An A Posteriori Error Estimate for Symplectic Euler Approximation of Optimal Control Problems

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Abstract

This work focuses on numerical solutions of optimal control problems. A time discretization error representation is derived for the approximation of the associated value function. It concerns Symplectic Euler solutions of the Hamiltonian system connected with the optimal control problem. The error representation has a leading order term consisting of an error density that is computable from Symplectic Euler solutions. Under an assumption of the pathwise convergence of the approximate dual function as the maximum time step goes to zero, we prove that the remainder is of higher order than the leading error density part in the error representation. With the error representation, it is possible to perform adaptive time stepping. We apply an adaptive algorithm originally developed for ordinary differential equations.

1. Optimal Control

The optimal control problem is to minimize the functional

$$\int_0^T h(X(t),\alpha(t))\,\mathrm{d}t + g(X(T)), \tag{1}$$

with given functions $h : \mathbb{R}^d \times \mathcal{B} \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$, with respect to the state variable $X : [0,T] \to \mathbb{R}^d$ and the control $\alpha : [0,T] \to \mathcal{B}$, with control set, $\mathcal{B}$, a subset of some Euclidean space, $\mathbb{R}^{d'}$, such that the ODE constraint,

$$\begin{aligned} X'(t) &= f(X(t),\alpha(t)), \quad 0 < t \le T, \\ X(0) &= x_0, \end{aligned} \tag{2}$$

is fulfilled. This optimal control problem can be solved (globally) using the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} u_t + H(x,u_x) &= 0, \quad x \in \mathbb{R}^d, \quad 0 \le t < T, \\ u(\cdot,T) &= g(\cdot), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{3}$$

with u_t and u_x denoting the time derivative and spatial gradient of u, respectively, and the Hamiltonian, $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, defined by

$$H(x,\lambda) := \min_{\alpha\in\mathcal{B}} \Big\{ \lambda \cdot f(x,\alpha) + h(x,\alpha) \Big\}, \tag{4}$$

and value function

$$u(x,t) := \inf_{X:[t,T]\to\mathbb{R}^d,\ \alpha:[t,T]\to\mathcal{B}} \left\{ \int_t^T h(X(s),\alpha(s))\,\mathrm{d}s + g(X(T)) \right\}, \tag{5}$$

where

$$\begin{aligned} X'(s) &= f(X(s),\alpha(s)), \quad t < s \le T, \\ X(t) &= x. \end{aligned}$$

The global minimum to the optimal control problem (1)-(2) is thus given by $u(x_0,0)$.
If the Hamiltonian is sufficiently smooth, the bi-characteristics to the HJB equation (3) are given by the following Hamiltonian system:

$$\begin{aligned} X'(t) &= H_\lambda(X(t),\lambda(t)),\ 0 < t \le T, \\ X(0) &= x_0, \\ -\lambda'(t) &= H_x(X(t),\lambda(t)),\ 0 \le t < T, \\ \lambda(T) &= g_x(X(T)), \end{aligned} \tag{6}$$

where H_λ, H_x, and g_x denote gradients with respect to λ and x, respectively, and the dual variable, $\lambda : [0,T] \to \mathbb{R}^d$, satisfies $\lambda(t) = u_x(X(t),t)$ along the characteristic. It can be solved numerically using the Symplectic Euler method:

$$\begin{aligned} X_{n+1} - X_n &= \Delta t_n H_\lambda(X_n,\lambda_{n+1}),\ n = 0,\ldots,N-1, \\ X_0 &= x_0, \\ \lambda_n - \lambda_{n+1} &= \Delta t_n H_x(X_n,\lambda_{n+1}),\ n = 0,\ldots,N-1, \\ \lambda_N &= g_x(X_N), \end{aligned} \tag{7}$$

with $0 = t_0 < t_1 < \ldots < t_N = T$, $\Delta t_n := t_{n+1} - t_n$, and $X_n, \lambda_n \in \mathbb{R}^d$, see [5, 4]

