Abstract—In multiple-input multiple-out (MIMO) radar, for desired transmit beampatterns, appropriate correlated waveforms are designed. To design such waveforms, conventional MIMO radar methods use two steps. In the first step, the waveforms covariance matrix, \( R \), is synthesized to achieve the desired beampattern. While in the second step, to realize the synthesized covariance matrix, actual waveforms are designed. Most of the existing methods use iterative algorithms to solve these constrained optimization problems. The computational complexity of these algorithms is very high, which makes them difficult to use in practice. In this paper, to achieve the desired beampattern, a low complexity discrete-Fourier-transform based closed-form covariance matrix design technique is introduced for a MIMO radar. The designed covariance matrix is then exploited to derive a novel closed-form algorithm to directly design the finite-alphabet constant-envelope waveforms for the desired beampattern. The proposed technique can be used to design waveforms for large antenna array to change the beampattern in real time. It is also shown that the number of transmitted symbols from each antenna depends on the beampattern and is less than the total number of transmit antenna elements.

Index Terms—Multiple-input multiple-output radars, beampattern design, closed-form solution, waveform design, two-dimensional Discrete-Fourier-transform.

I. INTRODUCTION

Collocated multiple-input and multiple-output (MIMO) radar has a number of advantages over the classical phased-array radar. For example, it yields significant improvement in parameter estimation and target detection. In addition, it provides enhanced flexibility to achieve desired transmit beampatterns [1]–[6]. In order to maximize the signal to interference-plus-noise ratio (SINR) and the probability of detection for a given target, optimum transmitter/receiver designs are investigated in [7], [8]. The target detection performance can be further improved with knowledge aided processing [8].

If the target location is not precisely known and there is some ambiguity in the exact target location, transmit beampattern techniques can be used to focus the transmitted power into certain region-of-interest (ROI) [9]–[14]. This process turns out to be essential in a number of applications. For example, imaging radars are generally focus the transmitted power into the pre-defined ROI, which reduces the reception of transmit signal dependent interference from outside the ROI [15], [16]. In order to transmit power into the given ROI, the following two approaches are available in the literature:

- The first approach is based on waveform covariance matrix design. It is known that the transmit beampattern of a collocated antenna array depends on the cross-correlation between the transmitted waveforms from different antenna elements. Therefore, to design a variety of transmit beampatterns, early solutions have relied on a two-step process [9]–[14]. In the first step, the user designs the covariance matrix such that the theoretical transmitted power matches the desired beampattern as closely as possible. The second step involves the design of the actual waveforms that can realize the designed covariance matrix. Both of these steps require constrained optimization and most of the available literature uses iterative algorithms.

- In the second approach, the waveforms which realize the desired beampattern are directly designed without synthesizing the covariance matrix [17]. However, to the best of the authors’ knowledge, optimal solutions that directly design the waveforms for a given beampattern are not available yet.

For the first approach, efficient iterative algorithms are proposed in [10], [13], [18] to synthesize the covariance matrix for the given beampattern. These algorithms are computationally very expensive for real-time applications. In order to synthesize the covariance matrix, a reduced complexity closed-form solution is proposed in [19]. To reduce the computational complexity, this algorithm exploits the Discrete-Fourier-transform (DFT). Once the covariance matrix is synthesized, the corresponding waveforms fulfilling some practical constraints such as close to unity peak-to-average power ratio (PAPR) are designed. An iterative algorithm to design constant-envelope waveforms is proposed in [20]. The computational complexity of this algorithm is also very high: in addition, it generates non-finite alphabets that can be challenging to use in practice.
In [21], by mapping Gaussian random variables onto binary phase-shift keying (BPSK) symbols, a closed-form solution to generate BPSK waveforms to realize the given covariance matrix and an iterative algorithm to achieve best possible beampattern are proposed. The main drawback of this algorithm is that its performance depends on the beampattern.

Using the second approach, a sub-optimal algorithm to directly design the waveform for a uni-modal symmetric beampattern is presented in [17]. In this algorithm, a scalar coefficient controls the width of the beampattern. This method requires a large number of transmitting antenna elements in order to achieve a good approximation of the desired beampattern. In other works such as [22], the authors considered other optimization metrics such as auto- and cross-correlation while designing the waveforms. However, they have not been able to achieve any beampattern design since the designed waveforms are orthogonal which leads to an isotropic transmitted power.

