.

Transient growth of a Vlasov plasma in a weakly inhomogeneous magnetic field

Valeria Ratushnaya and Ravi Samtaney

Citation: Phys. Plasmas **23**, 122116 (2016); doi: 10.1063/1.4972083
View online: http://dx.doi.org/10.1063/1.4972083
View Table of Contents: http://aip.scitation.org/toc/php/23/12
Published by the American Institute of Physics

PHYSICS OF PLASMAS **23**, 122116 (2016)

Transient growth of a Vlasov plasma in a weakly inhomogeneous magnetic field

Valeria Ratushnaya and Ravi Samtaney
Mechanical Engineering, Physical Science and Engineering Division, King Abdullah University of Science and Technology, 4700 KAUST, Thuwal 23955-6900, Kingdom of Saudi Arabia

(Received 24 July 2016; accepted 25 November 2016; published online 16 December 2016)

We investigate the stability properties of a collisionless Vlasov plasma in a weakly inhomogeneous magnetic field using non-modal stability analysis. This is an important topic in a physics of tokamak plasma rich in various types of instabilities. We consider a thin tokamak plasma in a Maxwellian equilibrium, subjected to a small arbitrary perturbation. Within the framework of kinetic theory, we demonstrate the emergence of short time scale algebraic instabilities evolving in a stable magnetized plasma. We show that the linearized governing operator (Vlasov operator) is non-normal leading to the transient growth of the perturbations on the time scale of several plasma periods that is subsequently followed by Landau damping. We calculate the first-order distribution function and the electric field and study the dependence of the transient growth characteristics on the magnetic field strength and perturbation parameters of the system. We compare our results with uniformly magnetized plasma and field-free Vlasov plasma. *Published by AIP Publishing.*
[http://dx.doi.org/10.1063/1.4972083]

I. INTRODUCTION

In the 1990s, hydrodynamic stability theory underwent serious redevelopment. The main motivation was the existing discrepancies between the results of modal stability theory and experimental observations, e.g., discrepancies between experimental and calculated Reynolds numbers in wall-bounded shear flows.[1] These flows exhibit a finite period of rapid growth of disturbances, followed by the exponential decay. In other words, there exist generally stable systems with a finite period of transient growth of instabilities. The modal stability theory failed to capture these short-term characteristics of the flow giving only the asymptotic behavior at $t \to \infty$. It was discovered then that this behavior is driven by the non-normality of the governing operator, which has to be taken into account in order to get accurate results for all times.[2–4]

Since the eigenfunctions of a non-normal operator are mutually non-orthogonal, there might be a finite period of time when the disturbance grows if the eigenfunctions decay (stable system) at different rates.[5,6] The stability properties of a system on finite time scales missing in the classical stability theory were recovered in the framework of non-modal approach.[1–6] The main feature of the non-modal stability theory is that, in contrast to the modal stability theory, it does not assume an exponential time dependence of the disturbances.

Inspired by the examples of transient growth in fluid systems, it is a natural question whether similar transient effects take place in plasma subjected to infinitesimal perturbations. It is known that a Maxwellian collisionless Vlasov plasma is stable as $t \to \infty$, i.e., within the framework of a modal stability theory.[7] A non-modal approach was employed by Podesta[8] to study the transient growth in a 1D collisionless field-free Vlasov plasma. In a previous study, we demonstrated the emergence of the transient growth of the disturbances in a homogeneously *magnetized* Vlasov plasma.[9] This was the first step towards analyzing the stability behavior and possibilities of transient growth in plasma such as that encountered in tokamaks. We showed that linearized Vlasov operator is non-normal, leading to an algebraic growth of perturbations on the time scales of several plasma periods.

The objective of the present work is to consider a more realistic case, i.e., non-homogeneously magnetized plasma subjected to small perturbations. For the background equilibrium magnetic field, we consider a so-called Soloviev equilibrium,[10] which is a particular solution of the Grad-Shafranov equation.[11–13] Our approach remains mostly analytical, and in the following sections, we analyze the stability dynamics of collisionless Vlasov plasma subjected to a weakly non-homogeneous magnetic field. In Section II, we formulate the kinetic model and linearize the governing equations. Section III demonstrates the non-normality of the linearized Vlasov operator, leading to the emergence of algebraically growing instability. Given the background magnetic field described in Section IV, we solve the linearized Vlasov equation using the method of integration over unperturbed trajectories in Section V. We obtain the unperturbed velocities and trajectories and compare them with the corresponding results of the homogeneously magnetized plasma. Furthermore, in Section VI, we calculate the first order distribution function and the electric field and show that the evolution of the electric field is described by the Volterra integral equation of the second kind. We solve this equation and study the dependence of its solution, exhibiting the transient growth behavior, on the parameters of the system in Section VII. We finalize our results with conclusions in Section VIII.

II. GOVERNING EQUATIONS AND LINEARIZATION

The starting point of our analysis is the Vlasov equation for the evolution of the particle distribution function in a collisionless plasma[14]

$$\frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla)f + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \quad (1)$$

where q, m, $\mathbf{B}$, and $\mathbf{E}$ are the electric charge, mass, magnetic and electric fields, respectively. For the completeness of the electromagnetic system, this equation is solved self-consistently with the Maxwell equations

$$\nabla\cdot\mathbf{E} = \frac{q}{\epsilon_0}\int f(\mathbf{r},\mathbf{v},t)\mathbf{dv}, \quad (2)$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad (3)$$

$$\nabla\cdot\mathbf{B} = 0, \quad (4)$$

$$\nabla\times\mathbf{B} = \mu_0\left(\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + q\int\mathbf{v}f\mathbf{dv}\right), \quad (5)$$

where ϵ_0 and μ_0 are permittivity and permeability of free space, respectively.

According to the standard stability analysis procedure, we decompose the distribution function into a base (time-independent) distribution function f_0 and a small perturbation f_1

$$f(\mathbf{r},\mathbf{v},t) = f_0(v) + f_1(\mathbf{r},\mathbf{v},t), \quad f_1 \ll f_0. \quad (6)$$

We assume that the equilibrium state has a Maxwellian profile

$$f_0(v) = \frac{n_0}{(2\pi)^{3/2}v_T^3}\exp\left(-\frac{v^2}{2v_T^2}\right), \quad (7)$$

where n_0 is the particle density, $v_T = \sqrt{K_BT/m}$ is the thermal speed, T is the temperature, and K_B is the Boltzmann constant. The distribution function is normalized such that the particle density is $\int f_0 d\mathbf{v} = n_0$. In the classical modal stability analysis at this step, after the system is perturbed, the linearized governing equations are usually Fourier transformed and the whole system is studied in Fourier wavenumber space. From the computational point of view, it is more convenient to postpone the transformation to Fourier space in a general case of an arbitrary magnetic field.

