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A low-complexity interacting multiple model filter for maneuvering target tracking

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Abstract

In this work, we address the target tracking problem for a coordinate-decoupled Markovian jump-mean-acceleration based maneuvering mobility model. A novel low-complexity alternative to the conventional interacting multiple model (IMM) filter is proposed for this class of mobility models. The proposed tracking algorithm utilizes a bank of interacting filters where the interactions are limited to the mixing of the mean estimates, and it exploits a fixed off-line computed Kalman gain matrix for the entire filter bank. Consequently, the proposed filter does not require matrix inversions during on-line operation which significantly reduces its complexity. Simulation results show that the performance of the low-complexity proposed scheme remains comparable to that of the traditional (highly-complex) IMM filter. Furthermore, we derive analytical expressions that iteratively evaluate the transient and steady-state performance of the proposed scheme, and establish the conditions that ensure the stability of the proposed filter. The analytical findings are in close accordance with the simulated results.

Keywords: Tracking, Kalman filtering, interacting multiple model filter, maneuvering targets, low-complexity.

1. Introduction

Tracking algorithms require an accurate stochastic description of dynamical mobility model of the moving target. Moreover an observation model is required, that describes the relation between the states (i.e., position, velocity, etc.) to be estimated and the available measurements. The information contained in the mobility and observation models is combined, usually in a Bayesian filtering framework, to recursively estimate the current

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state vector. In many cases, however, these models depend upon some parameters that are known only partially. This necessitates some adaptive mechanism to be incorporated in the filtering process to jointly estimate the state and the unknown parameters [1]. For instance, in the case of tracking maneuvering targets, adaptive Bayesian filtering based on multiple models (MM) framework is widely accepted as an admissible solution [2]. In the MM approach, the unknown maneuvers are modeled as jump Markovian processes, and a bank of filters is operated in parallel, where each constituent filter corresponds to a particular state of the Markovian process. Among the large number of variants of the MM filters, the interacting multiple model (IMM) filter has shown great promise and has been successfully applied in a large number of applications, e.g. [3–5]. In the IMM approach, the members of the filter-bank interact with each other by a process termed as *mixing*, which significantly improves its performance over other filters of similar structure, such as autonomous multiple model filters (AMM) and the first-order Gaussian pseudo Bayesian (GPB-1) filters [3].

The IMM filter requires the set of models \mathcal{M} to be specified a priori. In many cases of practical interest, the jump Markovian parameter can be specified as a switching bias value of the acceleration in the mobility model. Furthermore, the maneuvers in the x and y coordinate can be considered independent of each other. Such models are termed as the *coordinate-decoupled Markovian jump-mean acceleration models* [2] and these have been extensively applied in tracking aircrafts [6, 7] and vehicles [8, 9]. However, in many cases, a large number of models are required to cover the entire range of the possible acceleration values which can become computationally infeasible. To reduce this complexity, in the classical work of Moose *et al.* [6], a non-interacting degenerated multiple model filter was proposed which used similar Kalman gain for the entire filter bank. Owing to the simplicity of the filter described in [6], it has been adopted as an admissible solution for vehicle tracking in cellular networks [8, 9], and preferred over the conventional IMM filter. Some other works have explored the possibility of a low cost IMM based on constant gain Kalman filtering. For instance, in [10], and more recently in [11], constant gain IMM based on the α - β and the α - β - γ filters have been described. In [12], a constant velocity-constant acceleration IMM with constant gain was presented. However, these works rely on the assumption that the maneuvering behavior of the target can be adequately described using only two models; one for the uniform and the other for the accelerated

motion. This assumption is too simplistic for manned-maneuvering targets that can undergo complex maneuvers. Another promising development is the recent introduction of variable-structure (VS) IMM filters [3, 13–16]. The VS-IMM algorithms are equipped with an adaptive mechanism that adjusts the set of multiple models according to the available measurements. Consequently, VS-IMM filters utilize a much smaller set of models and still achieve adequate performance. In our work, for the sake of simplicity, we focus only on fixed IMM; however, our proposed technique can be extended easily to the variable-structure framework using the mode augmentation approach [13, 14].

We develop a low-complexity IMM filter for the Markovian jump-mean acceleration models. The proposed filter achieves a significant reduction in the computational expense as compared to the conventional IMM, while both solutions have comparable performance in terms of root mean square error (RMSE). The fundamental idea behind the proposed filter is as follows: since the jump Markovian parameter appears only as a bias term; therefore, the error covariance matrix of each filter in the filter-bank is assumed to be the same. This results in the computation of same Kalman gain for each filter thereby allowing the reduction of the computational costs involved in the traditional matrix inversions. Furthermore, if the system is time-invariant, the steady-state Kalman gain can be computed off-line and inserted in the filtering algorithm to further simplify the resulting procedure. Note that this approach is similar in spirit to the classical work of Moose *et al.* [6]; however, our algorithm retains the mixing step of the IMM filter for state estimates, which significantly enhances the quality of our solution over the one adopted in [6]. Further, we discuss analytically the transient as well as the steady-state performances of the proposed filter, and also establish the conditions to ensure the stability of the proposed filter. Following are the main contributions in this work: 1) We address a low-cost interacting multiple model filter under the Bayesian framework. 2) We discuss analytically the transient as well as the steady-state performances of the proposed filter. 3) We establish the conditions to ensure the stability of the proposed filter.

The rest of the paper is organized as follows: Section 2 describes the system model and the proposed filter; Section 3 discusses the analytical performance of the proposed filter, and also provides the computational cost; Section 4 provides simulation results, and Section 5 draws the conclusions.

2. System model and derivation of the Proposed filter

Consider a representative linear time-invariant jump Markovian system with the maneuvering behavior modeled as a bias terms in the mobility model.

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\boldsymbol{\alpha}_{(r_{k+1})} + \mathbf{w}_{k+1}, \quad (1a)$$

$$\mathbf{z}_{k+1} = H\mathbf{x}_{k+1} + \mathbf{v}_{k+1}, \quad (1b)$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state vector, $\mathbf{z}_k \in \mathbb{R}^p$ is the observation vector, $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}_{n \times 1}; Q)$ and $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}_{p \times 1}; R)$ are plant and observation noise, respectively. Both \mathbf{w}_k and \mathbf{v}_k are stationary white noise processes and are assumed to be independent from each other. The matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times u}$, $H \in \mathbb{R}^{p \times n}$, $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{p \times p}$ are time-invariant and assumed to be known. The parameter $\boldsymbol{\alpha}_{(r_k)}$ belongs to a finite discrete sets $\mathcal{L}_\alpha := \{\boldsymbol{\alpha}_{(1)}, \boldsymbol{\alpha}_{(2)}, \dots, \boldsymbol{\alpha}_{(M)}\}$; each member of \mathcal{L}_α is a bounded $u \times 1$ vector. The dimensions n, p, u are arbitrary and depend upon the scenario. The random process r_k is modeled as a homogeneous first-order discrete Markov random process with known transition probabilities $\mathbf{p}(r_k = i | r_{k-1} = j) = \pi_{ji}$. The proposed algorithm is derived in the following.

Let \mathbf{Z}^k be the set of all available measurements at k th instant, $\mathbf{Z}^k := \{\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_k\}$. Let us denote the event $\{r_k = i\}$ as $r_k^{(i)}$, and $\boldsymbol{\alpha}_{(r_k=i)}$ as $\boldsymbol{\alpha}_k^{(i)}$; we expand the expression for $\mathbf{p}(\mathbf{x}_k | \mathbf{Z}^k)$ as:

$$\mathbf{p}(\mathbf{x}_k | \mathbf{Z}^k) = \sum_{i=1}^M \mathbf{p}(\mathbf{x}_k, r_k^{(i)} | \mathbf{Z}^k) = \sum_{i=1}^M \mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^k) \mathbf{p}(r_k^{(i)} | \mathbf{Z}^k). \quad (2)$$

