Estimation and Control of Networked Distributed Parameter Systems: Application to Traffic Flow

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The management of large-scale transportation infrastructure is becoming a very complex task for the urban areas of this century which are covering bigger geographic spaces and facing the inclusion of connected and self-controlled vehicles. This new system paradigm can leverage many forms of sensing and interaction, including a high-scale mobile sensing approach. To obtain a high penetration sensing system on urban areas more practical and scalable platforms are needed, combined with estimation algorithms suitable to the computational capabilities of these platforms.

The purpose of this work was to develop a transportation framework that is able to handle different kinds of sensing data (e.g., connected vehicles, loop detectors) and optimize the traffic state on a defined traffic network. The framework estimates the traffic on road networks modeled by a family of Lighthill-Whitham-Richards equations. Based on an equivalent formulation of the problem using a Hamilton-Jacobi equation and using a semi-analytic formula, I will show that the model constraints resulting from the Hamilton-Jacobi equation are linear, albeit with unknown integer variables. This general framework solve exactly a variety of problems arising in transportation networks: traffic estimation, traffic control (including robust control), cybersecurity and sensor fault detection, or privacy analysis of users in probe-based traffic monitoring systems. This framework is very flexible, fast, and yields exact results.

The recent advances in sensors (GPS, inertial measurement units) and microprocessors enable the development low-cost dedicated devices for traffic sensing in cities,
which are highly scalable, providing a feasible solution to cover large urban areas. However, one of the main problems to address is the privacy of the users of the transportation system, the framework presented here is a viable option to guarantee the privacy of the users by design.
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<td>Extended Kalman Filtering</td>
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<td>HJ</td>
<td>Hamilton-Jacobi</td>
</tr>
<tr>
<td>LP</td>
<td>Linear Programming</td>
</tr>
<tr>
<td>LWR</td>
<td>Lighthill-Whitham-Richards</td>
</tr>
<tr>
<td>MILP</td>
<td>Mixed Integer Linear Programming</td>
</tr>
<tr>
<td>MIP</td>
<td>Mixed Integer Programming</td>
</tr>
<tr>
<td>PeMS</td>
<td>Performance Measurement System</td>
</tr>
<tr>
<td>UN</td>
<td>United Nations</td>
</tr>
<tr>
<td>VDS</td>
<td>Vehicle Detection Station</td>
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<tr>
<td>$A$</td>
<td>One-directional physical domain</td>
</tr>
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<td>$w$</td>
<td>Congested velocity of the road segment</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>Critical density of the road segment</td>
</tr>
<tr>
<td>$v_f$</td>
<td>Free flow velocity of the road segment</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Maximal density of the road segment</td>
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<td>$\chi$</td>
<td>Downstream position of the road segment</td>
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<td>$\xi$</td>
<td>Upstream position of the road segment</td>
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<td>$T$</td>
<td>Temporal uniform segment of the computational domain</td>
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Chapter 1

Introduction

1.1 Motivation of this work

According to United Nations (UN) predictions, by 2050, 70% of worldwide population will live on urban areas [1]. This phenomenon is changing the way cities are conceived and is creating challenges that need to be addressed. In the last century, traffic congestion has emerged as one major problem on every urban area of the world; its consequences are perceived on the daily city life, affecting the economy, health and psychology of its habitants. The causes of traffic congestion are, among others, the increasing demand on transportation and the suboptimal use of the available infrastructure. The proposed solutions should involve a combination of applied technology and society assimilation of the disrupting changes on transportation.

Transportation research is currently at a tipping point; the emergence of new transformative technologies and systems, such as vehicle connectivity, automation, shared-mobility, and advanced sensing is rapidly changing the individual mobility and accessibility. This will fundamentally transform how transportation planning and operations should be conducted to enable smart and connected communities. The transport systems can be highly beneficiated and become safer, more efficient and reliable. Nowadays, dynamic routing and traffic-dependent navigation services are available for users. Such applications need to estimate the present traffic situation and that of the near future at a forecasting horizon based on data that are available in real-time. Traffic state estimation for a road network refers to estimate all the
traffic variables (e.g. cars density, speed) of the network at an instant of time based of traffic measurements. This is, for a limited amount of traffic data the estimator obtains a complete view of the traffic scenario. On the other hand, traffic control consists in using the current traffic scenario obtained by the estimator to influence the future traffic state. The level of influence depends on the scheme used: signal control strategies and ramp metering are some examples. In order to have a real-time system responding automatically to the current traffic conditions both modules (estimation and control) are needed [2].

The status quo on traffic estimation and control presents some limitations and challenges; the low penetration of sensors to recollect traffic data, this is due, in part, to the reluctance of drivers to participate in a privacy invasive system and the cost inefficiency of the available sensors. Another limitation is the computational complexity implied to estimate and control the traffic state on the growing urban areas. As mentioned before, due to the emerging technologies, important changes will be observed in the transportation field. In the nearby future, the fleet of vehicles will be a heterogeneous combination: vehicles with no communication to infrastructure, communicated vehicles and autonomous vehicles. This means that new agents will appear in the transportation network of a city with different degrees of data gathering and response to control laws.

Due to this complexity, the traffic estimation and control has to be understood as a computational problem over a large-scale network. The classical premise of having a central entity where all data is available and the computation is done is no longer suitable to large networked systems where typically each agent in the network has access to its private local information and has a local view of the network. A framework suitable for a distributed computing approach will be the backbone of this dissertation.
1.2 Partial differential equations modeling distributed systems

Large scale infrastructure systems, such as ground transportation networks (which is the focus of this dissertation) or air transportation networks are distributed parameter systems: their state is described by a function of space and time. Among the mathematical approaches to model such systems using partial differential equations (PDEs) is the most common one. The PDEs provide a succinct way of representing physical phenomena in a mathematically compact manner, which integrates the distributed features of the systems of interest [3].

Among PDEs, one class stands out, conservation laws [4 5], which model phenomena in which a balance equation governs the physics (for example mass balance). For example, water channels can be modeled using the Saint-Venant PDE [6], obtained from the conservation of water mass and momentum. For the interested reader, examples of applications of such models can be found in [7 8 9 7]. For ground transportation networks, the most common model is the Lighthill-Whitham-Richards (LWR) PDE [10 11], which is based on the conservation of vehicles on the road. Another alternative to model traffic flow are second order models [12 13 14 15 16], which are non-scalar conservation laws. However, all these PDEs used to describe distributed parameter systems are not necessarily conservation laws. For example, in structural engineering, beam deformation can be modeled by the Euler-Bernoulli beam PDE [17], which is not a conservation law. In electrical engineering, the Telegraph equation [18], which is also not a conservation law, can be used to model wave propagation in telecommunication lines.

1.3 Estimation and control of partial differential equations

State estimation and control for PDE-based systems is a complex problem due to the distributed nature of the state; it is more complex than for their ordinary differen-
tial equation (ODE) based counterparts. Different methods have been explored by researchers to address this problem.

1.3.1 Filtering based methods

The available tools for estimating [19] and controlling [17] the states of an Ordinary Differential Equation (ODE) can be extended to large scale systems modeled by a PDE; one option are the variations of Kalman Filtering (KF), for example the Extended Kalman Filtering (EKF) [20], which is a modification of KF for nonlinear systems. EKF techniques have been applied to traffic flow estimation problems in [21, 20].

The EKF can sometimes has a poor performance for specific nonlinear systems for which Monte Carlo techniques appears as a possible method [22]. For instance, when the system dynamics exhibits nondifferentiability or nonsmoothness, EKF is known to have problems [23]. Monte Carlo methods involve estimating the current probable value of the state, computing the state evolution, and comparing it against new measurement data to obtain a current estimate. By their nature, Monte-Carlo based methods can apply to any model, albeit with some computational cost penalty. Ensemble Kalman Filtering (EnKF) [24] is a Monte-Carlo based method that can be used for systems modeled by nonlinear PDEs, for instance the LWR [23] PDE, without approximating the model around the current estimate as done in EKF. The EnKF samples the possible current states of the system according to a probability distribution, computes the evolution of these samples, and combine these evolutions with new measurements to obtain the best estimate of the state. An example of the operational implementation of the EnKF for traffic flow modelling is the well known Mobile Millennium system [25]. More generally, the state of distributed parameter systems can be estimated using Particle Filtering (PF), which can be used for general nonlinear systems, albeit with a higher computational cost [26].
1.3.2 Other methods

For some classes of nonlinear systems, backstepping control methods [17] can be applied. They involve designing a controller for a known-stable system and “back out” new controllers that progressively stabilize each outer super-system.

Lyapunov methods [27] are based on the extension of the Lyapunov theory for ODE-based systems to the PDE case. They involve the use of a Lyapunov function associated with the state of the system, and which is either bounded or decreasing (similarly to ODE-based systems).

Machine learning methods can be implemented as a different approach [28], which rely on experimental datasets to learn how the state evolves. One of the main focuses of machine learning methods is to automatically learn to recognize specific patterns using statistical methods [29]. Machine learning methods can be applied to very different problems, including estimation problems [30] on systems modeled by PDEs.

One of the major difficulties arising when dealing with sensing problems on systems modeled by PDEs is the integration of the model constraints into the estimation problem. The PDEs investigated in this dissertation are nonlinear. Their solutions can be nonsmooth and even discontinuous, which makes the model constraints difficult to derive. This dissertation is based on previous work [31, 32] where the model constraints are expressed as convex inequalities which are both computationally tractable and explicit.

1.4 Hamilton-Jacobi equations

In one dimensional systems (for example to model a highway section), hyperbolic scalar conservation laws have a direct counterpart in Hamilton-Jacobi (HJ) theory [3], which is the main topic of this dissertation. HJ PDEs [33] were originally derived for variational problems and have a particular importance in optimal control.

The solutions to a given HJ PDE satisfy the HJ PDE in a generalized sense, and
are thus called weak solutions. Several classes of weak solutions to HJ PDEs exist. Historically, viscosity solutions \[34,35\] were the first class of weak solutions identified for HJ PDEs. Viscosity solutions are continuous, but not necessarily differentiable everywhere. Barron-Jensen/Frankowska (BJ-F) solutions \[36,37\] generalize the concept of viscosity solutions by allowing the solution to be discontinuous. A third concept of solutions is sometimes used, so called “nonsmooth solutions”, based on nonsmooth analysis \[38\].

HJ PDEs also integrated the framework of differential games \[39,40,41\], which model problems containing two actors (or players), a pursuer and an evader, with conflicting goals. They can be used to solve aircraft safety problems \[42\] by computing the set in which an evader aircraft is always safe from a pursuer aircraft attempting to collide with it.

The solutions used in the present work are obtained using a Lax-Hopf \[43\] formula, which expresses the solution at any given point as a minimization (or maximization) problem. The Lax-Hopf formula will be described in the next chapter.

