Quasi-optimal sparse-grid approximations for elliptic PDEs with stochastic coefficients

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Elliptic equations with stochastic coefficients

Let $\mathbf{y} \in \Gamma \subset \mathbb{R}^N$ be a N-dimensional random vector with i.i.d. components and pdf $\varrho(\mathbf{y})d\mathbf{y}$, $D \subset \mathbb{R}^d$ and $V = H_0^1(D)$. We look for a real-valued function $u(\mathbf{x},\mathbf{y}) : \overline{D} \times \Gamma \to \mathbb{R}$, $u \in V \otimes L^2_\varrho(\Gamma)$, such that $\varrho(\mathbf{y})d\mathbf{y}$-a.e.:

$$\begin{cases} -\nabla \cdot (a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})) = f(\mathbf{x}) & \mathbf{x} \in D, \\ +B.C. & \mathbf{x} \in \partial D. \end{cases} \quad (1)$$

A sparse grids approach

We aim at building an efficient sparse grid approx. of $u(\mathbf{x},\cdot) : \Gamma \to V$ exploiting the fact that not all the y_n equally affect the solution of (1). Let

- $\mathcal{U}_n^{m(i_n)}[u]$ be an **interpolant operator** along y_n over $m(i_n)$ points;
- $\Delta^{m(\mathbf{i})}[u] = \bigotimes_{n=1}^N \left(\mathcal{U}_n^{m(i_n)}[u] - \mathcal{U}_n^{m(i_n-1)}[u]\right)$ be a **hierarchical surplus**;
- $\mathcal{S}_\mathcal{I}^m[u](\mathbf{y}) = \sum_{\mathbf{i}\in\mathcal{I}\subset\mathbb{N}_+^N} \Delta^{m(\mathbf{i})}[u](\mathbf{y})$ be the **sparse grid** approx. of u;
- $W_{\mathcal{I},m}$ be the total number of points of the sparse grid.

Key idea: select carefully the set $\mathcal{I}$, with a mixed *a-priori/a-posteriori* approach, i.e. provide an a-priori ansatz for $\mathcal{I}$, depending on some constants g_n to be tuned with some preliminary cheap computation.

A knapsack formulation

Error contribution $\Delta E(\mathbf{i})$ such that $\left\|\Delta^{m(\mathbf{i})}[u]\right\|_{V\otimes L^2_\varrho(\Gamma)} \leq \Delta E(\mathbf{i})$

