An Efficient Simulation Method for Rare Events

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Abstract

Estimating the probability that a sum of random variables (RVs) exceeds a given threshold is a well-known challenging problem. Closed-form expressions for the sum distribution do not generally exist, which has led to an increasing interest in simulation approaches. A crude Monte Carlo (MC) simulation is the standard technique for the estimation of this type of probability. However, this approach is computationally expensive, especially when dealing with rare events. Variance reduction techniques are alternative approaches that can improve the computational efficiency of naive MC simulations. We propose an Importance Sampling (IS) simulation technique based on the well-known hazard rate twisting approach, that presents the advantage of being asymptotically optimal for any arbitrary RVs. The wide scope of applicability of the proposed method is mainly due to our particular way of selecting the twisting parameter. It is worth observing that this interesting feature is rarely satisfied by variance reduction algorithms whose performances were only proven under some restrictive assumptions. It comes along with a good efficiency, illustrated by some selected simulation results comparing the performance of our method with that of an algorithm based on a conditional MC technique.

Problem Setting

Objective

- Let $\{X_i\}_{i=1}^N$ be a sequence of independent but not necessarily identically distributed RVs with Probability Density Function (PDF) $\{f_i(\cdot)\}_{i=1}^N$. Our objective is to efficiently estimate:

$$\alpha = \mathbb{P}\left(\sum_{i=1}^{N} X_i > \gamma_{th}\right) = P\left(S_N > \gamma_{th}\right), \qquad (1)$$

for a sufficiently large threshold γ_{th}.

Applications

- The performance analysis of communication systems generally requires analyzing the statistics of sums of Random Variables (RVs)
- Total Interference power, often modeled as a sum of RVs, exceeds a certain threshold.
- Shed the light on the behaviour of the signal-to-interference-plus noise (SINR) ratio.
- The Control of the interference in cognitive radio systems is of paramount importance.

Naive Monte Carlo Simulations

- Naive Monte Carlo simulations is computationally expensive for small probabilities
- More than $100/\alpha$ simulation runs for 10% relative precision .

1. Proposed Approach

- Importance Sampling increases the computational efficiency of naive Monte Carlo simulations.
- Perform a suitable change of the probability measure

$$\begin{aligned}\alpha &= \int_{\mathbb{R}^N} \mathbf{1}_{(S_N>\gamma_{th})} \prod_{i=1}^{N} f_i(x_i) dx_1 dx_2 ... dx_N \\ &= \int_{\mathbb{R}^N} \mathbf{1}_{(S_N>\gamma_{th})} L\left(x_1, x_2, ..., x_N\right) \prod_{i=1}^{N} g_i(x_i) dx_1 dx_2 ... dx_N \\ &= \mathbb{E}_{p^*}\left[\mathbf{1}_{(S_N>\gamma_{th})} L\left(X_1, X_2, ..., X_N\right)\right] = \mathbb{E}_{p^*}\left[T_{\gamma_{th}}\right] \qquad (2)\end{aligned}$$

- L is the likelihood ratio defined as

$$L\left(X_1, X_2, ..., X_N\right) = \prod_{i=1}^{N} \frac{f_i(X_i)}{g_i(X_i)}. \qquad (3)$$

- The Importance Sampling estimator is defined as

$$\hat{\alpha}_{IS} = \frac{1}{M} \sum_{j=1}^{M} \mathbf{1}_{(S_N(\omega_j)>\gamma_{th})} L(X_1(\omega_j), ..., X_N(\omega_j)). \qquad (4)$$

Hazard Rate Twisting

- Define the hazard rate as

$$\lambda_i(x) = \frac{f_i(x)}{1 - F_i(x)}, \; x > 0, \qquad (5)$$

where $F_i(\cdot)$ is the CDF of X_i , $i = 1, ..., N$.

- Define also the hazard function as:

$$\begin{aligned}\Lambda_i(x) &= \int_0^x \lambda_i(t) dt \\ &= -\log\left(1 - F_i(x)\right), \; x > 0. \qquad (6)\end{aligned}$$

- The PDF of X_i is related to the hazard rate and function as:

$$f_i(x) = \lambda_i(x) \exp\left(-\Lambda_i(x)\right). \qquad (7)$$

- The change of probability measure is obtained by twisting the hazard rate of the underlying distribution by a quantity $0 < \theta < 1$ as follows:

$$\begin{aligned}g_i(x) \triangleq f_{i,\theta}(x) &= (1-\theta)\, \lambda_i(x) \exp\left(-(1-\theta)\, \Lambda_i(x)\right) \\ &= (1-\theta)\, f_i(x) \exp\left(\theta \Lambda_i(x)\right). \qquad (8)\end{aligned}$$