1.5 Numerical analysis for Hamilton-Jacobi equations

The solutions to HJ PDEs (and their conservation laws counterparts e.g. LWR PDE) can be computed numerically using various methods, relying either on the structure of the PDE (finite difference schemes), the structure of their solutions (wave-front tracking methods), a different expression of the problem (level set methods), or the Lax-Hopf formula (dynamic programming, Lax-Hopf algorithm). The most basic numerical schemes that can be thought of are finite difference schemes, such as the Godunov scheme \[44\], or the Lax-Friedrichs method \[45\]. Finite difference methods require the approximation of the PDE as a finite difference equation on a computational grid. The finite difference equation is then solved numerically. Finite difference schemes compute approximate solutions, and are often subject to stability conditions,
such as the Courant-Friedrichs-Levy (CFL) condition, which constrains the computational grid [4].

Level set methods [46, 47] rely on finite difference schemes to numerically approximate the solution with subgrid accuracy and avoid their high cost of grid refinement. They can be extended in some cases by fast marching methods [48], which are computationally efficient (but have specific restrictions in their possible applications).

Wave-front tracking methods [5, 49] in contrast rely on the structure of the mathematical solutions to hyperbolic conservation laws, which feature shockwaves and expansion waves. Wave-front tracking methods are event-based numerical methods that compute the location of these waves, and thus derive the expression of the solution everywhere because of its structure.

Finally, the Lax-Hopf formula used in the present work can be solved numerically to compute the solution as a minimization problem. Possible solution methods include dynamic programming [3, 50] or the Lax-Hopf algorithm derived in this dissertation, adapted from [51].

1.6 Dissertation Outline

On this dissertation, the most important results of my PhD work will be presented. In chapter 2 the Lax Hopf method, backbone of the framework, will be described. In order to do this, section 2.1 introduces the HJ PDE investigated on this dissertation. Section 2.2 presents the concept of value condition which includes the traditional concept of initial, boundary and the new concept of internal conditions. In section 2.3 a method for solving the HJ PDE is shown, it is based on the control framework of viability theory to enable the definition of a Lax-Hopf formula which characterizes the solution. In section 2.4 the properties of the solution derived from the Lax-Hopf formula are discussed, focusing mainly on the inf-morphism property, which allows to decompose the problem involving initial, boundary and internal conditions in to
a more tractable subproblems. Then, in section 2.5 the capability of solving this tractable subproblems in an exact and explicit form is shown.

The derivation of the model constraints as convex inequalities on systems modeled by HJ PDEs is presented on chapter 3. The explicit solutions are derived on section 3.1 using a triangular fundamental diagram. The Mixed Integer Linear Program (MILP) estimation framework is described in section 3.2 an example of an estimation problem on a single road is shown in section 3.3.

The applications of this framework on a single road are presented in chapter 4. In section 4.1 cybersecurity and privacy applications are shown. A traffic control problem is detailed in section 4.2. An extension of the LWR model is presented on section 4.3 in which the inclusion of the bounded acceleration phase is described, also in this section the new model constratins are detailed. A queue estimation application using the bounded acceleration framework is shown in section 4.4.

The framework extension to highway networks is shown in chapter 5. The junction model used for the framework is described in section 5.1 a highway network estimation example is also described on this section. The control implementation on networks is detailed in section 5.2. Using the bounded acceleration toolbox and the junction model, a queue estimation on networks application was created, the description is shown is section 5.3. All the examples shown in this dissertation were done using traffic data from the Performance Measurement Systems (PeMS) and the Mobile Century experiment in California. The framework presented on this dissertation has been used by researchers in the field and is available to be downloaded as per request to the author of this dissertation.

1.7 Main Contributions

The contributions presented on this dissertation are:

• The development of a general framework that leverages Hamilton Jacobi equa-
tions to solve exactly a variety of problems arising in transportation networks: traffic estimation, traffic control (including robust control), cybersecurity and sensor fault detection, or privacy analysis of users in probe-based traffic monitoring systems. This framework is very flexible, fast, and yields exact results [52].

- The extension of the framework to estimation of traffic networks. The main contribution of this work is the junction model, based on a supply-demand approach the flows in the junction are maximized at all time, which is a realistic assumption on traffic modeling [53].

- One of the disadvantages of the LWR model is the unrealistic infinite acceleration that vehicles are modeled with; in order to address this, the inclusion of a bounded acceleration phase was done, the results of this work were validated with some applications.
Chapter 2

Exact and semi-analytic solution for scalar Hamilton-Jacobi partial differential equations

This chapter is an introduction to the Hamilton Jacobi solution that will be used in the rest of the dissertation, this chapter has been previously published in [54], the interested reader can consult the reference for a more detailed explanation of the solution construction, which is not the scope of this dissertation. The solution structure was derived using the control framework of Viability Theory. The interested reader can consult [55, 32] for more information about Viability Theory.

2.1 Macroscopic traffic model

2.1.1 State of the art

Depending on the aggregation level (the way they describe the reality) traffic flow models can be classified into the following classes: microscopic, macroscopic and mesoscopic models. The microscopic models, such as the car following model [56] and most cellular automata, describe traffic at the individual vehicle level as a flow of particles. These models provide a relationship between the velocity of a given vehicle and its environment, so they involve the reaction of the driver to the surrounding traffic conditions. In contrast, macroscopic models [57, 10, 11] describe traffic flow using an analogy with fluid or gases in motion, that is why they are sometimes called hydrodynamic models. This dissertation is focused on the Lighthill-Whitham-Richards (LWR) model [10, 11], which is a first order macroscopic flow model. This model is
commonly used in transportation engineering \cite{58, 50, 20, 23} due to its simplicity and its robustness. It is important to mention that macroscopic models are not necessarily first order ones, see for example \cite{15}. There is an intermediate class on which traffic flow can also be described: using mesoscopic models \cite{59}. Mesoscopic models are an intermediate class: it combines microscopic and macroscopic approaches to a hybrid one. These models follow methods of statistical mechanics, and express the solution using an integro-differential equation such as the Boltzmann equation \cite{60}.

Similarly to other large scale infrastructure systems (i.e. water channel network) the highway transportation network is a very complex graph containing highway sections connected by junctions or splits. In this dissertation, we will consider network estimation and control, describing the traffic flow on more than one highway section connected through a junction. Extending the framework to the whole transportation network \cite{61, 62} requires the computation of the boundary conditions of each highway section.

2.1.2 First order scalar conservation laws

We can define the physical (and computational) domain as the one-dimensional set

\[ A := [\xi, \chi] \subset \mathbb{R}, \]

where \( \xi \) represents the upstream boundary and \( \chi \) represents the downstream boundary of the domain. The upstream and downstream boundaries can be understood as the locations at which traffic enters and exits the road section respectively.

Two macroscopic functions are used to describe the state of traffic flow on the highway section: the density function and flow function, defined as follows. The density \( \rho(t, x) \) corresponds to the number of vehicles per unit distance at location \( x \) and time \( t \). The flow \( q(t, x) \) is defined as the number of vehicles that cross the point \( x \) per unit time, at time \( t \). Both functions are related by a conservation equation
expressing the fact that vehicles do not appear or disappear inside the highway section:

\[ \frac{\partial \rho(t, x)}{\partial t} + \frac{\partial q(t, x)}{\partial x} = 0 \] (2.1)

Equation (2.1) alone cannot be solved since it involves two different functions. In order to compute the evolution of \( \rho(\cdot, \cdot) \) and \( q(\cdot, \cdot) \), one needs an additional equation relating these two functions. Greenshields [57] was a pioneer in identifying a direct relationship between density and flow of the form \( q(\cdot, \cdot) = \psi(\rho(\cdot, \cdot)) \), where \( \psi(\cdot) \) is a function identified since as Fundamental Diagram [63]. The fundamental diagram translates the fact that drivers adapt their speed to the density of vehicles that surround them. Adding this relationship into equation (2.1) yields a first order scalar conservation law involving the density function, known as Lighthill-Whitham-Richards [10, 11] PDE:

\[ \frac{\partial \rho(t, x)}{\partial t} + \frac{\partial \psi(\rho(t, x))}{\partial x} = 0 \] (2.2)

### 2.1.3 Hamilton Jacobi equations with concave Hamiltonians

Instead of describing traffic flow in terms of a density function [4, 64], a possible alternate formulation is known as the Moskowitz function, which uses a Hamilton-Jacobi equation for describing the evolution of an integral of the function \( \rho(\cdot, \cdot) \) [65, 51, 32, 66]. The physical interpretation of the Moskowitz function is defined as follows.

**Definition 1** [Moskowitz function] Let consecutive integer labels be assigned to vehicles entering the highway at location \( x = \xi \). The Moskowitz function \( \mathbf{M}(\cdot, \cdot) \) is a continuous function satisfying \( \lceil \mathbf{M}(t, x) \rceil = n \) where \( n \) is the label of the vehicle located in \( x \) at time \( t \) [60, 67, 68]. Hence, \( \mathbf{M}(t, x) \) represents the label of the vehicle located at \( x \) at time \( t \), counted from the reference point \((0, \xi)\) corresponding to the vehicle numbered 0.
The properties of the Moskowitz function have been extensively studied, the interested reader can consult the famous Newell trilogy [69]. The formal link between the density function $\rho(\cdot, \cdot)$, the flow function $q(\cdot, \cdot)$ and the Moskowitz function $M(\cdot, \cdot)$ is given by:

$$M(t_2, x_2) - M(t_1, x_1) = \int_{x_1}^{x_2} -\rho(t_1, x)dx + \int_{t_1}^{t_2} q(t, x_2)dt$$  \hspace{1cm} (2.3)$$

Conversely, the flow and density functions $q(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$ are related to the spatial and temporal derivatives of the Moskowitz function $M(\cdot, \cdot)$:

$$q(t, x) = \frac{\partial M(t,x)}{\partial t} \hspace{1cm} \rho(t, x) = -\frac{\partial M(t,x)}{\partial x}$$  \hspace{1cm} (2.4)$$

The Moskowitz function $M(\cdot, \cdot)$ solves the following equation, obtained by combining (2.4) and the LWR PDE (2.2):

$$\frac{\partial M(t, x)}{\partial t} - \psi \left( -\frac{\partial M(t, x)}{\partial x} \right) = 0$$  \hspace{1cm} (2.5)$$

Equation (2.5) is an Hamilton-Jacobi (HJ) PDE [35, 51]. In the context of HJ PDEs, the parameter $\psi(\cdot)$ is known as Hamiltonian, while it is known as fundamental diagram in the context of the LWR PDE (2.2) and traffic engineering [70].

### 2.1.4 Hamiltonian

The LWR PDE (2.2) and its associated HJ PDE (2.5) are both characterized by a Hamiltonian $\psi(\cdot)$, which describes the relationship between density and flow. For low densities, the average velocity of traffic $v(\cdot, \cdot) = \frac{q(\cdot, \cdot)}{\rho(\cdot, \cdot)}$ is close to maximal velocity allowed on the road section, denoted by $\nu^\flat$. As the density increases, traffic velocity progressively drops and vanishes for the maximal density $\omega$ that the highway section can contain and known as jam density. Hence, the Hamiltonian $\psi(\cdot)$ satisfies the following properties:
\[ \lim_{\rho \to 0} \frac{\psi(\rho)}{\rho} = \nu^b \]

- the function \( \rho \to \frac{\psi(\rho)}{\rho} \) is decreasing
- \( \psi(\omega) = 0 \)

An example of flow-density plot using experimental data from the Performance Measurement System (PeMS) [71] is shown in Figure 2.1.