The above expression can be interpreted as a weighted sum of a bank of M filters, where the weights $\mathbf{p}(r_k^{(i)} | \mathbf{Z}^k)$ are the *mode probabilities*. The manner, in which the i th filter probability $\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^k)$ is evaluated, differentiates between the proposed and the conventional IMM filter. We evaluate $\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^k)$ as follows:

$$\begin{aligned} \mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^k) &= \mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{z}_k, \mathbf{Z}^{k-1}) \\ &= \frac{\mathbf{p}(\mathbf{x}_k, \mathbf{z}_k | r_k^{(i)}, \mathbf{Z}^{k-1})}{\mathbf{p}(\mathbf{z}_k | r_k^{(i)}, \mathbf{Z}^{k-1})} = \frac{\mathbf{p}(\mathbf{z}_k | \mathbf{x}_k) \mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1})}{\mathbf{p}(\mathbf{z}_k | r_k^{(i)}, \mathbf{Z}^{k-1})}, \end{aligned} \quad (3)$$

where $\mathbf{p}(\mathbf{z}_k | \mathbf{x}_k)$ is called the *likelihood* and can be written as $\mathbf{p}(\mathbf{z}_k | \mathbf{x}_k) \sim \mathcal{N}_{(\mathbf{z}_k)}(H\mathbf{x}_k; R)$. The factor $\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1})$ is called *prediction*; the manner in which the prediction is evaluated determines the interaction between the members of the filter bank. The interaction is achieved using *re-initialization* which will be described shortly. We expand

$\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1})$ using the *total probability theorem* and the *Cox theorem* [17], i.e.,

$$\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) = \sum_{j=1}^M \mathbf{p}(\mathbf{x}_k | r_k^{(i)}, r_{k-1}^{(j)}, \mathbf{Z}^{k-1}) \mathbf{p}(r_{k-1}^{(j)} | r_k^{(i)}, \mathbf{Z}^{k-1}), \quad (4)$$

where $\mathbf{p}(r_{k-1}^{(j)} | r_k^{(i)}, \mathbf{Z}^{k-1}) =: \gamma_{k-1}^{(i,j)}$ is termed as the *mixing probability*. It can be evaluated recursively as follows:

$$\gamma_{k-1}^{(i,j)} = \frac{\mathbf{p}(r_k^{(i)} | r_{k-1}^{(j)}, \mathbf{Z}^{k-1}) \mathbf{p}(r_{k-1}^{(j)} | \mathbf{Z}^{k-1})}{\mathbf{p}(r_k^{(i)} | \mathbf{Z}^{k-1})} = \frac{\pi_{ji} \mu_{k-1}^{(j)}}{\sum_{l=1}^M \pi_{li} \mu_{k-1}^{(l)}}, \quad (5)$$

where $\mu_{k-1}^{(j)} := \mathbf{p}(r_{k-1}^{(j)} | \mathbf{Z}^{k-1})$, and we use the fact that the denominator is a normalization constant which makes total probability summed to unity. Now, to determine the expression for $\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, r_{k-1}^{(j)}, \mathbf{Z}^{k-1})$, the first factor of the summand in (4), we write,

$$\begin{aligned} \mathbf{p}(\mathbf{x}_k | r_k^{(i)}, r_{k-1}^{(j)}, \mathbf{Z}^{k-1}) &= \int \mathbf{p}(\mathbf{x}_k, \mathbf{x}_{k-1} | r_k^{(i)}, r_{k-1}^{(j)}, \mathbf{Z}^{k-1}) d\mathbf{x}_{k-1} \\ &= \int \mathbf{p}(\mathbf{x}_k | \mathbf{x}_{k-1}, r_k^{(i)}, \mathbf{Z}^{k-1}) \mathbf{p}(\mathbf{x}_{k-1} | r_{k-1}^{(j)}, \mathbf{Z}^{k-1}) d\mathbf{x}_{k-1}. \end{aligned} \quad (6)$$

From (1a), $\mathbf{p}(\mathbf{x}_k | \mathbf{x}_{k-1}, r_k^{(i)}, \mathbf{Z}^{k-1})$ can be written as $\mathcal{N}_{(\mathbf{x}_k)}(A\mathbf{x}_{k-1} + B\boldsymbol{\alpha}_k^{(i)}; Q)$. Furthermore, we use the approximation, $\mathbf{p}(\mathbf{x}_{k-1} | r_{k-1}^{(j)}, \mathbf{Z}^{k-1}) \approx \mathcal{N}_{(\mathbf{x}_{k-1})}(\hat{\mathbf{x}}_{k-1|k-1}^{(j)}; P_{k-1|k-1})$ to support that the covariance matrix $P_{k-1|k-1}$ is independent of the Markovian process r_k and is same for all filters in the bank. This is the defining assumption of the proposed filter, and its justification rests on the fact that $\boldsymbol{\alpha}_k^{(i)}$ affects merely the mean value of the model in (1a). Now, we invoke the re-initialization approximation of the IMM algorithm, i.e.,

$$\sum_{j=1}^M \mathcal{N}_{(\mathbf{x}_{k-1})}(\hat{\mathbf{x}}_{k-1|k-1}^{(j)}; P_{k-1|k-1}) \gamma_{k-1}^{(i,j)} \approx \mathcal{N}_{(\mathbf{x}_{k-1})}(\mathbf{x}_{0,k-1}^{(i)}; P_{k-1|k-1}). \quad (7)$$

The mean value $\mathbf{x}_{0,k-1}^{(i)}$ is termed as the *mixing estimate* which becomes the initial condition for the i th filter at k th instant. A conventional IMM requires the additional mixing of covariance matrices, which is a costly procedure in terms of computations. Using (5), (6), and (7), the expression in (4) can be written as,

$$\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) = \int \mathcal{N}_{(\mathbf{x}_k)}(A\mathbf{x}_{k-1} + B\boldsymbol{\alpha}_k^{(i)}; Q) \mathcal{N}_{(\mathbf{x}_{k-1})}(\mathbf{x}_{0,k-1}^{(i)}; P_{k-1|k-1}) d\mathbf{x}_{k-1}. \quad (8)$$

Using the *Gaussian Product Theorem* (GPT) [17], (8) can be written as $\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) = \mathcal{N}_{(\mathbf{x}_k)}(\hat{\mathbf{x}}_{k|k-1}^{(i)}; P_{k|k-1})$, where $\mathbf{x}_{k|k-1}^{(i)} = A\mathbf{x}_{0,k-1}^{(i)} + B\boldsymbol{\alpha}_k^{(i)}$ and $P_{k|k-1} = AP_{k-1|k-1}A^T + Q$. Substituting $\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1})$ in (3), we get,

$$\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^k) = \frac{\mathcal{N}_{(\mathbf{z}_k)}(H\mathbf{x}_k; R) \mathcal{N}_{(\mathbf{x}_k)}(\hat{\mathbf{x}}_{k|k-1}^{(i)}; P_{k|k-1})}{\mathbf{p}(\mathbf{z}_k | r_k^{(i)}, \mathbf{Z}^{k-1})}. \quad (9)$$

The denominator in (9) is a normalization constant and can be expanded as,

$$\mathbf{p}(\mathbf{z}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) = \int \mathbf{p}(\mathbf{z}_k | r_k^{(i)}, \mathbf{x}_k, \mathbf{Z}^{k-1}) \mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) d\mathbf{x}_k. \quad (10)$$

Applying GPT on (10), (9), we get $\mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^k) = \mathcal{N}_{(\mathbf{x}_k)}(\hat{\mathbf{x}}_{k|k}^{(i)}; P_{k|k})$, where

$$\mathbf{x}_{k|k}^{(i)} = \mathbf{x}_{k|k-1}^{(i)} + K_k(\mathbf{z}_k - H\mathbf{x}_{k|k-1}^{(i)}), \quad (11a)$$

$$P_{k|k} = (I - K_k H) P_{k|k-1}, \quad (11b)$$

$$K_k = P_{k|k-1} H^T (H P_{k|k-1} H^T + R)^{-1}. \quad (11c)$$

Hence, using (11a)-(11c), the estimate for each filter in the filter-bank described in (2), can be evaluated recursively. To evaluate the weights of each filter, i.e., $\mathbf{p}(r_k^{(i)} | \mathbf{Z}^k) = \mu_k^{(i)}$, we proceed as follows:

$$\mathbf{p}(r_k^{(i)} | \mathbf{Z}^k) = \mathbf{p}(r_k^{(i)} | \mathbf{z}_k, \mathbf{Z}^{k-1}) = \frac{\mathbf{p}(\mathbf{z}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) \mathbf{p}(r_k^{(i)} | \mathbf{Z}^{k-1})}{\mathbf{p}(\mathbf{z}_k | \mathbf{Z}^{k-1})}. \quad (12)$$

The first factor in the numerator of (12) can be written as,

$$\begin{aligned} \mathbf{p}(\mathbf{z}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) &= \int \mathbf{p}(\mathbf{z}_k | \mathbf{x}_k, r_k^{(i)}) \mathbf{p}(\mathbf{x}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) d\mathbf{x}_k \\ &= \int \mathcal{N}_{(\mathbf{z}_k)}(H\mathbf{x}_k; R) \mathcal{N}_{(\mathbf{x}_k)}(\hat{\mathbf{x}}_{k|k-1}^{(i)}; P_{k|k-1}) d\mathbf{x}_k. \end{aligned} \quad (13)$$

