Supplementary materials for "Tukey g-and-h Random Fields"

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Short title: Tukey g-and-h Random Fields

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Technical Proofs

Lemma A.1 *If $Z \sim N(\mu, \sigma^2)$, then for any $a < \frac{1}{2\sigma^2}$ and $b \in \mathbb{R}$, we have that*

$$E\{\exp(aZ^2 + bZ)\} = \frac{1}{\sqrt{1-2a\sigma^2}} \exp\left\{\frac{b^2\sigma^2 + 2b\mu + 2a\mu^2}{2(1-2a\sigma^2)}\right\}. \tag{A.1}$$

Let $(Z_1, Z_2)^{\mathrm{T}} \sim N_2(\mathbf{0}_2, \mathbf{\Sigma}_0)$ with $\mathbf{\Sigma}_0 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. Then for any $a < \frac{1}{2}$ and $b_1, b_2 \in \mathbb{R}$, we have

$$\begin{aligned} &E\left[\exp\left\{\frac{a}{2}(Z_1^2 + Z_2^2) + b_1 Z_1 + b_2 Z_2\right\}\right] \\ &= \frac{1}{\sqrt{(1-a)^2 - \rho^2 a^2}} \exp\left[\frac{(1-\rho^2)}{\{1-a(1-\rho^2)\}^2 - \rho^2} \times \frac{\{1-a(1-\rho^2)\}(b_1^2+b_2^2) + 2\rho b_1 b_2}{2}\right]. \end{aligned} \tag{A.2}$$

Proof: Since $Z \sim N(\mu, \sigma^2)$, then

$$\begin{aligned} \mathrm{E}\{\exp(aZ^2 + bZ)\} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{az^2 + bz - \frac{(z-\mu)^2}{2\sigma^2}\right\} dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\left(\frac{1}{2\sigma^2} - a\right) z^2 + \left(b + \frac{\mu}{\sigma^2}\right) z - \frac{\mu^2}{2\sigma^2}\right\} dz \\ &= \frac{1}{\sqrt{\frac{1}{\sigma^2} - 2a\sigma}} \exp\left\{-\frac{\mu^2}{2\sigma^2} + \frac{(b + \frac{\mu}{\sigma^2})^2}{4(\frac{1}{2\sigma^2} - a)}\right\} \\ &= \frac{1}{\sqrt{1-2a\sigma^2}} \exp\left\{\frac{b^2\sigma^2 + 2b\mu + 2a\mu^2}{2(1-2a\sigma^2)}\right\}. \end{aligned}$$

Next, define $\mathbf{u}_1 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $\mathbf{\Sigma}_1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, and the vector $\mathbf{z} = (z_1, z_2)^{\mathrm{T}}$. We have that

$$\mathrm{E}\left[\exp\left\{\frac{a}{2}(Z_1^2 + Z_2^2) + b_1 Z_1 + b_2 Z_2\right\}\right] = \frac{1}{2\pi|\mathbf{\Sigma}_0|^{1/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}}(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_1)\mathbf{z} + \mathbf{u}_1^{\mathrm{T}}\mathbf{z}\right\} dz_1 dz_2$$

$$\begin{aligned} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}}(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_1)\mathbf{z} + \mathbf{u}_1^{\mathrm{T}}(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_1)^{-1}(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_1)\mathbf{z} - \frac{1}{2}\mathbf{u}_1^{\mathrm{T}}(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_1)^{-1}\mathbf{u}_1\right\} dz_1 dz_2 \\ &\qquad \times \frac{1}{2\pi|\mathbf{\Sigma}_0|^{1/2}} \exp\left\{\frac{1}{2}\mathbf{u}_1^{\mathrm{T}}(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_1)^{-1}\mathbf{u}_1\right\} \\ &= \frac{1}{|\mathbf{I} - \mathbf{\Sigma}_1\mathbf{\Sigma}_0|^{1/2}} \exp\left\{\frac{1}{2}\mathbf{u}_1^{\mathrm{T}}(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_1)^{-1}\mathbf{u}_1\right\} \\ &\qquad = \frac{1}{\sqrt{(1-a)^2 - \rho^2 a^2}} \exp\left[\frac{\{1-a(1-\rho^2)\}(b_1^2+b_2^2) + 2\rho b_1 b_2}{2\{(1-a)^2 - \rho^2 a^2\}}\right], \end{aligned}$$

where the last equation follows from the facts that $|\mathbf{I} - \mathbf{\Sigma}_1\mathbf{\Sigma}_0|^{1/2} = \sqrt{(1-a)^2 - \rho^2 a^2}$ and that

$$(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_1)^{-1} = \tfrac{1}{(1-a)^2-\rho^2 a^2} \begin{pmatrix} 1 - a(1-\rho^2) & \rho \\ \rho & 1 - a(1-\rho^2) \end{pmatrix}. \qquad \square$$