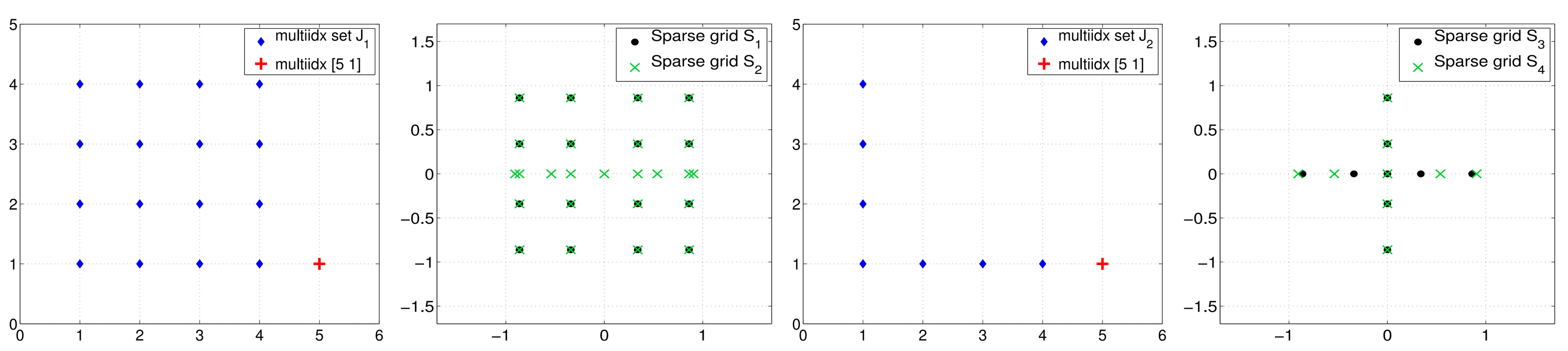
Work contribution

$$\Delta W(\mathbf{i}) = \begin{cases} \text{nested pts.: } \prod_{n=1}^N \left(m(i_n) - m(i_n-1)\right) & \Rightarrow W_{\mathcal{I}(w),m} = \sum \Delta W \\ \text{non-nested pts*.: } \prod_{n=1}^N m(i_n), & \Rightarrow W_{\mathcal{I}(w),m} \leq \sum \Delta W \end{cases}$$

Profit of a hierarchical surplus $P(\mathbf{i}) = \max_{\mathbf{j}\geq\mathbf{i}} \dfrac{\Delta E(\mathbf{j})}{\Delta W(\mathbf{j})}$,

Sequence of ordered profits $P_j^{ord} \geq P_{j+1}^{ord}$, and $\mathbf{i}(j)$ s.t. $P_j^{ord} = P(\mathbf{i}(j))$

We then consider the **set** $\mathcal{I}(w) = \{\mathbf{i}(1), \mathbf{i}(2), \ldots, \mathbf{i}(w)\}$ (which is necessarily downward closed i.e. if $\mathbf{j} \leq \mathbf{i}$ and $\mathbf{i} \in \mathcal{I}$, then $\mathbf{j} \in \mathcal{I}$)



*Using non-nested points the number of point added to $\mathcal{S}_\mathcal{I}^m[u]$ by $\Delta^{m(\mathbf{i})}$ depends on $\mathcal{I}$, in general.

Convergence theorem.
If the profits $P(\mathbf{i})$ satisfy the weighted summability condition

$$\left(\sum_{\mathbf{i}\in\mathbb{N}_+^N} P(\mathbf{i})^\tau \Delta W(\mathbf{i})\right)^{1/\tau} = C_P(\tau) < \infty$$

for some $0 < \tau < 1$, then

$$\left\|u - \mathcal{S}_{\mathcal{I}(w)}^m[u]\right\|_{V\otimes L^2_\varrho(\Gamma)} \leq W_{\mathcal{I}(w),m}^{1-1/\tau} C_P(\tau), \quad (2)$$

Note that given a linear functional $\Theta : V \to \mathbb{R}$, the associated error $\left\|\Theta(u) - \mathcal{S}_{\mathcal{I}(w)}^m[\Theta(u)]\right\|_{L^2_\varrho(\Gamma)}$ converges with the same rate.

Building error estimates $\Delta E(\mathbf{i})$

Given a ϱ-orthogonal expansion of u, $u(\mathbf{x},\mathbf{y}) = \sum u_\mathbf{q}(\mathbf{x})\Psi_\mathbf{q}(\mathbf{y})$, take

$$\left\|\Delta^{m(\mathbf{i})}[u]\right\| \leq \sum_{\mathbf{q}\geq m(\mathbf{i}-\mathbf{1})} \|u_\mathbf{q}\| \left\|\Delta^{m(\mathbf{i})}[\Psi_\mathbf{q}]\right\|_2 \leq \sum_{\mathbf{q}\geq m(\mathbf{i}-\mathbf{1})} 2\,\|u_\mathbf{q}\|\, \mathbb{M}^{m(\mathbf{i})} \|\Psi_\mathbf{q}\|_\infty$$

where $\mathbb{M}^{m(\mathbf{i})}$ is the $L^\infty(\Gamma)$-to-$L^2_\varrho(\Gamma)$ Lebesgue constant, and we need to provide estimates for $\|u_\mathbf{q}\|_V$ and $\mathbb{M}^{m(\mathbf{i})}$.