Using GPT, (13) becomes,

$$\begin{aligned} \mathbf{p}(\mathbf{z}_k | r_k^{(i)}, \mathbf{Z}^{k-1}) &= \mathcal{N}_{(\mathbf{z}_k)}(H\hat{\mathbf{x}}_{k|k-1}^{(i)}; H P_{k|k-1} H^T + R) \\ &= \mathcal{N}_{\binom{(i)}{k}}(\mathbf{0}, H P_{k|k-1} H^T + R) = \Lambda_k^{(i)}, \end{aligned} \quad (14)$$

where $\boldsymbol{\theta}_k^{(i)} = \mathbf{z}_k - H\hat{\mathbf{x}}_{k|k-1}^{(i)}$ is the residual vector. The second factor in the numerator of (12) is expanded as follows:

$$\mathbf{p}(r_k^{(i)} | \mathbf{Z}^{k-1}) = \sum_{j=1}^M \mathbf{p}(r_k^{(i)} | r_{k-1}^{(j)}) \mathbf{p}(r_{k-1}^{(j)} | \mathbf{Z}^{k-1}) = \sum_{j=1}^M \pi_{ji} \mu_{k-1}^{(j)}. \quad (15)$$

Substituting (14) and (15) in (12), we get a recursive evaluation of $\mu_k^{(i)}$

$$\mu_k^{(i)} = \frac{\Lambda_k^{(i)} \sum_{j=1}^M \pi_{ji} \mu_{k-1}^{(j)}}{\sum_{l=1}^M \Lambda_k^{(l)} \sum_{m=1}^M \pi_{ml} \mu_{k-1}^{(m)}}, \quad (16)$$

where the denominator in (12) has been evaluated as a normalization constant to ensure $\sum_i \mu_k^{(i)} = 1$. This completes our derivation of the proposed filter.

Note that the equations for the Kalman gain and error covariance matrices, i.e., (11b) and (11c) do not depend upon the Markovian parameter r_k . Furthermore, if the matrix pair (A, H) is completely observable and (A, \bar{Q}) is completely controllable (where $Q = \bar{Q}\bar{Q}^T$), then the error covariance matrix $P_{k|k}$ reaches a unique steady-state value P_∞ , independent of the initial condition $P_{0|-1}$, as $k \rightarrow \infty$ [18]. Moreover, the Kalman gain K_k reaches a steady-state value K_∞ , and all eigenvalues of matrix $A(I - K_\infty H)$ remain less than one [18]. The steady-state gain K_∞ can be evaluated, for a given value of A , Q and R , by simply propagating the Kalman gain using off-line iterations till it reaches a steady-state value. The proposed filter is summarized in **Algorithm 1** and the flow diagram is depicted in Figure 1.

Output

Mode Probability

Update

Filtering

Mixing

 $\hat{x}_{k|k}$ $P_k^{(1)}$ $P_k^{(M)}$ $\hat{x}_{k|k}^{(1)}$

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$\mathbf{x}_{0,k-1}^{(i)}$ using

$$\mathbf{x}_{0,k-1}^{(i)} = \sum_{l=1}^M \hat{\mathbf{x}}_{k-1|k-1}^{(l)} \gamma_{k-1}^{(i,l)}.$$

(b) Prediction:
$$\hat{\mathbf{x}}_{k|k-1}^{(i)} = A\mathbf{x}_{0,k-1}^{(i)} + B\boldsymbol{\alpha}_k^{(i)}.$$

(c) Measurement Update: Evaluate $\mu_k^{(i)}$ using (16), and $\hat{\mathbf{x}}_{k|k}^{(i)}$ using

$$\hat{\mathbf{x}}_{k|k}^{(i)} = \hat{\mathbf{x}}_{k|k-1}^{(i)} + K_\infty \boldsymbol{\theta}_k^{(i)}; \quad \boldsymbol{\theta}_k^{(i)} = \mathbf{z}_k - H\hat{\mathbf{x}}_{k|k-1}^{(i)}.$$

4. Output Estimate:
$$\hat{\mathbf{x}}_{k|k} = \sum_{i=1}^M \mathbf{x}_{k|k}^{(i)} \mu_k^{(i)}.$$

3. Performance Analysis of the Proposed Filter

Owing to the interactions involved among multiple Kalman filters in the bank, the analysis of IMM (or any of its variant) is not straight forward. In [19] and [20], iterative procedures are developed to evaluate the performance of the IMM filter for a given realization of Markovian process r_k . In our work, we essentially use the framework of [20] to analyze the performance of our proposed algorithm. Furthermore, we discuss the stability of the proposed algorithm and develop a novel iterative procedure that can readily evaluate the steady-state performance of the proposed algorithm.

Before we proceed with the analysis, let us describe a set of assumptions for tractability. **A1:** The true value of the Markovian sequence $r_k^{(t)}$ is assumed to be known for the analysis. **A2:** The error sequence $\mathbf{e}_k^{(i)} = \mathbf{x}_k^{(t)} - \hat{\mathbf{x}}_{k|k}^{(i)}$ is assumed to be uncorrelated with the mixing probability $\gamma_k^{(i,j)} \forall i, j, k$, where $\mathbf{x}_k^{(t)}$ is the true state vector at instant k . **A3:** The error sequence $\mathbf{e}_k^{(i)} = \mathbf{x}_k^{(t)} - \hat{\mathbf{x}}_{k|k}^{(i)}$ is assumed to be uncorrelated with the mode probability $\mu_k^{(i)} \forall i, k$. **A4:** The mean of the mixing probability is approximated as:

$$\mathbb{E}[\gamma_{k-1}^{(i,j)}] = \mathbb{E} \left[\frac{\pi_{ji} \mu_{k-1}^{(i)}}{\sum_{l=1}^M \pi_{li} \mu_{k-1}^{(l)}} \right] \approx \frac{\pi_{ji} \mathbb{E} \left[\mu_{k-1}^{(i)} \right]}{\sum_{l=1}^M \pi_{li} \mathbb{E} \left[\mu_{k-1}^{(l)} \right]}.$$

A5: The mean of the mode probability is approximated as:

$$\mathbb{E} \left[\mu_k^{(i)} \right] = \mathbb{E} \left[\frac{\Lambda_k^{(i)} \sum_{j=1}^M \pi_{ji} \mu_{k-1}^{(j)}}{\sum_{l=1}^M \Lambda_k^{(l)} \sum_{m=1}^M \pi_{ml} \mu_{k-1}^{(l)}} \right] \approx \frac{\mathbb{E} \left[\Lambda_k^{(i)} \right] \sum_{j=1}^M \pi_{ji} \mathbb{E} \left[\mu_{k-1}^{(j)} \right]}{\sum_{l=1}^M \mathbb{E} \left[\Lambda_k^{(l)} \right] \sum_{m=1}^M \pi_{ml} \mathbb{E} \left[\mu_{k-1}^{(l)} \right]}.$$

A1 states that the operating scenario is assumed to be known and hence the resulting performance analysis would be scenario-dependent. **A2** and **A3** are required to evaluate the expectations of the mixing and mode probabilities. The error sequence is only

weakly correlated with these probabilities and simulation results have been used to justify these assumptions in [19, 20]. **A4** and **A5** are based on the first-order Taylor series approximation of the mean of ratio of two random variables, as described in [21].

