Explicit Solutions for One-Dimensional Mean-Field Games

Thesis by
Mariana Oliveira Prazeres

In Partial Fulfillment of the Requirements
For the Degree of
Masters of Science

King Abdullah University of Science and Technology
Thuwal, Kingdom of Saudi Arabia

March, 2017
The thesis of Mariana Oliveira Prazeres is approved by the examination committee.

Committee Chairperson: Professor Diogo Gomes
Committee Members: Professor Peter Markowich, Professor Ganesh Sundaramoorthi
ABSTRACT

Explicit Solutions for One-Dimensional Mean-Field Games
Mariana Oliveira Prazeres

In this thesis, we consider stationary one-dimensional mean-field games (MFGs) with or without congestion. Our aim is to understand the qualitative features of these games through the analysis of explicit solutions. We are particularly interested in MFGs with a nonmonotonic behavior, which corresponds to situations where agents tend to aggregate.

First, we derive the MFG equations from control theory. Then, we compute explicit solutions using the current formulation and examine their behavior. Finally, we represent the solutions and analyze the results.

This thesis main contributions are the following: First, we develop the current method to solve MFG explicitly. Second, we analyze in detail non-monotonic MFGs and discover new phenomena: non-uniqueness, discontinuous solutions, empty regions and unhappiness traps. Finally, we address several regularization procedures and examine the stability of MFGs.
ACKNOWLEDGEMENTS

I would like to thank my advisor Professor Diogo Gomes for all the help and time during the past year - for the mathematics taught, for the endless tips on writing and for the help in planning my future. I would also like to thank Dr. Levon Nurbekyan for always being available to answer my questions.

I would also like to thank my friends for their support throughout my Masters. Especially the mathematicians - who understand what it takes - Dasha, Francisco, João and Ioanna. I thank you not only for the talks about mathematics but also for those about life. I would also like to thank Diana, my best friend, for the advice sent from far away - and for proving that friendships do not have to fade away with distance.

Finally, I want to thank my family for encouraging me to come to Saudi Arabia to study, although they would rather have me close by. Lastly, I would like to thank David and Rachel, for being the family away from home, and making my time in KAUST more enjoyable.

For all the others I did not mention, thank you.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Examination Committee Page</td>
<td>2</td>
</tr>
<tr>
<td>Copyright</td>
<td>3</td>
</tr>
<tr>
<td>Abstract</td>
<td>4</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>5</td>
</tr>
<tr>
<td>List of Figures</td>
<td>8</td>
</tr>
<tr>
<td><strong>1 Introduction</strong></td>
<td>10</td>
</tr>
<tr>
<td><strong>2 Derivation of mean-field games</strong></td>
<td>14</td>
</tr>
<tr>
<td>2.1 Derivation of time-dependent mean-field games</td>
<td>14</td>
</tr>
<tr>
<td>2.1.1 Derivation of the Hamilton-Jacobi equation</td>
<td>14</td>
</tr>
<tr>
<td>2.1.2 Derivation of the transport equation</td>
<td>17</td>
</tr>
<tr>
<td>2.1.3 Derivation of the mean-field game system</td>
<td>19</td>
</tr>
<tr>
<td>2.2 Derivation of stationary mean-field games</td>
<td>20</td>
</tr>
<tr>
<td><strong>3 The Current Formulation</strong></td>
<td>22</td>
</tr>
<tr>
<td>3.1 The Current Formulation</td>
<td>22</td>
</tr>
<tr>
<td>3.2 Non-Vanishing Current</td>
<td>25</td>
</tr>
<tr>
<td>3.2.1 Increasing Coupling</td>
<td>25</td>
</tr>
<tr>
<td>3.2.2 Decreasing Coupling</td>
<td>27</td>
</tr>
<tr>
<td>3.3 Vanishing Current</td>
<td>42</td>
</tr>
<tr>
<td>3.3.1 Increasing Coupling</td>
<td>42</td>
</tr>
<tr>
<td>3.3.2 Decreasing Coupling</td>
<td>45</td>
</tr>
<tr>
<td>3.4 Discontinuous Viscosity Solutions</td>
<td>49</td>
</tr>
<tr>
<td><strong>4 Regularizations</strong></td>
<td>53</td>
</tr>
<tr>
<td>4.1 The Current Formulation</td>
<td>53</td>
</tr>
<tr>
<td>4.2 Euler-Lagrange Equation</td>
<td>54</td>
</tr>
<tr>
<td>4.3 Gamma-Convergence</td>
<td>55</td>
</tr>
</tbody>
</table>
5 Properties of Solutions .................................................. 62
  5.1 Regularity Regimes ................................................. 62
  5.2 Asymptotic Behavior .............................................. 66
    5.2.1 The case \( j \to \infty \) ..................................... 66
    5.2.2 The case \( j \to 0 \) ......................................... 69
  5.3 Behavior of \( H_j \) .................................................... 72
  5.4 Behavior of \( p_j \) .................................................... 74
  5.5 Behavior of \( H(p) \) ................................................ 78

References ........................................................................ 80

Appendices ....................................................................... 84
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Solution $m$ for $\alpha = 0$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = m$.</td>
<td>26</td>
</tr>
<tr>
<td>3.2</td>
<td>Solution $m$ for $\alpha = 1$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = m$.</td>
<td>26</td>
</tr>
<tr>
<td>3.3</td>
<td>Solution $m$ for $\alpha = 5$, $j = 1$, $V(x) = \frac{3}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>28</td>
</tr>
<tr>
<td>3.4</td>
<td>Non positive solution $m$ for $\alpha = 5$, $j = 0.01$, $V(x) = \frac{3}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>28</td>
</tr>
<tr>
<td>3.5</td>
<td>(A) $F_2(m)$ for $m &gt; 0$ and $\overline{H}$ not critical. (B) $F_2(m)$ for $m &gt; 0$ and $\overline{H}$ critical.</td>
<td>29</td>
</tr>
<tr>
<td>3.6</td>
<td>Solution $m$ for $\alpha = 0$, $j = 0.001$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>33</td>
</tr>
<tr>
<td>3.7</td>
<td>Solution $m$ for $\alpha = 0$, $j = 0.5$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>33</td>
</tr>
<tr>
<td>3.8</td>
<td>Solution $m$ for $\alpha = 0$, $j = 10$, $V(x) = \frac{1}{2} \sin(2\pi(x+1/4))$, and $g(m) = -m$.</td>
<td>33</td>
</tr>
<tr>
<td>3.9</td>
<td>Two distinct solutions for $\alpha = 0$, $j = 0.5$, $V(x) = \frac{1}{2} \sin(4\pi(x + 1/8))$, and $g(m) = -m$.</td>
<td>33</td>
</tr>
<tr>
<td>3.10</td>
<td>Solution $m$ for $\alpha = 1$, $j = 0.01$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>35</td>
</tr>
<tr>
<td>3.11</td>
<td>Solution $m$ for $\alpha = 1$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>37</td>
</tr>
<tr>
<td>3.12</td>
<td>Solution $m$ for $\alpha = 1$, $j = 10$, $V(x) = \frac{1}{2} \sin(2\pi(x+1/4))$, and $g(m) = -m$.</td>
<td>37</td>
</tr>
<tr>
<td>3.13</td>
<td>Solution $m$ for $\alpha = 2.5$, $j = 0.01$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>40</td>
</tr>
<tr>
<td>3.14</td>
<td>Solution $m$ for $\alpha = 2.5$, $j = 2.5$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>40</td>
</tr>
<tr>
<td>3.15</td>
<td>Solution $m$ for $\alpha = 2.5$, $j = 10$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.</td>
<td>40</td>
</tr>
<tr>
<td>3.16</td>
<td>Solution $m$ for $\alpha = 4$, $j = 0.01$, $V(x) = \frac{1}{2} \sin(2\pi(x+1/4))$, and $g(m) = m$.</td>
<td>41</td>
</tr>
<tr>
<td>3.17</td>
<td>Non-positive solution $m$ for $\alpha = 4$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = m$.</td>
<td>41</td>
</tr>
<tr>
<td>3.18</td>
<td>Solution $m$ for $\alpha = 4$, $j = 10$, $V(x) = \frac{1}{2} \sin(2\pi(x+1/4))$, and $g(m) = m$.</td>
<td>41</td>
</tr>
<tr>
<td>3.19</td>
<td>$m_0$ as defined in (3.37) for $V(x) = 5 \sin(2\pi(x + \frac{1}{4}))$ with $d_2 = 0.5$ and $d_1$ such that (3.36) holds.</td>
<td>48</td>
</tr>
</tbody>
</table>
3.20 \( u_0 \) (left) and \((u_0)_x\) (right) as defined in (3.38) and (3.38) for \( V(x) = 5\sin(2\pi(x + \frac{1}{4})) \) with \( d_2 = 0.5 \) and \( d_1 \) such that (3.36) holds. 

4.1 Solution \( m \) when \( g(m) = m, \ j = 1, \ V(x) = \sin(2\pi(x + 1/4)) \) for \( \epsilon = 0.01 \) (dashed) and for \( \epsilon = 0 \) (solid). 

4.2 Solution \( m \) when \( g(m) = -m, \ j = 0.001, \ V(x) = \frac{1}{2}\sin(2\pi(x + 1/4)) \) for \( \epsilon = 0.01 \) (dashed) and for \( \epsilon = 0 \) (solid). 

4.3 Solution \( m \) when \( g(m) = -m, \ j = 1, \ V(x) = \frac{1}{2}\sin(2\pi(x + 1/4)) \) for \( \epsilon = 0.01 \) (dashed) and for \( \epsilon = 0 \) (solid). 

4.4 Solution \( m \) when \( g(m) = -m, \ j = 100, \ V(x) = \frac{1}{2}\sin(2\pi(x + 1/4)) \) for \( \epsilon = 0.01 \) (dashed) and for \( \epsilon = 0 \) (solid). 

5.1 \( \phi^+ \) and \( \phi^- \) for \( V(x) = A\sin(2\pi(x + 1/4)) \). \( j_{\text{lower}} = 0.218, \ j_{\text{upper}} = 1.750 \) (\( A = 0.5 \)); \( j_{\text{lower}} = 0, \ j_{\text{upper}} = 3.203 \) (\( A = 5 \)). 

5.2 \( H_j \) for \( V(x) = \frac{1}{2}\sin(2\pi(x + \frac{1}{4})) \). 

5.3 \( p_j \) for \( V(x) = \frac{1}{2}\sin(2\pi(x + \frac{1}{4})) \). 

5.4 \( H(p) \) for \( V(x) = \frac{1}{2}\sin(2\pi(x + \frac{1}{4})) \).
Chapter 1

Introduction

Mean-field game (MFG) theory models the behavior of large populations of identical agents interacting with each other. These agents are in infinite number and take rational decisions based on their observations. However, the agents’ information is limited; thus, their decisions rely on the statistical distribution of the remaining agents. MFGs were introduced independently around the same time by J.M. Lasry and P.L. Lions in [1, 2, 3] and by M. Huang, P.E. Caines and R.P. Malhamé in [4, 5].

The inspiration for MFG theory traces back to many sources. The expression “mean-field” derives from particle physics, where each particle is infinitesimal and collective interactions create a “mean-field.” Here, we do not study particles, but interacting rational agents.

In game theory, there exists extensive theory for $N$-player games. However, differential games with $N$ players are hard to analyze mathematically. MFG theory takes the limit $N \to \infty$ and, assuming invariance through permutation, gets simpler systems.

Multiple applications of MFG have emerged. Modeling in Economics is one of the most natural applications, [6, 7]. For instance, finite-state MFGs model phenomena in socio-economic sciences, [8], trade crowding, [9], growth theory, [10], mathematical finance, [11, 12], and energy [13].

A reduced MFG is a system of a Hamilton-Jacobi equation and a transport or Fokker-Plank equation. In this thesis, we focus on the following first-order stationary
MFG:
\[
\begin{cases}
\frac{(Du+p)^2}{2} + V(x) = g(m) + \overline{H}, \\
- \text{div}(m(Du + p)) = 0.
\end{cases}
\tag{1.1}
\]

Here, to simplify, we look for \( \overline{H} \in \mathbb{R} \) and \( m, u : \mathbb{T}^n \rightarrow \mathbb{R} \), where \( \mathbb{T}^n \) is the \( n \)-dimensional torus. The function \( u \) is the utility or preferences of the agents, whereas the function \( m \geq 0 \) is the distribution of agents and \( \int m = 1 \). Each point \( x \in \mathbb{T}^n \) is a possible state for the agents. The functions \( V, g : \mathbb{T}^n \rightarrow \mathbb{R} \) are given and \( C^\infty \). \( V(x) \) is the cost for the agents of state \( x \in \mathbb{T}^n \) and \( g(m) \) is the cost of state \( x \in \mathbb{T}^n \) if the distribution of other agents is \( m \). Finally, \( p \) is a fixed real number that determines a preferred direction in the associated control problem (see Chapter 2). In particular, we are interested in the following model
\[
\begin{cases}
\frac{(u_x+p)^2}{2} + V(x) = g(m) + \overline{H}, \\
-(m(u_x + p))_x = 0.
\end{cases}
\tag{1.2}
\]

Because of the one-dimensional nature, we can compute explicit solutions. We also consider the congestion case,
\[
\begin{cases}
\frac{(u_x+p)^2}{2m^\alpha} + V(x) = g(m) + \overline{H}, \\
-(m^{1-\alpha}(u_x + p))_x = 0.
\end{cases}
\tag{1.3}
\]

The prior system occurs when moving in high-density regions is difficult. Here, \( \alpha > 0 \) is the congestion strength.

The first equation in (1.1) is a stationary version of the Hamilton-Jacobi equation associated with the following control problem:
\[
\inf_{\nu \in W} \int_0^T \left( \frac{v(s)^2}{2} + pv(s) - V(x(s)) + g(m(x(s), s)) \right) ds + u(x(T), T),
\tag{1.4}
\]
where $\mathcal{W} = L^\infty ([t,T], \mathbb{R}^d)$ and $\dot{x} = v$. The derivation of the MFG (1.1) from (1.4) is presented in Chapter 2. In that chapter, we also address a similar derivation for the $n$-dimensional version of (1.3).

For (1.2), we study the solutions for different monotonicity properties of $g$. A usual assumption is that $g$ is an increasing function. When $g$ increases, agents prefer sparsely populated areas. This case has been widely studied: stationary problems were examined in [14, 15, 16, 17], weakly coupled systems in [18], obstacle problems in [19], extended games in [20], MFGs on networks in [21, 22, 23], and time-dependent MFGs in [24, 25, 26, 27, 28]. Existence and regularity for MFG were considered for weak solutions in [1, 29] and strong solutions in [14, 16, 17, 20, 30].

The case of $g$ non-monotonically increasing is substantially harder and has not been investigated systematically in the literature. A few examples considered previously rely on the particular structure of the equation, [11, 31]. In this thesis’ Chapters 3, 4 and 5, we describe the results from [32]. There, we considered a non-monotonically coupling $g$ motivated by applications where agents prefer to stay in high-density areas.

The primary goal of this thesis is to study the case where $g$ is decreasing by computing explicit solutions. Although MFGs have been extensively studied, there were few known explicit solutions. Chapter 2 of [33] describes previous examples that give important insight into the equation. These examples and explicit solutions are necessary for the continuation method, see Chapter 11 of [33], and used to prove the existence of solutions. Here, we search for explicit solutions of (1.2) and the congestion case (1.3). Although not as widely studied as the general case, the congestion problem was examined in [34, 35, 36]. The corresponding results in Chapter 3 follow [37].

To find explicit solutions of (1.3), we adapt the method used for (1.2). In Chapter 3 both problems, (1.2) and (1.3), are discussed simultaneous. Note that (1.2) is the same equation as above with $\alpha = 0$. To simplify the presentation, we focus first on
$\alpha = 0$ and then generalize to arbitrary $\alpha > 0$.

In Chapter 3, we use the current formulation to find explicit solutions for (1.2) and (1.3). We seek solutions for $g$ increasing and decreasing, and $j > 0$ and $j = 0$.

We study these semiconcave solutions by rewriting (1.2) in terms of a current variable,

$$j = m(u_x + p).$$

(1.5)

We replace $j$ in (1.2) and note that $j$ is constant. Then, we fix $j \in \mathbb{R}$ and get an algebraic equation for $m$. We proceed in a similar way for congestion problems. These methods are valid for any monotonicity of $g$. We illustrate both cases in Chapter 3.

Subsequently, for $\varepsilon > 0$, we consider elliptic regularizations in the non-congestion case

$$\begin{cases} 
-\varepsilon u_{xx} + \frac{(u_x + p)^2}{2} + V(x) = g(m) + \overline{H} \\
-\varepsilon m_{xx} - (m(u_x + p))_x = 0.
\end{cases}$$

(1.6)

We are interested in understanding the behavior of solutions of (1.6) as $\varepsilon \to 0$. This behavior depends on the monotonicity of $g$. If $g$ is increasing, we recover solutions for $\varepsilon = 0$ from the limit. However, this is not always the case as $g$ is decreasing.

In Chapter 4, we study the elliptic problem (1.6) using variational methods. We examine the behavior of solutions as $\varepsilon \to 0$ for the separate cases of $g$ increasing and decreasing.

Finally, in Chapter 5, we take the solutions found in Chapter 3 and study their properties. We examine the regularity and asymptotic behavior of solutions. Moreover, we analyze how the current impacts $\overline{H}$ and $p$. 
Chapter 2

Derivation of mean-field games

In this Chapter, we derive the mean-field games (MFGs) equations from an optimal control perspective. Originally, MFGs were derived from $N$-player games, and by writing the problem for $N$ agents and then letting $N \to \infty$, we would get a similar model, see [1, 3, 38].

The main idea behind MFGs is to describe the interactions among infinitesimal rational agents in competition. Here, each agent strives to optimize its control problem, while only having probabilistic information on the other agents.

The derivation consists of two steps. First, we consider time-dependent MFGs. Then, we transform time-dependent problems into stationary problems. We discuss problems with or without congestion. Note that these cases correspond to $\varepsilon = 0$ in the corresponding regularizations. For $\varepsilon > 0$ the derivation is similar, but uses stochastic control, see Chapter II in [6].