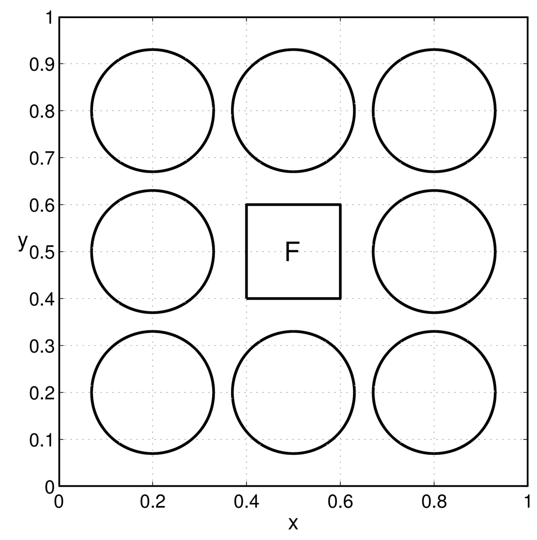
References

I) F.N., L.T., R.T., "*Convergence of quasi-optimal sparse grids approximation of Hilbert-valued functions: application to elliptic PDEs with random coefficients*", in preparation.

II) J. Beck, F.N., L.T., R.T.. "On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods", M^{3}AS, 22(09),2012.

III) J. Beck, F.N., L.T., R.T., "*A quasi-optimal sparse grids procedure for groundwater flows*". ICOSAHOM 2012 conference papers, also available as MATHICSE report 46/2012.

Example I: inclusion problem



Consider problem (1) on the unit square $D = (0,1)^2$, with hom. Dir. B.C. and $a(\mathbf{x},\mathbf{y}) = a_0 + \sum_{n=1}^N \gamma_n \chi_n(\mathbf{x}) y_n$, $y_n \sim \mathcal{U}(-0.99, 0.99)$, $\gamma_n \in \mathbb{R}$.

Orthog. exp.: Chebyshev, $\|u_\mathbf{q}\| \leq C \prod_n e^{-g_n q_n}$ (r.v. act on disjoint subdomains)

Coll. points (nested): Clenshaw–Curtis, $\mathbb{M}^{m(\mathbf{i})} \leq \prod_n \left(\frac{2}{\pi}\log(m(i_n) - 1) + 1\right)$

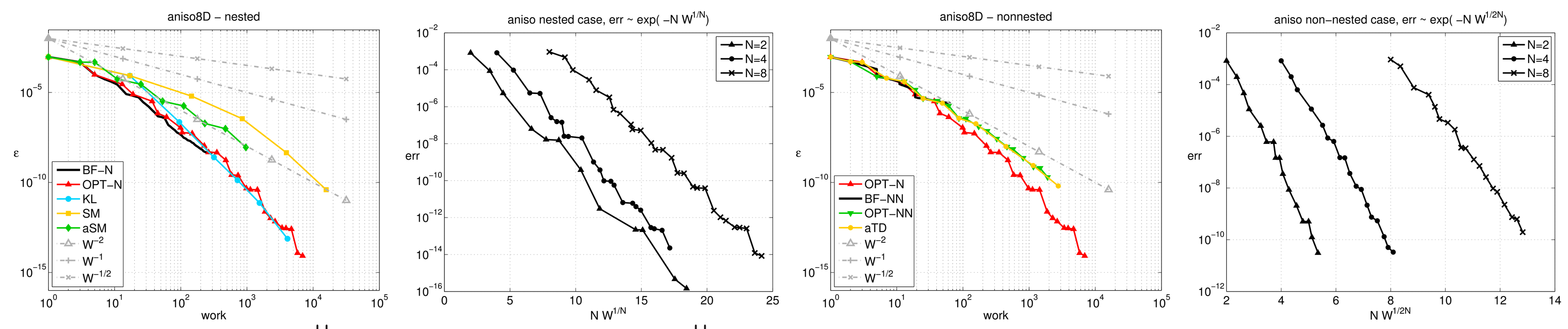
Coll. points (non nested): Gauss–Legendre, $\mathbb{M}_n^{m(\mathbf{i})} = 1$.