2. An A Posteriori Error Estimate

Theorem 2.1. *Assume that the following conditions are satisfied:*

- *The terminal cost $g : \mathbb{R}^d \to \mathbb{R}$ is a locally semiconcave function such that $g(x) \ge -k(1+|x|)$, for some constant k, and all $x \in \mathbb{R}^d$.*
- *There exists a convex, nondecreasing function $\mu : [0,\infty) \to \mathbb{R}$ and positive constants A and B such that*
$$-H(x,\lambda) \le \mu(|\lambda|) + |x|(A + B|\lambda|) \quad \text{for all } (x,\lambda) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{8}$$
- *$H(x,\cdot)$ is concave for every $x \in \mathbb{R}^d$.*
- *The Hamiltonian, H, is bounded in $C^2(\mathbb{R}^d \times \mathbb{R}^d)$.*
- *There exists a constant, C, such that for every discretization $\{t_n\}$ the difference between the discrete dual and the gradient of the value function is bounded as*
$$|\lambda_n - u_x(X_n,t_n)| \le C\Delta t_{\max},$$
where $\Delta t_{\max} := \max_n \Delta t_n$.

Assume further that either of the following two conditions holds:

1. *The value function, u, is bounded in $C^3((0,T) \times \mathbb{R}^d)$.*
2. *There exists a neighborhood in $C([0,T],\mathbb{R}^d)$ around the minimizer $X : [0,T] \to \mathbb{R}^d$ of $u(x_0,0)$ in (5) in which the value function, u, is bounded in C^3. Moreover, the discrete solutions $\{X_n\}$ converge to the continuous solution $X(t)$ in the sense that*
$$\max_n |X_n - X(t_n)| \to 0, \text{ as } \Delta t_{\max} \to 0.$$

If Condition 1 holds, then for every discretization $\{t_n\}$, the error $\bar{u}(x_0,0) - u(x_0,0)$ is given as

$$\bar{u}(x_0,0) - u(x_0,0) = \sum_{n=0}^{N-1} \Delta t_n^2 \rho_n + R, \tag{9}$$

with density

$$\rho_n := -\frac{H_\lambda(X_n,\lambda_{n+1}) \cdot H_x(X_n,\lambda_{n+1})}{2} \tag{10}$$

and the remainder term, $|R| \le C'\Delta t_{\max}^2$, for some constant C'.
If Condition 2 holds, then there exists a threshold time step, Δt_{thres}, such that for every discretization with $\Delta t_{\max} \le \Delta t_{\text{thres}}$ the error representation (9) holds.

3. An adaptive algorithm

We use an algoritm originally developed for adaptive solutions of ODE in [2].
Algorithm 3.1 (Adaptivity). *Choose the error tolerance TOL, the initial grid $\{t_n\}_{n=0}^N$, the parameters s and M, and repeat the following points:*

1. *Calculate $\{(X_n,\lambda_n)\}_{n=0}^N$ with the symplectic Euler scheme (7).*
2. *Calculate error densities $\{\rho_n\}_{n=0}^{N-1}$ and the corresponding approximate error densities*
$$\bar{\rho}_n := \operatorname{sgn}(\rho_n)\max(|\rho_n|, K\sqrt{\Delta t_{\max}}).$$
3. *Break if*
$$\max_n \bar{r}_n < \frac{\text{TOL}}{N}$$
where the error indicators are defined by $\bar{r}_n := |\bar{\rho}_n|\,\Delta t_n^2$.
4. *Traverse through the mesh and subdivide an interval (t_n,t_{n+1}) into M parts if*
$$\bar{r}_n > s\frac{\text{TOL}}{N}.$$
5. *Update N and $\{t_n\}_{n=0}^N$ to reflect the new mesh.*