We have noticed that the beampattern solutions proposed in the previous works deal only with the linear arrays and the ROI is defined only by one parameter, which is spatial angle $\theta$ [23]. In MIMO radar, with a planar-antenna-array at the input, a second called polar angle, $\phi$, is added to define locations in two-dimensional (2D) space and provide a larger radar aperture. In [14], various strategies for Hybrid MIMO phased-array radar, based on multiplication of signal sets by a pseudo-noise spreading sequence, are proposed for different transmit 2D beampatterns.

It is very expensive to synthesize the covariance matrix, for a large planar-antenna-array, using semi-definite-programming (SDP). In [24], the authors propose a simple low complexity sub-optimal method for direct waveform design in order to achieve the desired beampattern. In the proposed scheme, the specified region is uniformly divided into a number of grid points and corresponding to each grid point, snapshots are transmitted one by one to achieve the desired beampattern. The drawback of this scheme is that the beampattern depends on the number and the location of the chosen grid points. Moreover, this algorithm does not provide any relationship between the specified beampattern and the number and exact location of grid points. Consequently, the number of grid points can be more than the number of transmit antennas.

To deal with the above issues, our contributions are given below:
- We propose a closed-form solution to design the covariance matrix for a planar-antenna-array to achieve desired 2D beampatterns. In order to reduce the computational complexity of our algorithm, the 2D beampattern design problem is mapped onto the 2D-DFT. The algorithm in [19] can be considered as a special case of our proposed algorithm. To realise this covariance matrix, existing iterative algorithms can be used to design corresponding waveforms.
- By exploiting the derivations of the covariance matrix, for the desired 2D-beampattern, a novel algorithm to directly design the finite-alphabet constant-envelope waveforms is proposed. The proposed direct design of the waveforms yields a significant reduction in the computational complexity and can achieve the best possible performance among existing direct waveform design algorithms. Moreover, in contrast to the scheme presented in [24], our algorithm provides fixed number and exact location of the snapshots for the specified beampattern. Plus, the number of snapshots is guaranteed to be less than the number of transmit antennas.

The rest of this paper is organized as follows. In Sec. II, we present the signal model adopted for the planar-antenna-array and formulate the optimization problem for the beampattern design. In Sec. III, by exploiting 2D-DFT, an algorithm to design the covariance matrix for the desired beampattern is presented. The computational complexity of our method is studied and compared to the SDP method in Sec. IV. Direct design of the waveforms is presented in Sec. V. Simulation results are discussed in Sec. VI, and conclusions are finally drawn in Sec. VII.

**Notations:** Small letters, bold small letters, and bold capital letters respectively designate scalars, vectors, and matrices. If $A$ is a matrix, then $A^H$ and $A^T$ respectively denote the Hermitian transpose and the transpose of $A$. $v(i)$ denotes the $i$th element of vector $v$. $A(i,j)$ denotes the entry in the $i$th row and $j$th column of matrix $A$. The Kronecker product is denoted by $\otimes$. Modulo $M$ operation on an integer $i$ is denoted by $(i)_M$ and $[i]_M$ denotes the quotient of $i$ over $M$. Finally, the statistical expectation is denoted by $E\{\cdot\}$.

### II. System Model and Problem Formulation

Consider a MIMO radar system composed of $M \times N$ omni-directional collocated antenna elements, positioned at the origin of a unit radius sphere within a rectangular planar array as shown in Fig. 1. The inter-element-spacing between any two adjacent antenna elements in the $x$- and $y$-axis directions is $d_x$ and $d_y$, respectively. If a spatial location around this planar array has an azimuth angle $\theta$ and an polar angle $\phi$, the corresponding Cartesian coordinates of this location can be written as

$$
\begin{align*}
    x &= \sin(\phi) \cos(\theta), \\
    y &= \sin(\phi) \sin(\theta).
\end{align*}
$$

![Fig. 1: Planar array of $M \times N$ transmit antennas.](image-url)
Define the baseband transmitted signal vector containing the transmitted symbols from all antennas at time index \( n \) as

\[
\mathbf{x}(n) = [x_{0,0}(n), \ldots, x_{0,N-1}(n), \ldots, x_{M-1,N-1}(n)]^T,
\]

where \( x_{p,q}(n) \) denotes the transmitted symbol from the antenna element at the \((p,q)\)th location at time index \( n \). For narrow band signals, by assuming non-dispersive propagation and zero path-loss, the signal at a point in space defined by the azimuth angle \( \theta \) and polar angle \( \phi \) can be written as

\[
r(n; \theta, \phi) = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} x_{p,q}(n) e^{j2\pi p d_y(p,q) \sin(\phi) \cos(\theta)} e^{j2\pi q d_x(p,q) \sin(\phi) \sin(\theta)}.
\]