Minmax Twisting Parameter

- **First Step:** Construct an upper bound of the second moment of $T_{\gamma_{th}}$ which is achieved by solving the following maximization problem:

$$(P): \max_{X_1,...,X_N} L(X_1, X_2, ..., X_N)$$
$$\text{Subject to } \sum_{i=1}^{N} X_i \geq \gamma_{th},$$
$$X_i > 0, \; i = 1, ..., N,$$

Denote the optimal solution of (P) by $X_1^*(\gamma_{th}), X_2^*(\gamma_{th}), ..., X_N^*(\gamma_{th})$, it follows

$$\mathbb{E}_\theta\left[T_{\gamma_{th}}^2\right] \leq \frac{1}{(1-\theta)^{2N}} \exp\left(-2\theta \sum_{i=1}^{N} \Lambda_i(X_i^*(\gamma_{th}))\right). \qquad (9)$$

- **Second Step:** Minimize the previous upper bound with respect to θ

$$\theta^* = 1 - \frac{N}{\sum_{i=1}^{N} \Lambda_i(X_i^*(\gamma_{th}))}. \qquad (10)$$

Asymptotic Optimality Property

- For any arbitrary independent sum of RVs, the quantity of interest α is asymptotically optimally estimated using the proposed hazard rate twisting IS-based approach with the minmax optimal parameter θ^* given in (10). That is,

$$\lim_{\gamma_{th}\to\infty} \frac{\log\left(\mathbb{E}_{p^*}\left[T_{\gamma_{th}}^2\right]\right)}{\log\left(\mathbb{P}\left(S_N > \gamma_{th}\right)\right)} = 2. \qquad (11)$$

- Enlarges the framework of hazard rate twisting techniques for the general case of sums involving arbitrary independent RVs.
- Feature rarely observed in the field of variance reduction techniques: particular classes of distributions are generally assumed.

Algorithm

Algorithm 1: *Hazard rate twisting approach for the estimation of α*

Inputs: M, γ_{th}.
Outputs: $\hat{\alpha}_{IS}$.
Find the minmax value θ^* as in (10) by solving the maximization problem (P).
for $i = 1, ..., M$ **do**
  Generate independent realizations of $\{X_j(\omega_i)\}_{j=1}^N$ under the twisted PDF $\{f_{j,\theta^*}(\cdot)\}_{j=1}^N$
  Evaluate $T_{\gamma_{th}}(\omega_i)$.
end for
Compute the IS estimator $\hat{\alpha}_{IS}$.

Generating Realizations under the twisted PDF

- Sampling algorithms: Acceptance- Rejection technique, Inverse CDF method, Markov Chain Monte Carlo algorithm.
- Closed-form expression of the inverse CDF:

$$F_\theta^{-1}(y) = F^{-1}(1 - (1-y)^{-\frac{1}{\theta-1}}), \qquad (12)$$

- Applicable to many distributions: Log-normal, Weibull, Gamma.

Simulation Results

Efficiency of the Proposed IS Algorithm

Comparison is performed with a Conditional Monte Carlo based algorithm, J. Chan et al (2011).

$$\hat{\alpha}_{CMC} = \frac{1}{M} \sum_{k=1}^{M} T'_{\gamma_{th}}(\omega_k) \qquad (13)$$

where $T'_{\gamma_{th}} = \sum_{i=1}^{N} \bar{F}_i \left(\max(\gamma_{th} - \sum_{j\neq i} X_j, M_{-i})\right)$, $\bar{F}_i = 1 - F_i$, and $M_{-i} = \max(X_1, ..., X_{i-1}, X_{i+1}, ..., X_N)$, $i = 1, 2, ..., N$.

Table 1: *Sum of $N = 10$ independent Weibull Distribution with $\beta_i = 0.5 + i/10$, $k_i = 0.8$, $i = 1, 2, ..., 5$, $k_i = 0.9$, $i = 6, 7, ..., 10$, and $M = 10^7$. The Variance Reduction (VR) is computed with respect to the naive MC simulation.*

	Importance Sampling		Conditional Monte Carlo	
γ_{th}	$\hat{\alpha}_{IS}$	VR	$\hat{\alpha}_{CMC}$	VR
35	1.34×10^{-4}	200.30	1.34×10^{-4}	113.56
40	1.74×10^{-5}	1.05×10^{3}	1.74×10^{-5}	282.01
45	2.18×10^{-6}	5.42×10^{3}	2.18×10^{-6}	694.27
50	2.76×10^{-7}	2.44×10^{4}	2.76×10^{-7}	1.49×10^{3}
55	3.44×10^{-8}	1.08×10^{5}	3.40×10^{-8}	4.03×10^{3}

Table 2: *Sum of $N = 10$ independent Weibull Distribution with $\beta_i = 0.5 + i/10$, $k_i = 0.8$, $i = 1, 2$, $k_i = 1$, $i = 3, 7, ..., 10$, and $M = 10^7$. The Variance Reduction (VR) is computed with respect to the naive MC simulation.*