Proof of Lemma 1: A direct application of Lemma A.1, equation (A.1), with $\mu = 0$ and $\sigma^2 = 1$ yields that

$$\begin{aligned} \mathrm{E}\{T(\mathbf{s})\} &= \frac{1}{g}\mathrm{E}\left[\exp\left\{gZ(\mathbf{s}) + \frac{h}{2}Z^2(\mathbf{s})\right\}\right] - \frac{1}{g}\mathrm{E}\left[\exp\left\{\frac{h}{2}Z^2(\mathbf{s})\right\}\right] \\ &= \frac{1}{g\sqrt{1-h}}\exp\left\{\frac{g^2}{2(1-h)}\right\} - \frac{1}{g\sqrt{1-h}} = \frac{1}{g\sqrt{1-h}}\left[\exp\left\{\frac{g^2}{2(1-h)}\right\} - 1\right], \end{aligned}$$

proving equation (5).

Using equation (A.2), we have that

$$\begin{aligned} \mathrm{E}\{T(\mathbf{s}_1)T(\mathbf{s}_2)\} = &\frac{1}{g^2}\mathrm{E}\left(\exp\left[g\{Z(\mathbf{s}_1) + Z(\mathbf{s}_2)\} + \frac{h}{2}\{Z^2(\mathbf{s}_1) + Z^2(\mathbf{s}_2)\}\right]\right) \\ &- \frac{2}{g^2}\mathrm{E}\left(\exp\left[gZ(\mathbf{s}_1) + \frac{h}{2}\{Z^2(\mathbf{s}_1) + Z^2(\mathbf{s}_2)\}\right]\right) + \frac{1}{g^2}\mathrm{E}\left(\exp\left[\frac{h}{2}\{Z^2(\mathbf{s}_1) + Z^2(\mathbf{s}_2)\}\right]\right) \\ = &\frac{1}{g^2\sqrt{(1-h)^2 - \rho^2 h^2}}\left(\exp\left\{\frac{g^2(1+\rho)}{1-h(1+\rho)}\right\} - 2\exp\left[\frac{g^2\{1-h(1-\rho^2)\}}{2\{(1-h)^2-\rho^2h^2\}}\right] + 1\right), \end{aligned}$$

and thus equation (6) follows. $\square$

Proof of Theorem 1: If $Z(\mathbf{s})$ is second-order stationary, then its correlation function becomes $\rho_Z(\mathbf{s}_1, \mathbf{s}_2) = C_Z(\mathbf{s}_1 - \mathbf{s}_2)$, for some positive definite function $C_Z(d)$. By the expression of $C_T(\mathbf{s}_1, \mathbf{s}_2)$ in equation (6), it is straightforward to see that $C_T(\mathbf{s}_1, \mathbf{s}_2)$ only depends on $\mathbf{s}_1 - \mathbf{s}_2$ and thus $T(\mathbf{s})$ is also second-order stationary.

Define the covariance function of $T(\mathbf{s})$ as $K_T(d)$ such that $K_T(\mathbf{s}_1 - \mathbf{s}_2) = C_T(\mathbf{s}_1, \mathbf{s}_2)$. By Stein (1999, chapter 2.4), the mean-square continuity and the m-times mean-square differentiability of the random field $T(\mathbf{s})$ are equivalent to the continuity and $2m$-times differentiability of $K_T(d)$ at $d = 0$, respectively. Define the function

$$\varrho(x) = \frac{1}{g^2\sqrt{(1-h)^2 - x^2h^2}}\left[\exp\left\{\frac{g^2(1+x)}{1-h(1+x)}\right\} - 2\exp\left\{\frac{\frac{1}{2}g^2(1-h+hx^2)}{(1-h)^2-h^2x^2}\right\} + 1\right].$$

It is easy to see that $K_T(d) = \varrho\{C_Z(d)\} + c$ where c is some constant independent of d. For any $-1 \le x \le 1$ and $0 \le h < 1/2$, $\varrho(x)$ is a continuous and infinitely differentiable function. Hence, $K_T(d)$ is continuous at $d = 0$ if and only if $C_Z(d)$ is continuous at $d = 0$, which implies that $T(\mathbf{s})$ is mean-square continuous if and only if $Z(\mathbf{s})$ is mean-square continuous. Furthermore, the $2m$th derivative of $K_T(d)$ at $d = 0$, (i.e., $K_T^{(2m)}(0)$), exist if $C_Z^{(2m)}(0)$ exist, which implies that $T(\mathbf{s})$ is m-times mean-square differentiable if $Z(\mathbf{s})$ is m-times mean-square differentiable. $\square$