![Flow-density relationship example](image)

Figure 2.1: Flow-density relationship example

In this figure the horizontal axis represents the density of vehicles and the vertical axis corresponds to the flow of vehicles. Each point of this plot represents a simultaneous measurement of flow and density at a fixed location using an inductive loop detector [71].

For mathematical reasons, the Hamiltonian is often assumed to be either concave or convex [35, 51] in the HJ PDE theory, though this requirement is not dictated by the physics of the problem. In this dissertation, we assume once and for all that the Hamiltonian is a concave and upper semicontinuous function defined on \([0, \omega]\), where \(\omega\) is called jam density and that \(\psi(0) = \psi(\omega) = 0\). We also assume that \(\psi(\cdot)\) satisfies \(\psi'(0) = \nu^b\) and \(\psi'(\omega) = -\nu^s\), where \(\nu^b > 0\) and \(\nu^s > 0\), which implicitly assumes that \(\psi(\cdot)\) is differentiable at 0 and \(\omega\). However, we do not assume that \(\psi(\cdot)\) is differentiable on \([0, \omega]\) and construct our analysis for this general set of concave \(\psi(\cdot)\) functions.
Different choices of Hamiltonians satisfying these properties are possible, including the two examples presented below.

Example 1 [Greenshields Hamiltonian] [57 72]. Historically, one of the first Hamiltonian identified in the context of traffic-flow modeling is the Greenshields [57], defined by:

$$\forall \rho \in \mathbb{R}, \quad \psi(\rho) := \frac{\nu}{\omega} \rho (\omega - \rho)$$

where \(\omega\) and \(\nu\) are model parameters, respectively referred to as jam density and free flow velocity in the transportation literature. The Greenshields Hamiltonian depends only on two parameters, which makes it compact and easy to calibrate. The Greenshields Hamiltonian is however not used very often in practice, since it predicts unrealistically high maximal flows.

Another example of Hamiltonian is the Trapezoidal one, widely used in traffic flow modeling [70].

Example 2 [Trapezoidal Hamiltonian] [70 73 74]. The trapezoidal Hamiltonian is mostly used to model the hybrid nature of traffic flow propagation:

$$\psi(\rho) = \begin{cases} 
\nu^\flat \rho & \text{if } \rho \leq \gamma^\flat \\
\delta & \text{if } \rho \in [\gamma^\flat, \gamma^\sharp] \\
\nu^\sharp (\omega - \rho) & \text{if } \rho \geq \gamma^\sharp 
\end{cases}$$

where \(\nu^\flat, \nu^\sharp, \omega, \delta, \gamma^\flat\) and \(\gamma^\sharp\) are constants and satisfy the following relations: \(\delta \leq \frac{\omega \nu^\flat \nu^\sharp}{\nu^\flat + \nu^\sharp}\) (called capacity in the transportation engineering literature), \(\gamma^\flat := \frac{\delta}{\nu^\flat}\) (called lower critical density in the transportation engineering literature) and \(\gamma^\sharp := \frac{\nu^\flat \omega - \delta}{\nu^\flat}\) (called upper critical density in the transportation engineering literature). When \(\gamma^\flat = \gamma^\sharp\), the Hamiltonian is triangular, as used in the applications of Chapter 3.

The Greenshields and trapezoidal Hamiltonians can be observed in Figure 2.2.
Figure 2.2: Greenshields and trapezoidal Hamiltonians.
In the context of transportation some numerical values are defined, such as the variable $\rho$, which is homogeneous to the vehicle density. The Hamiltonian $\psi(\rho)$ unit is vehicles per hour. Left: Greenshields Hamiltonian example. Right: trapezoidal Hamiltonian example.

Solving the HJ PDE (2.5) requires the definition of value conditions, which we now define.

2.2 Value conditions

2.2.1 General definition

Value conditions encompass the traditional concepts of initial, boundary and internal conditions and are defined as follows.

Definition 2 [Value condition] A value condition $c(\cdot, \cdot)$ is a lower semicontinuous function defined on a subset of $[0, t_{\text{max}}] \times A$.

By convention, a value condition $c(\cdot, \cdot)$ as defined in definition 2 satisfies $c(t, x) = +\infty$ if $(t, x) \notin \text{Dom}(c)$. The domain of definition of a value condition represents the subset of the space-time domain $\mathbb{R}_+ \times A$ in which we want the value condition to apply. Different types of value condition exist, including the traditional initial, upstream and downstream boundary conditions [51, 70]. More complex value conditions do exist however. Internal conditions consist in value condition whose domains of definition
are connected and of empty interior. Hybrid conditions are the most general type of value condition, but are out of the scope of this dissertation.

### 2.2.2 Initial, boundary and internal conditions

The initial and boundary conditions are common in problems involving PDEs. Internal conditions are specific to the problem introduced in this thesis, though it applies to numerous other fields. These value conditions are defined as follows.

**Definition 3** [Initial condition] An initial condition is a value condition \( c(\cdot, \cdot) \) defined on \( \text{Dom}(c) := \{0\} \times A \).

Note that the traditional Cauchy problem consists in finding the solution to (2.5) associated with a value condition defined on \( \{0\} \times \mathbb{R} \), i.e. an initial condition defined on an infinite spatial domain.

In contrast, the upstream and downstream boundary conditions are related to the value of the state on the boundaries of the physical domain.

**Definition 4** [Upstream and downstream boundary conditions] An upstream boundary condition is a value condition \( c(\cdot, \cdot) \) defined on the set \( \text{Dom}(c) := [0, t_{\text{max}}] \times \{\xi\} \). A downstream boundary condition is a value condition \( c(\cdot, \cdot) \) defined on \( \text{Dom}(c) := [0, t_{\text{max}}] \times \{\chi\} \).

Note that the traditional mixed Initial-Boundary conditions problem consists in finding the solution to (2.5) associated with a value condition defined on \( \{0\} \times A \cup \mathbb{R}_+ \times \{\xi\} \cup \mathbb{R}_+ \times \{\chi\} \), i.e. an initial condition, an upstream boundary condition and a downstream boundary condition defined on an infinite temporal domain.

Note that the initial, upstream and downstream boundary conditions are all defined at the boundary of the computational domain \( [0, t_{\text{max}}] \times A \). Since probe measurements originate from the interior of the computational domain, a specific type of value condition, known as internal condition has to be defined as follows.
Definition 5 [Internal condition] An internal condition is a value condition \( c(\cdot, \cdot) \) defined on a domain of the form \( \text{Dom}(c) := \{(t, x_v(t)), \ t \in \text{Dom}(x_v)\} \), where \( x_v(\cdot) \) is a function of \([0, t_{\text{max}}]\).

In definition 5, the function \( x_v(\cdot) \) represents the velocity function associated with the internal condition. The set \( \{(t, x_v(t)), \ t \in \text{Dom}(x_v)\} \) is the trajectory associated with the internal condition.

Note that in the applications of this dissertation, measurement data alone is not sufficient to define the value conditions unambiguously, since some of coefficients used to build these value conditions are impossible to measure, or are not perfectly known due to measurement errors.

We now present a characterization of the solutions to the HJ PDE (2.5) associated with the value conditions defined earlier. This characterization uses Viability theory, an area of optimal control studying the evolution of dynamical systems evolving under state constraints [55, 76] known as viability constraints.

2.3 Formulation of the solution

2.3.1 Barron-Jensen/Frankowska solutions

In order to characterize the B-J/F solutions, we first need to define the Legendre-Fenchel transform of the Hamiltonian \( \psi(\cdot) \) as follows.

Definition 6 [Convex transform] Given a concave and upper semicontinuous function \( \psi(\cdot) \) with domain \( \text{Dom}(\psi) \), we define the convex transform \( \varphi^*(\cdot) \) of \( \psi(\cdot) \) as follows:

\[
\varphi^*(u) := \sup_{p \in \text{Dom}(\psi)} [p \cdot u + \psi(p)]
\]  

(2.7)

The inverse transform of a convex and lower semicontinuous function \( \varphi^*(\cdot) \) is
defined \cite{51} by:

\[ \psi(p) := \inf_{u \in \text{Dom}(\varphi^*)} [\varphi^*(u) - p \cdot u] \]  

(2.8)

Note that equation (2.7) in definition 6 differs from the traditional definition of the Legendre-Fenchel transform by a sign change.

As mentioned earlier, several classes of solutions to HJ PDEs exist. Viscosity solutions \cite{35} to HJ PDEs are continuous functions. The specific type of solutions to (2.5) that we consider in the present work is the Barron-Jensen/Frankowska (B-J/F) solutions \cite{36, 37}. B-J/F solutions extend the concept of viscosity solutions by allowing the solution to be lower semicontinuous. Note that both concepts are identical for mixed initial-boundary conditions problems involving Lipschitz-continuous initial and boundary conditions \cite{37}.

The B-J/F solutions to (2.5) can be derived using the control framework of Viability theory \cite{55}, the viability formulation of the Barron-Jensen/Frankowska solution is not the scope of this dissertation, the interested reader can consult \cite{32} for a detailed description.

**Theorem 1** [Barron-Jensen/Frankowska solution] \cite{51} For any lower semicontinuous value condition \( c_i \), the associated solution \( M_{c_i} \) is the unique lower semicontinuous function lower than \( c_i \) satisfying:

\[
\begin{cases}
(i) & \forall (t,x) \in \text{Dom}(M_{c_i}) \setminus \text{Dom}(c_i), \forall (p_t, p_x) \in d_- M_{c_i}(t,x), \quad p_t - \psi(-p_x) = 0 \\
(ii) & \forall (t,x) \in \text{Dom}(M_{c_i}) \setminus \text{Dom}(c_i), \forall (p_t, p_x) \in (\text{Dom}(D_+ M_{c_i}(t,x)))^+, \quad p_t - \sigma(\text{Dom}(\varphi^*), p_x) = 0
\end{cases}
\]

(2.9)

where the epiderivative \( D_+ \) is defined by its epigraph:

\[ \mathcal{E} p(D_+ M_{c_i}(t,x)) := T_{\mathcal{E} p(M_{c_i})}(t,x, M_{c_i}(t,x)) \]  

(2.10)

where in the formulae (2.10) and (2.9) \( T_{Z}(z) \) represents the contingent cone to \( Z \).
at $z$ (see [77]), $\sigma(\cdot, \cdot)$ is the support function (see [55, 77, 78]), the $+$ superscript denotes the normal cone (see [51]) and where the subdifferential $d_-$ of a function $u : X \to \mathbb{R} \cup \{+\infty\}$ is defined by $d_- u(x) = \{p \in X^* \mid \forall v \in X, \langle p, v \rangle \leq D^+ u(x)(v)\}$.