(2)

If the distance between any two adjacent antenna elements on the \( x \)- and \( y \)-axis directions is \( \lambda/2 \), \( d_x(p,q) = q \lambda/2 \), \( d_y(p,q) = p \lambda/2 \) and (2) simplifies to

\[
r(n; x, y) = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} x_{p,q}(n) e^{j2\pi q (f_x + p f_y)},
\]

(3)

where

\[
\left\{ \begin{array}{c}
  f_x = \frac{\sin(\phi) \cos(\theta)}{2} \\
  f_y = \frac{\sin(\phi) \sin(\theta)}{2}
\end{array} \right.
\]

are the normalized Cartesian coordinates of the spatial location. The received signal in (3) can be written in vector form as

\[
r(n; f_x, f_y) = \mathbf{a}_s^H(f_x, f_y) \mathbf{x}(n),
\]

(5)

where

\[
\mathbf{a}_s(f_x, f_y) = \left[ \begin{array}{c}
  e^{j2\pi f_y} \\
  \vdots \\
  e^{j2\pi (N-1)f_y}
\end{array} \right] \otimes \left[ \begin{array}{c}
  1 \\
  e^{j2\pi f_x} \\
  \vdots \\
  e^{j2\pi (M-1)f_x}
\end{array} \right].
\]

(6)

Using (5), the received power at the location \((f_x, f_y)\) can be easily written as

\[
B(f_x, f_y) = \mathbb{E}\{\mathbf{a}_s^H(f_x, f_y) \mathbf{x}(n) \mathbf{x}(n)^H \mathbf{a}_s(f_x, f_y)\} = \mathbf{a}_s^H(f_x, f_y) \mathbf{R} \mathbf{a}_s(f_x, f_y),
\]

(7)

where \( \mathbf{R} = \mathbb{E}\{\mathbf{x}(n)\mathbf{x}(n)^H\} \) is the \( MN \times MN \) covariance matrix of the transmitted waveforms, which have \( \frac{(MN)^2 - MN}{2} \) degrees of freedom. In the conventional transmit beampattern design problem, a covariance matrix, \( \mathbf{R} \), is synthesized to match the transmitted power \( B(f_x, f_y) \) to the desired beampattern which involves the minimization of the following cost function

\[
J(\mathbf{R}) = \sum_{l=1}^{L} \sum_{k=1}^{K} \left| \mathbf{a}_l^H(f_x(l), f_y(k)) \mathbf{R} \mathbf{a}_l(f_x(l), f_y(k)) \right|^2 - \alpha \mathbf{P}_d(f_x(l), f_y(k))^2,
\]

(8)

where \( \mathbf{P}_d(f_x(l), f_y(k)) \) is the desired beampattern defined over the 2D-grid \( \{(f_x(l), f_y(k)) \}_{l=1}^{L} \times_{k=1}^{K} \) and \( \alpha \) is a scaling factor. Since \( \mathbf{R} \) is a covariance matrix, it should be positive semi-definite. Moreover, radio-frequency power amplifiers (RFP A) have limited dynamic range and they can not transmit all power levels with the same power efficiency. If we want to design variety of transmit beampatterns without changing any hardware, the RFP A should transmit the same power levels for any beampattern. Therefore, to satisfy these constraints using the conventional methods, we define the following minimization problem:

\[
\begin{align*}
\min_{\mathbf{R}} & \quad J(\mathbf{R}) \\
\text{subject to} & \quad \mathbf{C}_1 : \mathbf{R} \succeq 0 \\
\mathbf{C}_2 : \quad & \mathbf{R}(n, n) = c, \ n = 1, 2, \ldots, MN,
\end{align*}
\]

(9)

where \( \mathbf{C}_1 \) represents the semi-definite constraint and \( \mathbf{C}_2 \) ensures each antenna element transmit same average power. The constrained problem in (9) can be optimally solved using SDP. However, for a large number of antennas the computational complexity of SDP becomes prohibitively high. Therefore, these approaches are not feasible for planar arrays consisting of a large number of antennas. In order to reduce the computational complexity to synthesize \( \mathbf{R} \), by exploiting 2D-DFT, a closed-form solution is proposed in the following section. The SDP algorithm is considered hereafter as a benchmark.