Following the linearization procedure, we obtain the linearized Vlasov equation describing the evolution of the distribution function perturbations

$$\frac{\partial f_1}{\partial t} + (\mathbf{v}\cdot\nabla)f_1 + \frac{q}{m}(\mathbf{v}\times\mathbf{B}(\mathbf{r}))\cdot\nabla_{\mathbf{v}}f_1 + \frac{q}{m}(\mathbf{E}_1\cdot\nabla_{\mathbf{v}})f_0 = 0. \quad (8)$$

We further omit subscript 1 for simplicity and refer to the perturbed distribution function and electric field by f and $\mathbf{E}$, respectively.

III. NON-NORMALITY OF THE LINEARIZED VLASOV OPERATOR

It is known that the emergence of the transient growth of initial perturbations is driven by the non-normality of the governing operator of a system.[5] Here, we show that that the linearized Vlasov operator of a non-homogeneously magnetized plasma is indeed non-normal. In case of the Vlasov plasma, the distribution function evolves according to Eq. (8), and the electric field follows the linearized Maxwellian-Ampère law, Eq. (5)

$$\frac{\partial\mathbf{E}}{\partial t} = -\frac{q}{\epsilon_0}\int\mathbf{v}f d\mathbf{v}. \quad (9)$$

Combining these equations results in the following canonical representation of a dynamical system:

$$\frac{\partial\mathbf{X}}{\partial t} = \hat{A}\mathbf{X}, \quad \text{where } \mathbf{X} = \begin{pmatrix} f \\ \mathbf{E}\end{pmatrix}, \hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}, \quad (10)$$

where $\hat{A}$ is the linearized Vlasov operator and its entries are given by

$$a_{11} = -(\mathbf{v}\cdot\nabla) - \frac{q}{m}(\mathbf{v}\times\mathbf{B})\cdot\frac{\partial}{\partial\mathbf{v}}, \quad a_{12} = -\frac{q}{m}\frac{\partial f_0}{\partial\mathbf{v}}\cdot,$$
$$a_{21} = -\frac{q}{\epsilon_0}\int\mathbf{v}\mathbf{dv}\cdot, \quad a_{22} = 0. \quad (11)$$

Here, "$(\cdot)$" denotes the inner product. Direct inspection shows that the Vlasov operator is non-normal, i.e., that $\hat{A}\hat{A}^+ \neq \hat{A}^+\hat{A}$, where $\hat{A}^+$ is the adjoint of the operator $\hat{A}$. The non-normality of the governing Vlasov operator has a crucial impact on the stability properties of plasma. Non-normal operators have non-orthogonal eigenfunctions which has implications for the emergence of short time scale instabilities evolving in a generally stable system.

IV. MAGNETIC FIELD CONFIGURATION

The equilibrium in an axisymmetric toroidal plasma in a tokamak is described by the Grad-Shafranov equation for the poloidal flux function Ψ in cylindrical coordinates (R, Z, ϕ), see Fig. 1, where R is the distance to the symmetry axis, Z is the vertical coordinate, and ϕ is the toroidal angle[11–13]

$$\Delta^*\Psi = -II' - R^2p', \quad (12)$$

where $p(\Psi)$ is the pressure, $I(\Psi) = RB_\phi$ is the poloidal current stream function, B_ϕ is a toroidal magnetic field, and the prime indicates the differentiation with respect to Ψ. Here, Δ^* is the elliptic operator given by

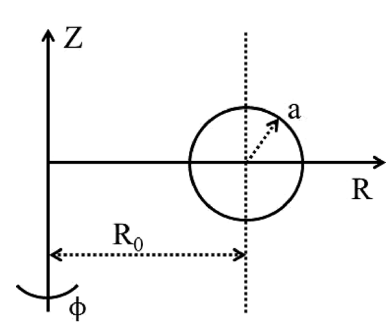

FIG. 1. Magnetic field geometry: a and R_0 are the minor and the major radii of a torus.

$$\Delta^*\Psi = R^2\nabla\cdot\left(\frac{1}{R^2}\nabla\Psi\right) = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right) + \frac{\partial^2\Psi}{\partial Z^2}. \quad (13)$$

Soloviev has considered a special case of the Grad-Shafranov equation, known as Soloviev equilibrium, assuming linear profiles for $I^2(\Psi)$ and $p(\Psi)$.[13] In this case, Eq. (12) is reduced to a linear nonhomogeneous PDE

$$\Delta^*\Psi = g + sR^2, \quad \text{where} \quad g, s = const \quad (14)$$

with a solution

$$\Psi(R,Z) = (c - dR^2)^2 + \frac{1}{2}\left[g + (s - 8d^2)R^2\right]Z^2. \quad (15)$$

Here, c and d are the parameters and their ratio $\sqrt{c/d}$ determines the position of the magnetic axis. The case $g=0$, i.e., constant current stream function ($I=I_0$), is one possible equilibrium in tokamak plasma. Furthermore, we will consider this case for thin tokamaks, i.e., when ($R_0 \gg a$), so that the toroidal direction $\mathbf{e}_\phi$ can be straightened (replaced by) into a Cartesian Z-direction $\mathbf{e}_z$. This implies that the equilibrium magnetic field in case of thin tokamak can be considered as

$$\mathbf{B} = \frac{I_0}{R}\mathbf{e_z} = \frac{B_0}{1+\beta x}\mathbf{e}_z = \frac{B_0}{1+\beta^* x^*}\mathbf{e}_z, \quad (16)$$

where $B_0 = I_0/R_0$, $x^* = x/a$ is a dimensionless distance, and $\beta^* = a/R_0$ is a degree of inhomogeneity and is considered to be a small quantity $\beta^* \ll 1$. We will now analyze the stability of plasma in the presence of this magnetic field.