Theorem 1 ensures that $M_{c_i}$ is a solution to the HJ PDE (2.5) in the B-J/F sense. In particular, since $d_- M_{c_i}(t,x) = \{(\frac{\partial M_{c_i}(t,x)}{\partial t}, \frac{\partial M_{c_i}(t,x)}{\partial x})\}$ whenever $M_{c_i}(t,x)$ is differentiable, equation (2.9) implies the following property:

$$\forall (t,x) \in \text{Dom}(M_{c_i}) \setminus \text{Dom}(c_i) \text{ such that } M_{c_i} \text{ is differentiable, } \frac{\partial M_{c_i}(t,x)}{\partial t} - \psi \left(-\frac{\partial M_{c_i}(t,x)}{\partial x}\right) = 0$$

(2.11)

The construction of the B-J/F solution to (2.5) as a capture basin [32] enables the definition of a Lax-Hopf formula.

**Remark 1** As mentioned before the B-J/F solutions are lower semicontinuous, however, we need to enforce continuity to have a realistic solution. Continuity follows directly from the fact that we have density and flow functions; integrating a function always yields something continuous.

### 2.3.2 The Lax-Hopf formula

The viability episolution $M_c(\cdot, \cdot)$ associated with a general value condition $c(\cdot, \cdot)$ can be computed using the following generalized Lax-Hopf formula. The classical Lax-Hopf formulae can be found in [51] for initial and upstream boundary conditions.

**Theorem 2** [Generalized Lax-Hopf formula] The viability episolution $M_c(\cdot, \cdot)$ associated with a target $\mathcal{C} := \mathcal{E}p\tilde{\iota}(c)$, for a given lower semicontinuous function $c(\cdot, \cdot)$ can be expressed as:

$$M_c(t,x) = \inf_{(u,T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+} (c(t - T, x + Tu) + T \varphi^*(u))$$

(2.12)
Equation (2.12) implies the existence of a B-J/F solution $M_c(\cdot, \cdot)$ for any value condition function $c(\cdot, \cdot)$. However, the solution itself may be incompatible with the value condition that we imposed on it, i.e. we do not necessarily have $\forall (t, x) \in \text{Dom}(c), M_c(t, x) = c(t, x)$.

2.4 Properties of the Barron-Jensen/Frankowska solutions to Hamilton-Jacobi equations

2.4.1 Domain of definition

Proposition 1 [Domain of definition] For a given value condition $c(\cdot, \cdot)$, the domain of definition of $M_c(\cdot, \cdot)$, also called domain of influence of $c(\cdot, \cdot)$, is defined by the following formula:

$$\text{Dom}(M_c) = \bigcup_{(t, x) \in \text{Dom}(c)} \left( \bigcup_{T \in \mathbb{R}_+} \{t + T\} \times [x - \nu^\flat T, x + \nu^\sharp T] \right)$$

(2.13)

Proof — The generalized Lax-Hopf formula (2.12) implies that

$$\text{Dom}(M_c) = \{(t, x) \in \mathbb{R}_+ \times X \text{ such that } \exists (T, u) \in \mathbb{R}_+ \times \text{Dom}(\varphi^*) \text{ and } (t - T, x + Tu) \in \text{Dom}(c)\}$$

Equation (2.13) is derived from the previous formula, observing that $u$ ranges in $\text{Dom}(\varphi^*) := [-\nu^\flat, \nu^\sharp]$.

Remark 2 The domain of influence of $c(\cdot, \cdot)$ is the union of the cones originating at $(t, x) \in \text{Dom}(c)$ and limited by the minimal $-\nu^\flat$ and maximal $\nu^\sharp$ slopes of the Hamiltonian. This property is illustrated in Figure 2.3.
Figure 2.3: Domain of influence of a value condition. For this example a value condition $c(\cdot)$ is defined on a domain represented by two black segments at $t = 0$. The domain of influence of $c(\cdot, \cdot)$ is highlighted in blue.

### 2.4.2 The inf-morphism property

It is well known [55, 76, 51] that for a given environment $\mathcal{K}$, the capture basin of a finite union of targets is the union of the capture basins of these targets.

$\text{Capt}_F \left( \mathcal{K}, \bigcup_{i \in I} C_i \right) = \bigcup_{i \in I} \text{Capt}_F(\mathcal{K}, C_i)$  \hspace{1cm} (2.14)

This property can be translated in epigraphical form as follows.

**Proposition 2** [Inf-morphism property] [51] Let $c_i$ ($i$ belongs to a finite set $I$) be a family of functions whose epigraphs are the targets $C_i$. Since the epigraph of the minimum of the functions $c_i$ is the union of the epigraphs of the functions $c_i$, the target $C := \bigcup_{i \in I} C_i$ is the epigraph of the function $c := \min_{i \in I} c_i$. Hence, equation (2.14) implies the following property, known as inf-morphism property:

$\forall t \geq 0, \ x \in X, \ M_c(t, x) = \min_{i \in I} M_{c_i}(t, x)$  \hspace{1cm} (2.15)

**Remark 3** The inf-morphism property enables us to decompose a complex problem into more tractable subproblems. For instance, a piecewise affine initial condition can
be decomposed as the minimum of a finite number of affine initial conditions. Hence, the solution associated with a piecewise affine initial condition is the minimum of a finite number of solutions associated with affine initial conditions.
2.5 Analytic solutions associated with affine initial, boundary and internal conditions

In this section, we will show the solutions associated with affine initial, boundary and internal conditions analytically. The detailed computation of these solutions can be consulted in [32, 66].

2.5.1 Analytic Lax-Hopf formula associated with an affine initial condition

**Definition 7** [Affine initial condition] We consider the following affine initial condition \( M_{0,i}(0,x) \), where \( i \) is an integer:

\[
M_{0,i}(0,x) = \begin{cases} 
  a_i x + b_i & \text{if } x \in [\alpha_i, \alpha_{i+1}] \\
  +\infty & \text{otherwise}
\end{cases}
\]  

(2.16)

Let \( u_0(a_i) \) be an element of \( -\partial_+ \psi(-a_i) \neq \emptyset \). The following formula expresses the Lax-Hopf formula (2.12) for the specific initial condition (2.16).

\[
M_{M_{0,i}}(t,x) = \begin{cases} 
  t\psi(-a_i) + a_i x + b_i & \text{if } u_0(a_i) \in [\frac{\alpha_i-x}{t}, \frac{\alpha_{i+1}-x}{t}] \\
  a_i \alpha_i + b_i + t \varphi^* \left( \frac{\alpha_i-x}{t} \right) & \text{if } u_0(a_i) \leq \frac{\alpha_i-x}{t} \\
  a_i \alpha_{i+1} + b_i + t \varphi^* \left( \frac{\alpha_{i+1}-x}{t} \right) & \text{if } u_0(a_i) \geq \frac{\alpha_{i+1}-x}{t}
\end{cases}
\]  

(2.17)

The different domains of equation (2.17) for the solution associated with an affine initial condition (equation 2.16) are illustrated in figure 2.4.
Figure 2.4: Development of the solution associated with an affine initial condition. Left: Illustration of the construction of a $u_0(a_i)$ from the knowledge of $a_i$. The transform $\varphi^*(u_0(a_i))$ corresponds to the value intercepted on the vertical axis by the tangent line of slope $-u_0(a_i)$ to the graph of $\psi$ in $-a_i$. Right: The $(t, x)$ domain of the solution corresponding to the affine initial condition (2.16) can be separated in three different areas. The domain highlighted in light gray corresponds to the case (i) in equation (2.17). The domain highlighted in medium gray corresponds to the case (iii) and the remaining domain in dark gray corresponds to the case (ii). The domain of the initial condition is represented by a dashed line.

2.5.2 Analytic Lax-Hopf formula associated with an affine upstream boundary condition

Definition 8 [Affine upstream boundary condition] We consider the following upstream boundary condition $\gamma_j(t, \xi)$:

$$
\gamma_j(t, \xi) = \begin{cases} 
c_j t + d_j & \text{if } t \in [\overline{\gamma}_j, \overline{\gamma}_{j+1}] \\
+\infty & \text{otherwise}
\end{cases} \quad (2.18)
$$

The following formula expresses the Lax-Hopf formula (2.12) for the specific up-
stream condition \( (2.18) \).

\[
M_{\gamma_j}(t,x) = \begin{cases} 
(i) & t\psi(\rho_j) + \rho_j(\xi - x) + d_j & \text{if } T_0(\rho_j, x) \\
(ii) & \psi(\rho_j)\tau_j + d_j + (t - \tau_j)\varphi^*(\frac{\xi - x}{t - \tau_j}) & \text{if } t - \tau_j \leq T_0(\rho_j, x) \\
(iii) & \psi(\rho_j)\tau_{j+1} + d_j + (t - \tau_{j+1})\varphi^*(\frac{\xi - x}{t - \tau_{j+1}}) & \text{if } T_0(\rho_j, x) \leq t - \tau_{j+1}
\end{cases}
\]

The different domains of equation \( (2.19) \) for the solution associated with an affine upstream boundary condition (equation \( 2.18 \)) are illustrated in figure 2.5.

Figure 2.5: Development of the solution associated with an affine upstream boundary condition. 
Left: Illustration of the construction of a \( u_0(\rho_j) \) from a known \( c_j \). The transform \( \varphi^*(u_0(\rho_j)) \) corresponds to the value intercepted on the vertical axis by the tangent line of slope \( -u_0(\rho_j) \) to the graph of \( \psi \) in \( \rho_j \). Right: The \( (t, x) \) domain of the solution corresponding to the affine upstream boundary condition \( (2.18) \) can be separated in three different areas. The domain highlighted in light gray corresponds to the case \((i)\) in equation \( (2.19) \). The domain highlighted in dark gray corresponds to the case \((ii)\) and the remaining domain in medium gray corresponds to the case \((iii)\). The domain of the upstream boundary condition is represented by a dashed line.
2.5.3 Analytic Lax-Hopf formula associated with an affine downstream boundary condition

Definition 9 [Affine downstream boundary condition] We consider the following downstream boundary condition $\beta_k(t, \chi)$:

$$
\beta_k(t, x) = \begin{cases} 
e_k t + f_k & \text{if } t \in [\beta_k, \beta_{k+1}] \\ +\infty & \text{otherwise} \end{cases}
$$

(2.20)

The Lax-Hopf formula (2.12) associated with the downstream boundary condition (2.20) can be expressed as:

$$
M_{\beta_k}(t, x) = \begin{cases} 
(i) & t\psi(\rho_k) + \rho_k(\chi - x) + f_k & \text{if } T_0(\rho_k, x) \\
(ii) & \psi(\rho_k)\beta_k + f_k + (t - \beta_k)\varphi^*(\frac{\chi - x}{t - \beta_k}) & \text{if } t - \beta_k \leq T_0(\rho_k, x) \\
(iii) & \psi(\rho_k)\beta_{k+1} + f_k + (t - \beta_{k+1})\varphi^*(\frac{\chi - x}{t - \beta_{k+1}}) & \text{if } T_0(\rho_k, x) \leq t - \beta_{k+1} \end{cases}
$$

(2.21)

The different domains of equation (2.21) for the solution associated with an affine downstream boundary condition (equation 2.20) are illustrated in figure 2.6.
Figure 2.6: Development of the solution associated with an affine downstream boundary condition.