2.1 Derivation of time-dependent mean-field games

2.1.1 Derivation of the Hamilton-Jacobi equation

As we mentioned in the introduction, in MFG theory, we consider interacting agents. We select an agent and represent its state by $x \in \mathbb{R}^d$. Suppose that this agent makes its decisions over a certain finite time, $t \in [0, T]$. So, the state of the agent is a curve $x(t) : [0, T] \to \mathbb{R}^n$.

The agent changes its state by selecting a control, a function $v \in \mathcal{W} = L^\infty([t, T])$. 
Then, the agent’s state evolves according to

\[ \dot{x}(t) = v(t). \] (2.1)

The minimization of a functional, \( J \), encodes the agent’s preferences and determines the choice of a control. By minimizing \( J \), the agent determines a value function,

\[ u(x, t) = \inf_{v \in W} J(v; x, t). \] (2.2)

The function \( J \) quantifies the cost of each control \( v \in W \). Here, \( J \) is

\[ J(v; x, t) = \int_t^T L(x(s), v(s), s) \, ds + u_T(x(T)). \] (2.3)

The Lagrangian, \( L : \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}, \ L := L(x, v, t) \), is uniformly convex in \( v \). The integral represents the cost of shifting from state \( x(t) \) to state \( x(T) \). The terminal cost, \( u_T : \mathbb{R}^d \to \mathbb{R} \), is bounded and continuous and gives the cost of ending in \( x(T) \). Here, \( x \) solves (2.1) with the initial condition \( x(t) = x \). By standard ordinary differential equations results, there is always a solution because \( v \in W \) is bounded.

For the model in (1.1), we use the Lagrangian

\[ L(x, v, t) = \frac{|v|^2}{2} + pv - V(x) + g(m(x, t)) \] (2.4)

and, for the congestion problem, the Lagrangian is

\[ L_\alpha(x, v, t) = m(x, t)^\alpha \frac{|v|^2}{2} + pv - V(x) + g(m(x, t)). \] (2.5)

The functional corresponding to the congestion problem is \( J_\alpha \), to prevent ambiguity. In both cases, \( g : \mathbb{R}_0^+ \to \mathbb{R} \) is a given continuous function.

Classical results in control theory (Chapter 3.3.1 in [39]) state that if \( u \in C^1(\mathbb{R}^d \times
$[t_0, T])$, then $u$ solves the Hamilton-Jacobi equation

$$
\begin{cases}
-u_t(x, t) + H(x, Du(x, t), t) = 0, \\
u(x, T) = u_T(x).
\end{cases}
$$

(2.6)

Here, the Hamiltonian, $H : \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T] \to \mathbb{R}$, is the Legendre transform of $L$:

$$
H(x, q, t) = \sup_{v \in \mathbb{R}^d} [-q \cdot v - L(x, v, t)].
$$

(2.7)

Because $L$ is uniformly convex in $v$, the supremum is attained for a particular value $v_{\text{max}}$. Fix $x \in \mathbb{R}^d$ and $t \in [t_0, T]$. Then, $v_{\text{max}}$ solves the equation

$$
q = -D_v L(x, v, t)|_{v = v_{\text{max}}}. 
$$

(2.8)

For $L$ as in (2.4), we get $v_{\text{max}} = -(q + p)$. For $L_\alpha$ in (2.5), we get $(v_{\alpha})_{\text{max}} = -m(x, t)^{-\alpha}(q + p)$. From these, the corresponding Hamiltonians are

$$
H = \frac{(q + p)^2}{2} + V(x) - g(m(x, t))
$$

(2.9)

and

$$
H_\alpha = \frac{(q + p)^2}{2m^\alpha} + V(x) - g(m(x, t)).
$$

(2.10)

Using the preceding Hamiltonians, $H$ and $H_\alpha$, in (2.6), we get the first equation for the time-dependent, first-order MFG,

$$
\begin{cases}
-u_t(x, t) + \frac{(Du(x, t) + p)^2}{2} + V(x) = g(m(x, t)), \\
u(x, T) = u_T(X).
\end{cases}
$$

(2.11)
and for the time-dependent, first-order MFG with congestion,

\[
\begin{aligned}
-u_t(x,t) + \frac{(Du(x,t)+p)^2}{2m^\alpha} + V(x) &= g(m(x,t)). \\
u(x,T) &= u_T(X)
\end{aligned}
\]  

(2.12)

Unfortunately, the value function in (2.2) is not always differentiable. However, if \(u\) is not differentiable, it is still a viscosity solution of (2.11) and (2.12). As in [39], Chapter 10.3.3., we recall that a bounded, uniformly continuous function \(u\) is a viscosity solution for (2.6) if

1. \(u(x,T) = u_T(x)\) on \(\mathbb{R}^n \times \{t = T\}\);

2. For all \(\phi \in C^\infty(\mathbb{R}^n \times (0, T))\), if \(u - \phi\) has a local maximum at \((x_0, t_0) \in \mathbb{R}^n \times [0, T]\), then

\[-\phi_t(x,t) + H(x,D\phi(x,t),t) \leq 0;\]

3. For all \(\phi \in C^\infty(\mathbb{R}^n \times (0, T))\), if \(u - \phi\) has a local minimum at \((x_0, t_0) \in \mathbb{R}^n \times [0, T]\), then

\[-\phi_t(x,t) + H(x,D\phi(x,t),t) \geq 0.\]

Moreover, if \(\tilde{u}\) is \(C^1\) and solves (2.11) (or (2.12)), with terminal condition \(\tilde{u}(x,T) = u_T(x)\), then \(\tilde{u}\) is the value function (2.2) with \(J\) (or \(J_\alpha\)). This result is called a verification theorem and is proved in Chapter 1.1.1 in [33]. Finally, if \(\tilde{u}\) is a viscosity solution of (2.11) (or (2.12)) then it is the value function. See Theorem 2 in Chapter 10.3.3 of [39].

2.1.2 Derivation of the transport equation

In the previous subsection, we derived equation (2.6) that determines how the state, \(x\), of each agent evolves according to its preferences. Now, as agents move, the
distribution of the agents, $m$, changes. Next, we derive the equation that represents this evolution.

Assume $v : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ is a Lipschitz vector field. The field $v$ determines the dynamics of the agents; that is,

$$\begin{align*}
\dot{x}(t) &= v(x(t), t), \quad t > 0 \\
\dot{x}(0) &= x.
\end{align*} \tag{2.13}$$

Now, we define a flow $\Psi : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ such that $\Psi(x, t) = x(t)$; that is, $x(t)$ is the solution at time $t > 0$ for \eqref{2.13} with initial condition $x \in \mathbb{R}^d$.

Take any probability measure $m_0 \in \mathcal{P}(\mathbb{R}^d)$. Let $m$ be such that

$$\int_{\mathbb{R}^d} \eta(x)m(x, t) \, dx = \int_{\mathbb{R}^d} \eta(\Psi(x, t)) \, m_0(x) \, dx \tag{2.14}$$

for any $t \in [0, T]$ and any function, $\eta : \mathbb{R}^d \times [0, T] \to \mathbb{R}$, $\eta \in C_c^\infty(\mathbb{R}^n)$. The function $m := m(x, t)$ is a probability measure for all $t \in [0, T]$.

Let $m$ be defined through \eqref{2.14}. Fix $\phi \in C_c(\mathbb{R}^n \times (0, T))$. Then,

$$\frac{d}{dt} \int_{\mathbb{R}^d} \phi(x, t) \, m(x, t) \, dx = \frac{d}{dt} \int_{\mathbb{R}^d} \phi(\Psi(x, t), t) \, m_0(x) \, dx$$

using \eqref{2.14} with $\eta(x) = \phi(x, t)$. Thus,

$$\frac{d}{dt} \int_{\mathbb{R}^d} \phi(x, t) \, m(x, t) \, dx = \int_{\mathbb{R}^d} \Psi_t(x, t) \, D\phi(\Psi(x, t), t) m_0(x) \, dx + \int_{\mathbb{R}^d} \phi_t(\Psi(x, t), t) m_0(x) \, dx$$

$$= \int_{\mathbb{R}^d} v(\Psi(x, t), t) \, D\phi(\Psi(x, t), t) m_0(x) \, dx + \int_{\mathbb{R}^d} \phi_t(x, t) \, m(x, t) \, dx.$$
using (2.14) as before and \( \Psi_t(x,t) = v(\Psi(x,t),t) \). Accordingly,

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \phi(x,t) \ m(x,t) \ dx = \int_{\mathbb{R}^d} v(x,t) \ D\phi(x,t) m(x,t) \ dx + \int_{\mathbb{R}^d} \phi_t(x,t) \ m(x,t) \ dx.
\]

Now, integrating in time in \([0,T]\), we conclude that

\[
\int_0^T \int_{\mathbb{R}^d} (\phi_t(x,t) + v(x,t) \ D\phi(x,t)) \ m(x,t) \ dx \ dt = 0. \quad (2.15)
\]

Therefore, \( m \) solves

\[
\begin{cases}
-m_t(x,t) - \text{div}(v(x,t) \ m(x,t)) = 0, \ (x,t) \in \mathbb{R}^d \times [0,T] \\
m(x,0) = m_0(x)
\end{cases}
\]

in the weak sense. Hence, we have derived the equation for the evolution of the mass \( m \) with dynamics \( v \).

### 2.1.3 Derivation of the mean-field game system

Next, we couple (2.6) and (2.16). Recall that when deriving (2.6), the state evolved according to a control \( v \in W \). We also showed that the optimal control was the solution \( v_{\text{max}} \) of (2.8). Since MFGs describe a situation where each agent follows an optimal trajectory, the agents choose the optimal control. Because all agents are rational and identical, they all use the same strategy. Hence, \( u \) is the same for each agent. Thus, \( u \), the value function of each agent, and \( m \), the distribution of agents per state, is determined by

\[
\begin{cases}
-u_t(x,t) + \frac{(Du(x,t)+p)^2}{2} + V(x) = g(m(x,t)) \\
m_t(x,t) - \text{div}(m(x,t)(Du(x,t) + p)) = 0 \\
u(x,T) = u_T(x), \ m(x,0) = m_0(x)
\end{cases}
\]

(2.17)
in the non-congestion case and

\[
\begin{cases}
-u_t(x,t) + \frac{(Du(x,t)+p)^2}{2m^\alpha} + V(x) = g(m(x,t)) \\
m_t(x,t) - \text{div}(m(x,t)^{1-\alpha}(Du(x,t) + p)) = 0 \\
u(x,T) = u_T(x), \ m(x,0) = m_0(x)
\end{cases}
\] (2.18)

for the non-congestion case and

\[
\begin{cases}
-u_t(x,t) + \frac{(Du(x,t)+p)^2}{2m^\alpha} + V(x) = g(m(x,t)) + \overline{H} \\

-m_t(x,t) - \text{div}(m(x,t)^{1-\alpha}(Du(x,t) + p)) = 0
\end{cases}
\] (2.20)

for the congestion case.

2.2 Derivation of stationary mean-field games

In the previous section, we derived the time-dependent MFGs (2.17) and (2.18). The structure is very similar to (1.1) and (1.3), except for the time-dependent terms $u_t$ and $m_t$, the initial-terminal conditions, and the constant $\overline{H}$. In this section, we demonstrate how to reduce time-dependent problems into stationary ones. Here, we look for solutions with $m$ independent of time, $m(x,t) = m(x)$. Moreover, we define $u(x,t) = \tilde{u}(x) + \overline{H}t$, where $\overline{H} \in \mathbb{R}$. The constant, $\overline{H}$, determines the long-time average running cost per unit of time or effective Hamiltonian.

Thus, we rewrite the MFGs (2.17) and (2.18) as

\[
\begin{cases}
\frac{(D\tilde{u}(x)+p)^2}{2} + V(x) = g(m(x)) + \overline{H} \\

- \text{div}(m(x)(D\tilde{u}(x) + p)) = 0
\end{cases}
\] (2.19)

for the non-congestion case and

\[
\begin{cases}
\frac{(D\tilde{u}(x)+p)^2}{2m^\alpha} + V(x) = g(m(x)) + \overline{H} \\

- \text{div}(m(x)^{1-\alpha}(D\tilde{u}(x) + p)) = 0
\end{cases}
\] (2.20)

for the congestion case.

In general, we cannot find truly stationary solutions; that is, solutions with $\overline{H} = 0$;
the first equation in both cases (2.19) and (2.20), may fail to have solutions, or the
solution $m$ may not be a probability measure.
Chapter 3

The Current Formulation

In this chapter, we use the current formulation to find explicit solutions for both (1.2) and (1.3). Here, we expand and simplify our results from [32] and [37]. While our methods are purely one-dimensional, we gain valuable insight on the qualitative properties of MFGs. In this chapter, we see that we may not have uniqueness nor continuity. Similar properties are expected in higher dimensional models. Moreover, we examine some special cases where there is a unique solution, depending on the properties of the potential and the current.

First, in Section 3.1, we introduce the current formulation and rewrite the MFGs (1.2) and (1.3) in terms of the currents, $j$ and $j_\alpha$. Then, for non-vanishing current, we find explicit solutions in Section 3.3; the vanishing current case is examined in Section 3.2. Finally, in Section 3.4, we find discontinuous viscosity solutions for MFGs (1.2) and (1.3).

3.1 The Current Formulation

We consider the one-dimensional problem (1.2). The second equation gives that $m(u_x + p)$ is a constant, $j$. We call $j$ the current. The current formulation consists in replacing the term $u_x + p$ by its equivalent in terms of $j$. If we have a non-vanishing
current, \( j \neq 0 \), (1.2) becomes:

\[
\begin{cases}
F_j(m) = H - V(x), \\
m > 0, \int_T m \, dx = 1, \\
\int_T \frac{1}{m} \, dx = \frac{p}{j}, \ j \neq 0,
\end{cases}
\tag{3.1}
\]

where \( F_j(m) = \frac{j^2}{2m^2} - g(m) \). We can solve (3.1) algebraically for \( m \).

If the current vanishes, \( j = 0 \), then (1.2) becomes:

\[
\begin{cases}
\frac{(u_x + p)^2}{2} - g(m) = H - V(x), \\
m > 0, \int_T m \, dx = 1, \\
m(u_x + p) = 0, \ j = 0.
\end{cases}
\tag{3.2}
\]

For the congestion problem, (1.3), we define \( j_\alpha \in \mathbb{R} \) as:

\[
j_\alpha = m^{1-\alpha}(u_x + p).
\tag{3.3}
\]

Define likewise \( F_{j_\alpha}(m) = \frac{j^{2}}{2m^{2-\alpha}} - g(m) \). Then, we get systems similar to (3.1) and (3.2):

\[
\begin{cases}
F_{j_\alpha}(m) = H - V(x), \\
m > 0, \int_T m \, dx = 1, \\
\int_T m^{1-\alpha} \, dx = \frac{p}{j_\alpha}, \ j_\alpha \neq 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
\frac{(u_x + p)^2}{2m^{\alpha}} - g(m) = H - V(x), \\
m > 0, \int_T m \, dx = 1, \\
m^{1-\alpha}(u_x + p) = 0, \ j_\alpha = 0.
\end{cases}
\tag{3.4}
\]

Because the systems obtained for \( j \neq 0 \) and \( j = 0 \) are distinct, we study these cases separately. Moreover, because the goal of this thesis is to study these MFGs for different monotonicity properties of \( g \), we consider separately the cases where \( g \) is increasing and \( g \) is decreasing. Often, to further simplify our exposition, we take
\( g(m) = m \) for the decreasing case and \( g(m) = -m \) for the increasing case.

Before moving on, we note when \( \alpha = 2 \), the existence of a solution is independent of \( g \) being increasing or decreasing. If we replace \( \alpha \) in (1.3), we get

\[
\begin{cases}
F_{j_2} (m) = \frac{(j_2)^2}{2} - g(m) = \overline{H} - V(x) \\
m > 0, \int_T m \, dx = 1 \\
\int_T m \, dx = \frac{p}{j_2}, \quad j_2 \neq 0
\end{cases}
\text{and}
\begin{cases}
\frac{(u_x + p)^2}{2m^2} - g(m) = \overline{H} - V(x) \\
m > 0, \int_T m \, dx = 1 \\
\frac{u_x + p}{m} = 0, \quad j_2 = 0.
\end{cases}
\tag{3.5}
\]

Hence, we get the following proposition:

**Proposition 3.1.1.** Let \( j \in \mathbb{R} \). Assume \( g \) has a monotone inverse \( g^{-1} \) and that

\[
g^{-1}\left( \frac{(j_2)^2}{2} + V(x) - \overline{H} \right) \geq 0. \tag{3.6}
\]

Then, the solution \( (u_{j_2}, m_{j_2}, \overline{H}_{j_2}) \) for (3.1) with \( \alpha = 2 \) is unique and smooth. Moreover,

\[
m_{j_2} (x) = g^{-1}\left( \frac{(j_2)^2}{2} + V(x) - \overline{H}_{j_2} \right), \quad u_{j_2} = \int_0^x j_2 \, m_{j_2}(y) \, dy - p_{j_2} x
\tag{3.7}
\]

where \( p_{j_2} = j_2 \) and \( \overline{H}_{j_2} \) is such that \( \int_T m_{j_2}(x) \, dx = 1 \).

**Proof.** The result for \( m_{j_2} \) follows from the monotonicity and positivity of the inverse \( g^{-1} \). The smoothness follows from \( g, V \in C^\infty \). The function \( u_{j_2} \) and the constants \( p_{j_2} \) and \( \overline{H}_{j_2} \) follow from elementary computations from (3.5). For \( j_2 = 0 \), note that we have \( u_x + p = 0 \) necessarily.

Now, we study the remaining possibilities with \( \alpha \neq 2 \).
3.2 Non-Vanishing Current

Here, we consider the case \( j > 0 \). The case \( j < 0 \) is analogous. We examine the case \( j = 0 \) in Section 3.3.

3.2.1 Increasing Coupling

As mentioned in the Introduction, it is usual to assume that \( g \) is increasing. This case usually has unique smooth solutions. In this section, we compute explicit solutions for (3.1) and (3.4). These solutions are smooth when the current does not vanish, as long as \( \alpha < 2 \), low congestion.