V. SOLUTION OF THE LINEARIZED VLASOV EQUATION

In order to solve the linearized Vlasov equation, Eq. (8), we introduce the following transformation involving integration over unperturbed orbits (see, e.g., Refs. 15 and 16)

$$\frac{d\mathbf{r}'}{dt'} = \mathbf{v}', \quad \frac{d\mathbf{v}'}{dt'} = \frac{q}{m}\mathbf{v}'\times\mathbf{B}(\mathbf{r}'), \quad (17)$$

$$\text{with} \quad \mathbf{r}'(t'=t) = \mathbf{r}, \quad \mathbf{v}'(t'=t) = \mathbf{v}. \quad (18)$$

The advantage of this transformation is that it reduces the initial PDE, Eq. (8), to the ordinary differential equation

$$\frac{df}{dt'} + \frac{q}{m}(\mathbf{E}\cdot\nabla_{\mathbf{v}'})f_0 = 0, \quad (19)$$

which has the following solution in Fourier space:

$$f(\mathbf{k},\mathbf{v},t) = f(\mathbf{k},\mathbf{v},0)e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - \frac{q}{m}\int_0^t (\mathbf{E}(\mathbf{k},t')\cdot\nabla_{\mathbf{v}'})f_0 e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,dt'. \quad (20)$$

Here, $f(\mathbf{k}, \mathbf{v}, t)$ is a Fourier-transformed distribution function. Note that due to symmetry of our system, i.e., homogeneity in Y and Z directions, we perform the Fourier transformation only in these directions ($k^2 = k_y^2 + k_z^2$). The unperturbed trajectories $\mathbf{r}'(t')$ are to be taken into account from Eqs. (17) with the magnetic field in thin tokamak plasma given by Eq. (16).

A. Charged particle in a non-homogeneous magnetic field

Mathematically, Eqs. (17) are the characteristics of the partial differential equation, Eq. (1), but from a physical point of view, they are the equations of motion of a charged particle in a magnetic field $\mathbf{B}(\mathbf{r})$. In general, finding trajectories of a charged particle in a non-homogeneous magnetic field is a challenging task and is analytically intractable for general magnetic fields. A variety of approaches have been developed to solve this problem approximately when the magnetic field is assumed to be only slightly non-uniform and/or non-stationary.[17–22] The common feature of all these approaches is the perturbation method they are based on. The approximate formulas for the particle trajectories were derived in Ref. 17 and later generalized by Hellwig,[18] who gave a few terms of a series in trajectories expansion, and Kruskal[19] who derived the complete series. The paper by Kruskal is one of the classical examples in this area, where he used asymptotic expansion of Eq. (17) to all orders in the radius and the period of particle gyration in a given electromagnetic field to calculate the trajectories. The approaches mentioned above are based on an elegant mathematical method, but they require rather involved calculations not necessary in our case. In order to obtain a solution to first order for weakly non-uniform magnetic fields we treat inhomogeneity using so-called guiding center approximation, also known as the adiabatic approximation, described in Ref. 23.

Introducing B^* to be a typical magnetic field and $L^*(=R_0)$ a typical length over which there is a noticeable change of the field, we can rewrite Eq. (17) in its non-dimensional form

$$\frac{d\hat{\mathbf{v}}'}{d\hat{t}'} = \frac{1}{\epsilon}\hat{\mathbf{v}}'\times\hat{\mathbf{B}}, \quad \text{where} \quad \epsilon = \frac{mv}{qB^*L^*}.$$

It can be seen that the parameter ϵ is the ratio between the charged particle gyroradius $\rho = mv/qB^*$ and the characteristic length L^*, $\epsilon = \rho/L^*$, and it has to be a small quantity in the framework of the guiding center approximation. For typical tokamak parameters, $B_0 = 1\text{T}$, $n_0 = 10^{19}/\text{m}^3$, $\text{T} = 10^8\,\text{K}$, and $L^* \approx 1\,\text{m}$, parameter $\epsilon \sim O(10^{-6})$.

We consider the effects of inhomogeneity as a perturbation of the motion in a homogeneous magnetic field. In zeroth order, the trajectory in the plane perpendicular to $\mathbf{B}$ is circular and the first order inhomogeneity in $\mathbf{B}$ is considered as a perturbation to the uniform motion. This means that Eq. (17) may be rewritten as follows:

$$\begin{aligned} \frac{dv_x'}{dt'} &= \omega v_y' - \beta\omega x_h' v_{yh}', \\ \frac{dv_y'}{dt'} &= -\omega v_x' + \beta\omega x_h' v_{xh}', \\ \frac{dv_z'}{dt'} &= 0, \end{aligned} \quad (21)$$

where $\omega = qB_0/m$ is the cyclotron frequency and terms containing β are the contributions due to inhomogeneity of the magnetic field. Here, $\mathbf{r}_h' = \{x_h', y_h', z_h'\}$ and $\mathbf{v}_h' = \{v_{xh}', v_{yh}', v_{zh}'\}$

are the trajectories and velocities in a homogeneous magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_z$, respectively, obtained in our previous paper[9]

$$\begin{cases} v'_{xh} = v_x \cos\alpha - v_y \sin\alpha, \\ v'_{yh} = v_x \sin\alpha + v_y \cos\alpha, \\ v'_{zh} = v_z, \end{cases} \tag{22}$$

$$\begin{cases} x'_h = x - \dfrac{v_x}{\omega}\sin\alpha + \dfrac{v_y}{\omega}(1-\cos\alpha), \\ y'_h = y - \dfrac{v_x}{\omega}(1-\cos\alpha) - \dfrac{v_y}{\omega}\sin\alpha, \\ z'_h = z - \dfrac{v_z}{\omega}\alpha, \end{cases} \tag{23}$$

where $\alpha = \omega(t-t')$. These velocities and trajectories satisfy conditions Eq. (18).

Substituting homogeneous velocities and trajectories, Eqs. (22), (23), into Eqs. (21), we obtain a system of linear non-homogeneous ordinary differential equations with the following solution:

$$\begin{cases} v'_x = B_1(\mathbf{r},\mathbf{v},\alpha)\cos\alpha - B_2(\mathbf{r},\mathbf{v},\alpha)\sin\alpha + \beta x'_h v'_{xh}, \\ v'_y = B_1(\mathbf{r},\mathbf{v},\alpha)\sin\alpha + B_2(\mathbf{r},\mathbf{v},\alpha)\cos\alpha, \\ v'_z = v_z, \end{cases} \tag{24}$$

where

$$B_1 = v_x + \beta[b_1\sin^2\alpha + b_2(0.5\sin 2\alpha + \alpha) + b_3\sin^3\alpha + b_4(3\sin\alpha - \sin^3\alpha) - b_5(\cos^3\alpha - 1) - xv_x],$$

$$B_2 = v_y + \beta[b_1(0.5\sin 2\alpha - \alpha) - b_2\sin^2\alpha + b_3(3\cos\alpha - \cos^3\alpha - 2) + b_4(\cos^3\alpha - 1) - b_5\sin^3\alpha],$$

$$b_1 = xv_x + \frac{v_x v_y}{\omega}, \quad b_2 = xv_y + \frac{v_y^2}{\omega}, \quad b_3 = \frac{1}{3\omega}\left(v_y^2 - 2v_x^2\right),$$

$$b_4 = \frac{1}{3\omega}\left(v_x^2 - 2v_y^2\right), \quad b_5 = -\frac{2v_x v_y}{\omega}.$$

Expressions for the homogeneous velocities, Eqs. (22), are the limiting case of Eqs. (24), when the degree of inhomogeneity β is set to zero.