Left: Illustration of the construction of a \( u_0(\rho_k) \) from a known \( e_k \). The transform \( \varphi^* (u_0(\rho_k)) \) corresponds to the value intercepted on the vertical axis by the tangent line of slope \( -u_0(\rho_k) \) to the graph of \( \psi \) in \( \rho_k \). Right: The \((t, x)\) domain of the solution corresponding to the affine downstream boundary condition (2.20) can be separated in three different areas. The domain highlighted in light gray corresponds to the case (i) in equation (2.21). The domain highlighted in dark gray corresponds to the case (ii) and the remaining domain in medium gray corresponds to the case (iii). The domain of the downstream boundary condition is represented by a dashed line.

2.5.4 Analytic Lax-Hopf formula associated with an affine internal condition

Definition 10 [Affine internal condition] We consider the following affine internal condition \( \mu_l(\cdot, \cdot) \), where \( l \) is an integer:

\[
\mu_l(t, x) = \begin{cases} 
  g_l(t - \delta_l) + h_l & \text{if } x = x_l + v_l (t - \delta_l) \\
  +\infty & \text{otherwise}
\end{cases} \quad \text{and } t \in [\delta_l, \delta_{l+1}] 
\]

(2.22)

The following formula expresses the Lax-Hopf formula (2.12) for the specific in-
ternal condition (2.22).

\[ M^\mu_l(t, x) = \begin{cases} 
(i) & \psi(p_1(v_l, g_l))(t - \bar{\delta}_l) + (x_l - x)p_1(v_l, g_l) + h_l \\
& \text{if } x_l + v_l(t - \bar{\delta}_l) \leq x \\
& \text{and } T_1(t, x, v_l, g_l) \in [t - \bar{\delta}_{l+1}, t - \bar{\delta}_l] \\
(ii) & \psi(p_2(v_l, g_l))(t - \bar{\delta}_l) + (x_l - x)p_2(v_l, g_l) + h_l \\
& \text{if } x_l + v_l(t - \bar{\delta}_l) \geq x \\
& \text{and } T_2(t, x, v_l, g_l) \in [t - \bar{\delta}_{l+1}, t - \bar{\delta}_l] 
\end{cases} \tag{2.23} \]

and

\[ M_{\mu_l}(t, x) = \begin{cases} 
(iii) & h_l + (t - \bar{\delta}_l)^* \left( \frac{x_l - x}{t - \bar{\delta}_l} \right) \\
& \text{if } x_l + v_l(t - \bar{\delta}_l) \leq x \text{ and } T_1(t, x, v_l, g_l) \geq t - \bar{\delta}_l \\
& \text{or if } x_l + v_l(t - \bar{\delta}_l) \geq x \text{ and } T_2(t, x, v_l, g_l) \geq t - \bar{\delta}_l \\
(iv) & g_l(\bar{\delta}_{l+1} - \bar{\delta}_l) + h_l + (t - \bar{\delta}_{l+1})^* \left( \frac{x_l + v_l(\bar{\delta}_{l+1} - \bar{\delta}_l) - x}{t - \bar{\delta}_{l+1}} \right) \\
& \text{if } x_l + v_l(t - \bar{\delta}_l) \leq x \text{ and } T_1(t, x, v_l, g_l) \leq t - \bar{\delta}_{l+1} \\
& \text{or if } x_l + v_l(t - \bar{\delta}_l) \geq x \text{ and } T_2(t, x, v_l, g_l) \leq t - \bar{\delta}_{l+1} 
\end{cases} \tag{2.24} \]

The different domains of equations (2.23) and (2.24) for the solution associated with an affine internal condition (equation 2.22) are illustrated in figure 2.7.
Figure 2.7: Development of the solution associated with an affine internal condition. Left: Illustration of the construction of a \( u_1(v_l, g_l) \) and \( u_2(v_l, g_l) \) from known \( v_l \) and \( g_l \). Right: The \((t, x)\) domain of the solution corresponding to the affine internal condition (2.22) can be separated in four different areas. The domains highlighted in dark gray correspond to the case \((iii)\) in equation (2.24). The domains highlighted in light gray correspond to the cases \((i)\) and \((ii)\) in equation (2.23). The remaining domain in medium gray corresponds to the case \((iv)\) in (2.24). The domain of the internal condition is represented by a dashed line.
Chapter 3

Formulation of the Mixed Integer Linear Programming framework for transportation applications

In the present work, the value conditions $c_j(\cdot, \cdot)$ are not known exactly, either because of measurement uncertainty (case of the upstream and downstream boundary condition) or because of the lack of measurements (case of the initial condition). However, even if the real values of $c_j(\cdot, \cdot)$ are not known exactly, they cannot be arbitrary as they have to apply in the strong sense (see [64] for a mathematical definition) to be compatible with the LWR model. In the remainder of this dissertation, we define the model constraints as the set of constraints that applies on the value conditions $c_j(\cdot, \cdot)$ to ensure that all value conditions apply in the strong sense.

The inf-morphism property is critical for the derivation of the LWR PDE model constraints, allowing us to instantiate these constraints as inequalities.

Proposition 3 [Model compatibility constraints for block value conditions] Let $c(\cdot, \cdot) = \min_{j \in J} c_j(\cdot, \cdot)$ be given, and let $M_c(\cdot, \cdot)$ be defined as in (2.12). The value condition $c(\cdot, \cdot)$ satisfies $\forall (t, x) \in \text{Dom}(c), M_c(t, x) = c(t, x)$ if and only if the following inequality constraints are satisfied:

$$M_{c_j}(t, x) \geq c_i(t, x) \forall (t, x) \in \text{Dom}(c_i), \forall (i, j) \in J^2$$ (3.1)

The proof of this proposition is available in [31]. Note that equation (3.1) represents an important improvement, as the model constraints are now semi-explicit. In order to solve the problem completely, we still need to evaluate the functions $M_{c_j}(\cdot, \cdot)$
explicitly. These explicit solutions were derived in \cite{66} for affine initial, boundary and internal conditions blocks which are presented in the next section.

For the rest of this dissertation we assume that the Hamiltonian $\psi(\cdot)$ is a continuous triangular function, this will enable the model constraints to be not only explicit but linear, the triangular function is defined by:

\begin{equation}
\psi(\rho) = \begin{cases} 
v_f \rho & \text{if } \rho \leq \rho_c \\
w(\rho - \kappa) & \text{otherwise}
\end{cases}
\end{equation}

where $v_f$, $w$, $\rho_c$ and $\kappa$ are model parameters satisfying $v\rho_c = w(\rho_c - \kappa)$ and representing the free flow speed ($v_f$), the critical density ($\rho_c$), the congestion speed ($w$) and the maximal density ($\kappa$).

### 3.1 Explicit solutions to piecewise affine initial, internal and boundary conditions

#### 3.1.1 Setting

Consider the previously prescribed uni-directional stretch of arterial road $A := [\xi, \chi] \subset \mathbb{R}$ where $\xi$ and $\chi$ denote respectively the upstream and downstream boundaries of the link. Assume that $A$ has a constant number of lanes and no lateral in- or out-flows. The time domain is defined by $[0, t_f]$ with a given $0 < t_f < +\infty$.

Consider the following assumptions:

- The spatio-temporal domain is discretized into $k_{max}$ uniform spatial segments $X_i$ with $i \in [0, k_{max} - 1]$ and $n_{max}$ discrete time segments $T_j$ with $j \in [0, n_{max} - 1]$ such that

$$\xi := X_0 < X_1 < \cdots < X_{k_{max} - 1} := \chi \quad \text{and} \quad 0 := T_0 < T_1 < \cdots < T_{n_{max} - 1} := t_f.$$
Notice that we do not need to require the discrete time and spatial steps to be uniform.

- The initial, upstream/downstream and internal conditions denoted respectively by $M_k$, $\gamma$, $\beta$, $\mu$ and $\Upsilon$ are piecewise affine on the discrete space and time segments $X_i$ and $T_j$.

We define by $\rho_{\text{ini}}(k)_{1\leq k \leq k_{\text{max}}} \in \mathbb{R}^{k_{\text{max}}}_+$ the set of initial densities, by $q_{\text{in}}(n)_{1\leq n \leq n_{\text{max}}} \in \mathbb{R}^{n_{\text{max}}}_+$ the set of upstream flows, by $q_{\text{out}}(n)_{1\leq j \leq n_{\text{max}}} \in \mathbb{R}^{n_{\text{max}}}_+$ the set of downstream flows, by $(L(m), r(m))_{1\leq m \leq m_{\text{max}}} \in (\mathbb{R} \times \mathbb{R}^+_0)^{m_{\text{max}}}$ the set of internal flow conditions and by $(L(u), \rho(u))_{1\leq u \leq u_{\text{max}}} \in (\mathbb{R} \times \mathbb{R}^+_0)^{u_{\text{max}}}$ the set of internal density conditions. These values are constant but they are not known exactly. The objective of this work is to determine these constants thanks to an optimization problem that boils down to a Mixed Integer Linear Programming (MILP). This optimization problem incorporates model constraints that arise from the explicit solutions to the considered traffic flow model and data constraints coming from direct measurements. These measurements (with some error) are obtained using different sensors, such as inductive loop detectors, magnetometers or GPS devices.

### 3.1.2 Definition of affine initial, upstream/downstream boundary and internal density conditions

The formal definition of initial, upstream/downstream boundary and internal conditions associated with the HJ PDE (2.5) is the subject of the following definition.