**Proposition 3.2.1.** Let \( g \) be an increasing function. Then, for all \( j > 0 \), the solution \((u_j, m_j, H_j)\) for (1.2) is unique and smooth. Moreover,

\[
m_j(x) = F_j^{-1}(H_j - V(x)), \quad u_j(x) = \int_0^x \frac{j(y)}{m_j(y)} \, dy - p_jx
\]

where \( p_j = \int_T \frac{j(y)}{m_j(y)} \, dy \), \( F_j \) is as defined for (3.1), and \( H_j \) is such that \( \int_T m_j(x) \, dx = 1 \).

Moreover, for all \( j_\alpha > 0 \), the solution \((u_{j_\alpha}, m_{j_\alpha}, H_{j_\alpha})\) for (1.3) with \( \alpha < 2 \) is also unique and smooth. Furthermore,

\[
m_{j_\alpha}(x) = F_{j_\alpha}^{-1}(H_{j_\alpha} - V(x)), \quad u_{j_\alpha}(x) = \int_0^x \frac{j_\alpha(y)}{m_{j_\alpha}(y)^{1-\alpha}} \, dy - p_{j_\alpha}x,
\]

where \( p_{j_\alpha} = \int_T \frac{j_\alpha(y)}{m_{j_\alpha}(y)^{1-\alpha}} \, dy \), \( F_{j_\alpha} \) as defined for (3.1), and \( H_{j_\alpha} \) is such that \( \int_T m_{j_\alpha}(x) \, dx = 1 \).

**Proof.** The result follows by elementary computations using that \( F_j \) and \( F_{j_\alpha} \) are strictly decreasing for \( m \geq 0 \) when \( g \) is monotonically increasing and \( \alpha < 2 \). \( \square \)

Next, we plot the solutions found in Proposition 3.2.1. First, we consider the non-congestion case, \( \alpha = 0 \). In Figure 3.1, we plot the solution for \( V(x) = \frac{1}{2} \sin \left(2\pi(x + \frac{1}{4})\right)\).
Figure 3.1: Solution $m$ for $\alpha = 0$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = m$.

Figure 3.2: Solution $m$ for $\alpha = 1$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = m$.

Next, in Figure 3.2, we plot the case $\alpha = 1$. In this case, we see that agents tend to avoid the minimum of $V$ as expected in terms of the control problem (2.1).

When $\alpha > 2$, the function $F_{j,\alpha} = \frac{j^2}{2} m^{a-2} - g(m)$ is no longer monotone. Hence, the above proof does not hold anymore. In fact, the results are similar to the non-decreasing case explained in the next section. Here, we do not detail this case and, instead, prove it after the discussion on the decreasing case.

**Proposition 3.2.2.** Let $g$ be an increasing function. Then, for all $j > 0$, the solution for (1.3) with $\alpha > 2$ is

$$m_{j,\alpha}(x) = m^-_{\Pi_{j,\alpha}}(x) \chi_{[0,d]} + m^+_{\Pi_{j,\alpha}}(x) \chi_{[d,1]}, \quad u_{j,\alpha}(x) = \int_0^x \frac{j}{m_{j,\alpha}(y)^{1-\alpha}} dy - p_{j,\alpha} x, \quad (3.10)$$

where $p_{j,\alpha} = \int_{\Pi} \frac{j}{m_{j,\alpha}(y)^{1-\alpha}} dy$, and $d$ is such that $\int_{\Pi} m_{j,\alpha}(x) dx = 1$. Here $m^-_{\Pi_{j,\alpha}}$ and $m^+_{\Pi_{j,\alpha}}$ are functions defined as in Remark 3.3.7 and $\chi$ the usual characteristic function.

**Proof.** See Remark 3.3.7 in the decreasing case, section 3.2.2.
3.2.2 Decreasing Coupling

Now, we focus on the decreasing case. In contrast with the results in Section 3.2.1, $m$ may not be unique and it may not be continuous. To simplify our presentation, we consider $g(m) = -m$. However, our methods can be generalized for other functions.

As $m$ may not be continuous, we need to look at the Hamilton-Jacobi equations in (1.2) and (1.3) as having discontinuous viscosity solutions. We discuss this type of solutions in Section (3.4). However, for this section, we are interested in semiconcave viscosity solutions.

**Definition 3.2.3.** We say a solution to (1.2) (or (1.3)) is a semiconcave viscosity solution if it satisfies the following conditions:

1. Let $x_0 \in \mathbb{T}$. If $u$ is differentiable at $x_0$ and $m$ is constant at $x_0$, then $u$ solves (1.2) (or (1.3)) for $x = x_0$;

2. Take $x_0 \in \mathbb{T}$. If $u$ is differentiable at $x_0$, then

$$\lim_{x \to x_0^-} u_x(x) \geq \lim_{x \to x_0^+} u_x(x).$$

In this section, we analyze existence and uniqueness of solutions. When it is clear from the context we drop the word “semiconcave”. First, we consider MFGs with congestion for $\alpha > 2$, whose analysis is elementary.

**Proposition 3.2.4.** Let $g$ be a decreasing function. Then, for all $j > 0$, the solution for (1.3) with $\alpha > 2$ is (3.9) where $p_{j\alpha}$ is as in Proposition 3.2.1, $F_{j\alpha}$ as defined for (3.1), and $\overline{H}$ is such that $\int_{\mathbb{T}} m_{j\alpha}(x) = 1$.

*Proof.* Note that $F'_{j\alpha}(m) = -(2 - \alpha)\frac{m^2}{2m^{2\alpha} - \alpha} - g'(m) > 0$, since $g$ is decreasing, $m > 0$ and $\alpha > 2$. Then, we can invert $F_{j\alpha}$ to obtain the result. \qed

However, the solutions obtained in Proposition 3.2.4 may not satisfy $m \geq 0$. In that case, we do not have existence. We plot the solutions for $\alpha = 5$ with $V(x) =$
Figure 3.3: Solution $m$ for $\alpha = 5$, $j = 1$, $V(x) = \frac{3}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.

Figure 3.4: Non positive solution $m$ for $\alpha = 5$, $j = 0.01$, $V(x) = \frac{3}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.

In Figure 3.3, we see an example where $m$ is a positive density. In Figure 3.4, $m$ has negative sections, and there is no solution for (1.3).

Next, we take $g(m) = -m$ in (1.2) and (1.3) and consider the case $\alpha < 2$. Finally, at the end of the section, we explain how $\alpha > 2$ can be tackled for an increasing $g$.

**Noncongestion case**

First, we replace $g$ in (1.2) and obtain

\[
\begin{cases}
F_j(m) = \frac{j^2}{2m^2} + m = \overline{H} - V(x), \\
m > 0, \int_T m \, dx = 1, \\
\int_T \frac{1}{m} \, dx = \frac{p}{j}.
\end{cases}
\tag{3.11}
\]

Now, fix $j > 0$. Here, $F_j$ is not monotone. In fact, $F_j(t)$ attains a minimum at
Figure 3.5: (A) $F_j(m)$ for $m > 0$ and $\bar{H}$ not critical. (B) $F_j(m)$ for $m > 0$ and $\bar{H}$ critical.

\[ t_{\min} = \left(j^{\frac{2}{3}}, \frac{3}{2}j^{\frac{2}{3}}\right), \text{ when } m > 0. \text{ Thus,} \]

\[ F_j(m) \geq F_j(t_{\min}) = \frac{3}{2}j^{\frac{2}{3}}. \]

So, from (3.11), we obtain the following bound for $\bar{H}$:

\[ \bar{H} \geq \frac{3}{2}j^{2/3} + \max_{\mathcal{T}} V := \bar{H}_{j}^{cr}, \]

where $^{cr}$ stands for critical.

The function $F_j$ decreases on $(0, t_{\min})$ and increases on $(t_{\min}, +\infty)$, as shown in Figure 3.5(A). In Figure 3.5, for a fixed $x \in \mathcal{T}$, $\bar{H} - V(x)$ intersects $F_j$ on two points as long as $\bar{H} \geq \bar{H}_{j}^{cr}$. These points, $m^-_{\bar{H}}(x)$ and $m^+_{\bar{H}}(x)$, satisfy $0 \leq m^-_{\bar{H}}(x) \leq t_{\min} \leq m^+_{\bar{H}}(x)$. Both $m^-_{\bar{H}}(x)$ and $m^+_{\bar{H}}(x)$ solve the first equation in (3.11). Hence, if $m$ solves (3.11), then $m(x)$ is either $m^-_{\bar{H}}(x)$ or $m^+_{\bar{H}}$.

Define $m^-_{j} := m^-_{\bar{H}_{j}^{cr}}(x)$ and $m^+_{j} := m^+_{\bar{H}_{j}^{cr}}(x)$. As in Figure 3.5(B), $m^-_{j}(x) \leq m^+_{j}(x)$ for all $x \in \mathcal{T}$, with equality for $x_{\max} = \arg\max_{\mathcal{T}} V$.

Next, we define some fundamental quantities.

**Definition 3.2.5.** We define the functions $\phi^+$ and $\phi^-$ as

\[
\begin{align*}
\phi^+(j) &= \int_{\mathcal{T}} m^+_j(x) \, dx, \\
\phi^-(j) &= \int_{\mathcal{T}} m^-_j(x) \, dx.
\end{align*}
\]
We have three possibilities,
\[
\phi^+ \leq 1, \quad \phi^- \geq 1, \quad \text{and} \quad \phi^- < 1 < \phi^+.
\]

If \( V \) is not constant, \( \phi^-(j) \leq \phi^+(j) \) for \( j > 0 \). Now, we find the solutions of (3.11) for each case. First, we suppose that \( V \) has a single maximum. In this case, the solution is unique. Later, we drop that assumption and find non-uniqueness.

**Proposition 3.2.6.** Suppose \( V \) has a unique maximum at \( x = 0 \). Then, for all \( j > 0 \), the solution \((u_j, m_j, \overline{H}_j)\) of (3.1) is unique. Furthermore, for each \( j > 0 \), there exists a unique \( p_j \) for which (1.2) has a solution. Moreover, we have the following cases:

**Case i.** If \( \phi^+(j) \leq 1 \),
\[
m_j(x) = m_j^+(x), \quad u_j(x) = \int_0^x \frac{j}{m_j(y)} \, dy - p_j x, \quad (3.13)
\]
where \( p_j = \int_{\overline{H}_j} \frac{j}{m_j(y)} \, dy \) and \( \overline{H}_j \) is such that \( \int_{\overline{H}_j} m_j(x) \, dx = 1 \).

**Case ii.** If \( \phi^-(j) \geq 1 \),
\[
m_j(x) = m_j^-(x), \quad u_j(x) = \int_0^x \frac{j}{m_j(y)} \, dy - p_j x, \quad (3.14)
\]
where \( p_j = \int_{\overline{H}_j} \frac{j}{m_j(y)} \, dy \) and \( \overline{H}_j \) is such that \( \int_{\overline{H}_j} m_j(x) \, dx = 1 \).

**Case iii.** If \( \phi^-(j) < 1 < \phi^+(j) \),
\[
m_j(x) = m_j^-(x)\chi_{[0,d_j]} + m_j^+(x)\chi_{[d_j,1)}, \quad u_j(x) = \int_0^x \frac{j}{m_j(y)} \, dy - p_j x, \quad (3.15)
\]
where \( p_j = \int_{\overline{H}_j} \frac{j}{m_j(y)} \, dy \), \( \overline{H}_j = \overline{H}_j^{\text{cr}} \), and \( d_j \) is such that
\[
\int_{\overline{H}_j} m_j(x) \, dx = \int_0^{d_j} m_j^-(x) \, dx + \int_{d_j}^1 m_j^+(x) \, dx = 1.
\]
Proof.

**Case i.** Referring back to Figure 3.5(B), $F_j$ is increasing in $(t_{\min}, +\infty)$. If $H$ increases, so does $m_j^+(x)$. Here, $m_j^+(x) \leq m_j^+(x)$ for all $x \in \mathbb{T}$. Since $\phi^+(j) \leq 1$ holds for $H = \bar{H}_j$ and \( \lim_{H \to \infty} \int_{\mathbb{T}} m_j^+(x) \, dx = +\infty \), there exists a unique $H_j \geq \bar{H}_j$ such that $\int_{\mathbb{T}} m_j^+(x) \, dx = 1$. Choose $m_j = m_j^+$. Then, $(u_j, m_j, H_j)$ is the unique solution with $p_j$ as defined.

**Case ii.** Referring back to Figure 3.5(B), $F_j$ is decreasing in $(0, t_{\min})$. If $H$ increases, $m_j^-(x)$ decreases. So, $m_j^-(x) \geq m_j^-(x)$ for all $x \in \mathbb{T}$. Since $\phi^-(j) \geq 1$ holds for $H = \bar{H}_j$ and \( \lim_{H \to \infty} \int_{\mathbb{T}} m_j^-(x) \, dx = 0 \), there exists a unique $H_j \geq \bar{H}_j$ such that $\int_{\mathbb{T}} m_j^-(x) \, dx = 1$. Choose $m_j = m_j^-$. Then, $(u_j, m_j, H_j)$ is the unique solution with $p_j$ as defined.

**Case iii.** We start by proving the following lemma.

**Lemma 3.2.7.** In the conditions of Case iii., $\bar{H}_j = \bar{H}_j^{cr}$.

*Proof.* We argue by contradiction. Assume that (1.2) has a solution $(u, m, H)$ for some $H > \bar{H}_j^{cr}$ and $p \in \mathbb{R}$.

As we established before, $m(x)$ is either $m_H^+(x)$ or $m_H^-(x)$. So,

$$m = m_H^+ \chi_E + m_H^- \chi_{\mathbb{T}\setminus E}$$

for some set $E \in \mathbb{T}$.

By the hypotheses on $\phi^-$ and $\phi^+$,

$$\int_{\mathbb{T}} m_H^-(x) \, dx \leq \phi^-(j) < 1 < \phi^+(j) \leq \int_{\mathbb{T}} m_H^+(x) \, dx. \quad (3.16)$$

Here, the first and last inequalities are explained in proofs for cases i. and ii. Assume
$E$ has Lebesgue measure 0, then $\int_T m = \int_T m^- < 1$. Assume $T \setminus E$ has Lebesgue measure 0, then $\int_T m = \int_T m^+ > 1$. So, neither $E$ or $T \setminus E$ can have zero Lebesgue measure. Therefore, there exists $z \in T$ such that,

$$\forall \epsilon > 0 \ (z - \epsilon, z) \cup E \neq \emptyset \text{ and } (z, z + \epsilon) \cup E^c \neq \emptyset. \quad (3.17)$$

Since $H > H^{cr}_j$, in Figure 3.5 we see that

$$\inf_{x \in T} m^+_H(x) - m^-_H(x) > 0.$$ 

Thus, $m$ has a negative jump at $x = z$, i.e., $m(z^-) - m(z^+) > 0$. Therefore, $u_x$ as defined in (3.15), has a positive jump at $x = z$. However, by the definition of solutions, the derivatives can only have negative jumps. This contradiction implies $H = H^{cr}_j$. \qed

Now, we construct the remaining elements of the solution, $m_j$ and $u_j$, and find the value of $p_j$.

We define $m_j$ as $m_j = m^-_j \chi_{[0,d)} + m^+_j \chi_{[d,1]}$. Note, that the negative jump in $m$ does not occur anymore as $m^+_j(1) = m^-_j(0)$. Since $V$ has one maximum, these points are the only possibility to switch from $m^+_j$ to $m^-_j$ without jumping.

Next, we select $d$ such that $\psi(d) = \int_T m_j(x) \, dx = 1$. If $d = 0$, $\psi(0) = \int_T m^+_j(x) \, dx > 1$. If $d = 1$, $\psi(1) = \int_T m^-_j(x) \, dx < 1$. Moreover, $\psi'(d) = m^-_j(d) - m^+_j(d) < 0$ for $d \in (0,1)$. Therefore, there exists a unique $d_j \in (0,1)$ such that $\psi(d_j) = 1$. Hence, $(u_j, m_j, H_j)$ is the unique solution of (3.1) for $p = p_j$. \qed

In Figures 3.6, 3.8, and 3.7 we plot the different cases of Proposition 3.2.6. We consider distinct values of $j$ and set $V(x) = \frac{1}{2} \sin(2\pi(x + \frac{1}{4}))$. In Figures 3.6 and 3.7 we see that $m$ is smooth as expected. Moreover, we have an “unhappiness trap” in Figure...
Figure 3.6: Solution $m$ for $\alpha = 0, j = 0.001$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$ and $g(m) = -m$.

Figure 3.7: Solution $m$ for $\alpha = 0, j = 0.5$, $V(x) = \frac{1}{2} \sin(2\pi(x+1/4))$, and $g(m) = -m$.

3.6 i.e., agents gather around the minimum of $V$. In Figure 3.7 the distribution is close to uniform. In Figure 3.8 we see that there is a discontinuity between low and high-density areas.

We found that the case where $g$ decreases is well-behaved and displays some regularity if $V$ has a single maximum. However, when $V$ has multiple maxima, that is not necessarily true. We see that in the next proposition for the case of two maxima. If $V$ has multiple maxima and Case iii. conditions hold then (3.1) has infinitely many solutions.

**Proposition 3.2.8.** Suppose $V$ has two maxima, at $x = 0$ and $x = x_0 \in (0, 1)$. Then,

Figure 3.8: Solution $m$ for $\alpha = 0, j = 10$, $V(x) = \frac{1}{2} \sin(2\pi(x+1/4))$, and $g(m) = -m$. 

for all $j$ such that $\phi^-(j) < 1 < \phi^+(j)$, there are infinitely many solutions $(u, m, \overline{H}_j)$ of (1.2).

Proof. Fix $j > 0$ such that $\phi^-(j) < 1 < \phi^+(j)$. We look for solutions of the form

$$m_j^{d_1, d_2}(x) = m_j^-(x) \chi_{[0, d_1] \cup [x_0, d_2]} + m_j^+(x) \chi_{[d_1, x_0] \cup [d_2, 1)},$$

where $0 < d_1 < x_0 < d_2 < 1$. The function $m_j^{d_1, d_2}$ has two discontinuities, which are positive jumps, $d_1$ and $d_2$.