Proof of Theorem 3 Without loss of generality, we assume that $\xi = 0$. Then, some straightforward calculation yields that

$$Y_k(\mathbf{s}) = \omega_k \frac{e^{g_k Z(\mathbf{s})} - 1}{g_k} e^{\frac{h_k}{2} Z_k(\mathbf{s})^2} = \frac{e^{g_k^* Z^*(\mathbf{s})} - 1}{g_k^*} e^{\frac{h_k^*}{2} Z_k^*(\mathbf{s})^2} = \tau_{g_k^*, h_k^*}\{Z_k^*(\mathbf{s})\},$$

where $g_k^* = g_k/\omega_k$, $h_k^* = h_k/\omega_k^2$ and $Z_k^*(\mathbf{s}) = \omega_k Z_k(\mathbf{s})$. Under the assumption that $g_1\omega_2 = g_2\omega_1$, $h_1\omega_2^2 = h_2\omega_1^2$, we have that $g_1^* = g_2^*$ and $h_1^* = h_2^*$. Define $g_0 = g_1^* = g_2^*$ and $h_0 = h_1^* = h_2^*$, then

$$Y_k(\mathbf{s}) = \tau_{g_0,h_0}\{Z_k^*(\mathbf{s})\}. \tag{A.3}$$

By definition $Z_k^*(\mathbf{s}) = \tau_{g_0,h_0}^{-1}\{Y_k(\mathbf{s}) = \omega_k Z_k(\mathbf{s})$, the image measure of $Z_k^*(\mathbf{s})$, denoted as $\mathbb{P}_{Z,k}$, induced from $\mathbb{P}_{Y,k}$ is stationary Gaussian with mean 0 and Matérn correlation function (17) in $\mathbb{R}^d$ with a variance ω_k^2, a scale parameter ϕ_k and the same smoothness parameter ν, $k = 1, 2$.

Let $\mathbb{R}^{\mathbf{D}} = \{f : f(\mathbf{s}) \in \mathbb{R}, \mathbf{s} \in \mathbf{D} \subset \mathbb{R}^d\}$ be the set of real-valued functions and $\mathfrak{B}(\mathbb{R})$ be the Borel subsets of $\mathbb{R}$. A cylinder set is of the form

$$C_{\mathbf{s}_1,\ldots,\mathbf{s}_n}^{B_1,\ldots,B_n} = \{f \in \mathbb{R}^{\mathbf{D}} : f(\mathbf{s}_1) \in B_1, \ldots, f(\mathbf{s}_n) \in B_n\}, \tag{A.4}$$

where $\mathbf{s}_1, \ldots, \mathbf{s}_n \in \mathbb{R}^{\mathbf{D}}$ and $B_1, \ldots, B_n \in \mathfrak{B}(\mathbb{R})$. Then the cylinder σ-algebra is defined as the σ-algebra generated by collection of cylinder sets

$$\mathfrak{R} = \sigma\left\{C_{\mathbf{s}_1,\ldots,\mathbf{s}_n}^{B_1,\ldots,B_n} : \mathbf{s}_1, \ldots, \mathbf{s}_n \in \mathbb{R}, B_1, \ldots, B_n \in \mathfrak{B}(\mathbb{R}), n = 1, 2, \ldots\right\} \tag{A.5}$$

Suppose $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space and $Z_k^*(\mathbf{s}) : \Omega \to \mathbb{R}^{\mathbf{D}}$ is a measurable map with respect to cylinder σ-algebra $\mathfrak{R}$. Then $Z_k^*(\mathbf{s})$ is a Gaussian random field equipped with triplet $(\mathbb{R}^{\mathbf{D}}, \mathfrak{R}, \mathbb{P}_{Z,k})$ as a probability space, $k = 1, 2$. Similarly, $Y_k(\mathbf{s})$ is a TGH random field equipped with probability space $(\mathbb{R}^{\mathbf{D}}, \mathfrak{R}, \mathbb{P}_{Y,k})$, $k = 1, 2$.

The equation (A.3) defines a map $\mathbf{L} : \mathbb{R}^{\mathbf{D}} \to \mathbb{R}^{\mathbf{D}}$ as

$$(\mathbf{L}f)(\mathbf{s}) = \tau_{g_0 h_0}\{f(\mathbf{s})\} \text{ for any } f \in \mathbb{R}^{\mathbf{D}}. \tag{A.6}$$

By above definition, when $h_0 > 0$, τ_{g_0,h_0} is a continuous strictly increasing function and hence $\mathbf{L}$ is a continuous bijection. In other words, for any $f \in \mathbb{R}^{\mathbf{D}}$, there exist a unique $\mathbf{L}^{-1}f \in \mathbb{R}^{\mathbf{D}}$.

We first show that for any set $A \in \mathfrak{R}$, the image set $\mathbf{L}(A) = \{\mathbf{L}f : f \in A\} \in \mathfrak{R}$. The proof is indirect. Let $\mathfrak{S} = \{A : \mathbf{L}(A) \in \mathfrak{R}\}$ be the collection of sets in $\mathfrak{R}$ whose image is also in $\mathfrak{R}$. The first step is to show that $\mathfrak{S}$ is a σ-algebra.