Integration of the velocities Eqs. (24) gives the trajectories $\mathbf{r}' = \{x', y', z'\}$ required to calculate the perturbed distribution function and the electric field. It can be shown that

$$\mathbf{r}' = \mathbf{r}'_h + \beta\mathbf{F}, \tag{25}$$

where $\mathbf{F} = \{F_x, F_y, 0\}$ is a vector modifying the circular trajectories $\mathbf{r}'_h$ due to non-uniform character of the magnetic field. This vector affects only the motion in the plane normal to the magnetic field $\mathbf{B}$

$$F_x = \frac{1}{\omega}[h_1b_1 + h_2b_2 + h_3(b_3 - b_4) + h_4b_5 + h_5b_6 + g_1],$$
$$F_y = \frac{1}{\omega}[h_6b_1 + h_7b_2 + h_8b_3 + h_9b_4 + h_{10}b_5 + h_{11}b_6], \tag{26}$$

where h_i are functions of time, functions b_i depend on velocities and coordinates, g_1 is a function of coordinates, velocities and time are given in Appendix A. These expressions mean that the charged particles oscillate in the plane perpendicular to the magnetic field and move uniformly in the direction parallel to the field. Moreover, the influence of the magnetic field inhomogeneity can be clearly seen: if the field was uniform, i.e., $\beta = 0$, one would obtain a circular gyration of constant radius in the plane perpendicular to the magnetic field, see Eqs. (23) and Eq. (25) with $\beta = 0$.

VI. KINETIC INSTABILITY IN THIN TOKAMAKS: DISTRIBUTION FUNCTION AND ELECTRIC FIELD

We now substitute the unperturbed trajectories, Eq. (25), into Eq. (20) to find the first order distribution function

$$f = f_1 + f_2, \tag{27}$$

where

$$f_1(\mathbf{k},\mathbf{v},t) = f(\mathbf{k},\mathbf{v},0)e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'_h)} - \frac{q}{m}\int_0^t (\mathbf{E}(\mathbf{k},t')\cdot\nabla_{\mathbf{v}'})f_0 e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'_h)}dt' \tag{28}$$

and

$$f_2(\mathbf{k},\mathbf{v},t) = i\beta(\mathbf{k}\cdot\mathbf{F})f(\mathbf{k},\mathbf{v},0)e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'_h)} - \frac{i\beta q}{m}\int_0^t (\mathbf{k}\cdot\mathbf{F})(\mathbf{E}\cdot\nabla_{\mathbf{v}'})f_0 e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'_h)}dt'. \tag{29}$$

To demonstrate the existence of the algebraically growing short time-scale instabilities, we consider the behavior of the perturbed magnetic field using the Fourier-transformed Maxwell equation for the divergence of the electric field

$$E(\mathbf{k},t) = \frac{q}{ik\epsilon_0}\int f(\mathbf{k},\mathbf{v},t)d\mathbf{v}. \tag{30}$$

Here, we have taken into account the fact that the electric field is parallel to the direction of propagation. Knowing the distribution function, Eqs. (27)–(29), we now obtain the equation for the evolution of the electric field

$$E(\mathbf{k},t) = J(\mathbf{k},t) + \int_0^t K(\mathbf{k},t-\tau)E(\mathbf{k},\tau)d\tau. \tag{31}$$

This is the Volterra integral equation of the second kind, where $J(\mathbf{k},t) = J_h + J_{nh}$ is a function which depends on the initial perturbation and $K(\mathbf{k},t-\tau) = K_h + K_{nh}$ is an integral nonlocal kernel depending on the equilibrium state of plasma. Here, J_h and K_h are the contributions corresponding to the case of homogeneous magnetic field[9]

$$J_h = \frac{q}{ik\epsilon_0}\int_{-\infty}^{+\infty} f(\mathbf{k},\mathbf{v},0)e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'_h)}d\mathbf{v}, \tag{32}$$

$$K_h = -\frac{q^2}{i\epsilon_0 mk^2}\int_{-\infty}^{+\infty}\frac{1}{v}\frac{df_0}{dv}(\mathbf{k}\cdot\mathbf{v}')e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'_h)}d\mathbf{v}. \tag{33}$$

Functions J_{nh} and K_{nh} are the contributions due to the inhomogeneity of the magnetic field

$$J_{nh} = \frac{\beta q}{\epsilon_0 k}\int_{-\infty}^{+\infty}(\mathbf{k}\cdot\mathbf{F})f(\mathbf{k},\mathbf{v},0)e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'_h)}d\mathbf{v},$$
$$K_{nh} = -\frac{\beta q^2}{\epsilon_0 m k^2}\int_{-\infty}^{+\infty}\frac{1}{v}\frac{df_0}{dv}(\mathbf{k}\cdot\mathbf{v}')(\mathbf{k}\cdot\mathbf{F})e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'_h)}d\mathbf{v}. \tag{34}$$

Note that we have freedom in the choice of initial condition, $f(\mathbf{k},\mathbf{v},0)$, that is limited only by the assumption of the smallness of the imposed perturbation. In order to study the effect of the magnetic field inhomogeneity, we consider the same initial condition as in Ref. 9, where we studied the transient growth in a homogeneously magnetized plasma

$$f(\mathbf{k},\mathbf{v},0) = \frac{v_T}{(2\pi)^{3/2}v_f^5}(\mathbf{C}\cdot\mathbf{v})\exp\left(-\frac{v^2}{2v_f^2}\right), \tag{35}$$

where $\mathbf{C}=(C_x,C_y,C_z)$ and v_f are parameters of the initial perturbation related to the amplitude and the duration of the transient growth in plasma. Similarly to a homogeneous magnetic field, we consider an isotropic case when $\mathbf{C}=C_0\mathbf{e}$ and $\mathbf{v}=\frac{v}{\sqrt{3}}\mathbf{e}$ with $\mathbf{e}=(1,1,1)$. A case somewhat similar to Eq. (35) was considered by Podesta for one-dimensional field-free Vlasov plasma.[8]