Let $p_j^{d_1, d_2} = \int_T \frac{j}{m_j^{d_1, d_2}(y)} \, dy$. We define $u_j$ as

$$u_j^{d_1, d_2}(x) = \int_0^x \frac{j}{m_j^{d_1, d_2}(y)} \, dy - p_j^{d_1, d_2} x$$

for $x \in \mathbb{T}$.

Next, we show there exists $(d_1, d_2) \in (0, x_0) \times (x_0, 1)$ such that

$$\psi(d_1, d_2) = \int_\mathbb{T} m_j^{d_1, d_2}(x) \, dx = 1$$

. We have $\psi(0, x_0) = \int_\mathbb{T} m_j^+(x) \, dx > 1$ and $\psi(x_0, 1) = \int_\mathbb{T} m_j^-(x) \, dx < 1$. Moreover, $\psi$ is continuous. Therefore, there are infinitely many pairs $(d_1, d_2) \in (0, x_0) \times (x_0, 1)$ such that $\psi(d_1, d_2) = 1$, because for every curve $\gamma$ connecting $(0, x_0)$ to $(x_0, 1)$ there exists at least one such pair $(d_1, d_2)$.

Therefore, for each pair $(d_1, d_2) \in (0, x_0) \times (x_0, 1)$ there exists a solution $(u_j^{d_1, d_2}, m_j^{d_1, d_2}, \overline{H}_j)$ that solves (3.1) for $p = p_j^{d_1, d_2}$.

Now, we consider $V(x) = \frac{1}{2} \sin(4\pi(x + \frac{1}{8}))$. Since $V$ has two maxima, we are in conditions of Proposition 3.2.8. In Figure 3.9 we plot two of the infinitely many solutions.
Figure 3.9: Two distinct solutions for $\alpha = 0, j = 0.5$, $V(x) = \frac{1}{2} \sin(4\pi(x + 1/8))$ and $g(m) = -m$.

**Congestion Case**

Now, we generalize the results above for $\alpha < 2$. Similarly to the non-congestion case, we choose $g(m) = -m$ and replace in (3.4). In this case, $t_{\text{min}} = c_\alpha \frac{2}{\alpha} j_{\alpha}$ where $c_\alpha = \frac{1 - \alpha^2}{2} \frac{1}{3 - \alpha}$.

We define our lower bound on $H$ as

$$
\overline{H}_{\alpha} = F_{\alpha}(t_{\text{min}}) + \max_T V = \frac{e^{2-2}}{2} j^{2+2-2} + c_\alpha j^{2-2} + \max_T V.
$$

Note $\frac{\overline{H}_{\alpha}}{\overline{H}_{\alpha}} = \frac{\overline{H}^{\alpha}}{\overline{H}^{\alpha}}$.

Now, using the new bound, define points $m_{{\overline{H}}_{\alpha}}$ and $m_{{\overline{H}}_{\alpha}}$ that solve

$$
F_{\alpha}(m) = \overline{H}_{\alpha} - V(x)
$$

and $0 \leq m_{{\overline{H}}_{\alpha}} \leq t_{\text{min}} \leq m_{{\overline{H}}_{\alpha}}$. Similarly, we set $m_{\alpha} := m_{{\overline{H}}_{\alpha}}$ and $m_{\alpha} := m_{{\overline{H}}_{\alpha}}$. Note that our fundamental quantities, $\phi^+$ and $\phi^-$, are the same but depend on $j_\alpha$.

In the single maximum case, we get a result similar to Proposition 3.2.6. When $V$ has multiple maxima, we obtain a result similar to Proposition 3.2.8.

**Proposition 3.2.9.** Suppose $V$ has a unique maximum at $x = 0$ and that $\alpha < 2$. Then, for all $j_\alpha > 0$, the solution $(u_{j_\alpha}, m_{j_\alpha}, \overline{H}_{j_\alpha})$ of (3.4) is unique. Furthermore, for each $j_\alpha > 0$, there exists a unique $p_{j_\alpha}$ for which (1.3) has a solution. Moreover, we have the following cases:
Case i. If $\phi^+(j_\alpha) \leq 1$,

$$m_{j_\alpha}(x) = m_{j_\alpha}^+(x), \quad u_{j_\alpha}(x) = \int_0^x \frac{j_\alpha}{m_{j_\alpha}(y)^{1-\alpha}} \, dy - p_{j_\alpha} x,$$

(3.19)

where $p_{j_\alpha} = \int_T \frac{j_\alpha}{m_{j_\alpha}(y)^{1-\alpha}} \, dy$ and $\overline{H}_{j_\alpha}$ is such that $\int_T m_{j_\alpha}(x) \, dx = 1$.

Case ii. If $\phi^-(j_\alpha) \geq 1$,

$$m_{j_\alpha}(x) = m_{j_\alpha}^-(x), \quad u_{j_\alpha}(x) = \int_0^x \frac{j_\alpha}{m_{j_\alpha}(y)^{1-\alpha}} \, dy - p_{j_\alpha} x,$$

(3.20)

where $p_{j_\alpha} = \int_T \frac{j_\alpha}{m_{j_\alpha}(y)^{1-\alpha}} \, dy$ and $\overline{H}_{j_\alpha}$ is such that $\int_T m_{j_\alpha}(x) \, dx = 1$.

Case iii. If $\phi^-(j_\alpha) < 1 < \phi^+(j_\alpha)$,

$$m_{j_\alpha}(x) = m_{j_\alpha}^-(x) \chi_{[0,d_{j_\alpha})} + m_{j_\alpha}^+(x) \chi_{[d_{j_\alpha},1)}, \quad u_{j_\alpha}(x) = \int_0^x \frac{j_\alpha}{m_{j_\alpha}(y)^{1-\alpha}} \, dy - p_{j_\alpha} x,$$

(3.21)

where $p_{j_\alpha} = \int_T \frac{j_\alpha}{m_{j_\alpha}(y)^{1-\alpha}} \, dy$, $\overline{H}_{j_\alpha} = \overline{H}_{j_\alpha}^{cr}$, and $d_{j_\alpha}$ is such that

$$\int_T m_{j_\alpha}(x) \, dx = \int_0^{d_{j_\alpha}} m_{j_\alpha}^-(x) \, dx + \int_{d_{j_\alpha}}^1 m_{j_\alpha}^+(x) \, dx = 1.$$

Proof. Same proof as in Proposition 3.2.6 but using the new definitions.}

Now, assume $\alpha = 1$. In Figures 3.10, 3.11, and 3.12, we do the same as before. We vary $j$ and fix $V(x) = \frac{1}{2} \sin(2\pi(x + \frac{1}{4}))$. These results are very similar to the ones in Figures 3.6, 3.8, and 3.7 for the case $\alpha = 0$. Once again, we see an “unhappiness trap” in the low current case.

**Proposition 3.2.10.** Suppose $V$ has two maxima, at $x = 0$ and $x = x_0 \in (0, 1)$. Then, for all $j_\alpha$ such that $\phi^-(j_\alpha) < 1 < \phi^+(j_\alpha)$, there are infinitely many solutions $(u, m, \overline{H}^{cr}_{j_\alpha})$ of (1.3).
Figure 3.10: Solution $m$ for $\alpha = 1$, $j = 0.01$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.

Figure 3.11: Solution $m$ for $\alpha = 1$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$.

Figure 3.12: Solution $m$ for $\alpha = 1$, $j = 10$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = -m$. 
Proof. Same proof as in Proposition 3.2.8 but replacing with the new definition. □

Remark 3.2.11. It turns out that the case for $\alpha > 2$ and $g(m) = m$ increasing has a similar behavior. Here, $F''_{j_\alpha}(m) = (\alpha - 2)(\alpha - 3)\frac{j_\alpha^2}{2m^{\alpha - 3}}$, i.e., whether $F_{j_\alpha}$ is concave or convex, depends on $\alpha$.

For $g(m) = m$, we have three cases: $2 < \alpha < 3$, $\alpha > 3$, and $\alpha = 3$. For a general $g$, the conditions on $\alpha$ may change, but the main ideas still hold. We use the subscript $\text{min}$ with $\alpha > 3$, when $F_{j_\alpha}$ is convex. The subscript $\text{max}$ is used for $2 < \alpha < 3$, when $F_{j_\alpha}$ is concave.

First, we replace $g(m)$ in (3.4) by $m$. We have $t = t_{\text{min}} = t_{\text{max}} = \left(j_\alpha^2 - \frac{\alpha j_\alpha^2}{2}\right)^{\frac{1}{\alpha - 3}}$. Then, we define a lower bound on $H$, $H_{j_\alpha,\text{min}} = F_{j_\alpha}(t_{\text{min}}) + \max \{V\}$, when $F_{j_\alpha}$ is convex. Moreover, we define an upper bound on $H$, $H_{j_\alpha,\text{max}} = F_{j_\alpha}(t_{\text{max}}) + \min \{V\}$, when $F_{j_\alpha}$ is concave.

Now, using the new bounds, we define points $m_{H_{j_\alpha}}^-$ and $m_{H_{j_\alpha}}^+$ as the two points that solve

$$F_{j_\alpha}(m) = H_{j_\alpha} - V(x)$$

and $0 \leq m_{H_{j_\alpha}}^- \leq t \leq m_{H_{j_\alpha}}^+$. Similarly, we set $m_{j_\alpha,\text{min}}^- := m_{H_{j_\alpha},\text{min}}^-$, $m_{j_\alpha,\text{min}}^+ := m_{H_{j_\alpha},\text{min}}^+$, $m_{j_\alpha,\text{max}}^- := m_{H_{j_\alpha},\text{max}}^-$, and $m_{j_\alpha,\text{max}}^+ := m_{H_{j_\alpha},\text{max}}^+$. Note that we redefine the fundamental quantities as $\phi_{j_\alpha,\text{min}}^+, \phi_{j_\alpha,\text{min}}^-$, $\phi_{j_\alpha,\text{max}}^+$, and $\phi_{j_\alpha,\text{max}}^-$, where $m_j^+$ and $m_j^-$ are replaced by the new functions in the most immediate way.

We get the following proposition, similar to 3.2.6 and 3.2.8

Proposition 3.2.12. Suppose $V$ has a unique minimum or maximum at $x = 0$, $\alpha > 2$, and $\alpha \neq 3$. Then, for all $j_\alpha > 0$, the solution $(u_{j_\alpha}, m_{j_\alpha}, H_{j_\alpha}^{\text{cr}})$ for (3.4) is unique. Moreover, for all cases

$$u_{j_\alpha}(x) = \int_0^x \frac{j_\alpha}{m_{j_\alpha}(y)^{1-\alpha}} \, dy - p_{j_\alpha} x, \quad \text{where } p_{j_\alpha} = \int_T \frac{j_\alpha}{m_{j_\alpha}(y)^{1-\alpha}} \, dy.$$  (3.22)
For each case,

**Case i.** If \( \phi_{\text{min}, \text{max}}^+ (j_\alpha) \leq 1 \),

\[
m_{j_\alpha}(x) = m_{\overline{H}_{j_\alpha}}^+ (x),
\]
(3.23)

where \( \overline{H}_{j_\alpha} \) is such that \( \int_T m_{j_\alpha}(x) \, dx = 1 \).

**Case ii.** If \( \phi_{\text{min}, \text{max}}^- (j_\alpha) \geq 1 \),

\[
m_{j_\alpha}(x) = m_{\overline{H}_{j_\alpha}}^- (x),
\]
(3.24)

where \( \overline{H}_{j_\alpha} \) is such that \( \int_T m_{j_\alpha}(x) \, dx = 1 \).

**Case iii.** If \( \phi_{\text{min}, \text{max}}^- (j_\alpha) \leq 1 \leq \phi_{\text{min}, \text{max}}^+ (j_\alpha) \), then

\[
m_{j_\alpha}(x)_{\text{min}} = m_{j_\alpha, \text{min}}^- (x) \chi_{[0, d_{j_\alpha, \text{min}}]} + m_{j_\alpha, \text{min}}^+ (x) \chi_{[d_{j_\alpha, \text{min}}, 1]},
\]
(3.25)

and

\[
m_{j_\alpha}(x)_{\text{max}} = m_{j_\alpha, \text{max}}^- (x) \chi_{[0, d_{j_\alpha, \text{max}}]} + m_{j_\alpha, \text{max}}^+ (x) \chi_{[d_{j_\alpha, \text{max}}, 1]},
\]
(3.26)

where \( d_{j_\alpha, \text{min}} \) is such that \( \int_T m_{j_\alpha, \text{min}}(x) \, dx = 1 \) and \( d_{j_\alpha, \text{max}} \) is such that \( \int_T m_{j_\alpha, \text{max}}(x) \, dx = 1 \). Furthermore, if \( 2 < \alpha < 3 \), set \( \overline{H}_{j_\alpha} = \overline{H}_{j_\alpha, \text{max}}^{cr} \) and \( m_{j_\alpha} = m_{j_\alpha, \text{min}} \). If \( \alpha > 3 \), set \( \overline{H}_{j_\alpha} = \overline{H}_{j_\alpha, \text{min}}^{cr} \) and \( m_{j_\alpha} = m_{j_\alpha, \text{max}} \).

**Proof.** See Proposition 3.2.6’s proof. These results follow in a similar way. \( \Box \)

Next, we plot the \( m \) for each case of proposition 3.2.12. First, we choose \( \alpha = 2.5 \). In Figures 3.13, 3.14, and 3.15 we plot the different cases of Proposition 3.2.12. Here, we are in the \( \text{max} \) case.
Figure 3.13: Solution $m$ for $\alpha = 2.5$, $j = 0.01$, $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$, and $g(m) = -m$.

Figure 3.14: Solution $m$ for $\alpha = 2.5$, $j = 2.5$, $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$, and $g(m) = -m$.

In Figures 3.13 and 3.15, we are in Cases i. and ii, respectively. We observe that $m$ is smooth as expected. When the current is small, agents avoid the minimum of $V$. However, when the current is large, the agents’ distribution is close to a uniform distribution. In Figure 3.14 we see a discontinuity between the high and low-density regions.

Secondly, we choose $\alpha = 4$. We are now in the $\min$ case. In Figures 3.16, 3.17, and 3.18, we plot each case of Proposition 3.2.12.

We see that Figures 3.16 and 3.18 exhibit similar behavior to the previous case,

Figure 3.15: Solution $m$ for $\alpha = 2.5$, $j = 10$, $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$, and $g(m) = -m$. 
Figure 3.16: Solution $m$ for $\alpha = 4$, $j = 0.01$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = m$.

Figure 3.17: Non-positive solution $m$ for $\alpha = 4$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = m$.

$\alpha = 2.5$. However, $m_j^+$ and $m_j^-$ switch places. In Figure 3.17, the solution is the same, but the transition from $m_j^+$ to $m_j^-$ occurs in the maximum of $V$, $x = 0$, instead of the minimum, $x = \frac{1}{2}$.

The solutions in Proposition 3.2.12 are solutions to the MFG (1.3) if $m \geq 0$. In some cases, this condition can fail. In Figure 3.17, $m$ takes negative values.

Moreover, for example, when $\alpha = 3$, the first equation of (3.4) can be solved explicitly. We get:

$$m_{j3}(x) = \frac{H_{j3} - V(x)}{\frac{1}{(j3)^2} - 1}, \quad u_{j3}(x) = \int_0^x j_3 m_{j3}(y)^2 \, dy - p_{j3}x,$$

(3.27)

Figure 3.18: Solution $m$ for $\alpha = 4$, $j = 10$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$, and $g(m) = m$. 
where \( p_{j3} = \int_T j_3 m_{j3}(y)^2 \, dy \) and \( \overline{H}_{j3} = \frac{(j_3)^2}{2} - 1 - \int_T V(x) \, dx \). However, \( m \geq 0 \) only holds if
\[
\dot{j}_3 \leq \sqrt{2} \quad \text{and} \quad \frac{(j_3)^2}{2} - 1 + \int_T V \leq \min V
\]
or if
\[
\dot{j}_3 \geq \sqrt{2} \quad \text{and} \quad \frac{(j_3)^2}{2} - 1 + \int_T V \geq \max V.
\]

3.3 Vanishing Current

In this section, we consider the MFGs (3.2) and (3.4) in the non-current form. First, we examine the problem without congestion, \( \alpha = 0 \). The congestion case is substantially simpler and we address it in Remark 3.3.7.

When \( \alpha = 0 \), we look at (3.2). The last equation in (3.2) implies that either \( m = 0 \) or \( u_x + p = 0 \). If \( m = 0 \), \( u \) solves
\[
\frac{(u_x + p)^2}{2} - g(0) = \overline{H} - V(x).
\]  

(3.28)

If \( u_x + p = 0 \), \( m \) solves
\[
-g(m) = \overline{H} - V(x).
\]  

(3.29)

In the next subsection, we compute solutions for \( g(m) = m \) and \( g(m) = -m \) using the preceding observation.

3.3.1 Increasing Coupling

To simplify, we consider \( g(m) = m \); our results can be extended for other choices of \( g \).

From (3.28) and (3.29), we either have \( m = 0 \) and \( \overline{H} - V(x) \geq 0 \) or \( m > 0 \) and, in this case, \( m = V(x) - \overline{H} \). So, \( m(x) = (V(x) - \overline{H})^+ \) for all \( x \in T \). Then, when \( \overline{H} \) increases, \( m(x) \) decreases. Moreover, when \( \overline{H} \) increases, \( \int_T m(x) \, dx \) increases.
Therefore, there exists a unique $\overline{H}$ such that $\int_T m(x) \, dx = 1$. Moreover, $\int_T V - 1 \leq \overline{H} < \max_T V$.

Thus, we obtain the following proposition,

**Proposition 3.3.1.** Let $g$ be an increasing function. Let $j = 0$. Then, there is a solution $(u, m, \overline{H})$ for (3.2). Moreover,

**Case i.** If $\overline{H} < \min_T V$, $m$ is smooth and positive. Furthermore,

$$m(x) = V(x) - \overline{H}, \ u(x) = 0,$$

where $p = 0$ and $\overline{H} = \int_T V - 1$.