(i) Suppose $A \in \mathfrak{S}$, then $\mathbf{L}(A) \in \mathfrak{R}$. For any $f \in A^c$, we must have $\mathbf{L}f \in \{\mathbf{L}(A)\}^c$ because otherwise there exists a $f^* \in A$ such that $(\mathbf{L}f^*)(\mathbf{s}) = (\mathbf{L}f)(\mathbf{s})$. By definition (A.6), since $\tau_{g_0,h_0}(\cdot)$ is continuous and strictly increasing from $\mathbb{R} \to \mathbb{R}$, $(\mathbf{L}f^*)(\mathbf{s}) = (\mathbf{L}f)(\mathbf{s})$ implies that $f = f^* \in A$, which contradicts the fact that $f \in A^c$. Therefore, $\mathbf{L}(A^c) \subseteq \{\mathbf{L}(A)\}^c$. On the other hand, for any $f \in \{\mathbf{L}(A)\}^c$, we must have $f^* = (\mathbf{L}^{-1}f) \in A^c$. Otherwise, if $f^* = (\mathbf{L}^{-1}f) \in A$, then $\mathbf{L}(f^*) = f \in \mathbf{L}(A)$, which contradicts the fact that $f \in \{\mathbf{L}(A)\}^c$. Therefore, $\{\mathbf{L}(A)\}^c \subseteq \mathbf{L}(A^c)$. We can then conclude that $A \in \mathfrak{S}$ implies that $\mathbf{L}(A^c) = \{\mathbf{L}(A)\}^c \in \mathfrak{R}$ and hence $A^c \in \mathfrak{S}$.

(ii) Suppose $A_1, \ldots, A_n$ is a sequence of sets in $\mathfrak{S}$. Then for any $f \in \bigcup_{i=1}^{\infty} A_i$, there exist an A_i such that $f \in A_i$ and thus $\mathbf{L}f \in \mathbf{L}(A_i)$. Hence we have that $\mathbf{L}(\bigcup_{i=1}^{\infty} A_i) \subseteq \bigcup_{i=1}^{\infty} \mathbf{L}(A_i)$. On the other hand, if $f \in \bigcup_{i=1}^{\infty} \mathbf{L}(A_i)$, there exist a $\mathbf{L}(A_i)$ such that $f \in \mathbf{L}(A_i)$. Using the same argument in part (i), we can show that $\mathbf{L}^{-1}f \in A_i \subseteq \bigcup_{i=1}^{\infty} A_i$ and hence $f = \mathbf{L}(\mathbf{L}^{-1}f) \in \mathbf{L}(\bigcup_{i=1}^{\infty} A_i)$. Hence, we have $\bigcup_{i=1}^{\infty} \mathbf{L}(A_i) \subseteq \mathbf{L}(\bigcup_{i=1}^{\infty} A_i)$. In summary, $\mathbf{L}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} \mathbf{L}(A_i) \in \mathfrak{R}$ since each $\mathbf{L}(A_i) \in \mathfrak{R}$, which further indicate $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{S}$. Therefore, for any sequence $A_1, \ldots, A_n, \cdots \in \mathfrak{S}$, we have $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{S}$.

Combing part (i) and (ii), we conclude that the collection of sets $\mathfrak{S}$ is a σ-algebra. The next step is to show that $\mathfrak{S}$ contains all cylinder sets defined in (A.4). By definition, let $f^* = \mathbf{L}f \in \mathbb{R}^{\mathbf{D}}$, by continuity and monotonicity of function $\tau_{g_0,h_0}(\cdot)$, we have that

$$\begin{aligned}\mathbf{L}\left(C_{\mathbf{s}_1,\ldots,\mathbf{s}_n}^{B_1,\ldots,B_n}\right) &= \{\mathbf{L}f : f \in \mathbb{R}^{\mathbf{D}}, f(\mathbf{s}_1) \in B_1, \ldots, f(\mathbf{s}_n) \in B_n\} \\ &= \{f^* \in \mathbb{R}^{\mathbf{D}} : f^*(\mathbf{s}_1) \in \tau_{g_0,h_0}(B_1), \ldots, f^*(\mathbf{s}_n) \in \tau_{g_0,h_0}(B_n)\},\end{aligned}$$

where $\mathbf{s}_1, \ldots, \mathbf{s}_n \in \mathbb{R}^{\mathbf{D}}$ and $B_1, \ldots, B_n \in \mathfrak{B}(\mathbb{R})$. By proposition C.1 in Burk (1998), page 273, since $\tau_{g_0,h_0}(\cdot)$ is a strictly increasing mapping of $\mathbb{R}$ onto $\mathbb{R}$, $\tau_{g_0,h_0}(B_i)$'s are all Borel sets as well. Hence $\mathbf{L}\left(C_{\mathbf{s}_1,\ldots,\mathbf{s}_n}^{B_1,\ldots,B_n}\right)$ is a cylinder set by definition. By definition, $\mathfrak{R}$ is generated by all cylinder sets, we must have $\mathbf{L}\left(C_{\mathbf{s}_1,\ldots,\mathbf{s}_n}^{B_1,\ldots,B_n}\right) \in \mathfrak{R}$. In other words, any cylinder set $C_{\mathbf{s}_1,\ldots,\mathbf{s}_n}^{B_1,\ldots,B_n} \in \mathfrak{S}$.