4. Numerical Examples

Example 4.1 (Hyper-sensitive optimal control). *This is a version of Example 6.1 in [1] and Example 51 in [3]. Minimize*

$$\int_0^{25} \left(X(t)^2 + \alpha(t)^2\right)\mathrm{d}t + \gamma(X(25)-1)^2,$$

subject to

$$\begin{aligned} X'(t) &= -X(t)^3 + \alpha(t), \qquad 0 < t \le 25, \\ X(0) &= 1, \end{aligned}$$

for some large $\gamma > 0$. The Hamiltonian is then given by

$$H(x,\lambda) := \min_\alpha \Big\{ -\lambda x^3 + \lambda\alpha + x^2 + \alpha^2 \Big\} = -\lambda x^3 - \lambda^2/4 + x^2.$$

Figure 1 shows the solution and final mesh when computed with the adaptive Algorithm 3.1.

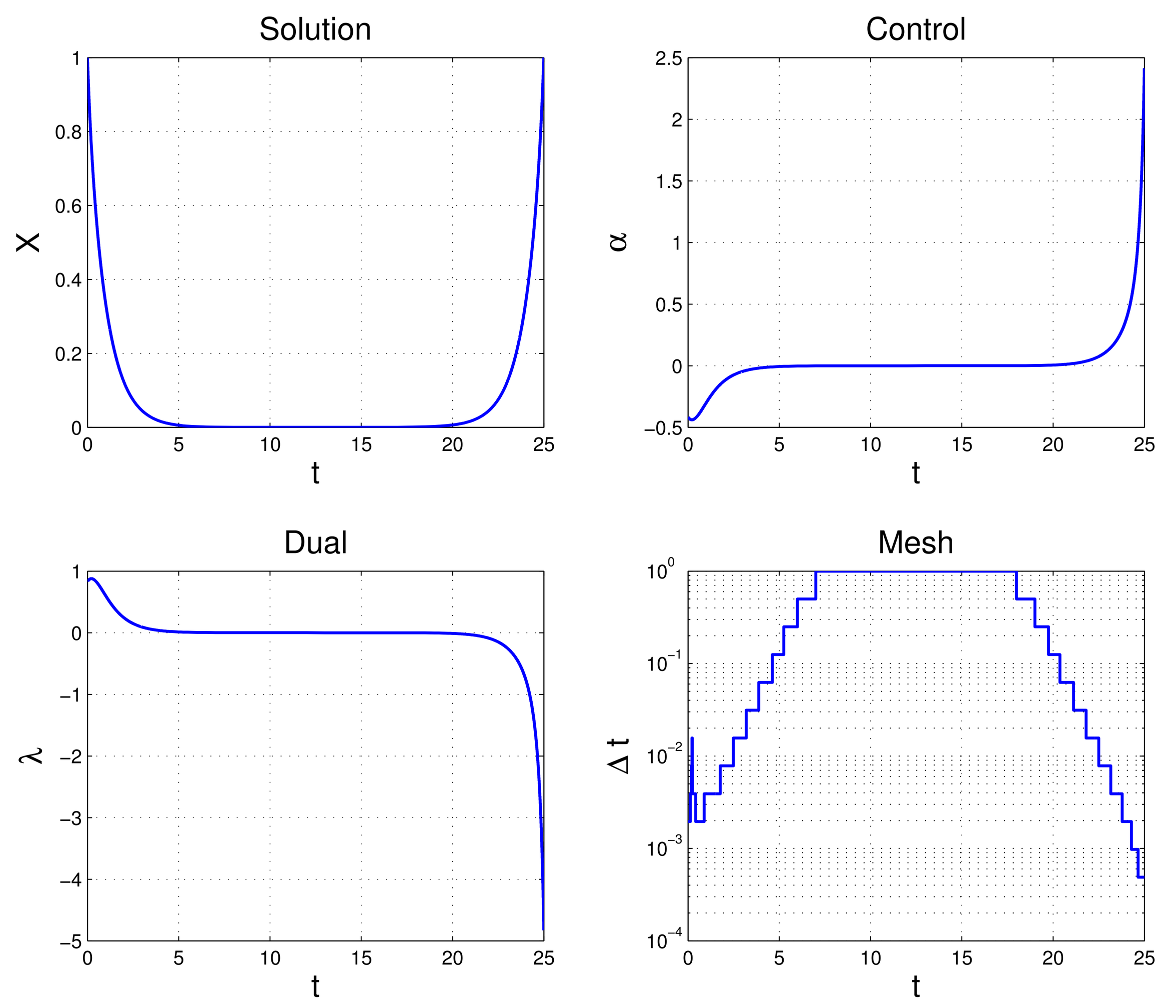

Figure 1: *Solution X, control α, dual λ and mesh Δt for the hyper-sensitive optimal control problem in Example 4.1, with $\gamma = 10^6$ and $\text{TOL} = 10^{-2}$.*

Example 4.2 (Singular optimal control problem). *This example is based on a singular ODE example in [2], suitable for adaptive refinement. Consider the optimal control problem to minimize*

$$\int_0^4 \big(\alpha(t) - X(t)\big)^2 \mathrm{d}t + \big(X(4) - X_{\text{ref}}(4)\big)^2 \tag{11}$$

under the constraint