**Case ii.** If $\min_T V < \overline{H} < \max_T V$, $m$ is not smooth, vanishes in some regions, and $m(x) = (V(x) - \overline{H})^+$. Moreover, there are $C^1$ solutions, namely,

$$u^\pm(x) = \pm \int_0^x \sqrt{2(\overline{H} - V(y))^+} \, dy - px,$$

where $p = \pm \int_T \sqrt{2(\overline{H} - V(y))^+} \, dy$ and $\overline{H} = \int_T V(x) \, dx - 1$.

Finally, there are also Lipschitz solutions with $u_x$ discontinuous:

$$(u^{x_0})_x(x) = \sqrt{2(\overline{H} - V(y))^+} \, \chi_{[0,x_0]} - \sqrt{2(\overline{H} - V(y))^+} \, \chi_{[x_0,1)} - px_0,$$

where $p^{x_0} = \int_{y < x_0} \sqrt{2(\overline{H} - V(y))^+} \, dy - \int_{y > x_0} \sqrt{2(\overline{H} - V(y))^+} \, dy$ and $\overline{H} = \int_T V - 1$.

**Proof.** Here, we have two different cases.

**Case i.** If $\overline{H} < \min_T V$, $m > 0$ everywhere. Hence, $m = V(x) - \overline{H}$. Then, $u_x + p = 0$ everywhere. Thus, $p = 0$, because $u$ is periodic and $u$ is constant everywhere. Because $\int m \, dx = 1$, we choose $\overline{H} = \int V \, dx - 1$. 


Case ii. If $\min_T V < \overline{H} < \max_T V$, $m$ vanishes in some regions. We discussed the definition of $m$ in the beginning of the section.

$C^1$ solutions

First, consider the region where $m > 0$. Here, $u_x + p = 0$ and $p = \int_T u_x(y) \, dy = 0$. Moreover, $u = 0$ everywhere. Furthermore, as mentioned before, $m = (V(x) - \overline{H})^+$. We integrate $m$ and get

$$\overline{H} = \int_T V(x) \, dx - 1.$$ 

Second, consider the region where $m = 0$. We solve the first equation in (3.2) and obtain

$$u^\pm(x) = \pm \int_0^x \sqrt{2(\overline{H} - V(y))^+} \, dy - px$$

for all $x \in \mathbb{T}$, where $p = \pm \int_T \sqrt{2(\overline{H} - V(y))^+} \, dy$. Hence, $(u^\pm, m, \overline{H})$ solves (3.2).

Lipschitz solutions

Now, we compute solutions with discontinuous $u_x$. Let $x_0 \in \mathbb{T}$ be such that $V(x_0) < \overline{H}$. Such a point always exists because $V$ is continuous and $\min V < \overline{H}$. Define,

$$(u^{x_0})_x(x) = \sqrt{2(\overline{H} - V(y))^+} \chi_{[0,x_0)} - \sqrt{2(\overline{H} - V(y))^+} \chi_{(x_0,1]} - p^{x_0}$$

for all $x \in \mathbb{T}$, where

$$p^{x_0} = \int_{y < x_0} \sqrt{2(\overline{H} - V(y))^+} \, dy - \int_{y > x_0} \sqrt{2(\overline{H} - V(y))^+} \, dy - p^{x_0}.$$ 

Here, $(u^{x_0})_x$ solves the first equation of (3.2) almost everywhere and has only negative jumps. Because $m$ is continuous, $u^{x_0}$ is a viscosity solution. Therefore, the $(u^{x_0}, m, \overline{H})$ solves (3.2). 

Corollary 3.3.2. The equation (3.2) has a smooth solution if and only if $u_x + p \equiv 0$
or \( m(x) = V(x) - \overline{H} \). The latter happens if \( \int_{\mathbb{T}} V(x) \, dx \leq 1 + \min_{\mathbb{T}} V \).

**Proof.** The result follows directly from the Proposition. \( \square \)

### 3.3.2 Decreasing Coupling

To simplify, we consider \( g(m) = -m \). Once again, the results can be extended for other choices of decreasing \( g \).

We start by proving that \( \overline{H} \) is

\[
\overline{H}_0 = \max \left( \max_{\mathbb{T}} V, \ 1 + \int_{\mathbb{T}} V \right).
\]  
(3.33)

**Proposition 3.3.3.** Let \( g(m) = -m \). The MFG (3.2) does not have solutions for \( \overline{H} > \overline{H}_0 \).

**Proof.** Assume \( \overline{H} > \overline{H}_0 \), and that \((u, m, H)\) is a solution of (3.2).

First, consider the region where \( m > 0 \). Here, \( u_x + p = 0 \), which gives \( m = \overline{H} - V(x) \).

Next, consider the region where \( m = 0 \). Then, we have \( \frac{(u_x + p)^2}{2} = \overline{H} - V(x) \). Thus,

\[
u_x(x) = \pm \sqrt{2 \left( \overline{H} - V(x) \right)} - p.
\]  
(3.34)

If the region where \( m = 0 \) has zero Lebesgue measure, then

\[
\int_{\mathbb{T}} m(x) \, dx > \overline{H}_0 - \int_{\mathbb{T}} V(x) \, dx \geq 1,
\]

which contradicts \( \int m = 1 \). Hence, the region has positive Lebesgue measure.

Note that \( \overline{H} - V(x) > \overline{H}_0 - V(x) \geq \max V - V \geq 0 \). Hence, \( u_x \) as defined in (3.34) is non-zero. Therefore, \( u_x \) take either the positive or negative root in (3.34) when \( m = 0 \).
Assume $u_x$ takes the negative value at $x = 0$, and define

$$z = \sup \left\{ x \in (0, 1) \text{ such that } u_x(x) = -\sqrt{2 \left( \bar{H} - V(x) \right) - p} \right\}.$$ 

Hence, $z$ is the “last point” in $(0, 1)$ where $u_x$ takes the negative value. So, there is a jump to zero (if $m > 0$) or the positive value (if $m = 0$). In both cases, it is a positive jump, which contradicts $u_x$ being a solution.

Therefore, $u_x$ only takes values $0$ and $\sqrt{2 \left( \bar{H} - V(x) \right) - p}$. So, a positive jump must exist some time from zero to the positive value, which also contradicts the regularity of $u_x$.

\[\square\]

Note that $\bar{H} \geq \bar{H}_0$. First, because $m \geq 0$, the first equation in (3.2) implies $\bar{H} - V(x) \geq 0$. Thus $\bar{H} \geq \max V$. Second, because $\frac{(u_x + p)^2}{2} \geq 0$, then $m \leq \bar{H} - V(x)$.

We integrate this expression and get $\bar{H} \geq 1 + \int_{\bar{T}} V$.

Since $\bar{H} \geq \bar{H}_0$ and there are no solutions for $\bar{H} > \bar{H}_0$, $\bar{H} = \bar{H}_0$. In fact, when $\bar{H} = \bar{H}_0$, there are solutions. We construct them in the next proposition.

**Proposition 3.3.4.** Let $g$ be a decreasing function. Let $j = 0$. Suppose $V$ has a single maximum in $x = 0$. Then, there is a solution $(u_0, m_0, \bar{H}_0)$ for (3.2). Moreover,

**Case i.** If $1 + \int_{\bar{T}} V \geq \max V$, $(u_0, m_0, \bar{H}_0)$ is a solution in the classical sense. Furthermore,

$$m_0(x) = \bar{H}_0 - V(x), \ u_0(x) = 0,$$  \hspace{1cm} (3.35)

where $p = 0$. 

Case ii. If \( 1 + \int_T V < \max \overline{V} \), there are triplets \((u_0^{d_1,d_2}, m_0^{d_1,d_2}, H_0^{d_1,d_2})\) that are solutions for each \((d_1,d_2)\) that satisfy

\[
\int_{d_1}^{d_2} \overline{H} - V(x) \, dx = 1. \tag{3.36}
\]

Moreover,

\[
m_0^{d_1,d_2}(x) = (\overline{H} - V(x)) \chi_{[d_1,d_2]}, \quad u_0^{d_1,d_2}(x) = \int_0^x (u_0^{d_1,d_2})_x(y) \, dy \tag{3.37}
\]

where

\[
(u_0^{d_1,d_2})_x(x) = \sqrt{2(\overline{H} - V(y))} \chi_{[0,d_1)} - \sqrt{2(\overline{H} - V(y))} \chi_{(d_2,1)} - p_0^{d_1,d_2} \tag{3.38}
\]

and \(p_0^{d_1,d_2} = \int_0^{d_1} \sqrt{2(\overline{H} - V(y))} \, dy - \int_{d_2}^{1} \sqrt{2(\overline{H} - V(y))} \, dy\).

Proof. We prove each case separately.

Case i. In this case, \( \overline{H}_0 = 1 + \int_T V \). Replacing \( m_0, p \) and \( u_0 \) in (3.2), we see that (3.2) holds in the classical sense.

Case ii. In this case, \( \overline{H}_0 = \max \overline{V} \). Moreover, \( \overline{H}_0 - \int_T V > 1 \). So, \( m = 0 \) in some region. Where \( m = 0 \), \((u_0)_x \) solves the first equation of (3.2) almost everywhere. Moreover, \((u_0)_x \) has only negative jumps as required. We have \( \int m = 1 \) as long as (3.36) holds. For \( d_1 = 0, d_2 = 1 \), (3.36) is greater than one, and for \( d_1 = d_2 \), (3.36) is zero, then there are infinitely many pairs \((d_1,d_2)\) for which (3.36) holds.

In Figures (3.19) and (3.20), we show the solutions of the non-vanishing case with \( g \) decreasing when \( V(x) = 5 \sin(2\pi(x + \frac{1}{4})) \).

Remark 3.3.5. If \( V \) has multiple maxima at \( x = 0 \) and \( x = x_0 \) and Case ii. in Proposition 3.3.4 holds, there is even a larger family of solutions. We define
Figure 3.19: \( m_0 \) as defined in (3.37) for \( V(x) = 5\sin(2\pi(x + \frac{1}{4})) \) with \( d_2 = 0.5 \) and \( d_1 \) such that (3.36) holds.

Figure 3.20: \( u_0 \) (left) and \((u_0)_x\) (right) as defined in (3.38) and (3.38) for \( V(x) = 5\sin(2\pi(x + \frac{1}{4})) \) with \( d_2 = 0.5 \) and \( d_1 \) such that (3.36) holds.

that family of solutions in the next proposition. The proof follows similar ideas to Proposition 3.3.4.

**Proposition 3.3.6.** Let \( g \) be a decreasing function. Let \( j = 0 \). Suppose \( V \) has a two maxima, \( x = 0 \) and \( x = x_0 \). Then, there is a solution \((u_0, m_0, \overline{H}_0)\) for (3.2)). Moreover, if \( 1 + \int_T V < \max_T V \), there are \((u_{0,d_1,d_2,e_1,e_2}, m_{0,d_1,d_2,e_1,e_2}, \overline{H}_{0,d_1,d_2,e_1,e_2})\) that are solutions for each \((d_1, d_2, e_1, e_2)\) such that

\[
\int_{d_1}^{d_2} \overline{H}_0 - V(x) \, dx + \int_{e_1}^{e_2} \overline{H}_0 - V(x) \, dx = 1. \tag{3.39}
\]

Moreover,

\[
m_{0,d_1,d_2,e_1,e_2}(x) = (\overline{H}_0 - V(x)) \chi_{[d_1,d_2]\cup[e_1,e_2]}, \tag{3.40}
\]

and

\[
u_{0,d_1,d_2,e_1,e_2}(x) = \int_0^x (u_{0,d_1,d_2,e_1,e_2})_x(y) \, dy. \tag{3.41}
\]
where

\[(u^{d_1,d_2,e_1,e_2}_0)_x(x) = \sqrt{2(H_0 - V(y))} \chi_{[0,d_1] \cup [x_0,e_1]} - \sqrt{2(H - V(y))} \chi_{[d_2,x_0] \cup [e_2,1]} - p^{d_1,d_2,e_1,e_2}_0 \]

and

\[p^{d_1,d_2,e_1,e_2}_0 = \int_{[0,d_1] \cup [x_0,e_1]} \sqrt{2(H_0 - V(y))} \, dy - \int_{[d_2,x_0] \cup [e_2,1]} \sqrt{2(H - V(y))} \, dy.\]

**Remark 3.3.7.** We consider the second system in (3.4) with \(\alpha > 0\) and make no assumptions on the monotonicity of \(g\). Moreover, we suppose \(V\) is not constant. The congestion case is simpler than the non-congestion case due to the \(\frac{1}{m}\) term.

First, we assume that \(m = 0\). Then, for the equation to hold, we need \(u_x + p = 0\). So, we get \(-g(0) = H - V(x)\). We have no solution \(H\) unless \(V\) is constant. Hence, \(m > 0\).

When \(m > 0\), for (3.4) to hold, \(u_x + p = 0\). Therefore, we get \(m = g^{-1}(V(x) - H)\), provided \(g^{-1}\) exists. Hence, we obtain the following proposition:

**Proposition 3.3.8.** Let \(g\) be an invertible function. Let \(j_\alpha = 0\). Suppose \(V\) is not constant. If there is \(H\) such that \(m_{0,j_\alpha} > 0\) and \(\int m = 1\), then there is a solution \((u_{0,j_\alpha}, m_{0,j_\alpha}, H)\) for (3.4). Moreover,

\[m_{0,j_\alpha} = g^{-1}(V(x) - H) > 0, \; u_{0,\alpha} = 0,\]

where \(p = 0\) and \(H\) is such that \(\int m = 1\).

### 3.4 Discontinuous Viscosity Solutions

In Proposition (3.2.6), we saw that \(m\) can be discontinuous when certain conditions hold. However, at that moment, we were looking for semiconcave solutions. Here, we drop that requirement and consider viscosity solutions for discontinuous
Hamilton-Jacobi equations. That is, the condition $\lim_{x \to x^-} u_x(x) \geq \lim_{x \to x^+} u_x(x)$ is not necessary anymore.

We use the same definitions as in [32]. First, we define upper and lower semiconcave continuous envelopes.

**Definition 3.4.1.** Given $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ locally bounded, we define the *lower semiconcave continuous envelope*

$$F_*(x, q) = \liminf_{(y, r) \to (x, q)} F(y, r), \quad (3.42)$$

and the *upper semiconcave continuous envelope*

$$F^*(x, q) = \limsup_{(y, r) \to (x, q)} F(y, r), \quad (3.43)$$

where $(x, q) \in \mathbb{T} \times \mathbb{R}$

Next, we define *discontinuous viscosity solutions.*

**Definition 3.4.2.** A function $u : \mathbb{T} \to \mathbb{R}$ is a *discontinuous viscosity solution* of $F(x, Du) = 0$ if

i. $u$ is locally bounded;

ii. For every smooth functions, $\phi : \mathbb{T} \to \mathbb{R}$, $F_*(x, \phi_x) \leq 0$ for all $x \in \text{argmax}(u - \phi)$.

iii. For every smooth functions, $\phi : \mathbb{T} \to \mathbb{R}$, $F^*(x, \phi_x) \geq 0$ for all $x \in \text{argmin}(u - \phi)$.

Now, define

$$\begin{align*}
m_*(x) &= \liminf_{y \to x} m(y) \\
m^*(x) &= \limsup_{y \to x} m(y)
\end{align*} \quad (3.44)$$
for \( m : \mathbb{T} \to \mathbb{R} \), with \( m \in L^\infty \).

For (1.2), we have

\[
F(x, q) = \frac{q^2}{2} + V(x) + m(x) - \overline{H}.
\]

Moreover,

\[
F^*(x, q) = \frac{q^2}{2} + V(x) + m^*(x) - \overline{H}, \quad F_*(x, q) = \frac{q^2}{2} + V(x) + m_*(x) - \overline{H}
\]

Now, we search for piecewise smooth solutions for this problem and find that there are infinitely many for all \( j \neq 0 \). Recall that we made no assumptions on \( V \) except continuity. So for \( V \) with a single maximum, the next result yields distinct solutions from the ones in Proposition (3.2.6).

**Proposition 3.4.3.** Let \( g(m) = -m \). Let \( j > 0 \). Suppose \( V \) is continuous. Then, there is a solution \((u, m, \overline{H})\) for (1.2). Moreover, fix arbitrary points \( 0 \leq x_0 < \cdots < x_n \leq 1 \) and assume

i. \( m > 0 \) is continuous on \((x_i, x_{i+1})\) and solves (3.1).

ii. \( \overline{H} \) is such that \( \int m = 1 \).

Then,

\[
u(x) = \int_0^x \frac{j}{m(y)} \, dy - px, \quad p = \int_{\mathbb{T}} \frac{j}{m(y)} \, dy. \tag{3.45}\]

**Proof.** The second equation of (1.2) holds in the sense of distributions since the definition of \( u \) yields \( u_x + p = \frac{j}{m} \) almost everywhere.

Observe that \( u \) is differentiable for \( x \neq x_i \) and the first equation in (1.2) is satisfied in the classical sense. So, \( u \) is a viscosity solution at points \( x \neq x_i \). We need to prove this is also true for \( x = x_i \).
We have two possible cases: \( m(x_i^-) > m(x_i^+) \) and \( m(x_i^-) < m(x_i^+) \). We prove only the first one, as the second is analogous.

Here, \( m^*(x_i) = m(x_i^-) \) and \( u_x(x_i^-) < u_x(x_i^+) \) This means there is no \( \phi \in C^\infty \) that touches \( u \) from above at \( x_i \).

So, take any \( \phi \) touching \( u \) from below at \( x_i \). Because \( j > 0 \) and \( \phi(x) \leq u(x) \), \( 0 < u_x(x_i^-) + p \leq \phi_x(x_i) + p \leq u_x(x_i^+) + p \). Hence,

\[
\frac{(\phi_x(x_i) + p)^2}{2} + V(x_i) + m(x_i^-) - \overline{H} \geq \frac{(u_x(x_i^-) + p)^2}{2} + V(x_i) + m(x_i^-) - \overline{H} = 0.
\]

since (1.2) is satisfied in the classical sense for \( x \neq x_i \).