Since by definition, $\mathfrak{R}$ is the smallest σ-algebra contains all cylinder sets and we have shown that $\mathfrak{S}$ is a σ-algebra contains all cylinder sets, it follows that $\mathfrak{R} \subseteq \mathfrak{S}$. Therefore, for any $A \in \mathfrak{R}$, we have $\mathbf{L}(A) \in \mathfrak{R}$. Following exactly the same arguments, we can show that for any $A \in \mathfrak{R}$, we also have $\mathbf{L}^{-1}(A) \in \mathfrak{R}$.

Finally, recall that $Z_k^*(\mathbf{s})$ is a Gaussian random field equipped with a probability space $(\mathbb{R}^{\mathbf{D}}, \mathfrak{R}, \mathbb{P}_{Z,k})$ and $Y_k(\mathbf{s})$ is a TGH random field equipped with triplet $(\mathbb{R}^{\mathbf{D}}, \mathfrak{R}, \mathbb{P}_{Y,k})$, $k = 1, 2$. And the map defined in (A.6) connects $Z_k(\mathbf{s})$ to $Y_k(\mathbf{s})$. Suppose that for any $A \in \mathfrak{R}$ such that $\mathbb{P}_{Y,1}(A) = 0$, since we have shown that $\mathbf{L}^{-1}(A) \in \mathfrak{R}$, $\mathbb{P}_{Z,1}\{\mathbf{L}^{-1}(A)\}$ is well defined and using the property of bijection (A.6) we have that $\mathbb{P}_{Z,1}\{\mathbf{L}^{-1}(A)\} = \mathbb{P}_{Y,1}(A) = 0$. Using Theorem 2 of Zhang (2004), when $d = 1, 2, 3$, if $\omega_1^2/\phi_1^{2\nu} = \omega_2^2/\phi_2^{2\nu}$, we have that $\mathbb{P}_{Z,1} \equiv \mathbb{P}_{Z,2}$. Hence $\mathbb{P}_{Y,2}(A) = \mathbb{P}_{Z,2}\{\mathbf{L}^{-1}(A)\} = \mathbb{P}_{Z,1}\{\mathbf{L}^{-1}(A)\} = 0$. Therefore, $\mathbb{P}_{Y,2}(A) = 0$ for any $A \in \mathfrak{R}$ such that $\mathbb{P}_{Y,1}(A) = 0$. In other words, $\mathbb{P}_{Y,2}(A) \ll \mathbb{P}_{Y,1}(A)$. Applying exactly the same argument, we can show that $\mathbb{P}_{Y,1}(A) \ll \mathbb{P}_{Y,2}(A)$. Therefore, under conditions of Theorem 3, we have that $\mathbb{P}_{Y,1}(A) \equiv \mathbb{P}_{Y,2}(A)$, which completes the proof. $\square$

Proof of Lemma 3: Lemma 3 follows from the fact that $\tau_{g,h}(z)$ is a monotone transformation when $h \geq 0$ and from the conditional distribution of a multivariate Gaussian random vector.

□

Proof of Theorem 4: The conditional distribution (11) follows directly from Lemma 3. For equation (12), notice that $\tilde{\mu}$ is the median of the distribution $GH_1(\tilde{\mu}, \tilde{\sigma}^2, g, h)$ since the function $\tau_{g,h}(z)$ is a monotone function of z when $h \geq 0$. Therefore, by the form of model (4), we have

$$\text{med}\{Y(\mathbf{s}_0)|\mathcal{D}_n\} = \xi + \mathbf{X}(\mathbf{s}_0)^{\mathrm{T}}\boldsymbol{\beta} + \omega\tau_{g,h}(\tilde{\mu}).$$