3.1. MSE Analysis

Define the error vector at k th instant as $\mathbf{e}_k = \mathbf{x}_k^{(t)} - \hat{\mathbf{x}}_{k|k}$. Using the fact that $\hat{\mathbf{x}}_{k|k} = \sum_{i=1}^M \mathbf{x}_{k|k}^{(i)} \mu_k^{(i)}$ and $\sum_{i=1}^M \mu_k^{(i)} = 1$, we can write $\mathbf{e}_k = \sum_{i=1}^M \mu_k^{(i)} \mathbf{e}_k^{(i)}$, where $\mathbf{e}_k^{(i)} = \mathbf{x}_k^{(i)} - \hat{\mathbf{x}}_{k|k}^{(i)}$. Using (1a) and inserting the expression of $\mathbf{x}_{k|k}^{(i)}$ from **Algorithm 1**, we write

$$\mathbf{e}_k^{(i)} = A\mathbf{e}_{0,k-1}^{(i)} + B\Delta\boldsymbol{\alpha}_k^{(i)} + \mathbf{w}_k - K_\infty\boldsymbol{\theta}_k^{(i)}, \quad (17)$$

where $\mathbf{e}_{0,k-1}^{(i)} = \mathbf{x}_{0,k-1}^{(i)} - \mathbf{x}_{0,k-1}^t$ and $\Delta\boldsymbol{\alpha}_k^{(i)} = \boldsymbol{\alpha}_k^t - \boldsymbol{\alpha}_k^{(i)}$. Now, using (1a) and (1b), we obtain $\boldsymbol{\theta}_k^{(i)} = H A \mathbf{e}_{0,k-1}^{(i)} + H B \Delta \boldsymbol{\alpha}_k^{(i)} + H \mathbf{w}_k + \mathbf{v}_k$. Taking expectation on both sides, we get

$$\bar{\boldsymbol{\theta}}_k^{(i)} = H A \bar{\mathbf{e}}_{0,k-1}^{(i)} + H B \Delta \boldsymbol{\alpha}_k^{(i)}, \quad (18)$$

where $\bar{\boldsymbol{\theta}}_k^{(i)} = E[\boldsymbol{\theta}_k^{(i)} | \boldsymbol{\alpha}_k^t]$ and $\bar{\mathbf{e}}_{0,k-1}^{(i)} = E[\mathbf{e}_{0,k-1}^{(i)} | \boldsymbol{\alpha}_k^t]$. Next using (17), we obtain

$$\mathbf{e}_k^{(i)} = (I - K_\infty H) A \mathbf{e}_{0,k-1}^{(i)} + (I - K_\infty H) B \Delta \boldsymbol{\alpha}_k^{(i)} + (I - K_\infty H) \mathbf{w}_k - K_\infty \mathbf{v}_k. \quad (19)$$

Taking expectation on both sides of (19), we get

$$\bar{\mathbf{e}}_k^{(i)} = (I - K_\infty H) A \bar{\mathbf{e}}_{0,k-1}^{(i)} + (I - K_\infty H) B \Delta \boldsymbol{\alpha}_k^{(i)}. \quad (20)$$

Owing to assumptions **A2** and **A4**, $\bar{\mathbf{e}}_{0,k}^{(i)}$ can be evaluated as

$$\bar{\mathbf{e}}_{0,k}^{(i)} \approx \sum_{j=1}^M \bar{\gamma}_k^{(i,j)} \bar{\mathbf{e}}_k^{(j)}. \quad (21)$$

Similarly, using assumptions **A3** and **A5**, we get an expression for the mean error $\bar{\mathbf{e}}_k$:

$$\bar{\mathbf{e}}_k \approx \sum_{i=1}^M \bar{\mu}_k^{(i)} \bar{\mathbf{e}}_k^{(i)}. \quad (22)$$

To evaluate the mean of the likelihood function $E[\Lambda_k^{(i)}]$, we note that $\Lambda_k^{(i)}$ is a function of $\boldsymbol{\theta}_k^{(i)}$, and $\boldsymbol{\theta}_k^{(i)} \sim \mathcal{N}(\bar{\boldsymbol{\theta}}_k^{(i)}, \text{Cov}[\boldsymbol{\theta}_k^{(i)}, \boldsymbol{\theta}_k^{(i)}])$. Hence $E[\Lambda_k^{(i)}]$ is obtained as follows:

$$E[\Lambda_k^{(i)}] = \int \mathcal{N}_{(i)}(\mathbf{0}, S_k) \mathcal{N}_{(i)}(\bar{\boldsymbol{\theta}}_k^{(i)}, \text{Cov}[\boldsymbol{\theta}_k^{(i)}, \boldsymbol{\theta}_k^{(i)}]) d\boldsymbol{\theta}_k^{(i)}, \quad (23)$$

where $S_k = H P_{k|k-1} H^T + R$ and the covariance matrix of the residual vector is

$$\text{Cov}[\boldsymbol{\theta}_k^{(i)}, \boldsymbol{\theta}_k^{(j)}] = H A \text{Cov}[\mathbf{e}_{0,k-1}^{(i)}, \mathbf{e}_{0,k-1}^{(j)}] A^T H^T + H Q H^T + R. \quad (24)$$

Exploiting the fact that the product of two Gaussian probability density functions (PDFs) is a Gaussian PDF [19], a simpler expression for (23) is obtained as

$$\mathbb{E} \left[\Lambda_k^{(i)} \right] = \frac{\left| \left(S_k^{-1} + \text{Cov} \left[\boldsymbol{\theta}_k^{(i)}, \boldsymbol{\theta}_k^{(i)} \right]^{-1} \right)^{-1} \right|^{1/2}}{2\pi |S_k|^{1/2} \left| \text{Cov} \left[\boldsymbol{\theta}_k^{(i)}, \boldsymbol{\theta}_k^{(i)} \right] \right|^{1/2}} \exp \left(-\frac{1}{2} \left[\bar{\boldsymbol{\theta}}_k^{(i)} \right]^T \left[\text{Cov} \left[\boldsymbol{\theta}_k^{(i)}, \boldsymbol{\theta}_k^{(i)} \right] + S_k \right]^{-1} \bar{\boldsymbol{\theta}}_k^{(i)} \right) \quad (25)$$

Using (18), (20), (21), (22), (25), **A4** and **A5**, we get a recursive evaluation of the mean of overall error $\bar{\mathbf{e}}_k, \forall k$. Note that the covariance of the $\mathbf{e}_k^{(i)}$ can be written as

$$\begin{aligned} \text{Cov} \left[\mathbf{e}_k^{(i)}, \mathbf{e}_k^{(j)} \right] &= (I - K_\infty H) A \text{Cov} \left[\mathbf{e}_{0,k-1}^{(i)}, \mathbf{e}_{0,k-1}^{(j)} \right] A^T (I - K_\infty H)^T \\ &\quad + (I - K_\infty H) Q (I - K_\infty H)^T + K_\infty R K_\infty^T. \end{aligned} \quad (26)$$

Furthermore,

$$\begin{aligned} \text{Cov} \left[\mathbf{e}_{0,k}^{(i)}, \mathbf{e}_{0,k}^{(j)} \right] &= \mathbb{E} \left[\left(\sum_{l=1}^M \gamma_k^{(i,l)} \mathbf{e}_k^{(l)} - \sum_{m=1}^M \bar{\gamma}_k^{(i,m)} \bar{\mathbf{e}}_k^{(m)} \right) \left(\sum_{n=1}^M \gamma_k^{(j,n)} \mathbf{e}_k^{(n)} - \sum_{o=1}^M \bar{\gamma}_k^{(j,o)} \bar{\mathbf{e}}_k^{(o)} \right) \right] \\ &\approx \sum_{l=1}^M \sum_{n=1}^M \left(\bar{\gamma}_k^{(l,i)} \bar{\gamma}_k^{(n,j)} \mathbb{E} \left[\mathbf{e}_k^{(l)} \mathbf{e}_k^{(n)} \right] - \bar{\gamma}_k^{(l,i)} \bar{\gamma}_k^{(n,j)} \bar{\mathbf{e}}_k^{(l)} \bar{\mathbf{e}}_k^{(n)} \right) \\ &= \sum_{l=1}^M \sum_{n=1}^M \bar{\gamma}_k^{(l,i)} \bar{\gamma}_k^{(n,j)} \text{Cov} \left[\mathbf{e}_k^{(l)}, \mathbf{e}_k^{(n)} \right]. \end{aligned} \quad (27)$$

Now if all interacting filters are initialized with same initial conditions then the initial error covariance matrix, $\text{Cov}[\mathbf{e}_0^{(i)}, \mathbf{e}_0^{(j)}], \forall i, j$, become similar. In such a case, the covariance expressions in (26) and (27) become similar for all i, j and k . Moreover, using the fact that $\sum_{l=1}^M \sum_{n=1}^M \bar{\gamma}_k^{(l,i)} \bar{\gamma}_k^{(n,j)} = 1$, we can write $\text{Cov}[\mathbf{e}_{0,k}^{(i)}, \mathbf{e}_{0,k}^{(j)}] \approx \text{Cov}[\mathbf{e}_k^{(i)}, \mathbf{e}_k^{(j)}]$. For the covariance matrix of the overall error, a similar reasoning would yield $\text{Cov}[\mathbf{e}_k, \mathbf{e}_k] \approx \text{Cov}[\mathbf{e}_k^{(i)}, \mathbf{e}_k^{(j)}]$, where $\text{Cov}[\mathbf{e}_k^{(i)}, \mathbf{e}_k^{(j)}]$ is similar for all i, j, k . Hence, the covariance matrix of the overall error can be evaluated as

$$\begin{aligned} \text{Cov} [\mathbf{e}_k, \mathbf{e}_k] &= (I - K_\infty H) A \text{Cov} [\mathbf{e}_{k-1}, \mathbf{e}_{k-1}] A^T (I - K_\infty H)^T \\ &\quad + (I - K_\infty H) Q (I - K_\infty H)^T + K_\infty R K_\infty^T. \end{aligned} \quad (28)$$

Finally, the mean square error (MSE) is evaluated using $\mathbb{E}[\mathbf{e}_k \mathbf{e}_k^T] = \text{Cov}[\mathbf{e}_k, \mathbf{e}_k] + \bar{\mathbf{e}}_k \bar{\mathbf{e}}_k^T$.