Note that the above proof can be easily adapted for \( \alpha > 0 \), so we get the following proposition.

**Proposition 3.4.4.** Let \( g(m) = -m \). Let \( j_\alpha > 0 \). Suppose \( V \) is continuous. Then, there is a solution \((u, m, \overline{H})\) for (1.3). Moreover, fix arbitrary points \( 0 \leq x_0 < \cdots < x_n \leq 1 \) and assume

i. \( m > 0 \) is continuous on \((x_i, x_{i+1})\) and solves (3.4).

ii. \( \overline{H} \) is such that \( \int m = 1 \).

Then,

\[
u(x) = \int_0^x \frac{j_\alpha}{m(y)^{1-\alpha}} \, dy - px, \quad p = \int_T \frac{j_\alpha}{m(y)^{1-\alpha}} \, dy.
\]

**Remark 3.4.5.** If we consider \( g \) to be increasing, then, \( m(x_i^-) = m(x_i^+) \), necessarily. Therefore, the solutions found in Proposition 3.2.1 are unique, even when considering nonsemiconcave solutions.
Chapter 4

Regularizations

In the previous chapter, we obtained explicit solutions for (1.2). In this chapter, we are interested in the solutions of the corresponding regularization, (1.6). Moreover, we examine the behavior of solution as \( \varepsilon \to 0 \), as defined in (1.6). We find that solutions of (1.6) converge to solutions of (1.2) when \( g \) is monotonically increasing. When \( g \) decreases, convergence does not hold. Here, in contrast with Chapter 3, we always consider \( j \in \mathbb{R} \).

First, in Section (4.1), we introduce the current formulation for the regularity problem (1.6). Then, in Section 4.2, we interpret the current form of (1.6) as an Euler-Lagrange equation. Finally, in sections 4.4 and 4.5, we study the cases of increasing and decreasing \( g \), respectively.

4.1 The Current Formulation

We define \( j_\varepsilon \in \mathbb{R} \) for (1.6) as

\[
j_\varepsilon = \varepsilon m_x + m(u_x + p).
\]  

(4.1)

Then, we replace \( u_x + p \) by its equivalent in terms of \( j_\varepsilon \). We set Define

\[
F_{j_\varepsilon}(m) = \varepsilon^2 \frac{m_{xx}}{m} - \frac{1}{2} \varepsilon^2 \frac{m_x^2}{m^2} + \frac{j_\varepsilon^2}{2m^2} - g(m).
\]
Then, (1.6) is equivalent to

\[
\begin{cases}
F_{j_\varepsilon}(m) = \mathcal{H} - V(x), \\
m > 0, \ \int_{\mathcal{T}} m \ dx = 1, \\
\int_{\mathcal{T}} \frac{1}{m} \ dx = \frac{p}{j_\varepsilon}.
\end{cases}
\] (4.2)

If we solve (4.1) for \( u_x \), we get

\[
u_x = \frac{j_\varepsilon - \varepsilon m_x}{m} - p.
\]

Then, we replace \( u_x \) in (1.6), and we get

\[-\varepsilon \left( \frac{j_\varepsilon - \varepsilon m x}{m} \right)_x + \frac{1}{2} \left( \frac{j_\varepsilon - \varepsilon m x}{m} \right)^2 - g(m) = \mathcal{H} - V(x),
\]

which is equivalent to the first equation in (4.2).

### 4.2 Euler-Lagrange Equation

As in Chapter 2, \( L \) is the Lagrangian. We define a functional, \( I \), as

\[
I[w] = \int_{\mathcal{T}} L(Dw(x), w(x), x) \ dx.
\] (4.3)

Suppose there exists a minimizer \( \bar{w} \in H^1_0(\mathbb{T}) \) with constraints \( \int_{\mathcal{T}} w = 0 \) and \( w \geq 0 \).

Define

\[
\mathcal{A} = \left\{ w \in H^1_0(\mathbb{T}) \text{ such that } \int_{\mathcal{T}} w = 1 \wedge w \geq 0 \right\}
\] (4.4)

Hence, define \( \bar{w} \) such that:

\[
I[\bar{w}] = \min_{w \in \mathcal{A}} I[w].
\]

Then, by a simple calculus of variations argument, we prove that \( \bar{w} \) is the solution
of a certain non-linear PDE. For the derivation see Chapter 8.1.2 and 8.4.1 in [39].

We call this PDE the Euler-Lagrange (EL) equation:

\[-(L_p(D\bar{w},\bar{w},x))_x + L_v(D\bar{w},\bar{w},x) = 0.\]

(4.5)

Next, we see that (4.2) is the EL equation of a functional, $I_{j\varepsilon}$. If we find minimizers for $I_{j\varepsilon}$, we solve (4.2).

To construct $I_{j\varepsilon}$, let $\Phi_j$ be such that $\Phi_j'(m) = \frac{j^2}{2m^2} - g(m)$. Then, if $G'(m) = g(m)$,

$$\Phi_j(m) = -\frac{j^2}{2m} - G(m).$$

Hence,

$$I_{j\varepsilon} = \int_T \frac{1}{2} \varepsilon^2 \frac{m_x^2}{m^2} - \Phi_{j\varepsilon}(m) - V(x)m \, dx.$$  

Equation (4.2) is the EL equation of the above functional with the constraints $\int m = 1$ and $m \geq 0$.

### 4.3 Gamma-Convergence

In the next two sections, we study the limit $\varepsilon \to 0$. Recall that when $\varepsilon = 0$, we have (3.1). We write the associated functional, $I_j$ as

$$I_j[w] = \int_T -\Phi_j(m) - V(x)m.$$

We examine the convergence of solutions by studying the limit of $I_{j\varepsilon}[w]$ to $I_j[w]$. Then, we choose to study the convergence using $\Gamma$-convergence. This is the appropriate notion to prove that minimizers of $I_{j\varepsilon}$ converge to minimizers of $I_j$ when $\varepsilon \to \infty$.

**Definition 4.3.1.** We say that a given sequence $J_\varepsilon : C^\infty(\mathbb{T}) \to \mathbb{R}$ $\Gamma$-converges to $J : C^\infty(\mathbb{T}) \to \mathbb{R}$, $J_\varepsilon \xrightarrow{\Gamma} J_0$, if for all $w \in C^\infty(\mathbb{T})$ we have
i. For every $w_\varepsilon \to w$,

$$J[w] \leq \liminf_{\varepsilon \to 0} J_\varepsilon[w_\varepsilon];$$

ii. There exists a sequence $w_\varepsilon \to w$ such that,

$$J[w] \geq \limsup_{\varepsilon \to 0} J_\varepsilon[w_\varepsilon].$$

**Remark 4.3.2.** For more results regarding $\Gamma$-convergence, see [40].

### 4.4 Increasing Coupling

Here, we consider $g(m) = m$ to simplify our presentation. We denote $I_{j_\varepsilon}^+$ by the functional $I_{j_\varepsilon}$ corresponding to $g(m) = m$,

$$I_{j_\varepsilon}^+[m] = \int_\mathbb{T} \left( \varepsilon^2 \frac{m^2}{2m} + \frac{j_\varepsilon^2}{2m} + \frac{m^2}{2} - V(x)m \right) \, dx.$$

Since $I_{j_\varepsilon}^+$ is a convex functional, there is a unique minimizer to the variational problem:

**Proposition 4.4.1.** For all $j_\varepsilon \in \mathbb{R}$, there exists a unique minimizer $m$ of $I_{j_\varepsilon}^+$ in

$$\mathcal{A} = \left\{ m \in W^{1,2}(\mathbb{T}) : m \geq 0 \land \int_\mathbb{T} m = 1 \right\}.$$

Moreover, $m > 0$ and solves $F_{j_\varepsilon}(m) = \overline{H} - V(x)$ for some constant $\overline{H} \in \mathbb{R}$.

**Proof.** To prove this result, we need to treat the cases $j \neq 0$ and $j = 0$ separately.

**Case i.** First, we consider $j \neq 0$. Take a minimizing sequence $\{m_n\} \in \mathcal{A}$. Then, there exists $C \in \mathbb{R}$ such that $I_{j_\varepsilon}^+ \leq C$. Hence,

$$\int_\mathbb{T} \varepsilon^2 \frac{(m_n)^2}{2m_n} + \frac{j^2}{2m_n} \, dx \leq C.$$
Note \( \frac{(m_n)^2}{m_n} = (\sqrt{m})^2 \) and \( \int m = 1 \). Hence, by Morrey’s inequality,

\[
||\sqrt{m}||_{C^{0,\frac{1}{2}}(\mathbb{R})} \leq ||\sqrt{m}||_{W^{1,2}(\mathbb{R})} = 1 + \int_T \frac{(m_n)^2}{m_n} \leq C.
\]

Therefore, the functions \( \sqrt{m_n} \) are equi-Hölder continuous of exponent \( \frac{1}{2} \). Since \( \int m = 1 \), the sequence is also equibounded. From the Arzela-Ascoli theorem, it follows that there exist a subsequence, \( m_{n_k} \), that converges uniformly, \( m_{n_k} \to m \). By Fatou’s lemma,

\[
\int \frac{1}{m} = \int_T \liminf_{k \to \infty} \frac{1}{m_{n_k}} \leq \liminf_{k \to \infty} \int_T \frac{1}{m_{n_k}} \leq C, \tag{4.6}
\]

where the limit \( m \) solves the corresponding EL equation.

Now, we argue by contradiction to prove \( m > 0 \). Assume \( \min_T m = m(x_0) = 0 \). Then, \( \left| \sqrt{m(x)} - \sqrt{m(x_0)} \right| \leq C|x - x_0|^\frac{1}{2} \). Hence, \( m(x) \leq C|x - x_0| \) for all \( x \in T \). From (4.6),

\[
\int_T \frac{1}{|x - x_0|} \, dx \leq C.
\]

However, this value cannot be finite. This contradiction implies \( m \) is a strictly positive minimizer.

**Case ii.** We consider \( j = 0 \). We rewrite the EL equation \( F_{j\epsilon}(m) = \overline{H} - V(x) \) as

\[
-\epsilon^2 (\ln m)_{xx} + m - V(x) = \overline{H}. \tag{4.7}
\]

Define

\[
\mathcal{P} = \{ f \in L^\infty(\mathbb{T}) : f \geq 0 \}.
\]

Consider the map \( \Xi : \mathcal{P} \to \mathcal{P} \). The function \( \Xi \) takes a function \( \eta \in \mathcal{P} \) and returns the solution to

\[
-\epsilon^2 w_{xx} + \eta - V(x) = -\overline{H}. \tag{4.8}
\]
Figure 4.1: Solution \( m \) when \( g(m) = m \), \( j = 1 \), \( V(x) = \sin(2\pi(x+1/4)) \) for \( \epsilon = 0.01 \) (dashed) and for \( \epsilon = 0 \) (solid).

In (4.8), \( \overline{H} \) satisfies the compatibility condition \( \overline{H} = \int V \, dx - 1 \) and \( w : \mathbb{T} \to \mathbb{R} \) is such that \( \int e^w = 1 \). Moreover, because of the constraints on \( m \), \( w \) is uniformly bounded from above and below.

Next, we set \( \Xi(\eta) = e^w \). Since \( \Xi \) is continuous and compact, by Schauder’s fixed point theorem, there is a fixed point, \( m \), that solves (4.11).

**Remark 4.4.2.** For \( j_\epsilon \in \mathbb{R} \), by the convexity of \( I_{\epsilon}^+ \), \( m \) is the unique positive minimizer.

Next, we study the convergence as \( \epsilon \to 0 \).

**Proposition 4.4.3.** Assume \( m \geq 0 \) and \( \int m = 1 \). Then, \( I_{j_\epsilon}^+ \overset{\Gamma}{\to} I_{j}^+ \), where

\[
I_{j}^+ = \int_\mathbb{T} \left( \frac{j^2}{2m} + \frac{m^2}{2} - V(x)m \right) \, dx.
\]

**Proof.** The proof follows the argument in Chapter 6 of [40]. \( \square \)

In Figure 4.1 we observe how the solution \( m \) of (1.6) when \( g \) increases tends to the solution of (1.2) when \( \epsilon \to 0 \).
4.5 Decreasing Coupling

Here, we consider $g(m) = -m$. The functional $I_{j\varepsilon}$ becomes

$$I_{j\varepsilon}[m] = \int_T \left( \varepsilon^2 \frac{m^2}{2m} + \frac{j^2}{2m} - \frac{m^2}{2} - V(x)m \right) \, dx.$$

Next, we prove that there is a minimizer to the associated variational problem.

**Proposition 4.5.1.** For all $j \in \mathbb{R}$, there exists a minimizer $m$ of $I_{j\varepsilon}$ in

$$\mathcal{A} = \left\{ m \in W^{1,2}(T) : m \geq 0 \land \int_T m = 1 \right\}.$$

Moreover, $m$ solves $F_{j\varepsilon}(m) = H - V(x)$ for some constant $H \in \mathbb{R}$.

**Proof.** We prove separately for $j \neq 0$ and $j = 0$.

**Case i.** First, we consider $j \neq 0$. Take a minimizing sequence $\{m_n\} \in \mathcal{A}$. There is $C \in \mathbb{R}$ such that $I_{j\varepsilon} \leq C$. Therefore,

$$\int_T \varepsilon^2 \frac{(m_n)^2}{2m_n} + \frac{j^2}{2m_n} \, dx \leq C + \int_T \frac{m^2}{2}.$$

We seek to control $\int m_n^2$ by some part of the expression on the left side. We use the Gagliardo-Nirenberg (GN) inequality to bound $\int m^2$. Remember that, by GN inequality,

$$||w||_{L^p} \leq ||w_x||_{L^r} ||w||_{L^q}^{1-a},$$

where $0 \leq a \leq 1$, $\int w = 0$ and

$$\frac{1}{p} = a \left( \frac{1}{r} - 1 \right) + (1 - a) \frac{1}{q}.$$
Figure 4.2: Solution $m$ when $g(m) = -m$, $j = 0.001$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$ for $\epsilon = 0.01$ (dashed) and for $\epsilon = 0$ (solid).

If $p = 4, r = q = 2$, then $a = \frac{1}{4}$. We use this values, choose $w = \sqrt{m}$, and get

$$\int_T m_n^2 \leq C + C \left(\int_T \frac{(m_n)_x^2}{2m_n}\right)^{\frac{1}{2}}. \tag{4.10}$$

Thus, using a weighed Cauchy inequality, we obtain

$$\int_T \frac{(m_n)_x^2}{2m_n} + \frac{j^2}{2m_n} dx \leq C,$$

the same result as in Proposition 4.4.1. We argue as in that proposition and prove existence.

**Case ii.** Now, we consider $j = 0$. We rewrite the EL equation $F_{\epsilon^j}(m) = H - V(x)$ as

$$-\epsilon^2 (\ln m)_{xx} - m - V(x) = \bar{H}. \tag{4.11}$$

Then, we use the Schauder’s fixed point theorem as in Proposition 4.4.1 and argue as before to prove existence.

**Remark 4.5.2.** Because $I_{\epsilon^j}$ is not convex, we cannot guarantee uniqueness.

Next, we study the convergence as $\epsilon \to 0$. Here, numerical evidence, such as in Figure 4.2 and 4.3 suggests there is no $\Gamma$-convergence to a minimizer when $j$ is small.
Figure 4.3: Solution $m$ when $g(m) = -m$, $j = 1$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$ for $\epsilon = 0.01$ (dashed) and for $\epsilon = 0$ (solid).

Figure 4.4: Solution $m$ when $g(m) = -m$, $j = 100$, $V(x) = \frac{1}{2} \sin(2\pi(x + 1/4))$ for $\epsilon = 0.01$ (dashed) and for $\epsilon = 0$ (solid).

However, in Figure 4.4, it seems that, for large $j$, we have $\Gamma$-convergence.
Chapter 5

Properties of Solutions

In this chapter, we focus on the behavior of solutions. Recall that, in Chapter 3, we developed explicit formulas for the solutions of (1.2) and (1.3). Here, we are interested in the case where $g$ decreases. Moreover, we consider (1.3) with $\alpha < 2$.

5.1 Regularity Regimes

Here, we analyze the regularity regimes of (3.1). We consider the case where $g$ is decreasing, using $g(m) = -m$ when appropriate. Furthermore, we focus on Case iii. of Proposition (3.2.6). We found in Chapter 3 that solutions may or may not be smooth. There, we had conditions on $\phi^+$ and $\phi^-$. Now, we find equivalent conditions on $j$.

First, we assume $j \geq 0$ and prove that $\phi^+$ and $\phi^-$ as defined in (3.12) are monotone functions.

**Proposition 5.1.1.** Suppose $V$ has a unique maximum at $x = 0$. Consider $\phi^+$ and $\phi^-$ as defined in (3.12). Let $\alpha < 2$. Then,

i. $\phi^+(j), \phi^-(j), \phi^+(j_\alpha)$ and $\phi^-(j_\alpha)$ are increasing on $(0, \infty)$;

ii. $\lim_{j \to +\infty} \phi^+(j) = \lim_{j_\alpha \to +\infty} \phi^-(j) = \lim_{j_\alpha \to +\infty} \phi^+(j_\alpha) = \lim_{j_\alpha \to +\infty} \phi^-(j_\alpha) = \infty$;

iii. $\lim_{j \to 0} \phi^+(j) = \lim_{j_\alpha \to 0} \phi^+(j_\alpha) = \max_{\mathbb{T}} V - \int_{\mathbb{T}} V(x) \, dx$, $\lim_{j \to 0} \phi^-(j) = \lim_{j_\alpha \to 0} \phi^-(j_\alpha) = 0$;

**Proof.** We prove each result separately.
i. The functions $\phi^\pm(j)$ increase if the integral of $m_j^\pm$ increase. First, we assume $\alpha = 0$. In fact, when $j$ increases, so does $m_j^\pm(x)$ for a fixed $x \in T$. To prove that, fix $x \in T$.

Recall that we are in Case iii of Proposition 3.2.6. Then, $\overline{H}_j = \overline{H}_j^{cr} = \frac{3}{2}j^2 + \max V$.

