Multivariate polynomial approximation by means of random discrete least squares

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Discrete least squares with random evaluations

In any dimension $d \in \mathbb{N}$, let $\Gamma_i := [-1,1] \subset \mathbb{R}$ be an interval and $\rho_i : \Gamma_i \to \mathbb{R}_0^+$ a univariate probability density for any $i = 1, \dots, d$. Denote by $\Gamma := \times_{i=1}^d \Gamma_i = [-1,1]^d$ the d-dimensional parameter space, and by Y a d-dimensional random variable distributed according to the multivariate probability density $\rho := \prod_{i=1}^d \rho_i : \Gamma \to \mathbb{R}_0^+$. Given a smooth real-valued (or Hilbert-valued) function $\phi = \phi(Y)$, we aim at approximating the function ϕ in the L^2-probability sense, using its pointwise noiseless evaluations in independent realizations of the random variable Y.

Definition of the polynomial space

A finite multi-index set $\Lambda \subset \mathbb{N}_0^d$ is downward closed if $(\nu \in \Lambda \wedge \mu \leq_L \nu)$ implies $\mu \in \Lambda$, where $\leq_L$ denotes the lexicographical ordering. For $i = 1, \dots, d$, we denote by $\{\varphi_j^i\}_{j \geq 0}$ the family of one-dimensional orthonormal polynomials w.r.t. the $L^2_{\rho_i}$ scalar product $\langle f_1, f_2 \rangle_{L^2_{\rho_i}} := \sqrt{\int_{\Gamma_i} f_1(y_i) f_2(y_i) \rho_i(y_i) dy_i}$. Given a downward closed multi-index set Λ, we define the polynomial space $\mathbb{P}_\Lambda(\Gamma)$ associated with the multi-index set Λ as $\mathbb{P}_\Lambda(\Gamma) := \{\psi_\nu : \nu \in \Lambda\}$, with $\psi_\nu := \prod_{i=1}^d \varphi_{\nu_i}^i(y_i)$ being the multivariate polynomial basis function associated with the multi-index $\nu \in \Lambda$.

The discrete L^2 projection on polynomial spaces

Denote by $\phi(y_1), \dots, \phi(y_M)$ the noiseless pointwise evaluations of the function ϕ in M independent realizations of the random variables $Y_1, \dots, Y_M \overset{\text{iid}}{\sim} \rho$. The discrete L^2 projection of ϕ onto the polynomial space $\mathbb{P}_\Lambda(\Gamma)$ is defined as

$$\varPi_\Lambda^M \phi := \underset{v \in \mathbb{P}_\Lambda(\Gamma)}{\text{argmin}} \frac{1}{M} \sum_{j=1}^M \left(\phi(\mathbf{y}_j) - v(\mathbf{y}_j)\right)^2 .$$

Given any finite dimensional polynomial space $\mathbb{P}_\Lambda$ associated with a downward closed multi-index set Λ, with basis $\{\psi_1, \dots, \psi_{\#\Lambda}\}$, of course it holds $\dim(\mathbb{P}_\Lambda) = \#\Lambda =: N$. We define the following quantity ([2,3]):

$$K(\Lambda) := \sup_{y \in \Gamma} \sum_{j=1}^N |\psi_j(y)|^2 .$$

Lemma 1 (from [3]) *For any downward closed set Λ, it holds that*

$$K(\Lambda) \leq N^2, \qquad \textit{with the tensorized Legendre polynomials},$$
$$K(\Lambda) \leq N^{\frac{\ln 3}{\ln 2}}, \qquad \textit{with the tensorized Chebyshev polynomials}.$$

Accuracy of discrete least squares

Given a truncation operator $T_\tau(t) := \text{sign}(t) \min\{\tau, |t|\}$ with threshold $\tau > 0$, denote by $\widetilde{\varPi}_\Lambda^M := T_\tau \circ \varPi_\Lambda^M$ the truncated discrete least-squares projection, and by $\varPi_\Lambda$ the exact L^2 projection over $\mathbb{P}_\Lambda$.