3.2. Stability

In this section, we discuss conditions under which the MSE of the proposed filter is guaranteed to remain bounded. We establish the stability results using a set of propositions, as follows:

Proposition 1: If the initial error is bounded, and the maximum difference between the true mode and any of the given models, i.e., $\Delta\boldsymbol{\alpha}_k^{(i)} = \boldsymbol{\alpha}_k^{(t)} - \boldsymbol{\alpha}_k^{(i)}$ is bounded, and eigenvalues of matrix $(I - K_\infty H)A$ are less than unity; then, the mean of the overall error $\bar{\mathbf{e}}_k$ of the proposed filter remains bounded for all k .

Proof: From (22), the mean of the overall error is given as $\bar{\mathbf{e}}_k \approx \sum_{i=1}^M \bar{\mu}_k^{(i)} \bar{\mathbf{e}}_k^{(i)}$. Using the fact that $\sum_{i=1}^M \bar{\mu}_k^{(i)} = 1$, we note that if $\bar{\mathbf{e}}_k^{(i)}$ is bounded for all k , then $\bar{\mathbf{e}}_k$ remains bounded as well. We introduce a slight change of notation and use η_i instead of i , where $\eta_i \in \{1, 2, \dots, M\}$ and $i \in \{0, 1, 2, \dots, k\}$. Using (20), we write

$$\bar{\mathbf{e}}_k^{(\eta_0)} = (I - K_\infty H) A \bar{\mathbf{e}}_{0,k-1}^{(\eta_0)} + (I - K_\infty H) B \Delta\boldsymbol{\alpha}_k^{(\eta_0)}. \quad (29)$$

Before proceeding further, we first define a function $\mathbf{f}_i(\mathbf{x}_k)$ for $i \geq 1$, i.e.,

$$\mathbf{f}_i(\mathbf{x}_k) = \sum_{\eta_1} \gamma_{k-1}^{(\eta_0, \eta_1)} \sum_{\eta_2} \gamma_{k-2}^{(\eta_1, \eta_2)} \dots \sum_{\eta_i} \gamma_{k-i}^{(\eta_{i-1}, \eta_i)} \mathbf{x}_{k-i}^{(\eta_i)}. \quad (30)$$

We also set $\mathbf{f}_i(\mathbf{x}_k) = 1$ for $i = 0$. We use $|\mathbf{x}_k|$ to denote $[|x_{k,1}|, |x_{k,2}|, \dots, |x_{k,n}|]^T$, where n is the dimension of the vector \mathbf{x}_k . With a slight abuse of notation, we use $|\mathbf{x}_k| \leq c_0$ to denote $|x_{k,l}| \leq c_0$ for $l = 1, 2, \dots, n$, where $c_0 \geq 0$ is some constant. We can show that if $|\mathbf{x}_k^{(\eta_i)}| \leq c_0$ for all η_i, k ; then, $|\mathbf{f}_i(\mathbf{x}_k)| \leq c_0, \forall i$, as follows:

$$\begin{aligned} |\mathbf{f}_i(\mathbf{x}_k)| &= \left| \sum_{\eta_1} \gamma_{k-1}^{(\eta_0, \eta_1)} \dots \sum_{\eta_i} \gamma_{k-i}^{(\eta_{i-1}, \eta_i)} \mathbf{x}_{k-i}^{(\eta_i)} \right| \leq \sum_{\eta_1} \gamma_{k-1}^{(\eta_0, \eta_1)} \dots \sum_{\eta_i} \gamma_{k-i}^{(\eta_{i-1}, \eta_i)} |\mathbf{x}_{k-i}^{(\eta_i)}| \\ &\leq c_0 \sum_{\eta_1} \gamma_{k-1}^{(\eta_0, \eta_1)} \dots \sum_{\eta_i} \gamma_{k-i}^{(\eta_{i-1}, \eta_i)} = c_0, \end{aligned} \quad (31)$$

where the last equation follows from the fact that $\sum_{\eta_i} \gamma_k^{(\eta_i, \eta_j)} = 1, \forall j, k$. Now we iteratively expand (29) for $k-1$ terms to get

$$\bar{\mathbf{e}}_k^{(\eta_0)} = [(I - K_\infty H) A]^k \mathbf{f}_{k-1}(\bar{\mathbf{e}}_{0,k-1}) + \sum_{j=0}^{k-1} [(I - K_\infty H) A]^j (I - K_\infty H) B \mathbf{f}_j(\Delta\boldsymbol{\alpha}_k). \quad (32)$$

Assuming $|\bar{\mathbf{e}}_{0,k-1}^{(\eta_i)}| \leq c_0$ and $|\Delta\boldsymbol{\alpha}_k^{(\eta_i)}| \leq c_1, \forall \eta_i, k$; we can write

$$|\bar{\mathbf{e}}_k^{(\eta_0)}| \leq c_0 \left| [(I - K_\infty H) A]^k \right| + c_1 \sum_{j=0}^{k-1} \left| [(I - K_\infty H) A]^j (I - K_\infty H) B \right|. \quad (33)$$

From elementary matrix algebra, we know that if all eigenvalues of $(I - K_\infty H) A$ lie inside the unit circle, then as $k \rightarrow \infty$, $[(I - K_\infty H) A]^k \rightarrow 0$ and $\sum_{j=0}^{k-1} [(I - K_\infty H) A]^j \rightarrow$

$(I - (I - K_\infty H)A)^{-1}$. Hence from (33), for all k , $\left| \bar{\mathbf{e}}_{0,k-1}^{(\eta_i)} \right|$ remains bounded, and consequently the overall error $\bar{\mathbf{e}}_k$ remains bounded too. \square

Proposition 2: If eigenvalues of matrix $(I - K_\infty H)A$ are less than unity, and the initial error is bounded, and the maximum difference between the true mode and any of the given models, i.e., $\Delta \boldsymbol{\alpha}_k^{(i)} = \boldsymbol{\alpha}_k^{(t)} - \boldsymbol{\alpha}_k^{(i)}$ is bounded; then the MSE of the proposed filter remains bounded for all k .

Proof: The MSE matrix is given as $E[\mathbf{e}_k \mathbf{e}_k^T] = \text{Cov}[\mathbf{e}_k, \mathbf{e}_k] + \bar{\mathbf{e}}_k \bar{\mathbf{e}}_k^T$. From (28), the expression for $\text{Cov}[\mathbf{e}_k, \mathbf{e}_k]$ is a standard *Lyapunov* equation that remains bounded if eigenvalues of matrix $(I - K_\infty H)A$ are less than unity. The second term, i.e., $\bar{\mathbf{e}}_k \bar{\mathbf{e}}_k^T$ is bounded by *Proposition 1*. \square

Remark: Note that $\|\Delta \boldsymbol{\alpha}_k^{(i)}\| = \|\boldsymbol{\alpha}_k^{(t)} - \boldsymbol{\alpha}_k^{(i)}\| \leq \|\boldsymbol{\alpha}_k^{(t)}\| + \|\boldsymbol{\alpha}_k^{(i)}\|$. This implies that if both the true mode $\boldsymbol{\alpha}_k^{(t)}$ and the set of given models $\boldsymbol{\alpha}_k^{(i)}$ are bounded for all k , then their difference remains bounded. We assume that $\boldsymbol{\alpha}_k^{(t)}$ is bounded owing to physical constraints¹. Also, $\boldsymbol{\alpha}_k^{(i)}$ belongs to the set \mathcal{L}_α , where \mathcal{L}_α has been defined as a set of bounded elements in Section 2.