## III. PROPOSED COVARIANCE MATRIX DESIGN

Given an \( M \times N \) time domain matrix \( \mathbf{H}_t \), the \( M \times N \) frequency domain matrix \( \mathbf{H}_f \) can be easily generated. The relationship between the time domain coefficients \( \mathbf{H}_t(m, n) \) and the frequency domain coefficients \( \mathbf{H}_f(k_1, k_2) \) is given by the following 2D-FFT formula

\[
\mathbf{H}_f(k_1, k_2) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \mathbf{H}_t(m, n) e^{-j2\pi k_1 m/M} e^{-j2\pi k_2 n/N}.
\]

(10)

Similarly, given frequency domain coefficients, the time domain coefficients are obtained with the 2D inverse discrete-Fourier-transform (2D-IDFT)

\[
\mathbf{H}_t(m, n) = \frac{1}{MN} \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{N-1} \mathbf{H}_f(k_1, k_2) e^{j2\pi k_1 m/M} e^{j2\pi k_2 n/N}.
\]

(11)

Using the 2D-DFT formula of (10), we obtain the following lemma which is proved in the appendix.

**Lemma 1:** Let \( \mathbf{H}_f \) be an \( M \times N \) matrix with real positive frequency domain coefficients and vectors \( \mathbf{e}_M(k_1) \) and \( \mathbf{e}_N(k_2) \)
are defined as

\[
e_M(k_1) = \left[ 1 \ e^{j2\pi k_1/M} \ldots e^{j2\pi k_1(M-1)/M} \right]^T,
\]

\[
e_N(k_2) = \left[ 1 \ e^{j2\pi k_2/N} \ldots e^{j2\pi k_2(N-1)/N} \right]^T,
\]

(12)

where \( k_1 = 0, 1, \ldots, M-1 \) and \( k_2 = 0, 1, \ldots, N-1 \). If we construct a matrix \( R_{hh} \) as

\[
R_{hh} = \frac{1}{MN^2} \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{N-1} H_f(k_1, k_2) e(k_1, k_2) e^H(k_1, k_2),
\]

where \( e(k_1, k_2) = e_N(k_2) \otimes e_M(k_1) \), then \( R_{hh} \) will be positive semi-definite and all of its diagonal elements will be equal. Moreover, the individual elements of \( H_f \) are related to the entries of \( R_{hh} \) using the following quadratic form

\[
H_f(l_1, l_2) = e^H(l_1, l_2) R_{hh} e(l_1, l_2). \tag{14}
\]

Finding \( R_{hh} \) using (13) can be computationally very expensive since it requires the outer product of \( MN \) vectors and the addition of \( MN \) corresponding matrices. To reduce the computational complexity, we use (13) to express the individual elements of \( R_{hh} \) as

\[
R_{hh}(i_1, i_2) = \frac{1}{MN} \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{N-1} H_f(k_1, k_2) \times e^{2\pi k_1(i_1-i_2)/M} e^{2\pi k_2(i_2-i_1)/N}, \tag{15}
\]

where \( i_1, i_2 = 0, 1, \ldots, MN - 1 \). Comparing (15) with (11) yields

\[
R_{hh}(i_1, i_2) = \frac{1}{MN} H_f \left( [i_1-i_2] M, [i_1 M - [i_2 M]. \right) \tag{16}
\]

As we know, for a given frequency domain matrix \( H_f \), the time domain matrix \( H_t \) can be found using the 2D-IDFT. Therefore, finding \( R_{hh} \) using \( H_t \) is computationally less expensive. It should also be noted here that since \( H_f \) is real, \( H_t(-m,-n) = H^*_t(m,n) \) and \( e^{-j2\pi k_1 m/M} = e^{j2\pi k_1 (M-m)/M} \), the matrix \( R_{hh} \) will be block Toeplitz.

Note that the case of uniform linear array, studied in [19], can be considered as the special case of our proposed planar array when \( N = 1 \). In this case, the frequency and time domain matrices \( H_f \) and \( H_t \) are reduced to \( M \times 1 \) vectors denoted respectively as \( h_f \) and \( h_t \). The correlation matrix \( R_{hh} \) becomes of dimension \( M \times M \) and by using formula (13) the individual elements of \( R_{hh} \) can be found as

\[
R_{hh}(i_1, i_2) = \frac{1}{M^2} \sum_{k_1=0}^{M-1} h_f(k_1) e^{2\pi k_1(i_1-i_2)/M},
\]

\[
= \frac{1}{M^2} \sum_{k_1=0}^{M-1} h_t(k_1) e^{2\pi k_1(i_2-i_1)/M}. \tag{17}
\]

Similarly, using the fact that \( h_f \) is real, the matrix \( R_{hh} \) can be found using the time domain coefficients of \( h_f \) as

\[
R_{hh}(i_1, i_2) = \frac{1}{M} h_t(i_1 - i_2). \tag{18}
\]

Since \( h_t(-i) = h_t^*(i) \), the matrix \( R_{hh} \) is the same Toeplitz matrix proposed in [19]. Thus, our generalized method for the 2D-beampatterns using planar arrays (defined by \( \theta \) and \( \phi \)) is valid also in the case of 2D-beampatterns using linear arrays as proposed in [19].