Theorem 1 (from [3]) *For any $\gamma > 0$ and $\beta \approx 0.15$, if M and $K(\Lambda)$ satisfy*

$$K(\Lambda) \leq \frac{\beta}{1+\gamma} \frac{M}{\log M}, \tag{1}$$

then for any $\phi \in L^\infty(\Gamma)$ with $\|\phi\|_{L^\infty} \leq \tau$, it holds that

$$\mathbb{E}\left(\|\phi - \widetilde{\varPi}_\Lambda^M \phi\|^2_{L^2_\rho}\right) \leq \left(1 + \frac{4\beta}{(1+\gamma)\log M}\right) \|\phi - \varPi_\Lambda \phi\|^2_{L^2_\rho} + 8\tau^2 M^{-\gamma},$$

$$Pr\left(\|\phi - \varPi_\Lambda^M \phi\|^2_{L^2_\rho} \leq (1 + \sqrt{2}) \inf_{v \in \mathbb{P}_\Lambda} \|\phi - v\|_{L^\infty}\right) \geq 1 - 2M^{-\gamma}.$$

Remark 1 *Lemma 1 states that, to fulfill condition* (1) (i.e. *to achieve an accurate least-squares approximation) it suffices to choose the number of evaluations M such that $M = N^2$ in the case of the uniform density or $M = N^{\frac{\ln 3}{\ln 2}}$ in the case of the Chebyshev density.*

Convergence estimate for elliptic PDEs

Theorem 2 (from [3]) *In a PDE isotropic model class (which includes the elliptic model and the linear elasticity model with "inclusion type" stochastic coefficients), with an a priori optimal choice $\widetilde{\Lambda}$ of the polynomial space of Total Degree type, the error of the random discrete least-squares approximation over the polynomial space $\mathbb{P}_{\widetilde{\Lambda}}$* ***converges exponentially fast*** *w.r.t. the number of evaluations M, provided M is proportiona to $N^{2+d^{-1}}$.*

The convergence rate is precisely quantified in terms of M, N, d in [3,Theorem 2]. The figures below show how the numerical results compare with the theoretical convergence rate displayed in black dashed line.

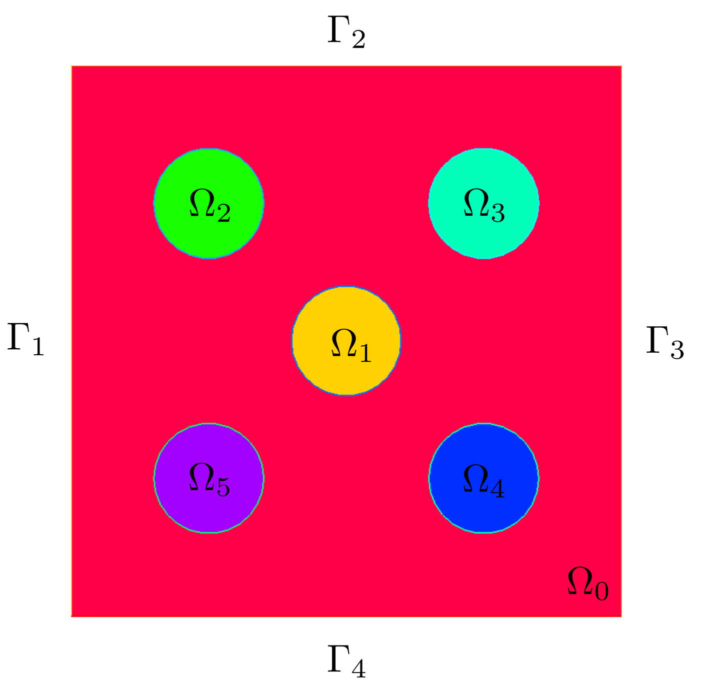
Darcy flow in a square domain with d inclusions

$$\begin{cases} -\nabla \cdot (a(x,y) \nabla u(x,y)) = 0, & x \in \Omega, y \in \Gamma, \\ u(x,y) = g_1(x), & x \text{ on } \Gamma_1 \cup \Gamma_3, y \in \Gamma, \\ \nabla u(x,y) \cdot \boldsymbol{n} = g_2(x), & x \text{ on } \Gamma_2 \cup \Gamma_4, y \in \Gamma. \end{cases}$$

