Problem Setting

We consider PDEs with a Gaussian random field coefficient $a: D \times \Omega \to \mathbb{R}$, $D = [0,1]^2$, with the objective of evaluating a quantity of interest $Q = Q(u|\omega)$. Given an orthonormal basis $(v_m)_{m \in \mathbb{N}}$ of $L^2(D)$, we expand $u$ in a series with random coefficients

$$a(x,\omega) = \sum_{m \in \mathbb{N}} \xi_m(\omega) v_m(x), \quad \xi_m(\omega) = \int_D a(x,\omega) v_m(x) \, dx. \quad (2)$$

For computations one typically truncates the series after a number $M$ terms chosen sufficiently large for an acceptable truncation error. A large value of $M$, however, constrains the feasibility of numerical methods for approximating $a = u(\xi)$ or $Q(u|\omega)$ based on stochastic Galerkin or stochastic collocation discretization. Thus, our goal is a sparse expansion (2) of $u$ which still allows a good approximation of $Q$.

Objective Setting and Basic Approach

Given a basis $(v_m)_{m \in \mathbb{N}}$ for expanding $u$ as in (2) we seek an optimal – i.e., minimal – selection $(v_m)_{m \in \mathbb{N}^*}$ of basis functions to approximate $Q$ with given tolerance $t$.

$$I^*_t = \arg\min_{I \subseteq \mathbb{N}} \{ I \} \in \mathbb{N}, \quad E \{ ||Q| - Q(I^*_t)\|_{L^2}\} < t,$$

where $\{ I \} \in \mathbb{N}$ denotes the restriction of $I \in \mathbb{R}^\mathbb{N}$ to $I \in \mathbb{N}$.

We do so by sequentially enriching an initial set $I_0$ by promising $(v_m)$, where the gain of including the $m$th direction in a given selection $I$ is evaluated by

$$\text{gain}(m,I) = E\{ \| Q(\xi_m|\omega) - Q(I^*_t) \|_{L^2} \} \approx E\{ \|\xi_m\|_{L^2} \| E\{ Q(\xi_m|\omega) \} \| E\{ Q(I^*_t) \} \| \}.$$

Here we denote by $\text{gain}(m,I)$ the sensitivity of $Q$ w.r.t. $\xi_m$ all $\xi_m \neq 0$.

Sequential Algorithm

Algorithm: Coefficient selection

1. $k = 0$, $I_0 = \{1\}$
2. repeat
3. $k = k + 1$
4. $I_k = I_{k-1}$
5. Determine new set of candidates $I_k^*$
6. Compute covariance matrix $C_{I_k}$ of random coefficients $\xi_{I_k}$
7. for $j = 1, N$ do
8. draw sample $\xi_j \in \mathbb{N}$ from $N(0, C_{I_k})$
9. compute $v(\xi_j)$ and $\xi_j(\omega)$
10. end
11. for $m \in I_k^*$ do
12. $\text{gain}(m,I_k) = E\{ \xi_m \| E\{ Q(\xi_m|\omega) \} \| E\{ Q(I^*_t) \} \|$ dz
13. end
14. Enrich active index set according to indicators $\text{gain}(m, I_k)$
15. until $\sum_{m \in I_k^*} \text{gain}(m, I_k) < \epsilon$
16. Compute $I_k$ leading eigenpairs $(\lambda_k, v_k)$ of $C_{I_k}$
17. Compressed uncorrelated coefficient vector $\eta_k = \lambda_k^{-1/2} v_k(\xi_m) \in \mathbb{R}^M$

Numerical Results

We consider problem (1) for a Gaussian random field $u$ with mean $\mu = 0$ and covariance $\sigma(x,y) = \exp(-|x-y|^2/0.17)$, with quantities of interest $Q(u)/Q = \int_D u(x) \, dx$, $i = 1,2$, where $D_i = [0,2]/[0,1]^i$ and $D_2 = [0,16]/[0,1]^2$. In view of the discretization we restrict ourselves to a finest wavelet level $V_{10}$. To illustrate the wavelets selected by the algorithm, we display in Fig. 3 a realization of $u$ along with its projection onto the selected wavelets for a tolerance $\varepsilon = 2^{-1}$. Fig. 4 confirms that our gain indicators and stopping criterion perform well, if somewhat conservative since the true error of the final selection is smaller than required by a factor of 2$^{-1}$, leaving some room for improvement. Fig. 5 shows the $L_1$-errors in $Q_1$ and $Q_2$, which representing $u$ by truncated KLEs of different lengths and by the expanding stopping criterion for a tolerance $\varepsilon = 0.002$. Note that the algorithm again stops too late, i.e., the final $I_{10}$ is too large. We observe further that wavelet expansions could only outperform KLEs for $\varepsilon$ above a certain lower bound and this bound increases if the quantity becomes less localized – compare the results for $Q_1$ and $Q_2$ in Fig. 5.