Mean-field Ensemble Kalman Filter

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Abstract

A proof of convergence of the standard EnKF generalized to non-Gaussian state space models is provided. A density-based deterministic approximation of the mean-field limiting EnKF (MFEnKF) is proposed, consisting of a PDE solver and a quadrature rule. Given a certain minimal order of convergence $\kappa$ between the two, this extends to the deterministic filter approximation, which is therefore asymptotically superior to standard EnKF for $\kappa > 2\kappa$. The fidelity of approximation of the true distribution is also established using an extension of total variance metric to random measures. This is limited by a Gaussian bias term arising from non-linearity/non-Gaussianity of the model, which arises in both deterministic and standard EnKF. Numerical results support and extend the theory.

1. Setting

Let $\mathbb{K} : \mathbb{R}^d \rightarrow \mathbb{P}(\mathbb{R}^d)$. Consider the Markov chain

$$y_j \sim \mathbb{K}(y_{j-1}), \quad j \in \mathbb{N},$$

and a sequence of noise $\{\eta_j\}_{j \in \mathbb{N}}$, where $\eta_j \sim \mathbb{P}(\mathbb{R}^d)$. Define $\eta = \{\eta_j\}_{j \in \mathbb{N}}$. The Gaussian error $\eta \sim \mathbb{P}(\mathbb{R}^d)$, with $\eta \sim \mathbb{N}(0, \Sigma)$. Define $\mathcal{L}$ as follows. For $u \sim \mathcal{L}$, observe $\mathcal{L} = \mathbb{F}(\mathbb{P}(\mathbb{R}^d), \eta \sim \mathbb{P}(\mathbb{R}^d)$, with $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d)$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$, and $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d)$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$. Define $\mathcal{L}$ as follows. For $u \sim \mathcal{L}$, observe $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d)$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$, and $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d)$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$. Define $\mathcal{L}$ as follows. For $u \sim \mathcal{L}$, observe $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d)$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$, and $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d)$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$.

1.1 Filtering distribution

Distribution of $y_j$ is gold standard. Likelihood function is $g(y_j) = \mathcal{L}(y_j | y_{j-1})$, $g(y_j) = \mathcal{L}(y_j | y_{j-1})$.

Optimal filter by

$$E(y_j | y_0, y_1, \ldots, y_{j-1}) = \sum_{j=0}^{\infty} \mathcal{L}(y_j | y_{j-1}),$$

$g(y_0) = \mathcal{L}(y_0 | y_{-1})$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$, and $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d)$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$.

1.2 Optimal Linear Filtering

$$m_j(y_j) = \text{argmin}_{m_j(y_j)} \mathcal{L}(y_j | y_{j-1}),$$

Optimizing with respect to $\mathcal{L}$, and $\mathcal{L}$ gives

$$m_j(y_j) = \text{argmin}_{m_j(y_j)} \mathcal{L}(y_j | y_{j-1}),$$

$g(y_0) = \mathcal{L}(y_0 | y_{-1})$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$, and $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d)$, $\eta \sim \mathbb{P}(\mathbb{R}^d)$.

2. EnKF

2.1 Finite mean EnKF

$$F = \mathbb{F}(y_j, y_{j-1}),$$

$$C_j = \mathbb{E}(y_j | y_{j-1}) \otimes \mathbb{E}(y_{j-1} | y_{j-2})$$

Analysis

$$K_j = \mathbb{E}(y_j - y_{j-1} | y_{j-1}) \otimes \mathbb{E}(y_{j-1} | y_{j-2}),$$

Prediction

$$\hat{y}_{j+1} = y_{j+1} + K_j \eta_{j+1},$$

Here $\eta_j$ are i.i.d. draws from $\mathcal{N}(0, \Sigma)$.

2.2 Finite ensemble EnKF

$$\hat{y}_{j+1} \sim \mathbb{F}(y_{j+1}, y_{j-1}),$$

$$\hat{y}_{j+1} = y_{j+1} + \sum_{i=1}^{N} \eta_{j+1} \otimes (\eta_{j+1} - \eta_{j+1}),$$

$$\hat{y}_{j+1} = y_{j+1} + \sum_{i=1}^{N} \eta_{j+1} \otimes (\eta_{j+1} - \eta_{j+1}),$$

Prediction

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Analysis

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$$\hat{y}_{j+1} = y_{j+1} + \sum_{i=1}^{N} \eta_{j+1} \otimes (\eta_{j+1} - \eta_{j+1}),$$

3. Fokker-Planck Filters

3.1 FPF Algorithms

$$\partial_t \rho = \mathcal{L}(\mathbb{F}(x(t)))$$

Density governed by Fokker-Planck equation

$$\partial_t \rho = \mathcal{L}(\mathbb{F}(x(t)))$$

3.2 EnKF Converges to MFEnKF

Let $\pi \in \mathbb{P}(\mathbb{R}^d)$ and $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d), \eta \sim \mathbb{P}(\mathbb{R}^d)$. Define

$$\rho \sim \mathcal{L}(\mathbb{P}(\mathbb{R}^d), \eta \sim \mathbb{P}(\mathbb{R}^d)$$

Now define

$$d(\pi, \rho) = ||\pi - \rho||.$$ (4)

• $\pi_j$ filtering density.
• $\pi_j$ the mean-field EnKF density.
• $\pi_j \Rightarrow \pi_j$ the deterministic approximation (EnKF).
• $\pi_j \Rightarrow \pi_j$ the standard Monte-Carlo EnKF.

THEOREM 1: It is true that $\pi_j \Rightarrow \pi_j \Rightarrow \pi_j \Rightarrow \pi_j$. Proof: See [1], Sec. 4.3.1, for extension of [2].

4. MFEnKF is not the Posterior

Assume (i) $\pi \in \mathbb{P}(\mathbb{R}^d)$ and $\mathcal{L} = \mathcal{L}(\mathbb{P}(\mathbb{R}^d), \eta \sim \mathbb{P}(\mathbb{R}^d)$. And (ii) for time $k$, $\pi_{k|k-1} = \pi_{k|k-1} = \pi_{k|k-1}$ and $\pi_{k|k-1} = \pi_{k|k-1} = \pi_{k|k-1}$.

THEOREM 2: Given observation increment $\pi$, under assumptions above, as $\lambda \rightarrow 0$ and $N \rightarrow \infty$,

$$d(\pi_j, \rho) \approx \mathcal{O}(N^{-1/2} + \lambda^2)$$

where $\lambda = 0$ if $\pi$ is linear and $\lambda = 1$ if $\pi$ is nonlinear.

Figure 1: $F(u) = \text{ax}(1-u^2)^2$ with $u = \text{ax}(1-u^2)^2$,

Full FPF

• Discretize the density at $t$ over space.
• Evolve with accurate time-stepper to time $t + 1$.
• Update $\pi_j$ by $\pi_j = \mathcal{L}(\pi_j)$.

Figure 2: $F(u) = \text{ax}(1-u^2)^2$, Error of FPF MFEnKF-G2 vs. EnKF.

Figure 3: Nonlinear case, RMSE of mean with respect to the true (unconditioned) signal.

Figure 4: Nonlinear case, RMSE of covariance with respect to the true posterior covariance.

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References