Hence, equation (12) is proven. For equation (13), by using equation (A.1) in Lemma A.1 repeatedly, some straightforward algebra yield that

$$E\{T(\mathbf{s}_0)|\mathcal{D}_n\} = \frac{1}{g\sqrt{1-h\tilde{\sigma}^2}} \exp\left\{\frac{h\tilde{\mu}^2}{2(1-h\tilde{\sigma}^2)}\right\}\left[\exp\left\{\frac{g^2\tilde{\sigma}^2 + 2g\tilde{\mu}}{2(1-h\tilde{\sigma}^2)}\right\} - 1\right],$$

and hence equation (14) follows. □

Lemma A.2 *If $Z \sim N(\mu, \sigma^2)$, then for any $a < \frac{1}{2\sigma^2}$ and $b \in \mathbb{R}$, we have that*

$$\int_{-\infty}^{z_0} \exp\{aZ^2 + bZ\} f_Z(z)\, dz = E\big\{\exp(aZ^2 + bZ)\big\} = \Phi\left\{\frac{\sqrt{1-2a\sigma^2}}{\sigma}\left(z_0 - \frac{\mu + b\sigma^2}{1-2a\sigma^2}\right)\right\}, \tag{A.7}$$

$$E\{\Phi(Z)\} = \int_{-\infty}^{\infty} \Phi(z) f_Z(z)\, dz = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right), \tag{A.8}$$

where z_0 is a fixed number, $f_Z(z)$ is the density function of $Z \sim N(\mu, \sigma^2)$ and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Proof: Straightforward algebra yields that

$$\begin{aligned}
&\int_{-\infty}^{z_0} \exp\{aZ^2 + bZ\} f_Z(z)\, dz \\
&= \int_{-\infty}^{z_0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{az^2 + bz - \frac{(z-\mu)^2}{2\sigma^2}\right\} dz \\
&= \int_{-\infty}^{z_0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\left(\frac{1}{2\sigma^2} - a\right) z^2 + \left(b + \frac{\mu}{\sigma^2}\right) z - \frac{\mu^2}{2\sigma^2}\right\} dz \\
&= \frac{1}{\sqrt{\frac{1}{\sigma^2} - 2a\sigma}} \exp\left\{-\frac{\mu^2}{2\sigma^2} + \frac{(b+\frac{\mu}{\sigma^2})^2}{4(\frac{1}{2\sigma^2} - a)}\right\} \Phi\left\{\sqrt{\frac{1}{2\sigma^2} - a}\left(z_0 - \frac{\mu + b\sigma^2}{1 - 2a\sigma^2}\right)\right\} \\
&= \frac{1}{\sqrt{1 - 2a\sigma^2}} \exp\left\{\frac{b^2\sigma^2 + 2b\mu + 2a\mu^2}{2(1-2a\sigma^2)}\right\} \Phi\left\{\sqrt{\frac{1-2a\sigma^2}{2\sigma^2}}\left(z_0 - \frac{\mu + b\sigma^2}{1 - 2a\sigma^2}\right)\right\}
\end{aligned}$$

Thus equation (A.7) is proved. The equation (A.8) follows readily from Lemma 2.1 in Arellano-Valle and Genton (2005). □

Proof of Lemma 4: Using the representation of CRPS in (14), we have that

$$CRPS\{F_{\mathbf{s}_0}, y(\mathbf{s}_0)\} = \mathrm{E}(|Y - y(\mathbf{s}_0)|) - \frac{1}{2}\mathrm{E}(|Y - Y^*|) = \omega\left\{\mathrm{E}(|T - t_0|) - \frac{1}{2}\mathrm{E}(|T - T^*|)\right\},$$

where $T = \frac{Y - \xi - \mathbf{x}(\mathbf{s}_0)^{\mathrm{T}}\boldsymbol{\beta}}{\omega}$, $T^* = \frac{Y^* - \xi - \mathbf{x}(\mathbf{s}_0)^{\mathrm{T}}\boldsymbol{\beta}}{\omega}$, and, $t_0 = \frac{y(\mathbf{s}_0) - \xi - \mathbf{x}(\mathbf{s}_0)^{\mathrm{T}}\boldsymbol{\beta}}{\omega}$. By Theorem 3, we can see that T and T^* are independent variables with $GH_1(\tilde{\mu}, \tilde{\sigma}^2, g, h)$ distribution, which means that $\tilde{Z} = \tau_{g,h}^{-1}(T)$ and $\tilde{Z}^* = \tau_{g,h}^{-1}(T^*)$ are independent random variables following $N(\tilde{\mu}, \tilde{\sigma}^2)$ distribution. Let $f_{\tilde{Z}}(z)$ and $f_{\tilde{Z}^*}(z)$ be their density functions, we have

$$\begin{aligned}
\mathrm{E}(|T - t_0|) &= \int_{-\infty}^{z_0} \{t_0 - \tau_{g,h}(z)\} f_{\tilde{Z}}(z) dz + \int_{z_0}^{\infty} \{\tau_{g,h}(z) - t_0\} f_{\tilde{Z}}(z) dz \\
&= t_0\left\{2\Phi\left(\frac{z_0 - \tilde{\mu}}{\tilde{\sigma}}\right) - 1\right\} + \mathrm{E}\{\tau_{g,h}(Z)\} - 2\int_{-\infty}^{z_0} \tau_{g,h}(z) f_{\tilde{Z}}(z) dz,
\end{aligned}$$

