Weierstrass Institute for Applied Analysis and Stochastics

Leibniz Association

Christian Bayer, Peter K. Friz, Jim Gatheral, Benjamin Stemper

Pricing under rough volatility

Introduction to stochastic volatility modeling

We are given a traded asset S_t satisfying

$$dS_t = \sqrt{v_t} S_t dZ_t.$$

Assume that S_t corresponds to the S&P500 Index (SPX). We define

- Realized variance: $w_{t,T} = \int_t^T v_s ds$ & Forward variance: $\xi_t(u) = E_t v_u$ for $t \le u$
- Variance swaps: swaps on realized variances; allow direct trades in volatility, not indirectly via options. Price: $E_t w_{t,T} = E_t \int_t^T v_s ds = \int_t^T \xi_t(s) ds$ can be observed in the market!
- CBOE introduced volatility index $\mathrm{VIX}_t \approx \sqrt{\frac{1}{\Delta} E_t w_{t,t+\Delta}}$ for Δ = one month, $\mathrm{VIX}_t \approx \sqrt{v_t}$ which is computed using the Log-strip formula: $E_t w_{t,T} = -2\left(\int_0^{S_t} \frac{P(K)}{K^2} dK + \int_{S_t}^\infty \frac{C(K)}{K^2} dK\right)$
- Implied volatility: volatility parameter σ implied by market price under Black-Scholes framework

Fundamental question: How to model instantaneous stochastic volatility v_t consistently across all strikes and maturities?

Model needs to be consistent with full SPX implied volatility surface. Now, VIX itself is not traded, but VIX futures, VIX Options and Variance swaps are.

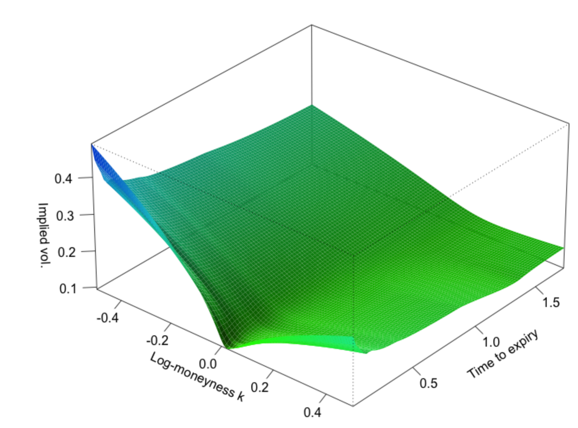

Figure 1: *SPX implied volatility surface as of 14th Aug 2014.*

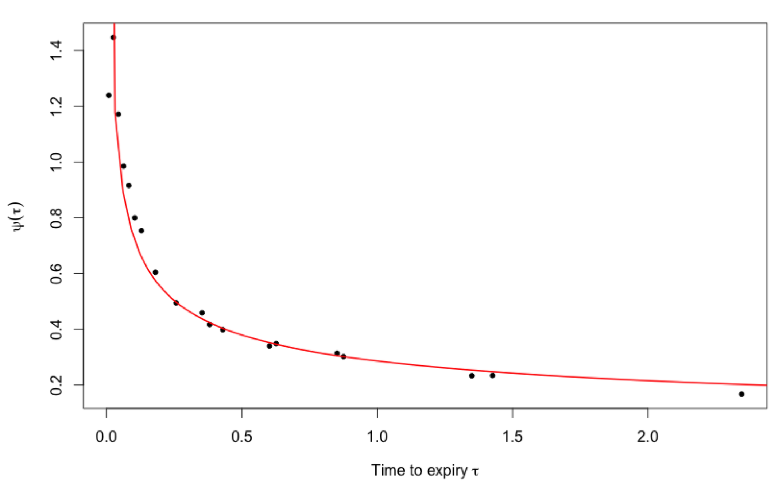

Figure 2: *SPX ATM volatility skew as of 14th Aug 2014.*

- Since the rough shape of the empirical volatility surfaces seems pretty stable, we look for time-homogeneous models.
- Term structure of ATM volatility skew for $k = \log(K/S_t), \alpha \in [0.3, 0.5]$

$$\psi(\tau) = \left|\frac{\partial}{\partial k}\sigma_{BS}(k,\tau)\right|_{k=0} \sim 1/\tau^\alpha$$

  constitutes important proxy for surface.
- Conventional stochastic volatility models produce ATM skews which are constant for $\tau \ll 1$ and of order $1/\tau$ for $\tau \gg 1$. Hence, conventional stochastic volatility models cannot fit the full volatility surface.

Volatility is rough: Gatheral, Jaisson and Rosenbaum (2014)

- Oxford-MAN Institute provides estimated realized variances v_t for numerous indices on a daily basis.
- For lag $\Delta > 0$ and $\langle \cdot \rangle$ denoting sample averages, define moment of log-differences by

$$m(q,\Delta) := \left\langle \left|\log\sqrt{v_{t+\Delta}} - \log\sqrt{v_t}\right|^q \right\rangle$$

  such that $m(q,\Delta)$ measures smoothness of realized volatility at various lags.