Convergence results.

nested pts. $\left\|u - \mathcal{S}_{\mathcal{I}(w)}^m[u]\right\|_{V\otimes L^2_\varrho(\Gamma)} \leq \alpha \exp\left(-\beta N \sqrt[N]{W_{\mathcal{I}(w),m}}\right)$,

non nested pts. $\left\|u - \mathcal{S}_{\mathcal{I}(w)}^m[u]\right\|_{V\otimes L^2_\varrho(\Gamma)} \leq \alpha \exp\left(-\beta N \sqrt[2N]{W_{\mathcal{I}(w),m}}\right)$

Idea of proof. The profits are τ-summable for $0 < \tau < 1$. Use (2) and minimize over the range of feasible τ.



Convergences for $\left\|\mathcal{S}_{\mathcal{I}(w)}^m[\Theta(u)] - \Theta(u)\right\|_{L^2_\varrho(\Gamma)}$, with $\Theta(u) = \int_F u(\mathbf{x})d\mathbf{x}$, $N = 4, 8$.

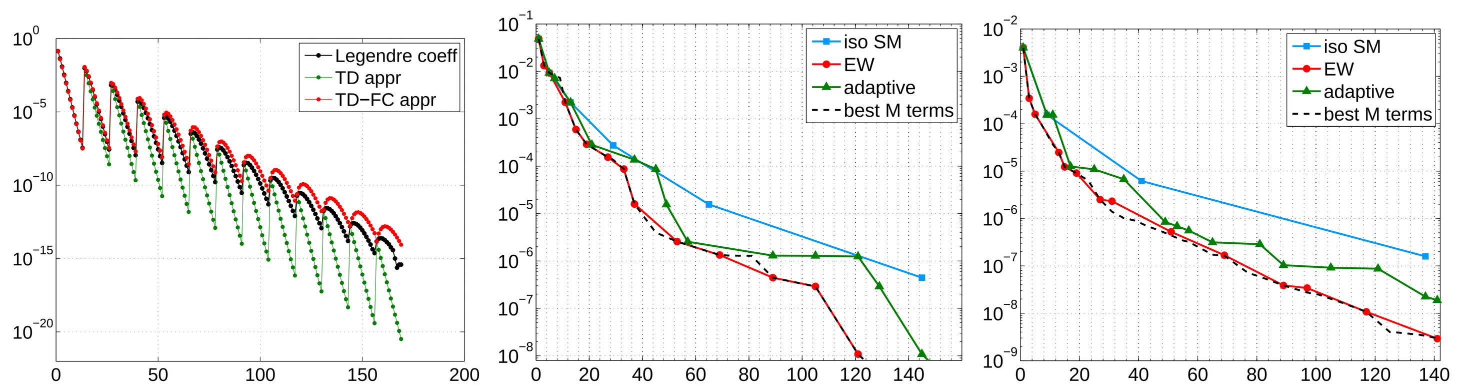
Example II: uniform variables

Consider problem (1) on the interval $D = (0,1)$, with hom. Dir. B.C. , let $y_n \sim \mathcal{U}(-1,1)$ and consider the following forms of a

A) $a = 1 + 0.1y_1 + 0.5y_2$,
B) $a(x,\mathbf{y}) = 4 + y_1 + 0.2\sin(\pi x)y_2 + 0.04\sin(2\pi x)y_3 + 0.008\sin(3\pi x)y_4$,

Orthog. exp.: Legendre, with $\|u_\mathbf{q}\|_V \leq Ce^{-\sum_n g_n q_n}\frac{|\mathbf{q}|!}{\mathbf{q}!}$ (r.v. act on the whole domain)

Coll. points (nested): Clenshaw–Curtis, $\mathbb{M}^{m(\mathbf{i})} \leq \prod_n \left(\frac{2}{\pi}\log(m(i_n)-1)+1\right)$



$\|u_\mathbf{q}\|$ estimate, case (A) $\mathcal{S}_{\mathcal{I}(w)}^m$ convergence-(A) $\mathcal{S}_{\mathcal{I}(w)}^m$ convergence-(B)