**Definition 11** [Affine initial, upstream/downstream boundary and internal conditions]

Let us define $\mathbb{K} = \{0, \ldots, k_{\text{max}} - 1\}$, $\mathbb{N} = \{0, \ldots, n_{\text{max}} - 1\}$, $\mathbb{M} = \{0, \ldots, m_{\text{max}} - 1\}$ and $\mathbb{U} = \{0, \ldots, u_{\text{max}} - 1\}$. For all $k \in \mathbb{K}$, $n \in \mathbb{N}$, $m \in \mathbb{M}$ and $u \in \mathbb{U}$, we define the following functions, respectively called initial, upstream, downstream internal flow
and internal density conditions:

\[
M_k(t, x) = \begin{cases} 
- \sum_{i=0}^{k-1} \rho_{ini}(i) X \\
- \rho_{ini}(k)(x - kX) & \text{if } t = 0 \\
+\infty & \text{otherwise}
\end{cases}
\quad \text{and } x \in [kX, (k+1)X]
\]

(3.3)

\[
\gamma_n(t, x) = \begin{cases} 
\sum_{i=0}^{n-1} q_{in}(i)T \\
+q_{in}(n)(t - nT) & \text{if } x = \xi \\
+\infty & \text{otherwise}
\end{cases}
\quad \text{and } t \in [nT, (n+1)T]
\]

(3.4)

\[
\beta_n(t, x) = \begin{cases} 
\sum_{i=0}^{n-1} q_{out}(i)T \\
+q_{out}(n)(t - nT) \\
- \sum_{k=0}^{k_{\text{max}}} \rho(k)X & \text{if } x = \chi \\
+\infty & \text{otherwise}
\end{cases}
\quad \text{and } t \in [nT, (n+1)T]
\]

(3.5)

\[
\mu_m(t, x) = \begin{cases} 
L(m) + r(m)(t - t_{\text{min}}(m)) & \text{if } x = x_{\text{min}}(m) \\
+\nu_{\text{meas}}(m)(t - t_{\text{min}}(m)) & \text{and } t \in [t_{\text{min}}(m), t_{\text{max}}(m)] \\
+\infty & \text{otherwise}
\end{cases}
\]

(3.6)

\[
\Upsilon_u(t, x) = \begin{cases} 
L(u) - \rho(u)(x - x_{\text{min}, u}(u)) & \text{if } x \in [x_{\text{min}, u}(u), x_{\text{max}, u}(u)] \\
+\infty & \text{and } t = t_{\rho}(u)
\end{cases}
\quad \text{otherwise}
\]

(3.7)

where \(\nu_{\text{meas}}(m) = \frac{x_{\text{max}}(m) - x_{\text{min}}(m)}{t_{\text{max}}(m) - t_{\text{min}}(m)}\)

In the above definition, internal density conditions (3.7) are specific to model density sensors that are inside our computational domain. Regarding flow sensors also located inside the computational domain, they can be thought as an internal flow condition (3.6) associated with a zero velocity \(\left(\nu_{\text{meas}}(m) = 0\right)\). Note that the affine initial, upstream/downstream boundary and internal conditions defined above for the HJ PDE (2.5) are equivalent to constant initial, upstream/downstream boundary
and internal conditions for the LWR PDE \([2,2]\). The domains of definitions of these functions are illustrated in Figure [3.1].

![Domains of the initial, upstream/downstream boundary and internal conditions.](image)

3.1.3 Analytical solutions to affine initial, upstream/downstream boundary and internal conditions

Given the affine initial, upstream/downstream boundary and internal conditions defined above, the corresponding solutions \(M_{M_k}(\cdot, \cdot)\), \(M_{\gamma_n}(\cdot, \cdot)\), \(M_{\beta_n}(\cdot, \cdot)\) and \(M_{\Upsilon_u}(\cdot, \cdot)\) are given \([31, 79]\) by the following formulas:
\[ \begin{align*}
M_{M_k}(t,x) = \begin{cases} 
+\infty & \text{if } x \leq kX + wt \\
-\sum_{i=0}^{k-1} \rho_{\text{ini}}(i)X & \text{or } x \geq (k+1)X + v_f t \\
+\rho_{\text{ini}}(k)(tv_f + kX - x) & \text{if } kX + tv_f \leq x \\
& \quad \text{and } (k+1)X + tv_f \geq x \\
& \quad \text{and } \rho_{\text{ini}}(k) \leq \rho_c \\
-\sum_{i=0}^{k-1} \rho_{\text{ini}}(i)X & \text{if } kX + tv_f \geq x \\
+\rho_c(tv_f + kX - x) & \text{if } kX + tw \leq x \\
& \quad \text{and } kX + tw \leq x \\
& \quad \text{and } \rho_{\text{ini}}(k) \leq \rho_c \\
-\sum_{i=0}^{k} \rho_{\text{ini}}(i)X & \text{if } kX + tw \leq x \\
-\rho_m tw & \text{if } (k+1)X + tw \geq x \\
& \quad \text{and } (k+1)X + tw \leq x \\
& \quad \text{and } \rho_{\text{ini}}(k) \geq \rho_c \\
-\sum_{i=0}^{k} \rho_{\text{ini}}(i)X & \text{if } (k+1)X + tv_f \geq x \\
\rho_c(tw + (k+1)X - x) & \text{if } (k+1)X + tw \geq x \\
& \quad \text{and } (k+1)X + tw \leq x \\
& \quad \text{and } \rho_{\text{ini}}(k) \geq \rho_c \\
\end{cases}
\end{align*} \]

\[ \begin{align*}
M_{\gamma_n}(t,x) = \begin{cases} 
+\infty & \text{if } t \leq nT + \frac{x-\xi}{v_f} \\
\sum_{i=0}^{n-1} q_{\text{in}}(i)T & \text{if } nT + \frac{x-\xi}{v_f} \leq t \\
+q_{\text{in}}(n)(t - \frac{x-\xi}{v_f} - nT) & \text{and } t \leq (n+1)T + \frac{x-\xi}{v_f} \\
\sum_{i=0}^{n} q_{\text{in}}(i)T & \text{if } (n+1)T + \frac{x-\xi}{v_f} \leq t \\
+\rho_c v_f(t - (n+1)T - \frac{x-\xi}{v_f}) & \text{otherwise}
\end{cases}
\end{align*} \]
\[
M_{\beta_n}(t, x) = \begin{cases} 
+\infty & \text{if } t \leq nT + \frac{x-\chi}{w} \\
- \sum_{k=0}^{k_{\text{max}}} \rho_{\text{ini}}(k)X + \sum_{i=0}^{n-1} q_{\text{out}}(i)T \\
+ q_{\text{out}}(n)(t - \frac{x-\chi}{w} - nT) \\
- \kappa(x - \chi) & \text{if } nT + \frac{x-\chi}{w} \leq t \\
\text{and } t \leq (n+1)T + \frac{x-\chi}{w} \\
- \sum_{k=0}^{k_{\text{max}}} \rho(k)X + \sum_{i=0}^{n} q_{\text{out}}(i)T \\
+ \rho_c v_f (t - (n+1)T - \frac{x-\chi}{v_f}) & \text{otherwise} 
\end{cases} 
\] (3.10)

\[
M_{\mu_m}(t, x) = \begin{cases} 
L_m + \\
\frac{r_m(t - x - x_{\text{min}}(m) - v_{\text{max}}(m)(t - t_{\text{min}}(m)) - t_{\text{min}}(m))}{v_f - v_{\text{max}}(m)} & \text{if } x \geq x_{\text{min}}(m) + v_{\text{max}}(m)(t - t_{\text{min}}(m)) \\
\text{and } x \geq x_{\text{max}}(m) + v_f(t - t_{\text{max}}(m)) \\
\text{and } x \leq x_{\text{min}}(m) + v_f(t - t_{\text{min}}(m)) \\
L_m + \\
\frac{r_m(t - x - x_{\text{min}}(m) - v_{\text{max}}(m)(t - t_{\text{min}}(m)) - t_{\text{min}}(m))}{v_f - v_{\text{max}}(m)} \\
+ k_c(v_f - w) \frac{x - x_{\text{min}}(m) - v_{\text{max}}(m)(t - t_{\text{min}}(m))}{w - v_{\text{max}}(m)} & \text{if } x \leq x_{\text{min}}(m) + v_{\text{max}}(m)(t - t_{\text{min}}(m)) \\
\text{and } x \leq x_{\text{max}}(m) + w(t - t_{\text{max}}(m)) \\
\text{and } x \geq x_{\text{min}}(m) + w(t - t_{\text{min}}(m)) \\
L_m + r_m(t_{\text{max}}(m) - t_{\text{min}}(m)) + \\
(t - t_{\text{max}}(m)) k_c \left(v_f - \frac{x - x_{\text{max}}(m)}{t - t_{\text{max}}(m)}\right) & \text{if } x \leq x_{\text{max}}(m) + v_f(t - t_{\text{max}}(m)) \\
\text{and } x \geq x_{\text{max}}(m) + w(t - t_{\text{max}}(m)) \\
+ \infty & \text{otherwise} 
\end{cases} 
\] (3.11)
\[
M_{\Psi_{p}}(t, x) = \begin{cases} 
+\infty & \text{if } x \leq x_{\min_p}(u) + w(t - t_{p}(u)) \\
\quad \quad \text{or } x \geq x_{\max_p}(u) + v_{f}(t - t_{p}(u)) \\
L(u) & \text{or } t \leq t_{p}(u)
\end{cases}
\]

\[
+\rho(u)(v_{f}(t - t_{p}(u)) + x_{\min_p}(u) - x) \quad \text{if } x_{\min_p}(u) + v_{f}(t - t_{p}(u)) \leq x
\]

\[
\quad \quad \text{and } x_{\max_p}(u) + v_{f}(t - t_{p}(u)) \geq x
\]

\[
\quad \quad \text{and } \rho(u) \leq \rho_{c}
\]

\[
L(u)
\]

\[
+\rho_{c}(v_{f}(t - t_{p}(u)) + x_{\min_p}(u) - x) \quad \text{if } x_{\min_p}(u) + w(t - t_{p}(u)) \leq x
\]

\[
\quad \quad \text{and } x_{\max_p}(u) + w(t - t_{p}(u)) \geq x
\]

\[
\quad \quad \text{and } \rho(u) \leq \rho_{c}
\]

\[
L(u)
\]

\[
+\rho(u)(w(t - t_{p}(u)) + x_{\min_p}(u) - x)
\]

\[
-\rho_{m}w(t - t_{p}(u)) \quad \text{if } x_{\min_p}(u) + w(t - t_{p}(u)) \leq x
\]

\[
\quad \quad \text{and } x_{\max_p}(u) + w(t - t_{p}(u)) \geq x
\]

\[
\quad \quad \text{and } \rho(u) \geq \rho_{c}
\]

\[
L(u)
\]

\[
+\rho_{c}(w(t - t_{p}(u)) + x_{\max_p} - x)
\]

\[
-\rho_{m}w(t - t_{p}(u)) \quad \text{if } x_{\max_p}(u) + w(t - t_{p}(u)) \leq x
\]

\[
\quad \quad \text{and } x_{\max_p}(u) + v_{f}(t - t_{p}(u)) \geq x
\]

\[
\quad \quad \text{and } \rho(u) \geq \rho_{c}
\]

3.1.4 Properties of the affine initial, upstream/downstream boundary and internal conditions

In this section, we show that the LWR model constraints (3.1) are convex in the variable \(\rho_{c}(1, \ldots, \rho(k_{\max}), q_{\in}(1, \ldots, q_{\in}(n_{\max}), q_{\out}(1, \ldots, q_{\out}(n_{\max}))\).
Proposition 4 [Linearity property of the initial, upstream/downstream boundary and internal conditions] Let us fix \((t, x) \in \mathbb{R}_+ \times [\xi, \chi]\). The initial, upstream and downstream boundary condition functions \(M_k(t, x), \gamma_n(t, x)\) and \(\beta_n(t, x)\) are linear functions of the coefficients \((\rho(1), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}))\).

The proof of this proposition is straightforward and follows directly from equations (3.3), (3.7) and (3.5).

Proposition 5 [Concavity property of the solution associated with the initial condition] Let us fix \((t, x) \in \mathbb{R}_+ \times [\xi, \chi]\). The solution \(M_{M_k}(t, x)\) associated with the initial condition (3.3) is a concave function of the coefficients \(\rho(\cdot)\).