Two startingly simple regularities were uncovered:

- Monofractal scaling behaviour: Let $H \approx 0.1$. For each q,

$$m(q,\Delta) \sim \Delta^{Hq}.$$

- Normality of log-volatility increments.

Ubiquitous phenomenon for all 21 indices in the Oxford-MAN database (with varying H between different markets).

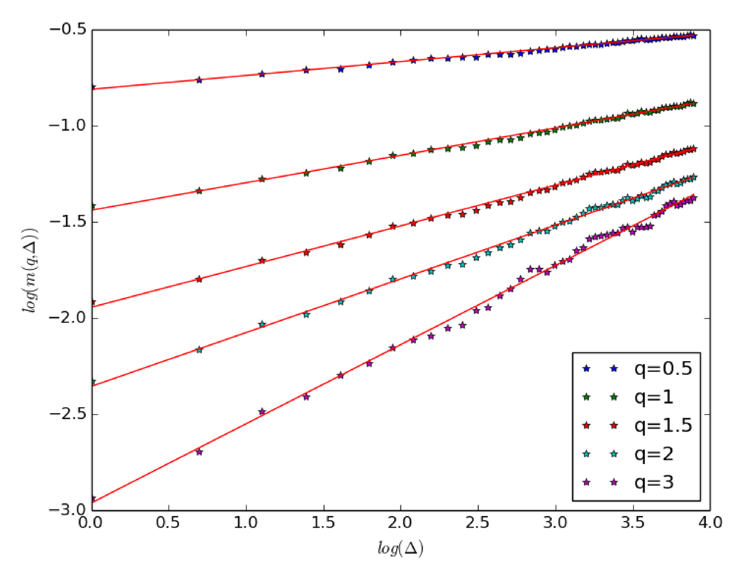

Figure 3: *Monofractal scaling of* $m(q,\Delta)$

Strongly suggests to model log-volatility as a fractional Brownian motion with Hurst parameter H, so that

$$\log v_u - \log v_t = 2\nu\left(W_u^H - W_t^H\right). \qquad (1)$$

Pricing under rough volatility: Bayer, Friz and Gatheral (2015)

Plugging the Mandelbrot – Van Ness representation of fBm (with $\gamma = 1/2 - H$) into (1), we arrive at

$$\log v_u - \log v_t = 2\nu C_H \left(\int_t^u \frac{dW_s^P}{(u-s)^\gamma} + \int_{-\infty}^t \left[\frac{1}{(u-s)^\gamma} - \frac{1}{(t-s)^\gamma}\right] dW_s^P\right)$$

Then, setting

$$\widetilde{W}_t = \sqrt{2H}\int_0^t \frac{dW_s}{(t-s)^\gamma},$$

we obtain after some calculations the rough Bergomi model:

$$dS_t = \sqrt{v_t} S_t dZ_t.$$
$$v_t = \xi_0(t)\mathcal{E}\left(\eta \widetilde{W}_t\right)$$

with covariance structure

$$E\left[\widetilde{W}_v \widetilde{W}_u\right] = \frac{2H}{1/2+H}\frac{u^{1/2+H}}{v^{1/2-H}}\,{}_2F_1(1, 1/2-H, 3/2+H, u/v), \; u \le v,$$
$$E\left[\widetilde{W}_v Z_u\right] = \rho\frac{\sqrt{2H}}{1/2+H}\left(v^{1/2+H} - [v - \min(u,v)]^{1/2+H}\right).$$

Non-markovianity of rBergomi model (resolved)

- The instantaneous variance v_t inherits non-Markovianity from fBM:

$$\xi_t(u) = \mathbb{E}(v_u \mid \mathcal{F}_t) \neq \mathbb{E}(v_u \mid v_t) \quad \text{for} \quad u > t.$$

  which is very bad from a simulation point of view.
- Fortunately, $\xi_t(u)$ may be observed in the market.