3.3. Steady-State Performance

Here, we develop an iterative procedure that readily evaluates the steady-state performance for a given jump in the Markovian process r_k . Let us assume that at some instant \bar{k} , r_k takes on the value r^t , which remains same $\forall k > \bar{k}$; consequently, the proposed filter achieves a steady-state. Let us denote the mean steady-state variables as $\bar{\boldsymbol{\theta}}_\infty^{(i)}$, $\bar{\mathbf{e}}_\infty^{(i)}$, $\bar{\boldsymbol{\mu}}_\infty^{(i)}$ and $\bar{\gamma}_\infty^{(i,j)}$. Using (18)-(22), we note that these steady-state values can be obtained as follows:

By choosing an initial guess and a stopping criteria, we can iteratively solve this set of equations to determine the steady-state error. For instance, we initialize all errors to be zero and all mode probabilities to be equal. Then, we iteratively solve the set of equation mentioned above, until, either the maximum difference in the residuals between two iterations is less than a given threshold, or a per-defined number of iterations is reached. A detailed procedure is described below:

Procedure 1: Steady-State MSE of the Proposed Filter

¹An unbounded $\boldsymbol{\alpha}_k^{(t)}$ would result in an unbounded state vector $\mathbf{x}_k^{(t)}$ which is not possible in most practical scenarios.

1. Off-line computations: Use an arbitrary initial condition and iteratively solve (24) and (28) to determine the steady-state error covariance matrices $\text{Cov}[\boldsymbol{\theta}_\infty, \boldsymbol{\theta}_\infty]$ and $\text{Cov}[\mathbf{e}_\infty, \mathbf{e}_\infty]$. Use an arbitrary initial condition $P_{0|-1}$, and iteratively solve, (11b) and (11c) to determine $S_\infty = HP_\infty H^T + R$.
2. Initialization: Set $\bar{\mathbf{e}}_{\infty,0}^{(i)} = 0$ and $\bar{\mu}_{\infty,0}^{(i)} = \frac{1}{M}, \forall i$.
3. Iterations: Let n be the iterations index, perform the following iterations, for all i, j till the stopping criteria is achieved:

$$\begin{aligned} \bar{\gamma}_{\infty,n}^{(i,j)} &= \frac{\pi_{ji} \bar{\mu}_{\infty,n}^{(j)}}{\sum_{l=1}^M \pi_{li} \bar{\mu}_{\infty,n}^{(l)}}, \\ \bar{\boldsymbol{\theta}}_{\infty,n+1}^{(i)} &= A \sum_{l=1}^M \bar{\gamma}_{\infty,n}^{(i,l)} \bar{\mathbf{e}}_{\infty,n}^{(i)} + HB\Delta\boldsymbol{\alpha}^{(i)}, \\ \bar{\mathbf{e}}_{\infty,n+1}^{(i)} &= (I - KH) A \sum_{l=1}^M \bar{\gamma}_{\infty,n}^{(i,l)} \bar{\mathbf{e}}_{\infty,n}^{(i)} + (I - KH) B\Delta\boldsymbol{\alpha}^{(i)}, \\ \bar{\mu}_{\infty,n+1}^{(i)} &= \frac{\bar{\Lambda}_{\infty,n+1}^{(i)} \sum_{l=1}^M \pi_{li} \bar{\mu}_{\infty,n}^{(l)}}{\sum_{m=1}^M \bar{\Lambda}_{\infty,n+1}^{(m)} \sum_{o=1}^M \pi_{om} \bar{\mu}_{\infty,n}^{(o)}}, \\ \bar{\Lambda}_{\infty,n+1}^{(i)} &= \frac{\left| \left(S_\infty^{-1} + \text{Cov}[\boldsymbol{\theta}_\infty, \boldsymbol{\theta}_\infty]^{-1} \right)^{-1} \right|^{1/2}}{2\pi |S_\infty|^{1/2} |\text{Cov}[\boldsymbol{\theta}_\infty, \boldsymbol{\theta}_\infty]|^{1/2}} \exp \left(-\frac{1}{2} \left[\bar{\boldsymbol{\theta}}_{\infty,n+1}^{(i)} \right]^T [\text{Cov}[\boldsymbol{\theta}_\infty, \boldsymbol{\theta}_\infty] + S_\infty]^{-1} \bar{\boldsymbol{\theta}}_{\infty,n+1}^{(i)} \right). \end{aligned}$$

4. Stopping Criteria: Stop when $\max_i \left\| \boldsymbol{\theta}_{\infty,n+1}^{(i)} - \boldsymbol{\theta}_{\infty,n}^{(i)} \right\| \leq \epsilon$, where $\epsilon \geq 0$ is a small number or when $n = N_{max}$.
5. Output: $\bar{\mathbf{e}}_\infty \approx \sum_{i=1}^M \bar{\mu}_\infty^{(i)} \bar{\mathbf{e}}_\infty^{(i)}$, and $\text{E}[\mathbf{e}_\infty \mathbf{e}_\infty^T] = \text{Cov}[\mathbf{e}_\infty, \mathbf{e}_\infty] + \bar{\mathbf{e}}_\infty \bar{\mathbf{e}}_\infty^T$.

3.4. Computational Performance

In this section, we discuss the computational complexity of the proposed filter and compare it with that of conventional IMM filter. We consider the number of addition and multiplication operations involved in the algorithms. The on-line calculations, at each instant k , required for the operations common to both the proposed filter and the conventional IMM filter are tabulated in Table 1. Note that M is the number of filters in the filter-bank and n, p , and u are the dimensions of vectors $\mathbf{x}_k, \mathbf{z}_k$, and $\boldsymbol{\alpha}_k$, respectively. The IMM filter also requires the on-line calculation of the Kalman gain and the covariance mixing, prediction and update at every instant. The additions and multiplications required for these operations are listed in Table 2 where we have assumed that the matrix inversion operations, required for the calculation of the Kalman gains, are carried out using Cholesky decomposition [22]. To compare the computational expense

Table 1: Computations required in the proposed filter and in the IMM filter.

Process	No. of Additions	No. of Multiplications
Mixing Probabilities	$2M^2$	$M(M-1)$
Mode Probabilities	$M(M+2)$	M^2-1
Mixing Estimate	nM	$n(M-1)$
Prediction step	$M(n^2+nu)$	$M(n^2+nu-2n)$
Update step	$2Mnp$	$2Mnp$
Output estimation	Mn	$n(M-1)$

Table 2: Computations required in the IMM filter only.

Process	No. of Additions	No. of Multiplications
Covariance mixing	$M^2(n^3+n)$	$M^2(n^3+2n)$
Covariance prediction	$2Mn^3$	$M(2n^3-n^2)$
Covariance update	$M(n^2p+n^3)$	$M[n^3+n^2(p-1)]$
Kalman Gain	$M(2n^2p+2p^2n+\frac{1}{2}p^3+\frac{3}{2}p^2)$	$M[np(2n-3)+2p^2n+\frac{1}{2}p^3-\frac{1}{2}p^2]$

of the proposed filter and the IMM filter, let us consider an example case where $n = 6$, $p = 2$, $u = 2$, and $M = 10$. Furthermore, we assume that one floating-point operation (flops) is required to carry out a single addition or a multiplication operation. In this case, 2057 flops are required by the proposed filter; whereas, IMM requires 64337 flops. We observe a significant reduction of complexity in the proposed filter.