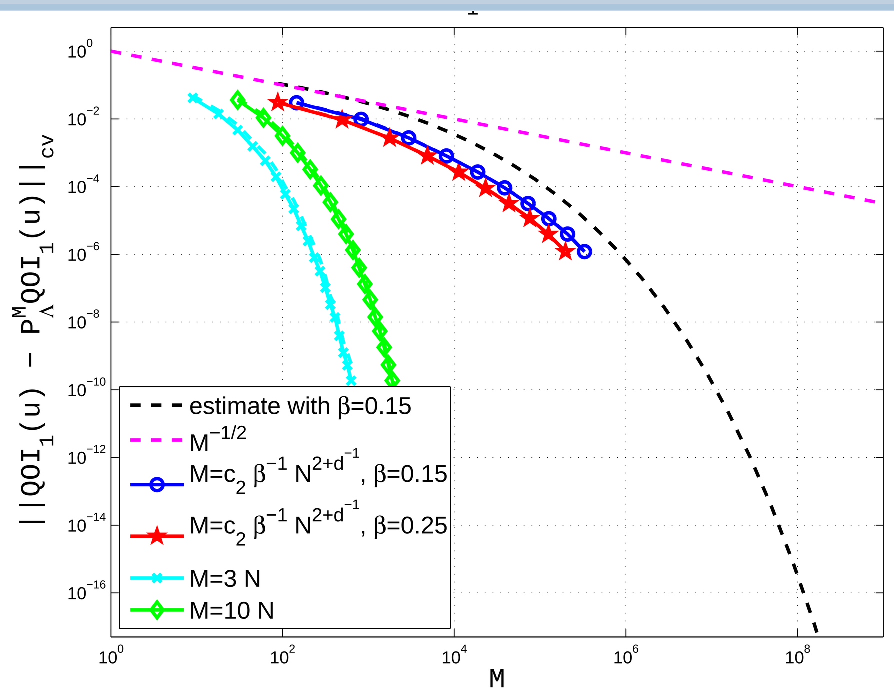
$$a(x,y) = \begin{cases} 0.395(y_i + 1) + 0.01, & x \in \Omega_i, \quad i = 1, \dots, d, \\ 1, & x \in \Omega_0 = \Omega \setminus \left(\bigcup_{i=1}^d \Omega_i\right), \end{cases}$$

with y being a realization of the random variable $Y \sim \rho = \mathcal{U}([-1,1]^d)$, $g_1|_{\Gamma_1} \equiv 10, \quad g_1|_{\Gamma_3} \equiv 0, \quad g_2 \equiv 0, \quad \text{QOI}_1(y) = |\Omega|^{-1} \int_\Omega u(x,y)\, dx$.