Proof — The Lax-Hopf formula (2.12) associated with the solution \(M_{M_k}(\cdot, \cdot)\) can be written as [32, 66]:

\[
M_{M_k}(t, x) = \inf_{u \in \text{Dom}(\varphi^*) \text{ s. t. } (x + tu) \in [kX, (k+1)X]} \left( - \sum_{i=0}^{k-1} \rho(i)X - \rho(k)(x - kX) + t\varphi^*(u) \right)
\]

(3.13)

Let us fix \(u \times \text{Dom}(\varphi^*)\). The function \(f\) defined as:

\[
f(\rho(1), \ldots, \rho(k_{\text{max}})) = - \sum_{i=0}^{k-1} \rho(i)X - \rho(k)(x - kX) + t\varphi^*(u)
\]

Is concave (indeed, affine). Hence, the function \(M_{M_k}(t, x)\) is a concave function of \((\rho(1), \ldots, \rho(k_{\text{max}}))\), since it is the infimum of concave functions [80] [81].

Proposition 6 [Concavity property of the solutions associated with upstream and downstream boundary conditions] Let us fix \((t, x) \in \mathbb{R}_+ \times [\xi, \chi]\). The solutions \(M_{\gamma_n}(t, x)\) and \(M_{\beta_n}(t, x)\) respectively associated with the upstream and downstream boundary conditions (3.7) and (3.5) are concave functions of the coefficients \(\rho(\cdot), q_{\text{in}}(\cdot)\) and \(q_{\text{out}}(\cdot)\).
Proof — The Lax-Hopf formula (2.12) associated with the solution $M_{\gamma_n}(\cdot,\cdot)$ can be written \cite{32,66} as:

$$M_{\gamma_n}(t,x) = \inf_{s \in \mathbb{R}^+ \cap [t - (n+1)T, t - nT]} \sum_{i=0}^{n-1} q_{in}(i)T + q_{in}(n)(t - s) + s\phi^*(\frac{\xi - x}{s})$$ (3.14)

Let us fix $s \in \mathbb{R}^+ \cap [t - (n+1)T, t - nT]$. The function $d$ defined as:

$$d(q_{in}(1), \ldots, q_{in}(n_{\text{max}})) = \sum_{i=0}^{n-1} q_{in}(i)T + q_{in}(n)(t - s) + s\phi^*(\frac{\xi - x}{s})$$

is concave (indeed, affine). Hence, the solution $M_{\gamma_n}(t,x)$ is a concave function of $(q_{in}(1), \ldots, q_{in}(n_{\text{max}}))$, since it is the infimum of concave functions \cite{80,81}. The same property applies for $M_{\beta_n}(t,x)$, which is a concave function of $(\rho(1), \ldots, \rho(k_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}))$.

Propositions 4, 5 and 6 thus imply the following convexity property:

**Proposition 7** [Convexity property of model constraints] The model constraints (3.1) are convex functions of $(\rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}))$.

Proof — The set of inequality constraints (3.1) can be written as:

$$M_{c_i}(t,x) \geq c_j(t,x), \forall (t,x) \in \text{Dom}(c_j)$$ (3.15)

$$\forall j \in I \text{ such that } (t,x) \in \text{Dom}(c_j), \forall i \in I$$

Note that Proposition 4 implies that the term $c_j(t,x)$ in (3.15) is a linear function (labeled $l_{j,t,x}(\cdot)$) of $(\rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}))$. In addition, by Propositions 5 and 6, the term $M_{c_i}(t,x)$ is a concave function (labeled $c_{i,t,x}(\cdot)$) of $(\rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}))$. 
Hence, the equality (3.15) can be written as:

\[ -c_{i,t,x}(\rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), \\
q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}})) + l_{j,t,x}(\rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), \\
q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}})) \leq 0, \]

\[ \forall j \in I, \forall (t, x) \in \text{Dom}(c_j), \ \forall i \in I \] (3.16)

This last inequality is a convex inequality in \((\rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \\
\ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}))\), that is, an inequality of the form \(f(\cdot) \leq 0\) where \(f(\cdot)\) is a convex function [80].

The above property is very important, and can be thought of as follows. Consider the vector space \(V\) of all parameters of the initial, upstream and downstream boundary conditions. Each point of this vector space corresponds to a known value condition (encompassing initial, upstream and downstream boundary conditions). However, the solution to the LWR PDE (2.2) associated with this arbitrary value condition will satisfy the value condition itself on its boundaries if and only if the model constraints (3.1) are satisfied. Proposition 7 essentially states that the set of value conditions compatible with the LWR PDE model is convex.

### 3.2 Formulation of the density estimation problem as a Mixed Integer Linear Program

#### 3.2.1 Decision variable

As outlined in Proposition 7, the variable \((\rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), \\
q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}))\) plays an important role in our estimation problem, and will be defined as the decision variable of our optimization framework.

**Definition 12 [Decision variable]** Let us consider a finite set of, initial, upstream and downstream boundary conditions be defined as in (3.3) (3.7) and (3.5). The decision
variable $y$ associated with this finite set of value conditions is defined by:

\[
y := \left( \rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}) \right)
\]  

We denote by $V$ the vector space of the decision variables $y$ defined by equation (3.17).

### 3.2.2 Model and data constraints

Let $\overline{y}$ denote the value of the decision variable associated with the true state of the system (which is not known in practice, and can only be estimated). Because of model and data constraints, $\overline{y}$ must satisfy the set of constraints outlined in Propositions 8 and 10 below.

**Proposition 8 [Model constraints]** The model constraints (3.1) can be expressed as the following finite set of convex inequality constraints:

\[
M_{M_k}(t_0, x_p) \geq M_p(t_0, x_p) \quad \forall (k, p) \in \mathbb{K}^2 \quad (i)
\]

\[
M_{M_k}(pT, \chi) \geq \beta_p(pT, \chi) \quad \forall (k, p) \in \mathbb{K} \times \mathbb{N} \quad (ii)(a)
\]

\[
M_{M_k}(t_0 + \frac{x_{k+1}}{v_f}, \chi) \geq \beta_p(t_0 + \frac{x_{k+1}}{v_f}, \chi) \quad \forall (k, p) \in \mathbb{K} \times \mathbb{N} \text{ s. t. } t_0 + \frac{x_{k+1}}{v_f} \in [pT, (p + 1)T] \quad (ii)(b)
\]

\[
M_{M_k}(pT, \xi) \geq \gamma_p(pT, \xi) \quad \forall (k, p) \in \mathbb{K} \times \mathbb{N} \quad (iii)(a)
\]

\[
M_{M_k}(t_0 + \frac{\xi_{k+1}}{w}, \xi) \geq \gamma_p(t_0 + \frac{\xi_{k+1}}{w}, \xi) \quad \forall (k, p) \in \mathbb{K} \times \mathbb{N} \text{ s. t. } t_0 + \frac{\xi_{k+1}}{w} \in [pT, (p + 1)T] \quad (iii)(b)
\]
\[(3.19)\]

\[
\begin{align*}
M_{M_k}(t_{\min}(m), x_{\min}(m)) & \geq \mu_m(t_{\min}(m), x_{\min}(m)) \\
\forall k \in \mathbb{K}, \forall m \in \mathbb{M} & \quad (iv)(a) \\
M_{M_k}(t_{\max}(m), x_{\max}(m)) & \geq \mu_m(t_{\max}(m), x_{\max}(m)) \\
\forall k \in \mathbb{K}, \forall m \in \mathbb{M} & \quad (iv)(b) \\
M_{M_k}(t_1(m, k), x_1(m, k)) & \geq \mu_m(t_1(m, k), x_1(m, k)) \\
\forall k \in \mathbb{K}, \forall m \in \mathbb{M} \text{ s. t. } \forall t_1(m, k) \in [t_{\min}(m), t_{\max}(m)] & \quad (iv)(c) \\
M_{M_k}(t_2(m, k), x_2(m, k)) & \geq \mu_m(t_2(m, k), x_2(m, k)) \\
\forall k \in \mathbb{K}, \forall m \in \mathbb{M} \text{ s. t. } \forall t_2(m, k) \in [t_{\min}(m), t_{\max}(m)] & \quad (iv)(d) \\
M_{M_k}(t_3(m, k), x_3(m, k)) & \geq \mu_m(t_3(m, k), x_3(m, k)) \\
\forall k \in \mathbb{K}, \forall m \in \mathbb{M} \text{ s. t. } \forall t_3(m, k) \in [t_{\min}(m), t_{\max}(m)] & \quad (iv)(e) \\
M_{M_k}(t_4(m, k), x_4(m, k)) & \geq \mu_m(t_4(m, k), x_4(m, k)) \\
\forall k \in \mathbb{K}, \forall m \in \mathbb{M} \text{ s. t. } \forall t_4(m, k) \in [t_{\min}(m), t_{\max}(m)] & \quad (iv)(f)
\end{align*}
\]

\[
(3.20)
\]

\[
\begin{align*}
M_{M_k}(t_\rho(u), x_{\min_\rho}(u)) & \geq \Upsilon_u(t_\rho(u), x_{\min_\rho}(u)) \\
\forall k \in \mathbb{K}, \forall u \in \mathbb{U} & \quad (v)(a) \\
M_{M_k}(t_\rho(u), x_{\max_\rho}(u)) & \geq \Upsilon_u(t_\rho(u), x_{\max_\rho}(u)) \\
\forall k \in \mathbb{K}, \forall u \in \mathbb{U} & \quad (v)(b) \\
M_{M_k}(t_\rho(u), x_5(u, k)) & \geq \Upsilon_u(t_\rho(u), x_5(u, k)) \\
\forall k \in \mathbb{K}, \forall u \in \mathbb{U} \text{ s. t. } x_5(u, k) \in [x_{\min_\rho}(u), x_{\max_\rho}(u)] & \quad (v)(c) \\
M_{M_k}(t_\rho(u), x_6(u, k)) & \geq \Upsilon_u(t_\rho(u), x_6(u, k)) \\
\forall k \in \mathbb{K}, \forall u \in \mathbb{U} \text{ s. t. } x_6(u, k) \in [x_{\min_\rho}(u), x_{\max_\rho}(u)] & \quad (v)(d) \\
M_{M_k}(t_\rho(u), x_7(u, k)) & \geq \Upsilon_u(t_\rho(u), x_7(u, k)) \\
\forall k \in \mathbb{K}, \forall u \in \mathbb{U} \text{ s. t. } x_7(u, k) \in [x_{\min_\rho}(u), x_{\max_\rho}(u)] & \quad (v)(e) \\
M_{M_k}(t_\rho(u), x_8(u, k)) & \geq \Upsilon_u(t_\rho(u), x_8(u, k)) \\
\forall k \in \mathbb{K}, \forall u \in \mathbb{U} \text{ s. t. } x_8(u, k) \in [x_{\min_\rho}(u), x_{\max_\rho}(u)] & \quad (v)(f)
\end{align*}
\]
\[ \begin{align*}
&M_{\gamma_n}(pT, \xi) \geq \gamma_p(pT, \xi) \quad \forall (n, p) \in \mathbb{N}^2 \\
&M_{\gamma_n}(pT, \chi) \geq \beta_p(pT, \chi) \quad \forall (n, p) \in \mathbb{N}^2 \\
&M_{\gamma_n}(nT + \frac{\xi - \chi}{v}, \chi) \geq \beta_p(nT + \frac{\xi - \chi}{v}, \chi) \quad \forall (n, p) \in \mathbb{N}^2 \text{ s. t. } nT + \frac{\xi - \chi}{v} \in [pT, (p + 1)T] \\
\end{align*} \]