4. Simulation Results

Here, we use a constant-acceleration model [2, 7] with a number of possible acceleration levels. Such models mimic adequately the maneuvering mobility behavior of aircrafts [6, 7] as well as vehicles [8, 9]. The state vector \mathbf{x}_k in a constant-acceleration model is given as $\mathbf{x}_k = [x_k, \dot{x}_k, \ddot{x}_k, y_k, \dot{y}_k, \ddot{y}_k]^T$, where the components x_k and y_k represent the position coordinates of the moving target; whereas, \dot{x}_k , \dot{y}_k and \ddot{x}_k , \ddot{y}_k represent (Cartesian set of) velocity and acceleration values, respectively. Furthermore, we assume that the measurement vector \mathbf{z}_k contains noisy information regarding the position coordinates only. The corresponding matrices A , B , H , Q and R in the system models (1a)-(1b), are specified as follows: $A = I_2 \otimes A_1$, $B = I_2 \otimes B_1$, $Q = I_2 \otimes Q_1$, $Q_1 = \sigma_m^2 CC^T$, $R = \sigma_r^2 I_2$,

$$A_1 = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}; B_1 = \begin{bmatrix} T^2/2 \\ T \\ 0 \end{bmatrix}; C = \begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix}; \text{ and } H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

where \otimes denotes Kronecker product. Note that the Markovian process α_k in (1a) represents sudden shifts or jumps in the acceleration level (i.e., maneuvers) in the constant-acceleration model described above. The set of possible acceleration levels is defined as

$\mathcal{L}_\alpha = \mathcal{L}_x \times \mathcal{L}_y$, where \mathcal{L}_x and \mathcal{L}_y are possible acceleration levels in x and y coordinates, respectively, and are specified as $\mathcal{L}_x = \mathcal{L}_y = \{-25, -15, -5, 5, 15, 25\}$ m/sec². The sampling time is set to 0.5 seconds and the total simulation time is 200 seconds. As suggested in [13], the value of true maneuvering command process α_k^t is simulated deterministically, i.e., initially it is set to $[2, -5]^T$ m/sec². After one minute, the target takes a sharp maneuver in the y -coordinate and α_k^t becomes $[5, 20]^T$ m/sec², and after one more minute it takes another maneuver in the x -coordinate and α_k^t becomes $[-15, 15]^T$ m/sec². The standard deviation of the mobility noise is set to 1 dB and the standard deviation of the observation noise is set to 20 dB. A typical trajectory and the corresponding position estimates of the proposed filter are depicted in Fig. 2. We first discuss consistency of the proposed algorithm and then compare absolute error (i.e., RMSE) performance of the proposed filter with other schemes.

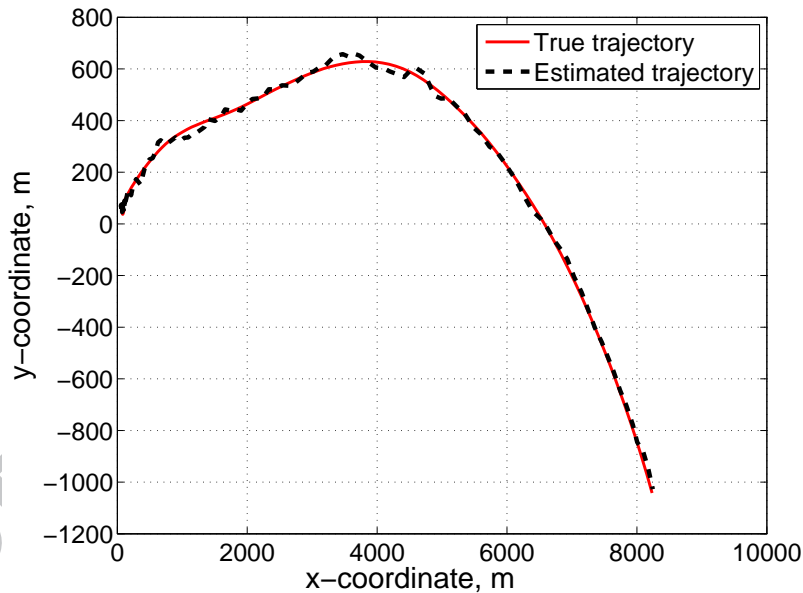


Figure 2: A typical trajectory and the corresponding position estimates of the proposed filter.

4.1. Consistency of the proposed scheme

A filter is said to be consistent if its self-assessment of estimation errors based on a finite number of samples is consistent with the theoretical statistical properties of the true filtering errors [23]. The criteria of filter consistency are 1) The filter generated covariance matrix should be same as (or close to) the true error covariance². 2) The filter residuals

²A number of measures to evaluate the similarity (or difference) of the filter generated covariance and true error covariance has been discussed in [24, 25], i.e., NEES, Log-NEES, NC, UIT etc., Their relative merits and demerits have been reported in [25].

should be acceptable as zero-mean and white random processes. From **Algorithm 1**, we note that proposed scheme utilizes an off-line computed gain matrix, and hence does not attempt to estimate the covariance matrix at each instant. In this work, we establish the consistency of our proposed filter using the *whiteness test* of filter residuals as specified in [23]. In simple words, we evaluate the following M -run sample autocorrelation statistic of the residuals:

$$\bar{\rho}_l(i, j) = \sum_{m=1}^M \theta_l^{(m)}(i) \theta_l^{(m)}(j) \left[\sum_{m=1}^M [\theta_l^{(m)}(i)]^2 \sum_{m=1}^M [\theta_l^{(m)}(j)]^2 \right]^{(-1/2)}, \quad (34)$$

where the residual vector $\boldsymbol{\theta}_k = \mathbf{z}_k - \hat{\mathbf{x}}_{k|k} = [\theta_l(k)]^T$ for $l = 1, 2, \dots, n_z$ and $n_z = \dim\{\mathbf{z}_k\}$.

Let H_0 be the hypothesis that the filter is consistent, then for large M and $i \neq j$, $\bar{\rho}_l(i, j)$ under H_0 can be sufficiently approximated a zero mean Gaussian random variable with variance equal to $1/N$ [23]. The hypothesis H_0 is accepted if $\bar{\rho}_l(i, j)$ lies within the acceptance interval (ρ_1, ρ_2) , where the interval is defined using the probability $p\{\bar{\rho}_l(i, j) \in (\rho_1, \rho_2) | H_0\} = 1 - \gamma$. Usually, γ is set to 0.05 which gives the 95% probability concentration region. We set $M = 50$, $i - j = 1$, $l = 1$ and $\gamma = 0.05$ for which the corresponding 95% acceptance region is equal to $[-0.277, 0.277]$. We evaluate the values of $\bar{\rho}_l(i, j)$ for the simulation scenario described in Section 4 and the result is depicted in Figure 3.

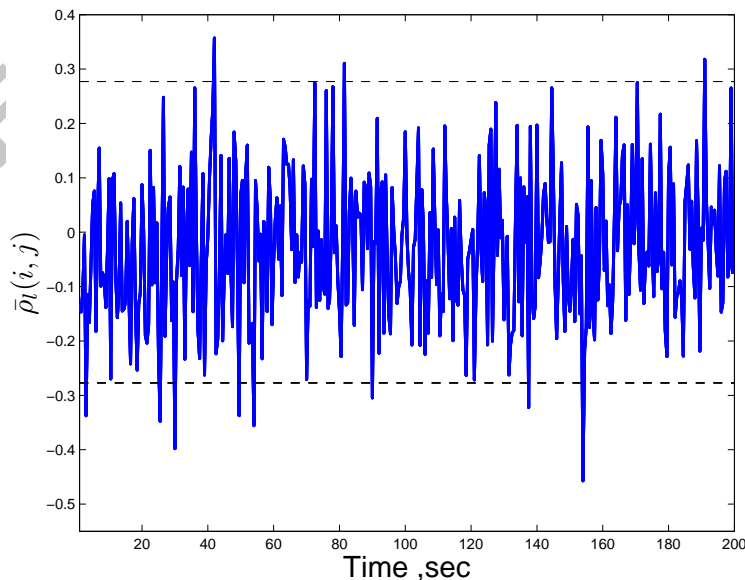


Figure 3: Sample autocorrelation of the residuals of the proposed scheme (CG-IMM) and the corresponding acceptance interval for the whiteness test.

We note that, only a small fraction of points, i.e., mere 12 out of 400, have been found

to breach the acceptance region which amounts to less than three percent of the observation window; thus, the proposed filter may be considered confidently to be consistent.³

4.2. Comparison of RMSE

We compare the RMSE performance of the proposed filter against the conventional IMM filter [3] and the degenerated non-interacting multiple model (DNMM) algorithm adopted in [6, 8, 9]. The DNMM filter has complexity similar to the proposed filter; however, it does not contain the mixing process and therefore is non-interacting. The expression for the RMSE for position estimates is given in (35).

$$\text{RMSE}_{pos,k} = \sqrt{\frac{1}{N_{mc}} \sum_{m=1}^{N_{mc}} \left(([\hat{x}_{k|k,m}]_1 - [x_k^t]_1)^2 + ([\hat{x}_{k|k,m}]_4 - [x_k^t]_4)^2 \right)} \quad (35)$$

where N_{mc} is the number of Monte-Carlo runs, $[\hat{x}_{k|k,m}]_1$ and $[\hat{x}_{k|k,m}]_4$ are the first and fourth elements of the estimated state vector that represent the position estimates at the k th instant for the m th Monte-Carlo run. Similarly, $[x_k^t]_1$ and $[x_k^t]_4$ are the true position values at k th instant. A similar expression for the RMSE velocity can easily be obtained by simply replacing position with velocity in (35).

