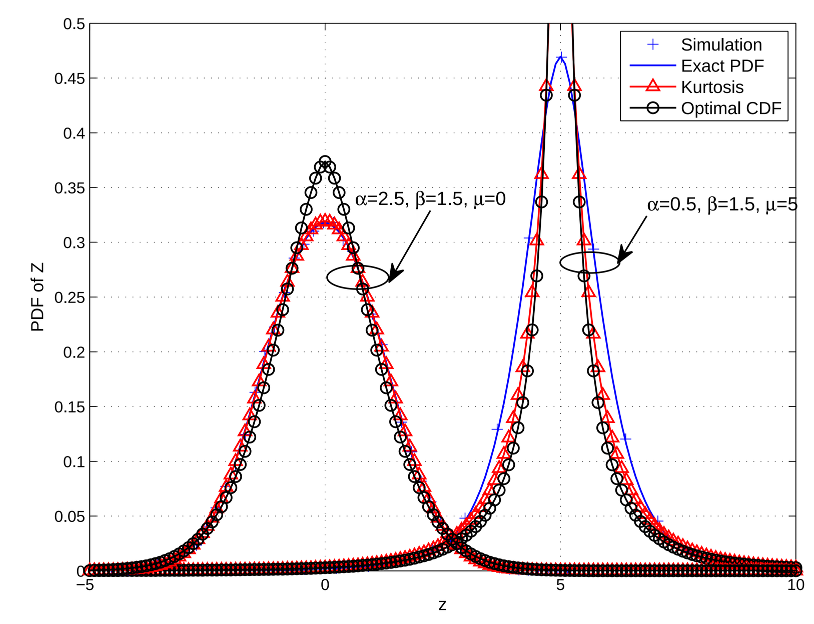

Figure 2: *Exact and approximated PDF of the sum of two GGRV. $\beta=1.5$, $\delta=2$, and $\sigma_1=1$.*

- Perfect match b/w simulated and exact PDF
- Good tail approximation for both methods
- $\alpha<2$: Tail matching for both methods, but big difference around the mean
- $\alpha>2$: Better PDF matching using Kurtosis approach than using CDF approach which approximates the CDF rather than the PDF.

Impact of estimation approaches on the CDF

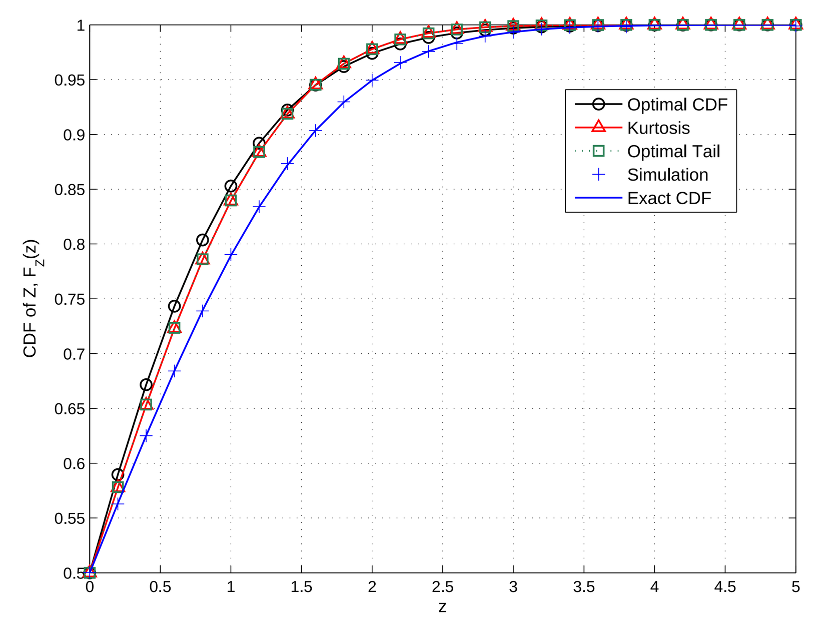

Figure 3: *CDF of the sum using Kurtosis, optimal tail, and optimal CDF approaches. $\alpha=2.5$, $\beta=1.5$, $\delta=2$, and $\sigma_1=1$*

- Kurtosis and Tail ($n=3$) methods give very close results to the exact CDF
- In linear scale, all methods are close to the CDF at saturation, i.e. $F_Z(z)\approx 1$.
- Plot the gap in Log scale.

Comparison of the Complementary CDF

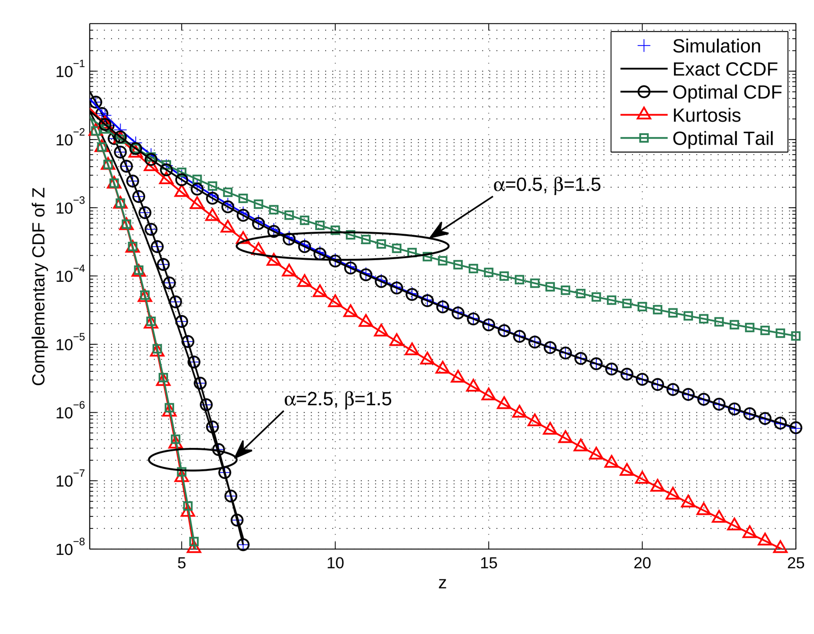

Figure 4: *Complementary CDF of the sum of two GGRV. $\beta=1.5$ and $\delta=2$.*

- CDF approach matches exact CDF of the sum for both values of α
- $\alpha<2$ Kurtosis and Tail methods are far from the exact CDF
- $\alpha>2$ The gap decreases and these methods become close to the exact (as seen in Fig.2).

Summary

- Expression of the statistics of the GGD
- Derivation of the distribution of the sum of two independent GG random variables
- Expression of the statistics of the sum distribution in terms of the FHF and BFHF
- Approximation of the distribution of the sum by single GGD using 3 methods
- Different behaviors of these 3 approaches regarding the exact distribution
- Choose of the suitable approach depends on the application in case

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