Then, if $x = 0$, $V(x) = \max V$. The function $m_j^\pm(0) = j^2$ is the minimum of $F_j$.

This function is increasing in $j$.

If $x \neq 0$, $V(x) < \max T$. Then, $m_j^-(x) < j^2 < m_j^+(x)$. Rewrite $F_j(t(j)) = \overline{H}_j^{cr} - V(x)$ as

$$\frac{j^2}{2t(j)^2} + t(j) = \frac{3}{2}j^2 + \max V - V(x).$$

By the implicit function theorem, $t(j)$ is differentiable and

$$t'(j) = \frac{j^{-\frac{1}{2}} - \frac{i}{t(j)^2}}{1 - \frac{j^2}{t(j)^2}}.$$

Because $m_j^-(x) < j^2 < m_j^+(x)$, $t'(j) > 0$. Therefore, $t(j)$ is increasing.

Next, we consider $\alpha < 2$. Thus, $c_\alpha > 0$. We are in Case iii. of Proposition 3.2.9.

Then, we repeat the above reasoning with the appropriate replacements. For $x \neq 0$, we write $\frac{j^2}{2t(j)^{2-\alpha}} + t(j) = \overline{H}_j^{cr} - V(x)$ and get:

$$t'(j) = \left(\frac{\partial}{\partial J_\alpha} \overline{H}_j^{cr} - \frac{j_\alpha}{t(j)^{2-\alpha}}\right) \frac{1}{1 - \frac{j_\alpha}{t(j)^{2-\alpha}}}.$$

Because $m_j^-(x) < c_\alpha j_\alpha^{\frac{2}{2-\alpha}} < m_j^+(x)$, by tedious, but simple computations, $t'(j) > 0$. Therefore, $t(j)$ is increasing.

ii. First, consider $\alpha = 0$. As seen before, $m_j^+ \geq j^2$. So, $\lim_{j \to \infty} m_j^+(x) = \infty$ for all $x \in T$. Hence, we get the limit for $\phi^+$. 
Moreover, recall $m_j^- \leq j^{\frac{2}{3}}$. Then, $\frac{j^2}{(m_j^-)^2} \geq j^{\frac{2}{3}}$. Therefore,

$$\lim_{j \to \infty} m_j^- (x) = \lim_{j \to \infty} \frac{3}{2} j^{\frac{2}{3}} - \frac{j^2}{(m_j^-(x))^2} + \max V - V (x) \geq \lim_{j \to \infty} \frac{1}{2} j^{\frac{2}{3}} + \max V - V (x) = \infty,$$

and we get the result for $\phi^-$. 

Next, consider the congestion case. For $\alpha < 2$, we have $m_j^+ \geq c_\alpha j^{\frac{2}{3-\alpha}}$. So, 

$$\lim_{j \to \infty} m_j^+ (x) = \infty$$

since $c_\alpha > 0$ and $\alpha < 2$ and we get the result for $\phi^+$. Furthermore, we have $m_j^- \leq c_\alpha j^{\frac{2}{3-\alpha}}$. Then, $\frac{j^2}{(m_j^-)^2} \geq c_\alpha^{\alpha-2} j^{2-2\frac{2}{3-\alpha}}$. Therefore, $\lim_{j \to \infty} m_j^- (x) = \infty$, since the exponents of $j$ are always positive, and we get the limit for $\phi^-$. 

**iii.** First, assume $\alpha = 0$. As seen before, $m_j^- \leq j^{\frac{2}{3}}$. So, $\lim_{j \to 0} m_j^- (x) = 0$ for all $x \in \mathbb{T}$, and we get the limit for $\phi^-$. Next, recall $m_j^+ \geq j^{\frac{2}{3}}$. Then, $0 \leq \frac{j^2}{(m_j^+)^2} \leq j^{\frac{2}{3}}$. Therefore,

$$\lim_{j \to 0} m_j^+ (x) = \lim_{j \to 0} \frac{3}{2} j^{\frac{2}{3}} - \frac{j^2}{(m_j^+(x))^2} + \max V - V (x) = \max V - V (x),$$

and we get the result for $\phi^-$ by integration.

For $\alpha < 2$, the proof is analogous. \qed

Now, we define two fundamental quantities that characterize the regularity regimes.

**Definition 5.1.2.** We define numbers $j_{\text{lower}}, j_{\text{upper}}, j_{\alpha,\text{lower}}$ and $j_{\alpha,\text{upper}}$ as

$$
\begin{align*}
j_{\text{lower}} &= \inf \{ j > 0 : \phi^+ (j) > 1 \} \\
j_{\text{upper}} &= \inf \{ j > 0 : \phi^- (j) > 1 \} \\
j_{\alpha,\text{lower}} &= \inf \{ j_\alpha > 0 : \phi^+ (j_\alpha) > 1 \} \\
j_{\alpha,\text{upper}} &= \inf \{ j_\alpha > 0 : \phi^- (j_\alpha) > 1 \}.
\end{align*}

(5.1)

Then, we examine how these numbers behave.

**Corollary 5.1.3.** Let $j_{\text{lower}}, j_{\text{upper}}, j_{\alpha,\text{lower}}$ and $j_{\alpha,\text{upper}}$ be given by (5.1). Then,

i. $0 \leq j_{\text{lower}} < j_{\text{upper}} < \infty$ and $0 \leq j_{\alpha,\text{lower}} < j_{\alpha,\text{upper}} < \infty$;
ii. for $j \geq j_{\text{upper}}$, the system (1.2) has smooth solutions; for $j_{\alpha} \geq j_{\alpha,\text{upper}}$, the system (1.3) has smooth solutions;

iii. for $j_{\text{lower}} < j < j_{\text{upper}}$, the system (1.2) has only discontinuous solutions; for $j_{\alpha,\text{lower}} < j < j_{\alpha,\text{upper}}$, the system (1.3) has only discontinuous solutions;

iv. if $j_{\text{lower}} > 0$, the system (1.2) has smooth solutions for $0 < j \leq j_{\text{lower}}$. If $j_{\alpha,\text{lower}} > 0$, the system (1.3) has smooth solutions for $0 < j_{\alpha} \leq j_{\alpha,\text{lower}}$.

**Proof.** This follows from Proposition 5.1.1 combined with Proposition 3.2.6 and 3.2.9.

Finally, we study what happens when $j = 0$:

**Proposition 5.1.4.** Assume $j = 0$. Let $g$ be decreasing. Then, equation (3.2) admits smooth solutions if and only if $\phi^+(0) \leq 1$.

**Proof.** The proof follows from Proposition 5.1.1, and Proposition 3.3.4.

If we define $V(x) = A \sin(2\pi(x + 1/4))$, we can plot $\phi^+$ and $\phi^-$ in terms of $j$. In Figure (5.1), we see that, if $A = 0.5$, $j_{\text{lower}} > 0$. Moreover, if $A = 5$, then $j_{\text{lower}} = 0$. Hence, for the first case, (3.1) has solutions for small enough current $j$. In contrast, in the second case, (3.1) has no solutions for small $j$.

Moreover, we bound $j$ by bellow:
**Proposition 5.1.5.** Assume \( \max T > 1 + \int T V \). Let \((u_j, m_j, \Pi_j)\) be a solution of (1.2) with \( m > 0 \). Then, there exists a constant, \( c(V) > 0 \), such that

\[
\inf_T j \geq c(V).
\]

**Proof.** We know \( \phi^+(0) > 1 \) from Proposition 5.1.1. Then, by Proposition 5.1.4, (1.2) has no smooth solutions for \( j = 0 \). Moreover, \( j_{\text{lower}} = 0 \). We get the result by taking \( c(V) = j_{\text{upper}} \) and by Corollary 5.1.3.

---

### 5.2 Asymptotic Behavior

Now, we examine the behavior of semiconcave solutions when the current tends to infinity and when it tends to zero. We study the asymptotic behavior for both (1.2) and (1.3).

#### 5.2.1 The case \( j \to \infty \)

First, we prove the following lemma.

**Lemma 5.2.1.** Suppose \( V \) has a unique maximum at \( x = 0 \). Let \( m_j \) be the solution of (3.1). Then, for all \( j > j_{\text{upper}} \), \( m_j \) and \( \frac{1}{m_j} \) are bounded.

**Proof.** When \( j > j_{\text{upper}} \), we are in Case ii. of Proposition 3.2.6. Then, \( m_j(x) = m^-_j(x) \leq j^{\frac{2}{3}} \). Hence, we have \( \frac{j^2}{m_j(x)^2} \geq m_j(x) \), which implies

\[
\frac{j^2}{2m_j(x)^2} \leq \frac{j^2}{2m_j(x)^2} + m_j(x) \leq \frac{3j^2}{2m_j(x)^2}.
\] (5.2)
Thus, using the first equation in (3.1) in inequality (5.2), we get

\[
\frac{j}{\sqrt{2(H_j - V(x))}} \leq m_j(x) \leq \frac{\sqrt{3}j}{\sqrt{2(H_j - V(x))}}.
\]

(5.3)

Next, we integrate the inequality above and get

\[
\int_T \frac{j}{\sqrt{2(H_j - V(x))}} \, dx \leq 1 \leq \int_T \frac{\sqrt{3}j}{\sqrt{2(H_j - V(x))}} \, dx,
\]

(5.4)

since \(\int_T m = 1\). Recall that \(\lim_{j \to \infty} H_j = \infty\) and that \(V\) is bounded. Then,

\[
\lim_{j \to \infty} \frac{\sqrt{2(H_j - V(y))}}{\sqrt{2(H_j - V(x))}} = 1.
\]

(5.5)

Therefore, if \(j\) is large enough, \(\sqrt{2(H_j - V(x))} \leq 2\sqrt{2(H_j - V(y))}\) for any \(x,y \in T\).

Let \(\bar{x} \in T\) be such that

\[
\frac{j}{\sqrt{2(H_j - V(\bar{x}))}} = \int_T \frac{j}{\sqrt{2(H_j - V(x))}} \, dx.
\]

Then, from (5.3), (5.4), and (5.5), we get

\[
m_j(x) \leq \frac{\sqrt{3}j}{\sqrt{2(H_j - V(x))}} \leq \frac{2\sqrt{3}j}{\sqrt{2(H_j - V(\bar{x}))}} \leq 2\sqrt{3}
\]

and

\[
m_j(x) \geq \frac{j}{\sqrt{2(H_j - V(x))}} \geq \frac{j}{2\sqrt{2(H_j - V(\bar{x}))}} \geq \frac{1}{2\sqrt{3}}.
\]

Proposition 5.2.2. Suppose \(V\) has a unique maximum at \(x = 0\). Then, for all \(j > 0\), let \((u_j, m_j, H_j)\) be the solution of (3.1). For all \(j_\alpha > 0\), let \((u_{j_\alpha}, m_{j_\alpha}, H_{j_\alpha})\) be
the solution of (3.4). Then,

i. \( \lim_{j \to \infty} H_j = \lim_{j, \alpha \to \infty} H_{j, \alpha} = \infty; \)

ii. For \( x \in \mathbb{T}, \) \( \lim_{j \to \infty} m_j(x) = 1, \) \( \lim_{j \to \infty} u_j(x) = 0, \) \( \lim_{j \to \infty} p_j = \infty. \)

Moreover, \( \lim_{j \to \infty} m_{j, \alpha}(x) = 1, \) \( \lim_{j \to \infty} u_{j, \alpha}(x) = 0, \) \( \lim_{j \to \infty} p_{j, \alpha} = \infty. \)

Proof. We prove each case separately.

i. Fix \( j, j, \alpha > 0. \) Let \( H_j \) solve (3.1) and let \( H_{j, \alpha} \) solve (3.4). Then, recall that, in Chapter 3, we bounded \( H_j \) by \( H_{j, cr} = \frac{3}{2} j^{\frac{3}{2}} + \max_{\mathbb{T}} V. \) Moreover, \( H_{j, \alpha} \geq H_{j, cr} \) and \( c_\alpha \geq 0. \) Thus, as \( j, j, \alpha \to \infty, \) \( H_{j, cr}, H_{j, cr, \alpha} \to \infty, \) and so does \( H_j, H_{j, \alpha} \to \infty. \)

ii. Here, we consider the noncongestion case. The congestion case is analogous.

Let \( (u_j, m_j, H_j) \) be a solution for (3.1). For \( j \) large enough, we have \( j > j_{upper}, \) and we are in Case ii. of Proposition 3.2.6. We compute each limit separately.

1. Limit of \( m_j. \)

First, we compute the limit for \( \frac{\partial m_j(x)}{\partial x}. \) We differentiate (3.1) in respect to \( x \) and get:

\[
-\frac{j^2}{m_j(x)^3} \frac{\partial m_j(x)}{\partial x} + \frac{\partial m_j(x)}{\partial x} = -V'(x).
\]

Then, from (5.6), we have

\[
\frac{\partial m_j(x)}{\partial x} = -\frac{V'(x)}{1 - \frac{j^2}{m_j(x)^3}}.
\]

We saw, in Lemma 5.2.1, that \( m_j(x) \) is bounded. Hence, \( \lim_{j \to \infty} \frac{\partial m_j(x)}{\partial x} = 0. \) Thus, taking into account that \( \int_{\mathbb{T}} m = 1, \) we have \( \lim_{j \to \infty} m_j(x) = 1. \)

2. Limit of \( p_j. \) Recall \( p_j = \int_{\mathbb{T}} \frac{j}{m_j(y)} \, dy. \) Moreover, for \( j \) large enough, \( m_j(x) \leq j^{\frac{23}{3}} \leq j. \) Then, \( \lim_{j \to \infty} p_j = \infty. \)
3. **Limit of** \(u_j\). We first compute the limit of \((u_j)_x = \frac{j}{m_j(x)} - p_j\). From (3.1), we have

\[
\frac{j}{m_j(x)} = \sqrt{2(\overline{H}_j - V(x) - m_j(x)}.
\]

Therefore, by the mean-value theorem:

\[
\left| \frac{j}{m_j(x)} - \frac{j}{m_j(y)} \right| = \left| \sqrt{2(\overline{H}_j - V(x) - m_j(x)} - \sqrt{2(\overline{H}_j - V(y) - m_j(y)} \right| \tag{5.7}
\]

\[
= \frac{|V(x) - V(y) + m_j(x) - m_j(y)|}{\sqrt{2(\overline{H}_j - V(z) - m_j(z)}} \tag{5.8}
\]

for some \(z \in \mathbb{T}\).

We have that \(V(z) \leq \max_{\mathbb{T}} V, \overline{H}_j \geq \frac{3}{2} j^\frac{3}{2} + \max_{\mathbb{T}} V\) and \(m_j(z) \leq j^\frac{2}{3}\). Therefore,

\[
\left| \frac{j}{m_j(x)} - \frac{j}{m_j(y)} \right| \leq \frac{|V(x) - V(y)| + |m_j(x) - m_j(y)|}{j^\frac{3}{2}} \tag{5.9}
\]

\[
\leq \frac{\text{osc}V + |m_j(x) - m_j(y)|}{j^\frac{1}{2}}, \tag{5.10}
\]

Hence,

\[
\lim_{j \to \infty} |(u_j)_x| = \lim_{j \to \infty} \left| \int_{\mathbb{T}} \frac{j}{m_j(x)} - \frac{j}{m_j(y)} \, dy \right| \leq \int_{\mathbb{T}} \left| \frac{j}{m_j(x)} - \frac{j}{m_j(y)} \right| \, dx = 0.
\]

Consequently, \(\lim_{j \to \infty} u_j(x) = 0\).

\[\square\]

**5.2.2 The case** \(j \to 0\)

Here, we focus on the noncongestion case. For \(\alpha \neq 0\), see Remark [5.2.4](#).
Proposition 5.2.3. Suppose \( V \) has a unique maximum at \( x = 0 \). Then, for all \( j > 0 \), let \((u_j, m_j, \overline{H}_j)\) solve (3.1). Then,

i. \( \lim_{j \to 0} \overline{H}_j = \max \left( \max_T V, 1 + \int_T V \right) = \overline{H}_0; \)

ii. If \( 1 + \int_T V > \max_T V \), then,

\[
\lim_{j \to 0} m_j(x) = 1 + \int_T V - V(x), \quad \lim_{j \to 0} u_j(x) = 0, \quad \text{and} \quad \lim_{j \to 0} p_j = 0,
\]

for all \( x \in \mathbb{T} \);

iii. If \( 1 + \int_T V \leq \max_T V \), then, recalling that \( \overline{m}_{d,1} \) and \( u_{d,1} \) are given by (3.37) and (3.38),

\[
\lim_{j \to 0} m_j(x) = \overline{m}_{d,1}(x), \quad \lim_{j \to 0} u_j(x) = \overline{u}_{d,1}(x), \quad \text{and} \quad \lim_{j \to 0} p_j = \int_0^d \sqrt{2(\max_T V - V(x))} \, dx
\]

for all \( x \in \mathbb{T} \).

Proof. We prove each case separately.

i. We have two possible cases \( j_{\text{lower}} > 0 \) and \( j_{\text{lower}} = 0 \).