We further introduce the following dimensionless quantities:

$$\hat{t} = \frac{t}{\tau_p} = \frac{\omega_p t}{2\pi}, \quad \hat{\omega} = 2\pi\frac{\omega}{\omega_p}, \quad \mu = \frac{v_f}{v_T}, \quad \hat{\mathbf{k}} = \mathbf{k}\lambda_D, \tag{36}$$

where $\omega_p = \sqrt{n_0 q^2/m\epsilon_0}$ is the plasma frequency, τ_p is plasma period, and $\lambda_D = \sqrt{\epsilon_0 K_B T/n_0 q^2}$ is the Debye length. We use these quantities to non-dimensionalize functions J and K as follows:

$$J(\mathbf{k},t) = \left(\frac{q v_T \tau_p}{\epsilon_0}\right)\hat{J}(\hat{\mathbf{k}},\hat{t}), \tag{37}$$

$$K(\mathbf{k},t-\tau) = \frac{1}{\tau_p}\hat{K}(\hat{\mathbf{k}},\hat{t}-\hat{\tau}). \tag{38}$$

In fact, the dynamics of the Vlasov plasma is driven by the ratio of two frequencies $\hat{\omega}$, and the other two parameters μ and $\hat{\mathbf{k}}$ characterize the perturbation. Taking into account the equilibrium distribution Eqs. (7) and the initial condition profile, Eqs. (35), we obtain the expressions for the dimensionless functions $\hat{J}(\mathbf{k},t)$ and $\hat{K}(\mathbf{k},t-\tau)$

$$\hat{J}_h(\hat{\mathbf{k}},\hat{t}) = \frac{C_0}{\hat{\omega}\sqrt{2}}[\sin\hat{\omega}\hat{t} + \cos\hat{\omega}\hat{t} - 1 + \hat{\omega}\hat{t}] \times \exp\left\{-2\left(\frac{\pi\mu\hat{k}}{\hat{\omega}}\right)^2\left[2(1-\cos\hat{\omega}\hat{t}) + (\hat{\omega}\hat{t})^2\right]\right\},$$

$$\hat{K}_h(\hat{\mathbf{k}},\hat{t}-\hat{\tau}) = -\frac{(2\pi)^2}{\hat{\omega}}[\hat{\alpha}+\sin\hat{\alpha}] \times \exp\left\{-2\left(\frac{\pi\hat{k}}{\hat{\omega}}\right)^2\left[2(1-\cos\hat{\alpha}) + \hat{\alpha}^2\right]\right\},$$

and complex functions $\hat{J}_{nh} = \mathrm{Re}\hat{J}_{nh} + i\mathrm{Im}\hat{J}_{nh}$ and $\hat{K}_{nh} = \mathrm{Re}\hat{K}_{nh} + i\mathrm{Im}\hat{K}_{nh}$ are given by

$$\mathrm{Re}\hat{K}_{nh}(\hat{\mathbf{k}},\hat{t}-\hat{\tau}) = -(2\pi)^4\frac{\beta^* x^* \hat{k}^2}{\hat{\omega}^3}\hat{G}_1(\hat{\alpha}) \times \exp\left\{-2\left(\frac{\pi\hat{k}}{\hat{\omega}}\right)^2\left[2(1-\cos\hat{\alpha}) + \hat{\alpha}^2\right]\right\},$$

$$\mathrm{Im}\hat{K}_{nh}(\hat{\mathbf{k}},\hat{t}-\hat{\tau}) = -(2\pi)^4\frac{\beta^* \hat{k}}{\hat{\omega}^3}\hat{G}_3(\hat{\alpha}) \times \exp\left\{-2\left(\frac{\pi\hat{k}}{\hat{\omega}}\right)^2\left[2(1-\cos\hat{\alpha}) + \hat{\alpha}^2\right]\right\},$$

$$\mathrm{Re}\hat{J}_{nh} = -(2\pi)^2\frac{\beta^* x^* \hat{k}^2 \mu^2}{\omega^3}\hat{G}_2(\hat{\omega}\hat{t}) \times \exp\left\{-2\left(\frac{\pi\mu\hat{k}}{\hat{\omega}}\right)^2\left[2(1-\cos(\hat{\omega}\hat{t})) + (\hat{\omega}\hat{t})^2\right]\right\},$$

$$\mathrm{Im}\hat{J}_{nh} = (2\pi)^2\frac{\beta^* \mu^2 \hat{k}}{\hat{\omega}^3}\hat{G}_4(\hat{\omega}\hat{t}) \times \exp\left\{-2\left(\frac{\pi\mu\hat{k}}{\hat{\omega}}\right)^2\left[2(1-\cos(\hat{\omega}\hat{t})) + (\hat{\omega}\hat{t})^2\right]\right\},$$

$$\hat{\alpha} = \hat{\omega}(\hat{t}-\hat{\tau}).$$

The functions G_i are given in Appendix B. The dynamics of the perturbed Vlasov plasma in the non-homogeneous magnetic field given by Eq. (16) is determined by the non-dimensional equation of the form

$$\hat{E}(\hat{\mathbf{k}},\hat{t}) = \hat{J}(\hat{\mathbf{k}},\hat{t}) + \int_0^{\hat{t}}\hat{K}(\hat{\mathbf{k}},\hat{t}-\hat{\tau})\hat{E}d\hat{\tau}. \tag{39}$$

We solve this equation numerically and will study the dependence of its solution on the parameters of the system in Sec. VII.