$$\begin{aligned} X'(t) &= \frac{\alpha(t)}{\big((t-t_0)^2 + \varepsilon^2\big)^{\beta/2}}, \\ X(0) &= X_{\text{ref}}(0). \end{aligned}$$

where $t_0 = 5/3$. The reference $X_{\text{ref}}(t)$ solves

$$X'_{\text{ref}}(t) = \frac{X_{\text{ref}}(t)}{\big((t-5/3)^2 + \varepsilon^2\big)^{\beta/2}}$$

and is given explicitly by

$$X_{\text{ref}}(t) = \exp\left(\frac{t-t_0}{\varepsilon^\beta}\, {}_2F_1\Big(\frac{1}{2},\frac{\beta}{2};\frac{3}{2};-\frac{(t-t_0)^2}{\varepsilon^2}\Big)\right),$$

where ${}_2F_1$ is the hypergeometric function.

Figure 2 shows errors for adaptive and uniform time stepping versus total number of time steps.

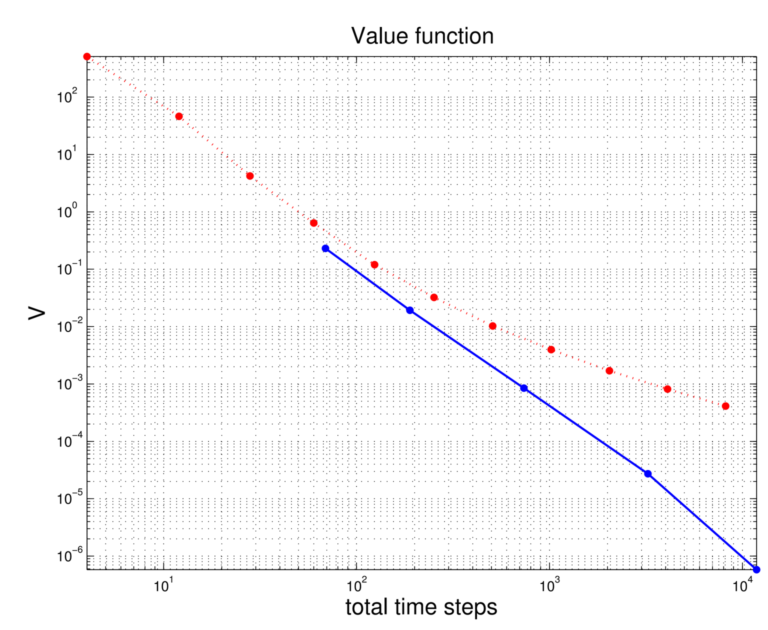

Figure 2: *The minimum value of the functional in (11) for the singular optimal control problem in Example 4.2, versus cumulative number of time steps on all refinement levels for the adaptive algorithm (solid) and uniform time steps (dotted). Since the true value of (11) is zero the graphs also indicate the respective errors. The regularization paramater $\varepsilon = 10^{-10}$ and $\beta = 3/4$.*

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