If \( j_{\text{lower}} = 0 \), then, by (5.1), \( \phi^+(j) > 1 \) for all \( j > 0 \). If \( j > 0 \) is small enough, then \( \phi^-(j) < 1 \). So, we are in Case iii. of Proposition 3.2.6. Recall that, \( \overline{H}_j = \overline{H}_j^r = \max_T V + \frac{3}{2} j^2 \). Hence, \( \lim_{j \to 0} \overline{H}_j = \max_T V \). Moreover, from Proposition 5.1.1,

\[
\max_T V - \int_T V(x) \, dx = \lim_{j \to 0} \phi^+(j) \geq 1.
\]

Therefore,

\[
\lim_{j \to 0} \overline{H}_j = \max_T V = \max \left( \max_T V, 1 + \int_T V \right) = \overline{H}_0.
\]
If \( \underline{j}_{\text{lower}} > 0 \), then, for small enough \( 0 < j < \underline{j}_{\text{lower}} \), we get \( \phi^+(j) < 1 \). So, we are in Case i. of Proposition 3.2.6. Here, \( \overline{H}_j \) is such that \( \int_{\mathbb{T}} m = 1 \). Since \( m_j(x) = m_j^+(x) \), \( m_j(x) \geq j^2 \). Hence,

\[
\lim_{j \to 0} \overline{H}_j = \lim_{j \to 0} \int_{\mathbb{T}} \frac{j^2}{2m_j(x)^2} + \int_{\mathbb{T}} m_j + \int_{\mathbb{T}} V \leq \lim_{j \to 0} \int_{\mathbb{T}} \frac{j^2}{2} + 1 + \int_{\mathbb{T}} V = 1 + \int_{\mathbb{T}} V.
\]

Moreover, from Proposition 5.1.1

\[
\max_{\mathbb{T}} V - \int_{\mathbb{T}} V(x) \, dx = \lim_{j \to 0} \phi^+(j) \leq 1.
\]

Therefore,

\[
\lim_{j \to 0} \overline{H}_j = 1 + \int_{\mathbb{T}} V = \max \left( \max_{\mathbb{T}} V, \, 1 + \int_{\mathbb{T}} V \right) = \overline{H}_0.
\]

ii. By hypothesis, \( 1 + \int V > \max V \). Then, \( \lim_{j \to 0} \phi^+(j) = \max V - \int V < 1 \). So, \( \underline{j}_{\text{lower}} > 0 \). From the proof for i., \( \overline{H}_0 = 1 + \int V \). Now, we compute the limits for the remaining elements of \((u_j, m_j, \overline{H}_j)\). We get

\[
\lim_{j \to 0} m_j(x) = \lim_{j \to 0} \overline{H}_0 - V(x) - \frac{j^2}{2m_j^2(x)} = 1 + \int V - V(x) \tag{5.11}
\]

since we are in Case i. of Proposition 3.2.6. Furthermore,

\[
\lim_{j \to 0} u_j(x) = \lim_{j \to 0} \int_0^x \frac{j}{m_j(y)} \, dy - \int_{\mathbb{T}} \frac{j}{m_j(z)} \, dz = 0, \tag{5.12}
\]

and

\[
\lim_{j \to 0} p_j(x) = \lim_{j \to 0} \int_{\mathbb{T}} \frac{j}{m_j(z)} \, dz = 0. \tag{5.13}
\]

iii. By hypothesis, \( 1 + \int V < \max V \). Then, \( \lim_{j \to 0} \phi^+(j) = \max V - \int V > 1 \). So, \( \underline{j}_{\text{lower}} = 0 \). From the proof for i., \( \overline{H}_0 = \max V \). Now, we compute the limits for the
remaining values of the solution, \((u_j, m_j, \mathcal{H}_j)\). Since \(0 < m_j^- \leq j^{\frac{2}{3}} \leq m_j^+\),

\[
\lim_{j \to 0} m_j^-(x) = 0, \quad \lim_{j \to 0} m_j^+(x) = \lim_{j \to 0} \mathcal{H}_0 - V(x) - \frac{j^2}{2(m_j^+)^2(x)} = \max V - V(x).
\]

Hence, \(m_j(x) = m_j^+(x)\chi_{[d_{j,1}]} = m_0^{d,1}\) since we are in Case iii. of Proposition 3.2.6. Moreover, \(\lim_{j \to 0} m_j(x) = (\max V - V(x))\chi_{[d_{0,1}]}\). Let \(d_j \to d\) be a convergent subsequence. Then, \(\int_d^1 \max V - V(x) \, dx\) is defined uniquely because \(V\) has a single maximum. So, \(m_j \to m_0^{d,1}\) and \(d_j \to d\) globally. Furthermore,

\[
\lim_{j \to 0} u_j(x) = u_0^{d,1}, \quad \lim_{j \to 0} p_j(x) = p_0^{d,1}.
\]

(5.14)

\[\square\]

**Remark 5.2.4.** In the above proposition, we compute the limit of solutions for \(j > 0\) as \(j \to 0\). These solutions are only part of what we found for \(j = 0\) in Chapter 3. In that chapter, when we computed solutions, we neglected negative solutions, \(m(x) < 0\). If we take the limit of these negative solutions, we recover all solutions for \(j = 0\).

**Remark 5.2.5.** Recall Remark 3.3.7. For the congestion case, since \(m > 0\) for all \(x \in T\), Proposition 5.2.3 holds for \(j_\alpha\). Thus, \(d = 0\) everywhere, because otherwise \(m = 0\). Then, \(\mathcal{H} = 1 + \int V\). Therefore, we are always in Case ii of Proposition 5.2.3.

### 5.3 Behavior of \(\mathcal{H}_j\)

Because the results are similar, we focus on the case \(\alpha = 0\) from now on. In this section, we analyze several properties for \(\mathcal{H}_j\). Recall that from Section 5.2, \(\mathcal{H}_j \to \infty\).
as \( j \to \infty \) and \( \mathcal{H}_j \to \mathcal{H}_0 \) as \( j \to 0 \).

**Proposition 5.3.1.** Let \( g(m) = -m \). Suppose \( V \) has a unique maximum \( x = 0 \). Let \((u_j, m_j, \mathcal{H}_j)\) be the solution of (3.1). Then,

i. For every \( j \in \mathbb{R} \), there exists a unique number, \( \mathcal{H}_j \), such that (1.2) has solutions with a current level \( j \);

ii. \( \mathcal{H}_j \) is even; that is, \( \mathcal{H}_j = \mathcal{H}_{-j} \);

iii. \( \mathcal{H}_j \) is continuous;

iv. \( \mathcal{H}_j \) increasing on \((0, \infty)\) and decreasing on \((-\infty, 0)\);

v. \( \min_{j \in \mathbb{R}} \mathcal{H}_j = \mathcal{H}_0 = \max \left( \max_{\mathbb{T}} V, 1 + \int_{\mathbb{T}} V(x) dx \right) \);

vi. \( \lim_{|j| \to \infty} \frac{\mathcal{H}_j}{j^{2/2}} = 1 \).

**Proof.** We prove each point separately.

i. Follows from Proposition 3.3.4 and 3.2.6.

ii. Follows since \( j \to \frac{j^2}{2t^2} \) is an even function for all \( t > 0 \).

iii. Follows from the continuity of \((j, t) \to \frac{j^2}{2t^2}\).

iv. Since \( \mathcal{H}_j \) is even by ii., we only need to prove that \( \mathcal{H}_j \) increases on \((0, +\infty)\).

First, we prove that \( \mathcal{H}_j \) increases on \((j_{\text{upper}}, \infty)\). We fix \( j_0 > j_{\text{upper}} \). Then, \( \mathcal{H}_{j_0} \geq \mathcal{H}^{cr}_{j_0} \). If we take \( j_{\text{upper}} < j < j_0 \), then \( \mathcal{H}^{cr}_j = \max_{\mathbb{T}} V + \frac{3}{2} j_0 \frac{j^2}{2} < \max_{\mathbb{T}} V + \frac{3}{2} j_0 \frac{j^2}{2} = \mathcal{H}^{cr}_{j_0} \leq \mathcal{H}_{j_0} \).

Let \( \tilde{m}_j \) solve

\[
\frac{j^2}{2\tilde{m}_j(x)^2} + \tilde{m}_j(x) = \mathcal{H}_{j_0} - V(x)
\] (5.15)
for all $j_{\text{upper}} < j < j_0$. Since $\overline{H}_{j_0} > \overline{H}_j^*, \tilde{m}(x)$ is well defined. Moreover, since $j > j_{\text{upper}}, \tilde{m}_j(x) \leq j^2$. By the implicit function theorem, $\tilde{m}_j$ is differentiable and

$$
\frac{d\tilde{m}_j(x)}{dj} = -\frac{j\tilde{m}_j(x)}{j^2 - \tilde{m}_j(x)}.
$$

The derivative of $\tilde{m}_j(x)$ is negative since $\tilde{m}_j(x) \leq j^2$. Hence, $\tilde{m}(x) < \tilde{m}_{j_0}(x) = m_{j_0}(x)$. Accordingly,

$$
\int_T \tilde{m}_j(x) \, dx \leq \int_T m_{j_0}(x) \, dx = 1.
$$

Thus, $\overline{H}_j < \overline{H}_{j_0}$.

**v.** Follows from the previous properties in Proposition 5.2.3 and Proposition 5.2.2.

**vi.** From (3.1), we get

$$
\overline{H}_j = \frac{j^2}{2m_j(x)^2} + m_j(x) + V(x).
$$

Recall, from Lemma 5.2.1 that for $j$ large enough, $j > j_{\text{upper}}, m_j(x)$ is bounded. Moreover, $V$ is bounded for all $x \in T$. Then,

$$
\lim_{j \to \infty} \frac{2\overline{H}_j}{j^2} = \lim_{j \to \infty} \frac{1}{2m_j(x)^2} + \frac{m_j(x)}{j^2} + \frac{V(x)}{j^2} = \lim_{j \to \infty} \frac{1}{2m_j(x)^2}.
$$

From Proposition 5.2.2 we have that $\lim_{j \to \infty} m_j(x) = 1$. Hence, we prove vi.

In Figure 5.2 we plot $\overline{H}(j)$ and confirm the results of Proposition 5.3.1

### 5.4 Behavior of $p_j$

In this section, we analyze the properties and behavior of $p$ in terms of $j$. 


Proposition 5.4.1. Suppose $V$ has a unique maximum at $x = 0$. Let $g(m) = -m$.

i. Then, for all $j \neq 0$, there exists a unique $p_j$ such that (1.2) has a solution. Moreover, $j \to p_j$ is increasing for $j \neq 0$;

ii. If $1 + \int V \geq \max T V$, then, (1.2) has solutions with $j = 0$ for $p_j = p_0 = 0$. Moreover $\lim_{j \to 0} p_j = 0$.

iii. If $1 + \int V < \max \frac{V}{T}$, then,

\[
\begin{cases}
    p_j > \int_0^{d_1} \sqrt{2(\max \frac{V}{T} - V(x))} \, dx, & \text{if } j > 0, \\
    p_j < -\int_1^{d_2} \sqrt{2(\max \frac{V}{T} - V(x))} \, dx, & \text{if } j < 0,
\end{cases}
\]

where $d_1, d_2 \in (0, 1)$ are such that

\[
\int_0^{d_1} \left( \max \frac{V}{T} - V(x) \right) \, dx = \int_0^{d_2} \left( \max \frac{V}{T} - V(x) \right) \, dx = 1.
\]

Moreover, (1.2) has semiconcave solutions with $j = 0$ for $p_j$ such that

\[
- \int_{d_2}^{1} \sqrt{2(\max \frac{V}{T} - V(x))} \, dx \leq p_j \leq \int_{0}^{d_1} \sqrt{2(\max \frac{V}{T} - V(x))} \, dx.
\]

(5.16)

Proof. We prove each case separately.
In Proposition 3.2.6, we proved that, for all \( j > 0 \), there exists a solution, \((u_j, m_j, \Pi_j)\), with \( p_j = \int_T \frac{j}{m_j(y)} \, dy \). Hence, to prove that \( j \to p_j \) is an increasing function we prove that \( j \to \frac{j}{m_j(x)} \) is increasing.

Define \( n_j(x) = \frac{j}{m_j(x)} \). Then, we have

\[
\frac{n_j(x)^2}{2} + \frac{j}{n_j(x)} = H_j - V(x).
\] (5.17)

We differentiate (5.17) for \( x \in \mathbb{T} \) where \( m_j \) is continuous and get

\[
\frac{\partial n_j(x)}{\partial j} = \frac{H'_j - \frac{1}{n_j(x)}}{n_j(x) - \frac{j}{n_j(x)^2}}.
\] (5.18)

We prove the \( n_j \) is increasing by separating \( j \) into three different cases.

1. \( j_{\text{lower}} < j < j_{\text{upper}} \). Here, we are in Case iii. of Proposition 3.2.6. Then,

\[
n_j(x) = \frac{j}{m_j(x)} \chi_{[0,d_j)} + \frac{j}{m_j^+(x)} \chi_{(d_j,1]}.
\]

Since \( m_j^- \) and \( m_j^+ \) are increasing (see proof of Proposition 5.1.1.), so is \( d_j \).

Define \( x = d_{j_x} \) for some \( j_{\text{lower}} < j_x < j_{\text{upper}} \). First, assume \( d_j \neq x \). Then, we use (5.18). Since \( H_j = H^{cr}_j \), we have \( H'_j = j^{-\frac{1}{4}} \). For \( j > j_x, d_j > x \), thus, \( n_j(x) = \frac{j}{m_j^-(x)} > j^{\frac{1}{4}} \), which implies \( \frac{\partial n_j(x)}{\partial j} > 0 \). Similarly, for \( j < j_x, \frac{\partial n_j(x)}{\partial j} > 0 \).

Now, assume \( d_j = d_{j_x} \). At a point, \( x \), \( n_j(x) \) takes a positive jump as \( \frac{j}{m_j^-(x)} - \frac{j}{m_j^+(x)} > 0 \). Thus, \( n_j \) increases when \( j \neq j_x \) and has a positive jump when \( j = j_x \).

Hence, \( n_j \) is increasing for all \( j_{\text{lower}} < j < j_{\text{upper}} \).

2. \( j > j_{\text{upper}} \). Here, we are in Case ii. of Proposition 3.2.6. Then, \( n_j(x) = \frac{j}{m_j(x)} \).

Since \( m_j^- < j^{\frac{3}{2}}, n_j > j^{\frac{3}{4}} \). Then, if \( H'_j \geq \frac{1}{n_j(x)} \), the map is increasing. The following lemma proves that \( H'_j \geq \frac{1}{n_j(x)} \).

**Lemma 5.4.2.** In the conditions of this proof, \( H'_j \geq \frac{1}{n_j(x)} \).
Proof. Fix \( j_0 \). For \( j > j_0 \), define \( \tilde{H}_j = \mathcal{H}_{j_0} + (j - j_0) \frac{\min m_{j_0}(x)}{j_0} \). Let \( \tilde{m}_j(x) \) solve

\[
\frac{j^2}{2\tilde{m}_j(x)^2} + \tilde{m}_j(x) = \tilde{H}_j - V(x),
\]

(5.19)

Since \( \tilde{H}_j \geq \mathcal{H}_{j_0} \geq \mathcal{H}_{j_0}^{cr} \), we have \( \tilde{m}_j(x) \leq j_0^2 < j^2 \). Then, the derivative of \( \tilde{m}_j \),

\[
\left. \frac{d\tilde{m}_j(x)}{dj} \right|_{j=j_0} = \frac{\tilde{H}'_j - \frac{j}{\tilde{m}_j}}{1 - \frac{j^2}{\tilde{m}_j^2}} \left|_{j=j_0}\right. = \frac{\min m_{j_0} - \frac{j_0}{m_{j_0}}}{1 - \frac{j_0^4}{m_{j_0}^4}},
\]

is positive everywhere. Hence, \( \int_{\mathbb{T}} \tilde{m}_j(x) \, dx \) is also increasing near \( j_0 \). Note that \( \tilde{m}_{j_0} = m_{j_0} \) and \( \tilde{H}_{j_0} = \mathcal{H}_{j_0} \). Therefore, for \( j > j_0 \) close to \( j_0 \),

\[
\int_{\mathbb{T}} \tilde{m}_j(x) \, dx > \int_{\mathbb{T}} m_{j_0}(x) \, dx = 1.
\]

Consequently, \( \mathcal{H}_j > \tilde{H}_j \). Therefore, since \( \tilde{H}_{j_0} = \mathcal{H}_{j_0} \), we have:

\[
\mathcal{H}'_{j_0} \geq \tilde{H}'_{j_0} = \frac{\min m_{j_0}}{j_0} = \max \frac{1}{n_{j_0}},
\]

which completes the proof of the lemma. \( \square \)

3. \( j < j_{\text{lower}} \). Analogous to the proof of item 2.

ii. and iii. Follows from the monotonicity of \( j \to p_j \) and Proposition 5.2.3 and 3.3.4. \( \square \)

Note that, by Propositions 3.2.8 if \( V \) has more than one maximum and it is not constant, there are multiple values of \( p_j \) for a certain \( j \). To conclude, we plot \( p(j) \) in Figure 5.3 to confirm the results of Proposition 5.4.1.
5.5 Behavior of $\overline{H}(p)$

Finally, to complete our analysis of the solutions, we study $\overline{H}$ in terms of $p$.

**Proposition 5.5.1.** Suppose $V$ has a unique maximum at $x = 0$. Let $g(m) = -m$. Then,

i. For every $p \in \mathbb{R}$, there exists a unique number, $\overline{H}(p)$, for which (1.2) has a semiconcave solution;

ii. For every $p \in \mathbb{R}$, (1.2) has a unique semiconcave solution;

iii. If $1 + \int_T V(x) \, dx < \max_T V$, $\overline{H}(p)$ is flat at the origin;

iv. $\overline{H}(p)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. Thus,

$$\min_{p \in \mathbb{R}} \overline{H}(p) = \overline{H}(0) = \max \left( \max_T V, 1 + \int_T V(x) \, dx \right);$$

v. $\lim_{|p| \to \infty} \frac{\overline{H}(p)}{p^{2/2}} = 1$.

**Proof.**

i. and ii. Proposition 5.4.1 implies there is a unique $j$ such that (1.2) has semiconcave solutions. Given $j$, Proposition 5.3.1 implies there is an unique $\overline{H}$ such
that (1.2) has a semiconcave solution. Therefore, for each $p$ there is a unique $\bar{H}$ for which (1.2) has a semiconcave solution. Moreover, given $\bar{H}$, there is a unique semiconcave solution for (1.2) by Proposition 3.2.6.

iii. From Proposition 5.4.1-iii, $p$ satisfies (5.16). Then $j = 0$ has a regular solution, and $\bar{H}(p) = \bar{H}_0$.

iv. Follows from Proposition 5.4.1-i. and Proposition 5.3.1-iv,v.

v. Follows from Proposition 5.3.1-vi, Proposition 5.2.2 and $p_j = \int_T \frac{j}{m_j(y)} dy$. 

Finally, we plot $H(p)$. In Figure 5.4, we check the results of Proposition 5.5.1.
REFERENCES


APPENDICES

A Papers Submitted and Under Preparation