VII. TRANSIENT GROWTH AND ITS PROPERTIES

As mentioned above, there are three parameters relevant to the dynamics of the perturbations: system parameter $\hat{\omega}$, initial condition parameter μ, and perturbation parameter $\hat{k}$. In Fig. 2, we plot a solution of Eq. (39), the perturbed electric field in a non-homogeneously magnetized plasma with $\hat{k}=0.25$, $\mu=0.05$, $x^*=0.25$, and we choose magnetic field, temperature, and the density typical for tokamak plasma, e.g., $B_0 = 1\mathrm{T}$, $\mathrm{T}=10^8\mathrm{K}$, and $n_0 = 10^{19}/\mathrm{m}^3$, corresponding to the dimensionless frequency ratio $\hat{\omega}=2\pi$. We fix the degree of inhomogeneity β^* to be equal 0.05. The electric field growths algebraically over about 10–15 plasma periods and is dominated by the Landau damping at later stages. This is a transient growth phenomenon which we also observed in the homogeneous case when the plasma was subjected to a homogeneous and stationary magnetization (red curve). We compare both cases in this figure as well and see that inhomogeneity enhances the amplitude of the transient instability (black curve). The duration and the amplitude of the transient growth can be controlled by the parameters of the system $(\hat{\omega},\hat{k},\mu)$. We study the influence of

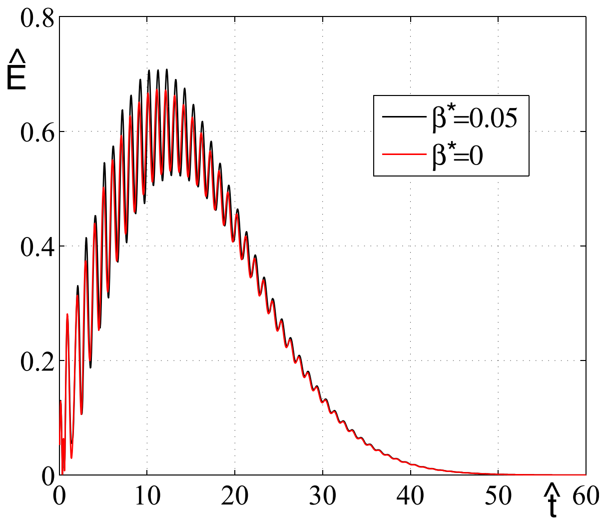

FIG. 2. Influence of the inhomogeneity of the magnetic field on the transient growth for $\hat{\omega} = 2\pi$, $\hat{k} = 0.25$, $\mu = 0.05$, $x^* = 0.25$.

the magnetic field in Fig. 3, where we see that the growth of $\hat{\omega}$ leads to the decrease in the electric field maximum and increase in the transient growth duration. This indicates that higher magnetic fields somewhat suppress the instability but prolong its duration. In Fig. 4, we plot the perturbed magnetic field for higher magnetic fields ($\hat{\omega} = 8\pi$) and see that the transient growth is less affected by the magnetic field growth when $\hat{\omega} > 2\pi$. Below $\hat{\omega} = 2\pi$, electric field behaves qualitatively similar to the corresponding homogeneous case. Comparing Figs. 3 and 4, we also observe that as $\hat{\omega}$ increases, the difference between the homogeneous and non-homogeneous cases becomes larger.

We study the dependence of the transient growth on the wave number in Fig. 5. We are interested in a range of the wave numbers between 0 (perturbations with very long wavelength) and $\hat{k}_{max} = \pi\sqrt{2}$, eliminating the perturbations with the wavelength shorter than the Debye length. Figure 5 shows that the disturbances of the Debye length scale decay very rapidly after 1–2 plasma periods, whereas the amplitude

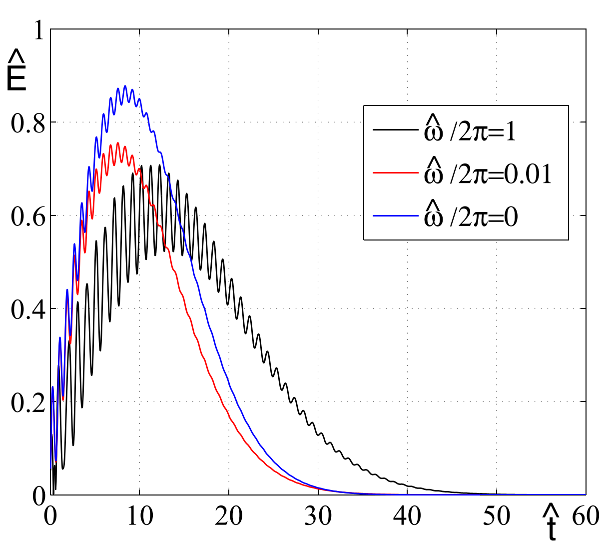

FIG. 3. Dependence of $\hat{E}$ on the magnetic field ($\hat{\omega}$) for $\hat{k} = 0.25$, $\mu = 0.05$, $\beta^* = 0.05$, $x^* = 0.25$.

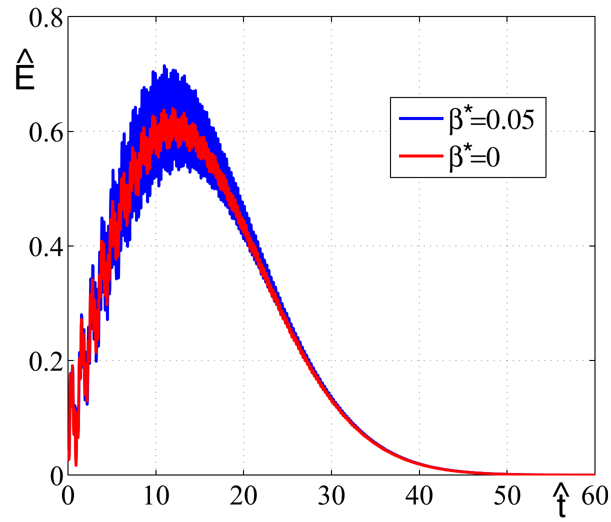

FIG. 4. Influence of the inhomogeneity on the transient growth for $\hat{\omega} = 8\pi$, $\hat{k} = 0.25$, $\mu = 0.05$, $x^* = 0.25$.

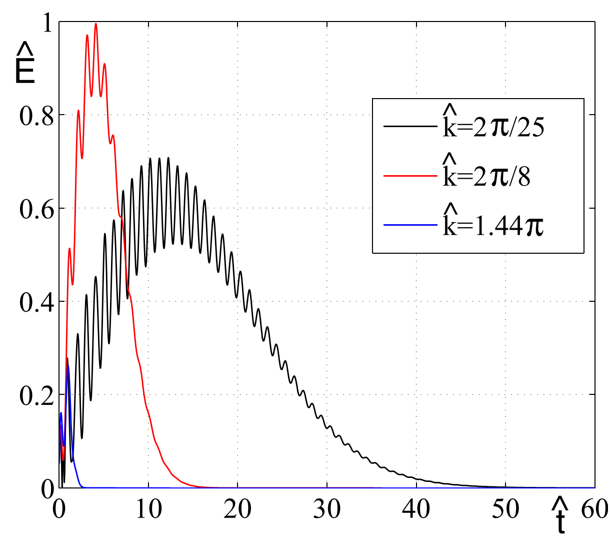

FIG. 5. Dependence of $\hat{E}$ on the wave number $\hat{k}$ for $\hat{\omega} = 2\pi$, $\mu = 0.05$, $\beta^* = 0.05$, $x^* = 0.25$.