Since the matrix \( R_{hh} \) is positive semi-definite and all of its diagonal elements are equal, it satisfies both the \( C_1 \) and \( C_2 \) constraints in (9). Therefore, if \( R_{hh} \) is considered to be the covariance matrix, by comparing (7) with (14), it can be easily seen that the problem of transmit beampattern design can be mapped to the result obtained in the Lemma 1. This transformation only requires the mapping of the steering vector \( a_s(f_x, f_y) \) to \( e(k_1, k_2) \). This can be done by mapping the values of \( f_x \) and \( f_y \) respectively onto \( k_1 \) and \( k_2 \) using the following expressions

\[
\begin{align*}
& f_x \mapsto -0.5 + \frac{k_1}{M-1}, \quad k_1 = 0 \ldots M-1, \\
& f_y \mapsto -0.5 + \frac{k_2}{N-1}, \quad k_2 = 0 \ldots N-1.
\end{align*} \tag{19}
\]

It should be noted here that \(-0.5 \leq \{f_x, f_y\} \leq +0.5 \). Since the beampattern is mapped on \( f_x \) and \( f_y \) that respectively have \( M \) and \( N \) discrete values, for small sized planar-antenna-array the spatial resolution will be low. Although, the desired beampattern is defined in terms of \( f_x \) and \( f_y \), the beampattern in terms of spherical coordinates can be found using (4).

Using the proposed technique, to achieve the desired beampattern, the 2D-space can be divided into 2D-grid points \( \{f_x(l)\}_{l=1}^{M}, \{f_y(k)\}_{k=1}^{N} \) represented by a \( M \times N \) matrix \( H_f \). The entry \( H_f(m,n) \) corresponds to \( f_x = -0.5 + \frac{m}{M-1} \) and \( f_y = -0.5 + \frac{n}{N-1} \). In order to define the ROI of the desired beampattern, we just have to assign 1 to the entries of \( H_f \) which are inside the ROI and 0 everywhere else. The different steps of our method are summarized in the Table I.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 0</td>
<td>Define ( H_f ) according to the ROI</td>
</tr>
<tr>
<td>Step 1</td>
<td>( H_f \leftarrow 2D-IDFT(H_f) )</td>
</tr>
<tr>
<td>Step 2</td>
<td>Compute ( R_{hh} ) using (16)</td>
</tr>
<tr>
<td>Step 3</td>
<td>Use ( R_{hh} ) as the covariance matrix ( R )</td>
</tr>
</tbody>
</table>

**TABLE I: Steps to compute \( R \)**

IV. COMPUTATIONAL COMPLEXITY

It can be noticed in Table I, the only computational complexity of the proposed method comes from the IDFT computation step. The \( NM \) IDFT coefficients can be computed using famous FFT algorithm, the computational complexity of which is \( O(MN \log(MN)) \). In contrast to this, for a given accuracy of \( \eta \), the computational complexity of SDP is \( O(\log(\frac{1}{\eta}) \ (MN)^{3.5}) \) [21]. The computational complexity comparison of our proposed and SDP algorithms is shown in Fig. 3, it can be seen in the figure that the computational complexity gap between our proposed and SDP algorithm is increasing with the number of antennas in the planar array. This makes our method more suitable for real time applications.
Fig. 2: Circular shaped beampattern.

Fig. 3: Computational complexity comparison between the DFT-based algorithm and the SDP method.

In the next step, to realize the synthesized covariance matrix, actual waveforms are designed. To design the waveforms, most of the proposed algorithms are iterative and their computational complexity is very high. In the following section, by exploiting the derivations of Sec. III, we provide a closed-form solution to directly design the waveforms for the desired beampattern. Thus, the covariance matrix design step can be avoided, which can significantly reduce the computational complexity of the beampattern design.