In Fig. 4, the RMSE of position estimates of IMM and DNMM filters are compared with that of the proposed filter. Moreover, the analytical value of the RMSE is evaluated using the procedure outlined in Section 3. We observe that there is no considerable difference in the performance of IMM and the proposed filters. A closer inspection of the transient error reveals that the peak estimation error of the IMM filter is slightly better than that of the proposed filter; however, the difference is not significant. On the other hand, the DNMM filter suffers from a large transient error during maneuvers as compared to IMM filters. Also observe that the analytical results are in close agreement with the simulated ones. In Fig. 5, the RMSE in velocity is shown for all filters. Again we observe that the performances of IMM and the proposed filter are similar; whereas, the DNMM filter gives a large peak error during each maneuver. However, the steady-state error of the DNMM filter is slightly better than the interacting filters.

In Fig. 6 and 7, the steady-state performances are compared for a range of standard deviation values of observation noise. We observe that the steady-state performances are quite similar for the entire range. The analytical steady-state performance of the proposed

³With $T = 0.5$ seconds, we have 400 points in the observation window of 200 seconds.

filter is also evaluated using **Procedure 1**. For position estimates, the analytical results are in close agreement with the simulated results; however, the analytical results for the velocity estimates are slightly biased. The bias, however, remains small for the entire range of simulated observation noise.

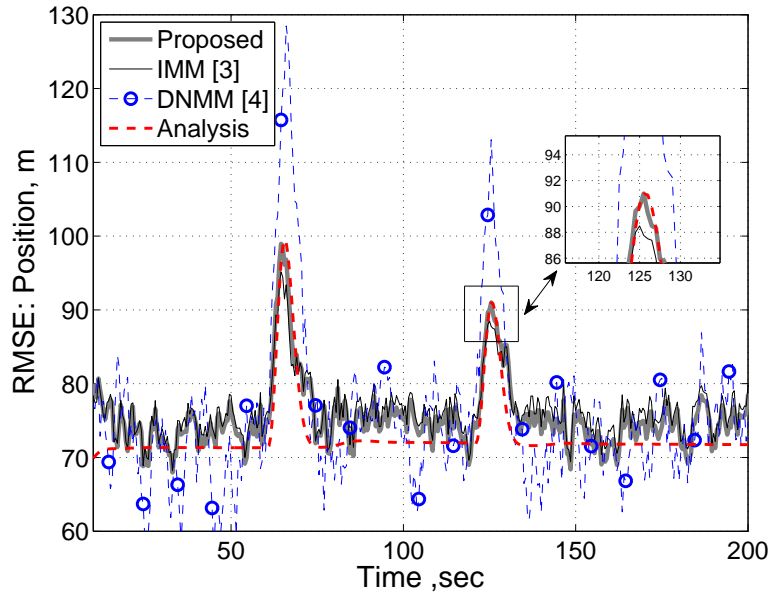


Figure 4: Comparison of the conventional IMM and the proposed filter in terms of RMSE in position estimate.

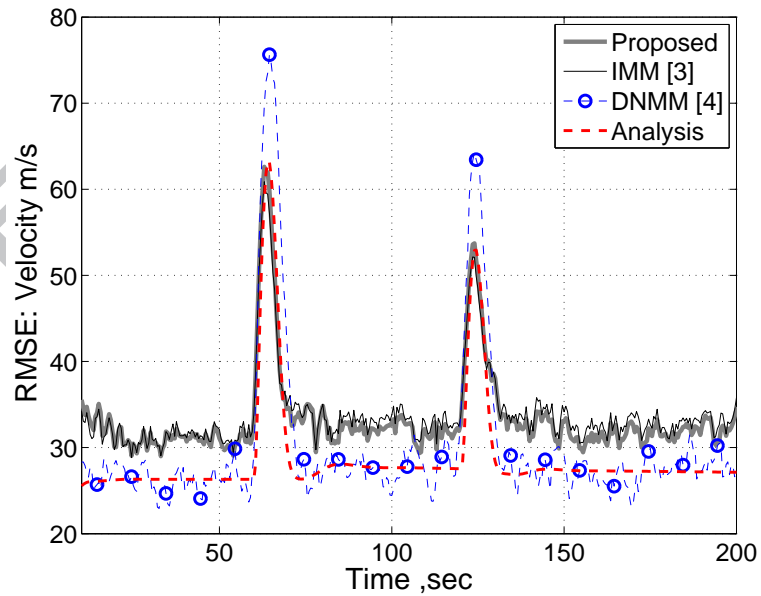


Figure 5: Comparison of the conventional IMM and the proposed filter in terms of RMSE in velocity estimate.

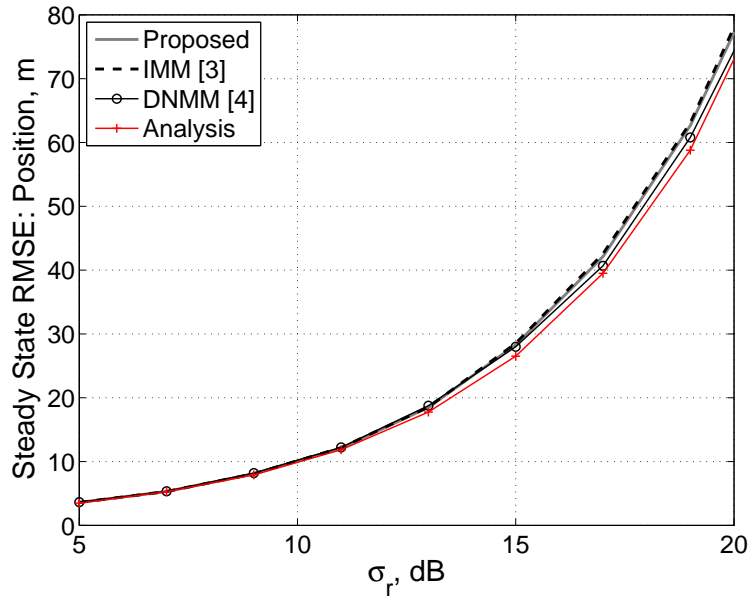


Figure 6: Steady-state position RMSE versus noise deviation.

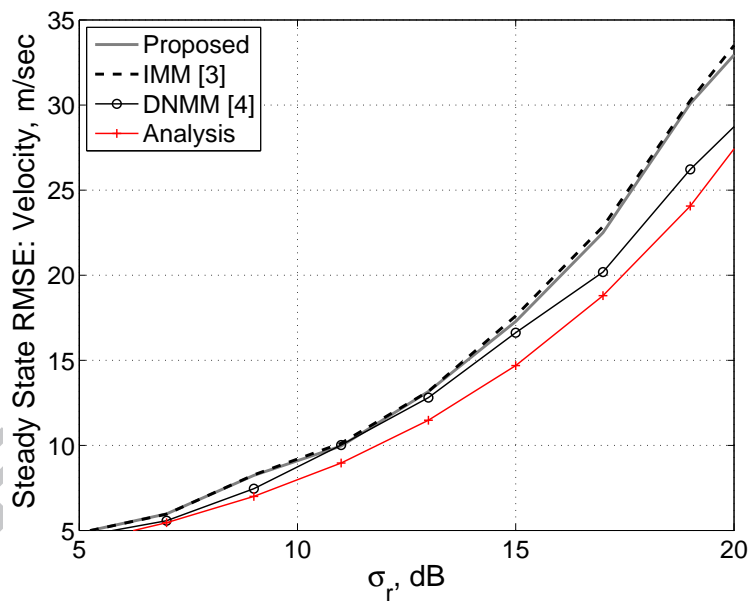


Figure 7: Steady-state velocity RMSE versus noise deviation.

5. Conclusions

We proposed a low-complexity alternate to the conventional interacting multiple model tracking filter for a class of maneuvering targets. The proposed filter limits the interactions within the filter-bank to the estimates of mean values, and exploits a fixed Kalman gain for the entire filter-bank. Accordingly, it circumvents the high-complexity matrix inversion operations required in the filtering algorithm. Tracking performance of the proposed filter was compared with two state-of-the-art filters (the conventional IMM

and a low-complexity degenerated non-interacting MM) in terms of position and velocity root-mean-squared error values. Results show that, during the maneuvering process, the proposed tracker performed as good as the traditional IMM filter while the degenerated non-interacting MM maintained the largest error. Moreover, iterative procedures are derived to analytically evaluate the transient and steady-state performances of the proposed scheme. Simulation results are found to be in agreement with our analytical findings. In our future research, we intend to extend the proposed scheme into a variable-structure multiple model filter using mode-augmentation schemes.

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