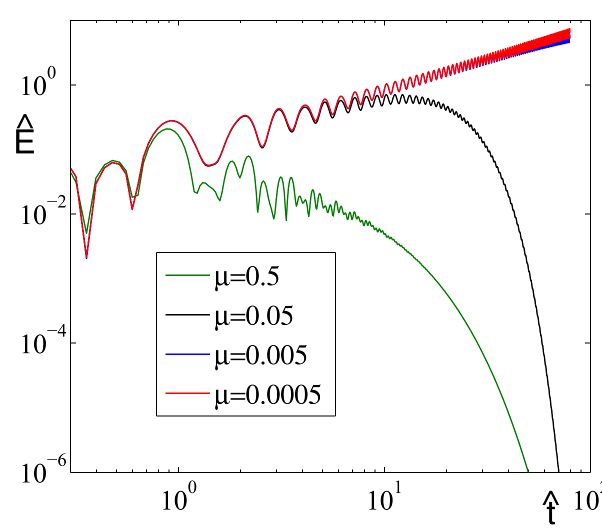

FIG. 6. Dependence of $\hat{E}$ on μ for $\hat{\omega} = 2\pi$, $\hat{k} = 0.25$, $\beta^* = 0.05$, $x^* = 0.25$.

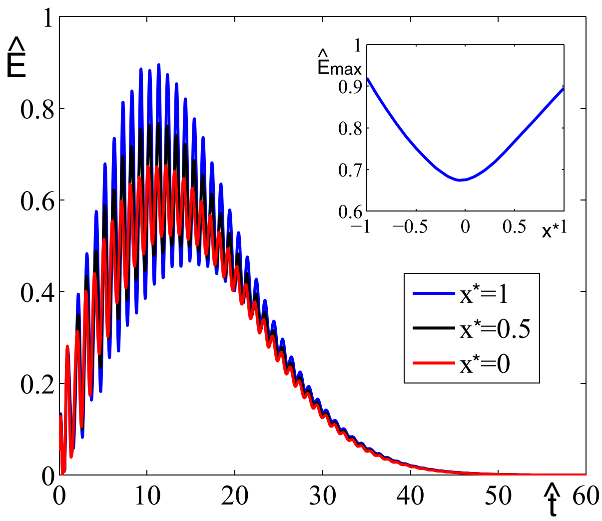

FIG. 7. Dependence of $\hat{E}$ on x^* for $\hat{\omega} = 2\pi$, $\hat{k} = 0.25$, $\mu = 0.05$, $\beta^* = 0.05$. Inset: Maximum of the electric field $\hat{E}_{max}$ as a function of position x^*.

of the transient growth at shorter wave lengths behaves non-monotonically: first, it increases to its maximum value and then decreases shifting to the right. Figure 5 also reveals a result known from the modal stability analysis of plasma, namely, the faster (slower) decay of the perturbations of smaller (larger) wavelengths. The third parameter determining the dynamics of the perturbations is μ. As we can see in Fig. 6, small μ can prolong the duration of the transient growth to hundreds of plasma periods and enhance its amplitude significantly, whereas perturbations with higher μ decay very rapidly. In Figure 7, we plot the dependence of transient growth on the coordinate x^*. The figure shows that the distance does not affect the duration of the transient growth, and the large distances increase the amplitude of the instability.

VIII. CONCLUSION

In conclusion, we have studied the emergence of the transient growth in a weakly non-homogeneous magnetized collisionless Vlasov plasma. We considered the evolution of the electric field perturbations in a plasma subjected to a magnetization similar to the one in tokamak plasma. A Soloviev equilibrium for thin tokamak plasma was used to calculate the particles unperturbed trajectories, which we then used to obtain the first order distribution function. These steps comprised of rather involved calculations as they actually dealt with solving the Lorentz equation for a charged particle in a nonhomogeneous magnetic field. We showed that due to the non-normality of the governing linearized Vlasov operator, there emerge algebraically growing instabilities dominated by the classical Landau damping at later stages. Previously, we studied the transient growth in a homogeneously magnetized plasma, which is a limiting case of the present broader contribution. We compare our results with the previous findings and conclude that the inhomogeneity of the magnetic field enhances the amplitude of the transient growth. Qualitatively, the behavior of the electric field perturbations is similar to the homogeneous case. The significance of this work is determined by the importance of the understanding the nature and mechanisms of the kinetic instabilities in plasma, in general, and in tokamaks, in particular.

APPENDIX A: PARTICLE VELOCITIES AND TRAJECTORIES IN A NON-HOMOGENEOUS MAGNETIC FIELD

The expressions of h_i and b_i

$$h_1 = \alpha\cos\alpha, \quad h_2 = -\alpha\sin\alpha,$$
$$h_3 = \sin^2\alpha, \quad h_4 = \frac{\sin\alpha}{2}(2\cos\alpha - 1),$$
$$h_5 = 1 - \cos\alpha, \quad h_6 = \alpha\sin\alpha + \cos\alpha - 1,$$
$$h_7 = \alpha\cos\alpha - \sin\alpha, \quad h_8 = -\frac{3}{2}\left(\frac{\sin 2\alpha + \alpha}{6}\right),$$
$$h_9 = \frac{3}{2}\left(\frac{\sin 2\alpha}{6} - \alpha\right), \quad h_{10} = \sin\alpha\left(\frac{1}{2} + \sin\alpha\right),$$
$$h_{11} = -\sin\alpha, \quad b_6 = \frac{v_x^2}{\omega}.$$

Function g_1 is given by the following expression:

$$g_1 = \frac{\omega}{2}\left[x_h'^2 - \left(x + \frac{v_y}{\omega}\right)^2 - \frac{v_y^2}{\omega^2} + 2\frac{v_y}{\omega}\left(x + \frac{v_y}{\omega}\right)\right],$$

where x_h' is x-coordinate of the trajectories of particles in a homogeneous magnetic field and is given by Eq. (23).