V. DIRECT DESIGN OF WAVEFORMS FOR THE DESIRED BEAMPATTERN

In this section, a closed-form expression to directly design the waveforms for the desired beampattern is proposed. We start with (13), which can also be written as

\[
R_{hh}(i_1, i_2) = \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{N-1} \left( \frac{\sqrt{H_f(k_1, k_2)}}{MN} e^{\frac{2\pi i (k_1 x_1) M}{M}} e^{\frac{2\pi i (k_2 x_2) N}{N}} \right)^*.
\]  

(20)

Assuming \( k = k_1 + M k_2 = \langle k \rangle_M + M \langle k \rangle_M \), both terms in the above equation can be considered as the \( k \)th symbols of the waveforms \( s_{i_1} \) and \( s_{i_2} \) that can be written as

\[
s_{i_1}(k) = \frac{\sqrt{H_f(\langle k \rangle_M, \langle k \rangle_M)}}{MN} e^{\frac{2\pi i (\langle k \rangle_M x_1) M}{M}} e^{\frac{2\pi i (\langle k \rangle_M x_2) N}{N}},
\]

\[
s_{i_2}(k) = \frac{\sqrt{H_f(\langle k \rangle_M, \langle k \rangle_M)}}{MN} e^{\frac{2\pi i (\langle k \rangle_M x_2) M}{M}} e^{\frac{2\pi i (\langle k \rangle_M x_1) N}{N}},
\]

where \( k = 0, 1, \ldots, MN - 1 \) represents the time index. Thus, the cross-correlation between the waveforms \( \{s_{i_1}(k)\} \) and \( \{s_{i_2}(k)\} \) can be written as

\[
R_{hh}(i_1, i_2) = \sum_{k=0}^{MN-1} s_{i_1}(k) s_{i_2}(k)^*.
\]

(21)

The corresponding waveform vector can be written as

\[
s_i = \left[ \begin{array}{c} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_{M-1} \end{array} \right] = \left[ \begin{array}{c} \frac{\sqrt{H_f(0,0)}}{MN} e^{\frac{2\pi i (0) x_1 M}{M}} e^{\frac{2\pi i (0) x_2 N}{N}} \\ \vdots \\ \frac{\sqrt{H_f(0,N-1)}}{MN} e^{\frac{2\pi i (N-1) x_1 M}{M}} e^{\frac{2\pi i (0) x_2 N}{N}} \\ \vdots \\ \frac{\sqrt{H_f(M-1,0)}}{MN} e^{\frac{2\pi i (0) x_1 M}{M}} e^{\frac{2\pi i (M-1) x_2 N}{N}} \\ \vdots \\ \frac{\sqrt{H_f(M-1,N-1)}}{MN} e^{\frac{2\pi i (N-1) x_1 M}{M}} e^{\frac{2\pi i (M-1) x_2 N}{N}} \end{array} \right],
\]

(22)

where

\[
\mathbf{v}_i = \left[ \begin{array}{c} \frac{1}{MN} \sqrt{H_f(p,0)} e^{\frac{2\pi i (0) x_1 M}{M}} e^{\frac{2\pi i (0) x_2 N}{N}} \\ \vdots \\ \frac{1}{MN} \sqrt{H_f(p,N-1)} e^{\frac{2\pi i (N-1) x_1 M}{M}} e^{\frac{2\pi i (0) x_2 N}{N}} \end{array} \right],
\]

(23)

where \( p = 0, 1, \ldots, M - 1 \). Therefore, for any transmitting element of the rectangular array at location \((m, n)\) where \(m = 0 \ldots M - 1\) and \(n = 0 \ldots N - 1\), we assign the waveform \( s_i \) defined in (22) with \( i = m + nM \). Note that each waveform \( s_i \) contains \( MN \) time domain symbols. Depending on the desired beampattern, some of these symbols may be equal to zero. If \( N_a \) is the number of non-zero elements in the matrix \( \mathbf{H}_f \), each waveform will transmit only \( N_a \) non-zero symbols. Therefore, to achieve the desired beampattern only \( N_a < MN \) snapshots or time domain symbols will be required.

Since the waveforms are derived from the synthesized covariance matrix, the covariance matrix of the directly designed
waveforms will be exactly the same to that of the synthesized covariance matrix. Therefore, in contrast to the conventional MIMO waveforms design algorithms, the designed beampattern of our synthesized covariance matrix and the directly designed waveforms will be the exactly same.