where $z_0 = \tau_{g,h}^{-1}(t_0)$. By equation (A.7) in Lemma A.2, we have that

$$\begin{aligned}\int_{-\infty}^{z_0} \tau_{g,h}(z) f_{\tilde{Z}}(z)dz &= \frac{1}{g}\int_{-\infty}^{z_0} \exp\left(\frac{h}{2}z^2 + gz\right) f_{\tilde{Z}}(z)dz - \frac{1}{g}\int_{-\infty}^{z_0} \exp\left(\frac{h}{2}z^2\right) f_{\tilde{Z}}(z)dz \\ &= \frac{1}{g}\mathrm{E}\left\{\exp\left(\frac{h}{2}\tilde{Z}^2 + g\tilde{Z}\right)\right\}\Phi\left\{\frac{\sqrt{1-h\tilde{\sigma}^2}}{\tilde{\sigma}}\left(z_0 - \frac{\tilde{\mu}+g\tilde{\sigma}^2}{1-h\tilde{\sigma}^2}\right)\right\} \\ &\quad - \frac{1}{g}\mathrm{E}\left\{\exp\left(\frac{h}{2}\tilde{Z}^2\right)\right\}\Phi\left\{\frac{\sqrt{1-h\tilde{\sigma}^2}}{\tilde{\sigma}}\left(z_0 - \frac{\tilde{\mu}}{1-h\tilde{\sigma}^2}\right)\right\},\end{aligned}$$

which yields that

$$\begin{aligned}\mathrm{E}(|T-t_0|) &= t_0\left\{2\Phi\left(\frac{z_0-\tilde{\mu}}{\tilde{\sigma}}\right)-1\right\} + \frac{1}{g}\mathrm{E}\left\{\exp\left(\frac{h}{2}\tilde{Z}^2\right)\right\}\left[2\Phi\left\{\frac{\sqrt{1-h\tilde{\sigma}^2}}{\tilde{\sigma}}\left(z_0-\frac{\tilde{\mu}}{1-h\tilde{\sigma}^2}\right)\right\}-1\right] \\ &\quad -\frac{1}{g}\mathrm{E}\left\{\exp\left(\frac{h}{2}\tilde{Z}^2+g\tilde{Z}\right)\right\}\left[2\Phi\left\{\frac{\sqrt{1-h\tilde{\sigma}^2}}{\tilde{\sigma}}\left(z_0-\frac{\tilde{\mu}+g\tilde{\sigma}^2}{1-h\tilde{\sigma}^2}\right)\right\}-1\right]. \qquad \text{(A.9)}\end{aligned}$$

Note that $\mathrm{E}(|T-T^*|) = \mathrm{E}_{T^*}\left\{\mathrm{E}_T\left(|T-T^*|\big|T^*\right)\right\}$, where E_T and E_{T^*} represent taking expectation with respect to T and T^*, respectively. The conditional expectation $\mathrm{E}_T\left(|T-T^*|\big|T^*\right)$ takes the same form as equation (A.9) by replacing t_0 and z_0 with T^* and $\tilde{Z}^*$, respectively. Therefore, to find $\mathrm{E}(|T-T^*|)$, we need to find four quantities: $\mathrm{E}\left\{T^*\Phi\left(\frac{\tilde{Z}^*-\tilde{\mu}}{\tilde{\sigma}}\right)\right\}$, $\mathrm{E}(T^*)$, $\mathrm{E}\left[\Phi\left\{\frac{\sqrt{1-h\tilde{\sigma}^2}}{\tilde{\sigma}}\left(\tilde{Z}^*-\frac{\tilde{\mu}+g\tilde{\sigma}^2}{1-h\tilde{\sigma}^2}\right)\right\}\right]$, and $\mathrm{E}\left[\Phi\left\{\frac{\sqrt{1-h\tilde{\sigma}^2}}{\tilde{\sigma}}\left(\tilde{Z}^*-\frac{\tilde{\mu}}{1-h\tilde{\sigma}^2}\right)\right\}\right]$. We find them one by one as follows:

$$\begin{aligned}\mathrm{E}\left\{T^*\Phi\left(\frac{\tilde{Z}^*-\tilde{\mu}}{\tilde{\sigma}}\right)\right\} &= \frac{1}{g}\int_{-\infty}^{\infty}\left\{\exp\left(gz+\frac{h}{2}z^2\right)-\exp\left(\frac{h}{2}z^2\right)\right\}\Phi\left(\frac{z-\tilde{\mu}}{\tilde{\sigma}}\right)f_{\tilde{Z}^*}(z)dz \\ &= \frac{1}{g\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1-h\tilde{\sigma}^2}{2}u^2+(g+h\tilde{\mu})\tilde{\sigma}u+\frac{h}{2}\tilde{\mu}^2+\tilde{\mu}g\right\}\Phi(u)du \\ &\quad - \frac{1}{g\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1-h\tilde{\sigma}^2}{2}u^2+h\tilde{\mu}\tilde{\sigma}u+\frac{h}{2}\tilde{\mu}^2\right)\Phi(u)du \qquad \text{(A.10)} \\ &= \frac{1}{g\sqrt{1-h\tilde{\sigma}^2}}\exp\left\{\frac{g^2\tilde{\sigma}^2+2\tilde{\mu}g+h\tilde{\mu}^2}{2(1-h\tilde{\sigma}^2)}\right\}\Phi\left\{\frac{(h\tilde{\mu}+g)\tilde{\sigma}}{\sqrt{2-3h\tilde{\sigma}^2+h^2\tilde{\sigma}^4}}\right\} \\ &\quad - \frac{1}{g\sqrt{1-h\tilde{\sigma}^2}}\exp\left\{\frac{h\tilde{\mu}^2}{2(1-h\tilde{\sigma}^2)}\right\}\Phi\left\{\frac{h\tilde{\mu}\tilde{\sigma}}{\sqrt{2-3h\tilde{\sigma}^2+h^2\tilde{\sigma}^4}}\right\},\end{aligned}$$