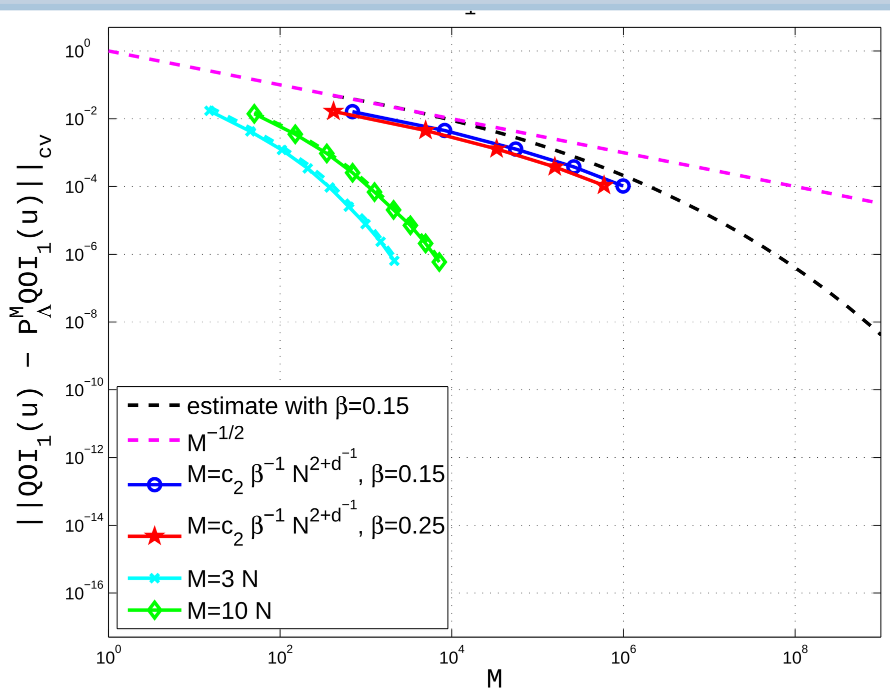
Example of geometry with d=5



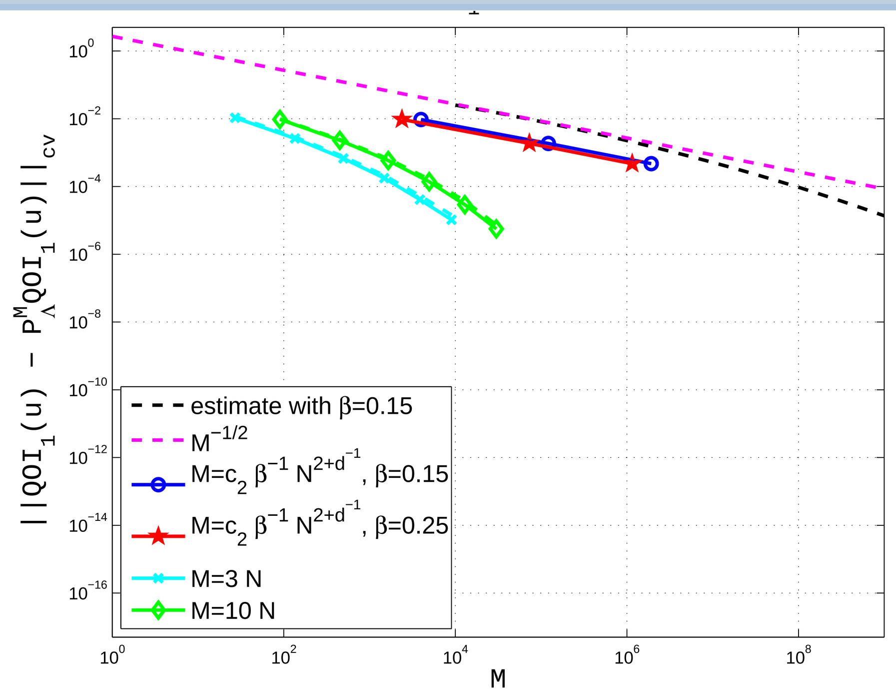
d=2



d=4



d=8



Remarks on the numerical results

- The figures show the crossvalidated error $\|\text{QOI}_1(u) - \varPi_{\widetilde{\Lambda}}^M \text{QOI}_1(u)\|_{cv}$ with the cross-validation procedure described in [1,Section 4], in the cases $d = 2$, $d = 4$ and $d = 8$.
- The theoretical convergence rate (black dashed line) sharply predicts the numerical convergence rate achieved when $M \propto N^{2+d^{-1}}$ as prescribed by the theory. In particular, the dependence on the dimension d is sharply predicted.
- As observed in [1,3,4,5], a faster convergence rate of the error w.r.t. M can be achieved by choosing $M \propto N$, but without a theoretical proof for the moment.
- A relation with the convergence rate of the Stochastic Galerkin method has been established, see [3, Section 5].
- Natural applications to moderately high-dimensional interpolation/integration.

References and email

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