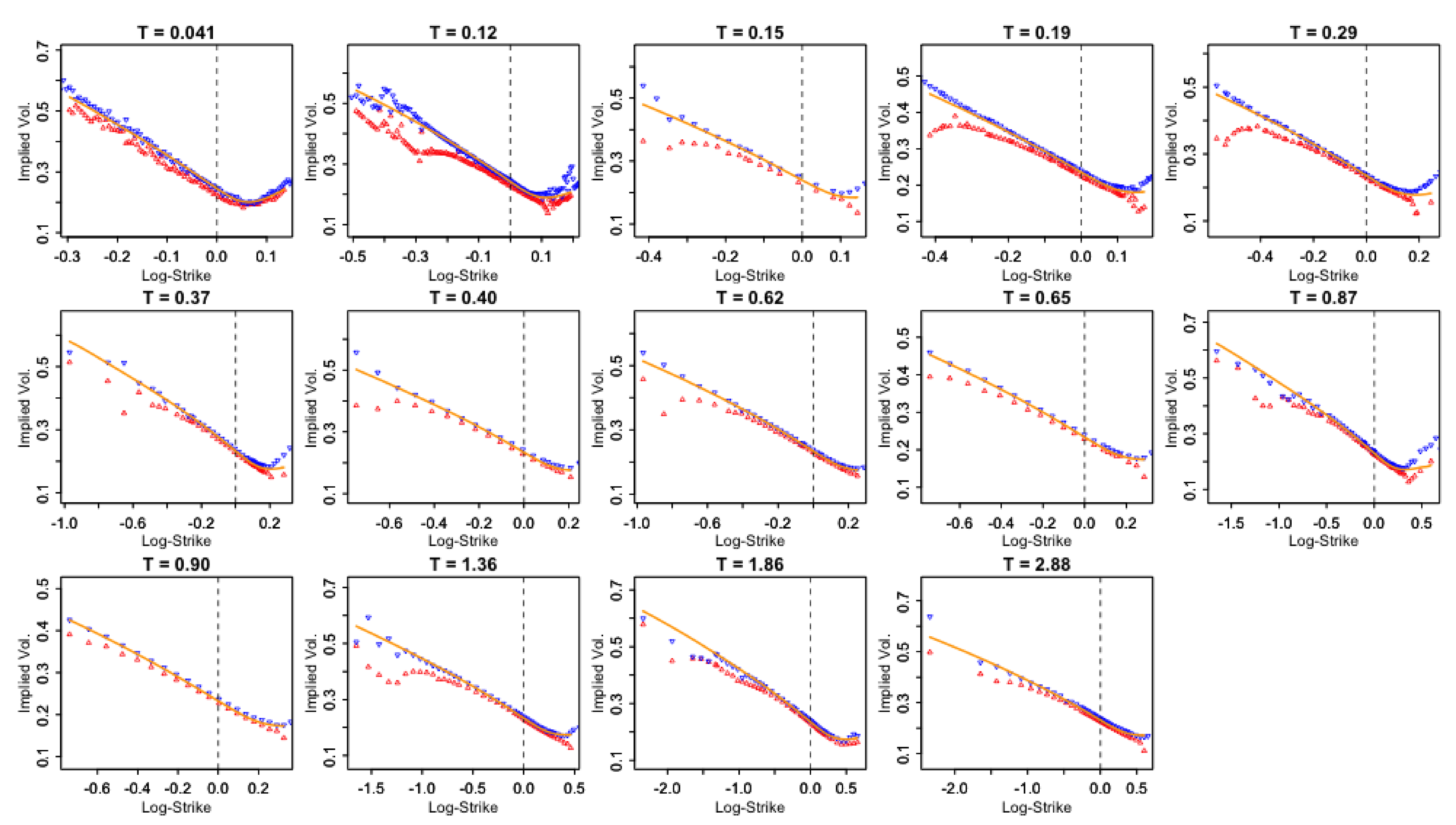

Figure 4: *SPX smiles as of 4th Feb 2010: Red and blue points represent bid and offer SPX implied volatilities; orange smilies are from rBergomi simulation.*

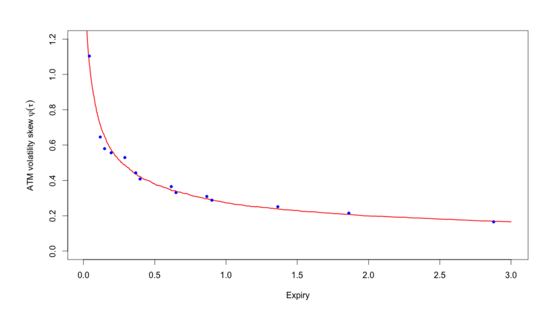

Figure 5: *As of 4th Feb 2010: Blue points are empirical skews; red line is from rBergomi simulation.*

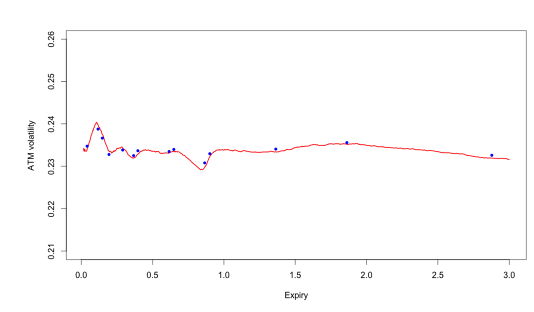

Figure 6: *As of 4th Feb 2010: Blue points are empirical ATM implied volatilities; red line is from rBergomi simulation.*

The rBergomi model achieves a fantastic fit to the real data! This is particularly impressive as it only contains three parameters: the Hurst parameter H of the fBM, the volatility of volatility: η and the correlation ρ between driving noises of asset and volatility.

Outlook & Challenges

- Derivation of fast & accurate numerical methods for option pricing in rough stochastic volatility models.
- Calibration of models & derivation of asymptotic formulas.
- Research market implied change of measure. Current deterministic change of measure doesn't give realistic smile for VIX options.

Funding

We gratefully acknowledge funding by the DFG Research Grants BA5484/1 and FR2943/2.

contact: C. Bayer, P.K. Friz & B. Stemper · Mohrenstraße 39, D-10117 Berlin, Germany · +49 30 203720 · christian.bayer@wias-berlin.de, peter.friz@wias-berlin.de & stemper@wias-berlin.de · www.wias-berlin.de