A. Peak-to-Average Power Ratio

Let us investigate the performance of our waveform design method in terms of its PAPR. The $i$th waveform will be transmitting $N_a$ non-zero time domain symbols. Therefore, the average transmitted power from the $(m,n)$th antenna element can be written as

$$P_i(\text{avg}) = \frac{1}{N_a} s_i^H s_i,$$

$$= \frac{1}{N_a} \sum_{k=0}^{MN-1} \frac{1}{(MN)^2} s_i(k) s_i^*(k),$$

$$= \frac{N_a}{N_a(MN)^2}. $$

Note the average transmitted power does not depend on the antenna location, which confirms that the uniform elemental power constraint is satisfied. Similarly, the peak power of the $i$th waveform can be derived as

$$P_i(\text{peak}) = \max_k \left| \frac{H_f(kM, [k]_M)}{(MN)^2} e^{j \frac{2\pi}{MN} \frac{2m}{MN} (iM)} \right|^2,$$

$$= \max_k \frac{H_f((kM, [k]_M)}{(MN)^2} = \frac{1}{(MN)^2}. \quad (24)$$

Therefore, the PAPR can be found as

$$\text{PAPR} = \frac{P_i(\text{peak})}{P_i(\text{avg})} = 1/(MN)^2 / 1/(MN)^2 = 1. \quad (25)$$

From (25), it can be noted that the PAPR is equal to unity for any antenna element in the planar array.

VI. NUMERICAL SIMULATIONS

A. Beampattern design

In this section, the performance of the proposed DFT-based algorithm is investigated. For simulation, a rectangular planar array composed of $M \times N$ antenna elements is considered. The spacing between any two adjacent antenna elements on the $x-$ and $y-$axis of the planar array is kept $\lambda/2$. The MSE between the desired and designed beampatterns is defined as

$$\text{MSE} = \sum_{l=1}^{L} \sum_{k=1}^{K} |a^H_l(f_x(l), f_y(k))R a_k(f_x(l), f_y(k)) - \alpha P_0(a_l(f_x(l), f_y(k)))|^2 / KL.$$

In the first simulation, the ROI is defined as $-0.1 \leq f_x \leq 0.1$ and $-0.1 \leq f_y \leq 0.1$. To design this beampattern, we set $N = M = 10$ and synthesize the covariance matrix, $R$, using SDP method proposed in [10]. The designed beampattern using synthesized $R$ is shown in Fig. 4, which is the best possible designed beampattern. Note that the beampattern is normalized by dividing $\alpha$. For this simulation, the total number of antenna elements is 100, therefore, to synthesize the covariance matrix, the simulation is very time consuming. Here, the actual waveforms to realize the synthesized covariance matrix are not designed as they also require iterative algorithms with very high computational complexity. The designed beampattern with the actual waveforms may be degraded too. In order to reduce the computational complexity to synthesize the covariance matrix, $R$, for the desired beampattern, in the second simulation our proposed algorithm with the same number of antenna elements is used. The corresponding designed beampattern, using the synthesized covariance matrix, is shown in Fig. 5.

![Fig. 4: The designed beampattern using SDP based method. Here the ROI is $-0.1 \leq f_x \leq 0.1$ and $-0.1 \leq f_y \leq 0.1$ and $M = N = 10$.](image-url)

In order to compare the performance of both algorithms in the previous two simulations for the same desired beampattern, the MSE of both methods for different planar-antenna-array dimensions is compared in Fig. 6. It can be seen in the figure, for small number of antennas, the performance of the DFT-based method is slightly poorer. This is due to the fact that the ROI (represented by the matrix $H_f$) is constructed using the 2D-grid points $\{(f_{x1}(l))_{l=1}^{M}, (f_{y2}(k))_{k=1}^{N}\}$, whose resolution is related to the number of antennas. However, as the dimensions of the rectangular array increases, the proposed method achieves lower MSE level approaching the SDP-based method with the advantage of being much lower computational complexity as shown in Fig. 3.

Next, for various desired beampatterns, by changing the ROI and plugging it into our proposed DFT-based algorithm described in Table I, corresponding covariance matrices are synthesized. Figs. 7-11 show the corresponding beampatterns obtained using the synthesized covariance matrices using a planar-antenna-array of dimensions $N = M = 20$. For display purposes, only a 2D-graph representing the projection of the designed beampattern in the $(f_x, f_y)$ plane is shown. In Fig. 7, the transmitted power is focused only in the corners. In Fig. 8,
the transmitted power is focused in the borders. In Fig. 9, the transmitted power is focused both in the borders and center. In Fig. 10, the transmitted power is focused in a non-symmetric region. Note that in the last case the covariance matrix will be complex.