	Importance Sampling		Conditional Monte Carlo	
γ_{th}	$\hat{\alpha}_{IS}$	VR	$\hat{\alpha}_{CMC}$	VR
30	8.26×10^{-5}	565.75	8.22×10^{-5}	99.52
35	4.88×10^{-6}	5.67×10^{3}	4.91×10^{-6}	292.22
40	2.64×10^{-7}	6.01×10^{4}	2.65×10^{-7}	956.52
45	1.36×10^{-8}	6.21×10^{5}	1.40×10^{-8}	1.99×10^{3}

Table 3: *Sum of $N = 10$ independent Weibull Distribution with $\beta_i = 0.5 + i/10$, $k_i = 2$, $i = 1, 2, ..., 10$, and $M = 10^7$. The Variance Reduction (VR) is computed with respect to the naive MC simulation.*

	Importance Sampling		Conditional Monte Carlo	
γ_{th}	$\hat{\alpha}_{IS}$	VR	$\hat{\alpha}_{CMC}$	VR
15	5.65×10^{-4}	92.47	5.64×10^{-4}	17.12
16	8.03×10^{-5}	429.41	8.05×10^{-5}	28.68
17	9.17×10^{-6}	2.47×10^{3}	9.18×10^{-6}	53.01
18	8.55×10^{-7}	1.76×10^{4}	8.74×10^{-7}	78.44
19	6.42×10^{-8}	1.55×10^{5}	6.94×10^{-8}	158.52

Sensitivity Analysis of the Minmax Approach

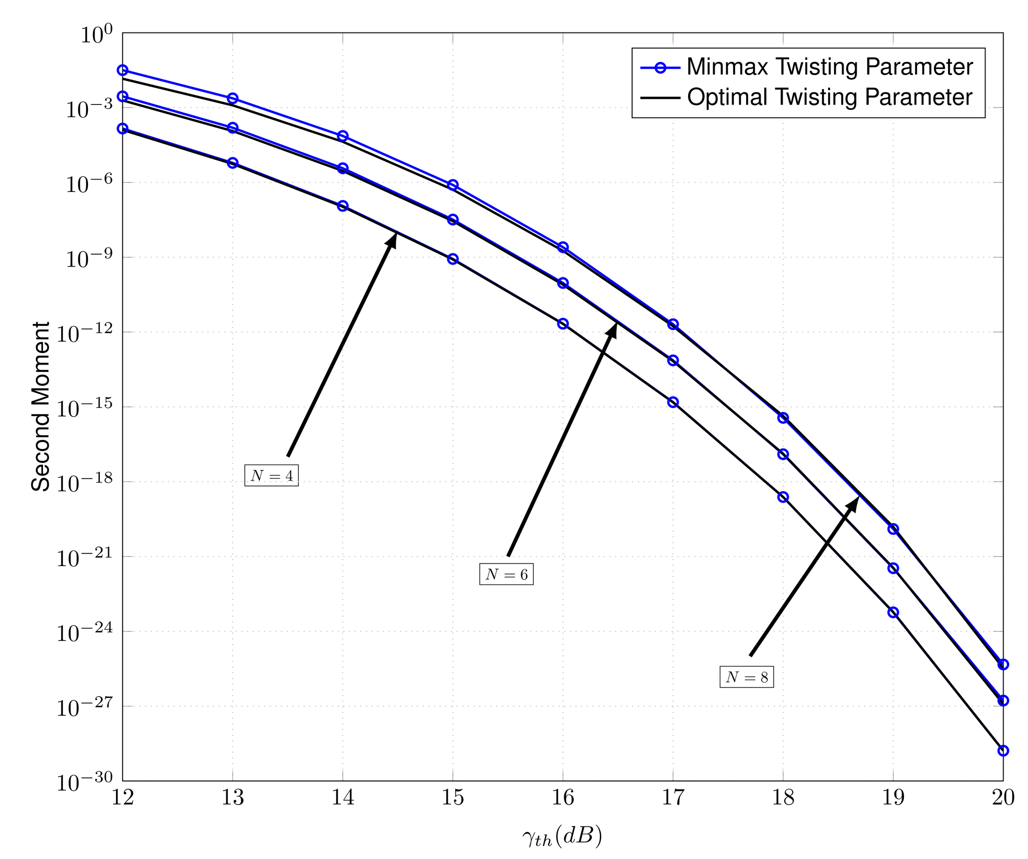

Figure 1: *Second moment of $T_{\gamma_{th}}$ with the minmax and the optimal twisting parameter for the sum of N Weibull RVs with shape parameters $k_i = 0.8$, scale parameters $\beta_i = 1$, $i = 1, 2, ..., N$, and $M = 10^7$.*

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References

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[2] N. Ben Rached, F. Benkhelifa, A. Kammoun, M.-S. Alouini, and R. Tempone, "Additional Results on the Hazard Rate Twisting-Based Simulation Approach", submitted to IEEE Transaction on Information Theory.