APPENDIX B: PERTURBED ELECTRIC FIELD

Functions $G_i(\alpha)$ are given by the following expressions:

$$\begin{aligned}G_1 = {} & (\alpha\cos\alpha - \sin\alpha)[\sin\alpha(\alpha + \sin\alpha(1-\cos\alpha)) \\ & + \cos\alpha(m_1 + \sin^2\alpha)] + (\alpha\sin\alpha + \cos\alpha - 1) \\ & \times [(1-\cos\alpha)(\alpha + 0.5\sin 2\alpha) \\ & + \sin\alpha(m_1 - (1-\cos\alpha)^2)],\end{aligned}$$

$$\begin{aligned}G_2 = {} & (-\omega t\cos\omega t + \sin\omega t) \\ & \times [-\sin\omega t(-\omega t + 1 - \cos\omega t - \sin\omega t) + m_2] \\ & + (\omega t\sin\omega t + \cos\omega t - 1) \\ & \times [(1-\cos\omega t)(-\omega t - 1 + \cos\omega t - \sin\omega t) + m_2],\end{aligned}$$

$$G_3 = q_1 p_1 + q_2 p_2 + q_3 p_3,$$
$$G_4 = q_4 p_4 + q_5 p_5 + q_6 p_6,$$

where $m_1 = -\left(\frac{\omega}{2\pi k}\right)^2$, $m_2 = -\left(\frac{\omega}{2\pi\mu k}\right)^2$, and

$$q_1 = -a_1(\alpha + \sin\alpha(1-\cos\alpha)) - 0.5a_2\sin 2\alpha,$$
$$q_2 = a_3\sin^2\alpha + a_1\cos\alpha(1-\cos\alpha) - a_5,$$
$$q_3 = -a_4\sin\alpha(1-\cos\alpha) - a_3(\alpha + 0.5\sin 2\alpha),$$
$$q_4 = a_6(-\omega t + 1 - \cos\omega t) - a_7\sin\omega t,$$
$$q_5 = -a_8\sin\omega t + a_6(1-\cos\omega t) - a_{10},$$
$$q_6 = -a_8(\omega t + \sin\omega t) + a_9(1-\cos\omega t),$$

$$
\begin{aligned}
p_1 &= -\frac{\alpha}{2} - \alpha\cos\alpha + \sin\alpha + \frac{\sin 2\alpha}{4},\\
p_2 &= \sin\alpha(\alpha - \sin\alpha),\\
p_3 &= -\frac{\alpha}{2} - \frac{\sin 2\alpha}{4} + \sin\alpha,\\
p_4 &= -\frac{\omega t}{2} - \omega t\cos\omega t + \sin\omega t + \frac{\sin 2\omega t}{4},\\
p_5 &= \sin\omega t(\omega t - \sin\omega t),\\
p_6 &= -\frac{\omega t}{2} - \frac{\sin 2\omega t}{4} + \sin\omega t,\\
a_1 &= 1 + m_1^{-1}\sin^2\alpha,\\
a_2 &= 3 + m_1^{-1}\sin^2\alpha,\\
a_3 &= 1 + m_1^{-1}(1-\cos\alpha)^2,\\
a_4 &= 3 + m_1^{-1}(1-\cos\alpha)^2,\\
a_5 &= m_1^{-1}\alpha\sin\alpha(1-\cos\alpha),\\
a_6 &= 1 + m_2^{-1}\sin^2\omega t,\\
a_7 &= 3 + m_2^{-1}\sin^2\omega t,\\
a_8 &= 1 + m_2^{-1}(1-\cos\omega t)^2,\\
a_9 &= 3 + m_2^{-1}(1-\cos\omega t)^2,\\
a_{10} &= m_2^{-1}\omega t\sin\omega t(1-\cos\omega t).
\end{aligned}
$$

[1]L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, Science **261**, 578 (1993).
[2]B. F. Farrell, J. Atmos. Sci. **39**, 1663 (1982).
[3]B. F. Farrell, J. Atmos. Sci. **42**, 2718 (1985).
[4]B. F. Farrell and P. J. Ioannou, J. Atmos. Sci. **53**, 2025 (1996); B. F. Farrell and P. J. Ioannou, J. Atmos. Sci. **53**, 2041 (1996).
[5]P. J. Schmid and D. S. Henningson, *Stability and Transition in Shear Flows* (Springer, 2001).
[6]P. J. Schmid, Annu. Rev. Fluid Mech. **39**, 129 (2007).
[7]L. Landau, J. Phys. **10**, 25 (1946).
[8]J. J. Podesta, Phys. Plasmas **17**, 122101 (2010).
[9]V. Ratushnaya and R. Samtaney, Europhys. Lett. **108**, 55001 (2014).
[10]L. S. Soloviev, “Hydromagnetic stability of closed plasma configurations,” in *Reviews of Plasma Physics* edited by M. A. Leontovich (New York, Consultants Bureau, 1975; original Russian edition, 1972), Vol. 6.
[11]H. Grad and H. Rubin, “Hydromagnetic equilibria and force-free fields,” in *Proceedings of the Second United Nation International Conference on Peaceful Uses of Atomic Energy* (New York, Columbia University Press, 1959), p. 31.
[12]V. D. Shafranov, Sov. Phys. JETP **33**, 710 (1957).
[13]J. P. Goedbloed, R. Keppens, and S. Poedts, *Advanced Magnetohydrodynamics with Applications to laboratory and Astrophysical Plasmas* (Cambridge University Press, 2010).
[14]A. Vlasov, Sov. Phys. JETP **8**, 291 (1938).
[15]D. R. Nicholson, *Introduction to Plasma Theory* (John Wiley & Sons, New York, 1983).
[16]P. M. Bellan, *Fundamentals of Plasma Physics* (Cambridge University Press, 2006).
[17]H. Alfvén, *Cosmical Electrodynamics* (Oxford at the Clarendon Press, 1950).
[18]G. Hellwig, Z. Naturforsch. **10a**, 508 (1955).
[19]M. Kruskal, *The Gyration of a Charged Particle* (Project Matterhorn, AEC Research and Development Report, Princeton University, 1958).
[20]J. Berkowitz and C. S. Gardner, Commun. Pure Appl. Math. **12**, 501 (1959).
[21]H. C. Brinkman, Physica **25**, 1016 (1959).
[22]D. M. Cook, J. Math. Phys. **11**, 986 (1970).
[23]M. Moisan and J. Pelletier, *Physics of Collisionless Plasmas* (Springer, 2012).