As mentioned in Sec. V, the covariance matrices of the directly designed waveforms, for the above desired beampatterns, will be exactly the same to that of the synthesized covariance matrices, the designed beampattern of the directly designed waveforms will also be the exactly same as shown in Figs. 7-11.

To clearly see the benefits of our proposed scheme, we take a simple desired beampattern defined by the azimuth angles between $-30^\circ$ and $30^\circ$. To achieve this beampattern, a linear array of 10 antennas is considered and waveforms to be transmitted using these antennas are directly designed using our proposed algorithm and the algorithm proposed in [14], [24]. Our proposed method indicates that for this beampattern, only five snapshots are required and yields location angles $-23.57, -11.53, 0, 11.53, 23.57$ to generate five snapshots. Moreover, our proposed algorithm can also tell the symbols for odd number of antennas. In contrast, for this beampattern, using the algorithm in [24], first we have to decide the number of grid points and then generate the same number of snapshots. A simulation is performed for both algorithms to achieve the desired beampattern with different number of snapshots/grid points chosen for the algorithm in [24]. As shown in Fig. 12, higher side-lobes and slower roll-off can be seen in the performance of the proposed algorithm in [24]. Another advantage of using our proposed algorithm is that it generates finite alphabet symbols for each waveform.

B. Target detection

For a MIMO radar with a planar-antenna-array at the input, the signal at the receiver can be written as

$$y(n; f_x, f_y) = \sum_{l=1}^{L} a_r(f_x, f_y) a_s^T(f_{x_l}, f_{y_l}) x(n) + z(n), \quad (26)$$

where $L$ is the number of targets, $a_r(f_x, f_y)$ is the receive steering vector and $z(n)$ is a vector of additive white Gaussian noise (AWGN) samples each of variance $\sigma_z$. In the last simulation, consider a scenario of two targets located at $(f_{x_1}, f_{y_1}) = (0.1, 0.1)$ and $(f_{x_2}, f_{y_2}) = (-0.1, -0.1)$. In order to detect these targets, conventional algorithms such as matched-filter, Capon, and APES can be used. The matched-filter is defined as

$$w_c(f_x, f_y) = a_s(f_x, f_y). \quad (27)$$
Fig. 8: DFT-based transmit beampattern, $N = M = 20$, ROI focused in the borders.

Fig. 9: DFT-based transmit beampattern design, $N = M = 20$. Here, the transmitted power needs to be focused in the center and the borders.

Fig. 10: DFT-based transmit beampattern design, $N = M = 20$. Here, the ROI has a circular shape.

Fig. 11: DFT-based non-symmetric transmit beampattern design, $N = M = 20$. Using the derivations of covariance matrix design, a method to directly design the actual waveforms is proposed. In contrast to the conventional MIMO methods of the waveform design, the covariance matrix of the directly designed waveforms is exactly the same to that of the synthesized covariance matrix. The numerical simulations presented confirm that the proposed method is computationally efficient and performs closely to the SDP-based method as the number of antennas increases. As a future work, we will study an On-Off scheme which will use the minimum number of antennas for a given beampattern.

VII. Conclusion

In this paper, we have presented a closed-form method to synthesize the waveform covariance matrix for a MIMO radar with a planar-antenna-array at the input to achieve the desired beampattern. The proposed algorithm exploits the 2D-DFT to synthesize the covariance matrix. The positive semi-definiteness and uniform elemental power constraints are verified by the designed matrix. In the second contribution, by next, we evaluate the quantity $|w_c(f_x, f_y)^H y|$ to detect the presence of the targets. In Fig. 13, $|w_c(f_x, f_y)^H y|$ is plotted as a function of $f_x$ and $f_y$. We can clearly notice the presence of high amplitude at the targets’ locations.

Appendix

The proof of Lemma 1 is straightforward. By exploiting the orthogonality of the vectors defined in (12), we have
positive semi-definite matrices, 

The Kronecker delta function. Thus, we obtain

\[
e_{(k_1, k_2)} = e^{H}(k_1, k_2) e^{H}(k_1, k_2) \]

Since \( e_{(k_1, k_2)} e^{H}(k_1, k_2) \) \( (i, i) = 1 \) for any index value \( i \), we can write

\[
R_{hh}(i, i) = \frac{1}{(MN)^2} \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{N-1} H_f(k_1, k_2) \]

where \( N_a \) is the number of non-zero elements in the frequency domain matrix \( H_f \).

**REFERENCES**


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