where the last equation follows from equation (A.8) in Lemma A.2. A straightforward application of equation (A.1) in Lemma A.1 yields that

$$\mathrm{E}(T^*) = \frac{1}{g\sqrt{1-h\tilde{\sigma}^2}}\exp\left\{\frac{h\tilde{\mu}^2}{2(1-h\tilde{\sigma}^2)}\right\}\left[\exp\left\{\frac{g^2\tilde{\sigma}^2+2g\tilde{\mu}}{2(1-h\tilde{\sigma}^2)}\right\}-1\right]. \qquad \text{(A.11)}$$

By using equation (A.8) in Lemma A.2, it is also straightforward to show that

$$\mathrm{E}\left[\Phi\left\{\frac{\sqrt{1-h\tilde{\sigma}^2}}{\tilde{\sigma}}\left(z_0-\frac{\tilde{\mu}+g\tilde{\sigma}^2}{1-h\tilde{\sigma}^2}\right)\right\}\right]=\Phi\left\{\frac{h\tilde{\mu}\tilde{\sigma}+g\tilde{\sigma}}{\sqrt{2-3h\tilde{\sigma}^2+h^2\tilde{\sigma}^4}}\right\},\ \text{and} \tag{A.12}$$

$$\mathrm{E}\left[\Phi\left\{\frac{\sqrt{1-h\tilde{\sigma}^2}}{\tilde{\sigma}}\left(z_0-\frac{\tilde{\mu}}{1-h\tilde{\sigma}^2}\right)\right\}\right]=\Phi\left\{\frac{h\tilde{\mu}\tilde{\sigma}}{\sqrt{2-3h\tilde{\sigma}^2+h^2\tilde{\sigma}^4}}\right\}. \tag{A.13}$$

In addition, by equation (A.2) in Lemma A.1, we have that

$$\mathrm{E}\left\{\exp\left(\frac{h}{2}\tilde{Z}^2+g\tilde{Z}\right)\right\}=\frac{1}{\sqrt{1-h\tilde{\sigma}^2}}\exp\left\{\frac{g^2\tilde{\sigma}^2+2\tilde{\mu}g+h\tilde{\mu}^2}{2(1-h\tilde{\sigma}^2)}\right\},\quad\text{and} \tag{A.14}$$

$$\mathrm{E}\left\{\exp\left(\frac{h}{2}\tilde{Z}^2\right)\right\}=\frac{1}{\sqrt{1-h\tilde{\sigma}^2}}\exp\left\{\frac{h\tilde{\mu}^2}{2(1-h\tilde{\sigma}^2)}\right\}. \tag{A.15}$$

By combining equations (A.9)-(A.15), we have that

$$\begin{aligned}\mathrm{E}(|T-T^*|)=&\frac{2}{g\sqrt{1-h\tilde{\sigma}^2}}\exp\left\{\frac{h\tilde{\mu}^2}{2(1-h\tilde{\sigma}^2)}\right\}\left[1-2\Phi\left\{\frac{h\tilde{\mu}\tilde{\sigma}}{\sqrt{2-3h\tilde{\sigma}^2+h^2\tilde{\sigma}^4}}\right\}\right]\\&-\frac{2}{g\sqrt{1-h\tilde{\sigma}^2}}\exp\left\{\frac{g^2\tilde{\sigma}^2+2\tilde{\mu}g+h\tilde{\mu}^2}{2(1-h\tilde{\sigma}^2)}\right\}\left[1-2\Phi\left\{\frac{h\tilde{\mu}\tilde{\sigma}+g\tilde{\sigma}}{\sqrt{2-3h\tilde{\sigma}^2+h^2\tilde{\sigma}^4}}\right\}\right].\end{aligned} \tag{A.16}$$

Using equations (A.9) and (A.16), equation (16) follows from the form of $CRPS\{F_{\mathbf{s}_0}, y(\mathbf{s}_0)\}$. □

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