Convergences for $\left\|\mathcal{S}_{\mathcal{I}(w)}^m[\Theta(u)] - \Theta(u)\right\|_{L^2_\varrho(\Gamma)}$, with $\Theta(u) = u(0.6)$

Example III: lognormal field

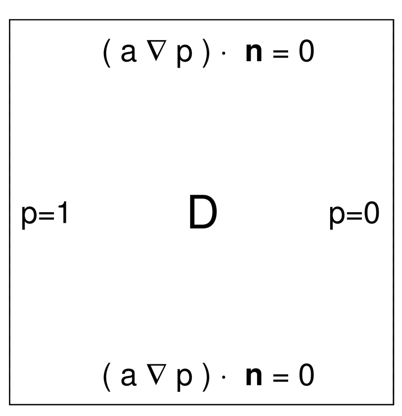
Consider problem (1) on the unit square $D = (0,1)^2$, (see picutre below for B.C.) and $a(\mathbf{x},\mathbf{y}) = e^{\gamma(\mathbf{x},\mathbf{y})}$, γ being a random field with Gaussian covariance, $cov(x,x') = \sigma^2 \exp\left(-\frac{|x_1-x_1'|^2}{L_c^2}\right)$, approximated with Fourier exp.($N = 11, 21, 33$ r.v.)

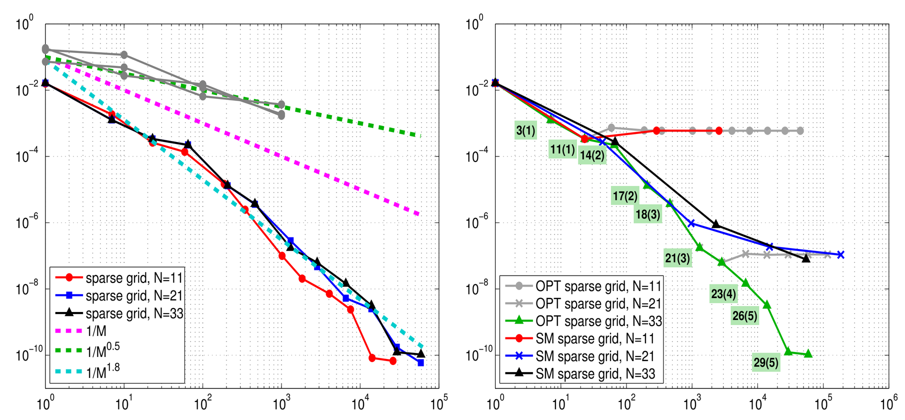
$$\gamma \approx \mu(x) + \sigma a_0 y_0 + \sigma \sum_{k=1}^K a_k \left[y_{2k-1}\cos\left(\frac{\pi}{L}kx_1\right) + y_{2k}\sin\left(\frac{\pi}{L}kx_1\right)\right]$$

$a_k = a_k(k, L_c, L)$, exponentially decaying w.r.t. k, $y_i \sim \mathcal{N}(0,1)$ $i.i.d.$

Orthogonal exp.: Hermite, with $\|u_\mathbf{q}\|_V \approx \prod_{n=1}^N \frac{e^{-g_n q_n}}{\sqrt{q_n!}}$ (heuristic)

Coll. points (nested): nested Gauss–Hermite, $\mathbb{M}^{m(\mathbf{i})}$ evaluated numerically: $\mathbb{M}^{m(\mathbf{i})} \approx \left\|\Delta^{m(\mathbf{i})}[\Psi_{m(\mathbf{i}-\mathbf{1})}]\right\|_{L^2_\varrho(\Gamma)}$





Convergences for $\left|\mathbb{E}\left[\mathcal{S}_{\mathcal{I}(w)}^m[\Theta(u)]\right] - \mathbb{E}[\Theta(u)]\right|$, where $\Theta(u)$ is the water flux through the right boundary (effective permeability).