\[ (3.21) \]

\[ \begin{align*}
&M_{\gamma_n}(t_{\min}(m), x_{\min}(m)) \geq \mu_m(t_{\min}(m), x_{\min}(m)) \\
&M_{\gamma_n}(t_{\max}(m), x_{\max}(m)) \geq \mu_m(t_{\max}(m), x_{\max}(m)) \\
&M_{\gamma_n}(t_9(m, n), x_9(m, n)) \geq \mu_m(t_9(m, n), x_9(m, n)) \\
\end{align*} \]

\[ (3.22) \]

\[ \begin{align*}
&M_{\gamma_n}(t_\rho(u), x_{\min,\rho}(u)) \geq \Upsilon_u(t_\rho(u), x_{\min,\rho}(u)) \\
&M_{\gamma_n}(t_\rho(u), x_{\max,\rho}(u)) \geq \Upsilon_u(t_\rho(u), x_{\max,\rho}(u)) \\
&M_{\gamma_n}(t_\rho(u), x_{10}(u, n)) \geq \Upsilon_u(t_\rho(u), x_{10}(u, n)) \\
\end{align*} \]

\[ (3.23) \]

\[ \begin{align*}
&M_{\beta_n}(pT, \xi) \geq \gamma_p(pT, \xi) \quad \forall (n, p) \in \mathbb{N}^2 \\
&M_{\beta_n}(nT + \frac{\xi - \chi}{v}, \xi) \geq \gamma_p(nT + \frac{\xi - \chi}{v}, \xi) \quad \forall (n, p) \in \mathbb{N}^2 \text{ s. t. } nT + \frac{\xi - \chi}{v} \in [pT, (p + 1)T] \\
&M_{\beta_n}(pT, \chi) \geq \beta_p(pT, \chi) \quad \forall (n, p) \in \mathbb{N}^2 \\
\end{align*} \]

\[ (3.24) \]
\[
\begin{align*}
\mathbf{M}_{\beta_n}(t_{\min}(m), x_{\min}(m)) & \geq \mu_m(t_{\min}(m), x_{\min}(m)) \\
& \quad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \quad (xii)(a) \\
\mathbf{M}_{\beta_n}(t_{\max}(m), x_{\max}(m)) & \geq \mu_m(t_{\max}(m), x_{\max}(m)) \\
& \quad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \quad (xii)(b) \\
\mathbf{M}_{\beta_n}(t_{11}(m, n), x_{11}(m, n)) & \geq \mu_m(t_{11}(m, n), x_{11}(m, n)) \\
& \quad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \text{ s. t. } t_{11}(m, n) \in [t_{\min}(m); t_{\max}(m)] \quad (xii)(c)
\end{align*}
\]

\[
\begin{align*}
\mathbf{M}_{\beta_n}(t_p(u), x_{\min_p}(u)) & \geq \Upsilon_u(t_p(u), x_{\min_p}(u)) \\
& \quad \forall n \in \mathbb{N}, \forall u \in \mathbb{U} \quad (xiii)(a) \\
\mathbf{M}_{\beta_n}(t_p(u), x_{\max_p}(u)) & \geq \Upsilon_u(t_p(u), x_{\max_p}(u)) \\
& \quad \forall n \in \mathbb{N}, \forall u \in \mathbb{U} \quad (xiii)(b) \\
\mathbf{M}_{\beta_n}(t_p(u), x_{12}(u, n)) & \geq \Upsilon_u(t_p(u), x_{12}(u, n)) \\
& \quad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \text{ s. t. } x_{12}(m, n) \in [x_{\min_p}(u); x_{\max_p}(u)] \quad (xiii)(c)
\end{align*}
\]

\[
\begin{align*}
\mathbf{M}_{\mu_m}(pT, \xi) & \geq \gamma_p(pT, \xi) \\
& \quad \forall (m, p) \in \mathbb{M} \times \mathbb{N} \quad (xiv)(a) \\
\mathbf{M}_{\mu_m}(t_{13}(m), \xi) & \geq \gamma_p(t_{13}(m), \xi) \\
& \quad \forall (m, p) \in \mathbb{M} \times \mathbb{N} \text{ s. t. } t_{13}(m) \in [pT, (p + 1)T] \quad (xiv)(b) \\
\mathbf{M}_{\mu_m}(t_{14}(m), \xi) & \geq \gamma_p(t_{14}(m), \xi) \\
& \quad \forall (m, p) \in \mathbb{M} \times \mathbb{N} \text{ s. t. } t_{14}(m) \in [pT, (p + 1)T] \quad (xiv)(c)
\end{align*}
\]

\[
\begin{align*}
\mathbf{M}_{\mu_m}(pT, \chi) & \geq \beta_p(pT, \chi) \\
& \quad \forall (m, p) \in \mathbb{M} \times \mathbb{N} \quad (xv)(a) \\
\mathbf{M}_{\mu_m}(t_{15}(m), \chi) & \geq \beta_p(t_{15}(m), \chi) \\
& \quad \forall (m, p) \in \mathbb{M} \times \mathbb{N} \text{ s. t. } t_{9}(m) \in [pT, (p + 1)T] \quad (xv)(b) \\
\mathbf{M}_{\mu_m}(t_{16}(m), \chi) & \geq \beta_p(t_{16}(m), \chi) \\
& \quad \forall (m, p) \in \mathbb{M} \times \mathbb{N} \text{ s. t. } t_{10}(m) \in [pT, (p + 1)T] \quad (xv)(c)
\end{align*}
\]
\begin{align}
M_{\mu_m}(t_{\text{min}}(p), x_{\text{min}}(p)) & \geq \mu_p(t_{\text{min}}(p), x_{\text{min}}(p)) \\
\forall (m, p) & \in M^2 \quad (\text{xvi}(a)) \\
M_{\mu_m}(t_{\text{max}}(p), x_{\text{max}}(p)) & \geq \mu_p(t_{\text{max}}(p), x_{\text{max}}(p)) \\
\forall (m, p) & \in M^2 \quad (\text{xvi}(b)) \\
M_{\mu_m}(t_{17}(m, p), x_{17}(m, p)) & \geq \mu_p(t_{17}(m, p), x_{17}(m, p)) \\
\forall (m, p) & \in M^2 \text{ s. t. } t_{17}(m, p) \in [t_{\text{min}}(p), t_{\text{max}}(p)] \quad (\text{xvi}(c)) \\
M_{\mu_m}(t_{18}(m, p), x_{18}(m, p)) & \geq \mu_p(t_{18}(m, p), x_{18}(m, p)) \\
\forall (m, p) & \in M^2 \text{ s. t. } t_{18}(m, p) \in [t_{\text{min}}(p), t_{\text{max}}(p)] \quad (\text{xvi}(d)) \\
M_{\mu_m}(t_{19}(m, p), x_{19}(m, p)) & \geq \mu_p(t_{19}(m, p), x_{19}(m, p)) \\
\forall (m, p) & \in M^2 \text{ s. t. } t_{19}(m, p) \in [t_{\text{min}}(p), t_{\text{max}}(p)] \quad (\text{xvi}(e)) \\
M_{\mu_m}(t_{20}(m, p), x_{20}(m, p)) & \geq \mu_p(t_{20}(m, p), x_{20}(m, p)) \\
\forall (m, p) & \in M^2 \text{ s. t. } t_{20}(m, p) \in [t_{\text{min}}(p), t_{\text{max}}(p)] \quad (\text{xvi}(f)) \\
M_{\mu_m}(t_{21}(m, p), x_{21}(m, p)) & \geq \mu_p(t_{21}(m, p), x_{21}(m, p)) \\
\forall (m, p) & \in M^2 \text{ s. t. } t_{21}(m, p) \in [t_{\text{min}}(p), t_{\text{max}}(p)] \quad (\text{xvi}(g)) 
\end{align}
\[
\begin{align*}
M_{\mu_m}(t_\rho(u), x_{\min_p}(u)) &\geq \Upsilon_u(t_\rho(u), x_{\min_p}(u)) \\
M_{\mu_m}(t_\rho(u), x_{\max_p}(u)) &\geq \Upsilon_u(t_\rho(u), x_{\max_p}(u)) \\
M_{\mu_m}(t_\rho(u), x_{22}(m, u)) &\geq \Upsilon_u(t_\rho(u), x_{22}(m, u)) \\
M_{\mu_m}(t_\rho(u), x_{23}(m, u)) &\geq \Upsilon_u(t_\rho(u), x_{23}(m, u)) \\
M_{\mu_m}(t_\rho(u), x_{24}(m, u)) &\geq \Upsilon_u(t_\rho(u), x_{24}(m, u)) \\
M_{\mu_m}(t_\rho(u), x_{25}(m, u)) &\geq \Upsilon_u(t_\rho(u), x_{25}(m, u)) \\
M_{\mu_m}(t_\rho(u), x_{26}(m, u)) &\geq \Upsilon_u(t_\rho(u), x_{26}(m, u)) \\
M_{\mu_m}(t_\rho(u), x_{\min_p}(u), x_{\max_p}(u)) &\geq \Upsilon_u(t_\rho(u), x_{\min_p}(u), x_{\max_p}(u)) \\
M_{\mu_m}(t_\rho(u), x_{\min_p}(u), x_{\max_p}(u)) &\geq \Upsilon_u(t_\rho(u), x_{\min_p}(u), x_{\max_p}(u))
\end{align*}
\]

\[(3.30)\]

\[
\begin{align*}
M_{\tau_u}(pT, \chi) &\geq \beta_p(pT, \chi) \\
M_{\tau_u}(t_\rho(u) + \frac{x_{-x_{\max_p}(u)}}{v_f}, \chi) &\geq \beta_p(t_\rho(u) + \frac{x_{-x_{\max_p}(u)}}{v_f}, \chi) \\
M_{\tau_u}(pT, \xi) &\geq \gamma_p(pT, \xi) \\
M_{\tau_u}(t_\rho(u) + \frac{x_{-x_{\min_p}(u)}}{w}, \xi) &\geq \gamma_p(t_\rho(u) + \frac{x_{-x_{\min_p}(u)}}{w}, \xi)
\end{align*}
\]

\[(3.31)\]
\[
M_{T_u}(t_{\min}(p), x_{\min}(p)) \geq \mu_p(t_{\min}(p), x_{\min}(p))
\]
\[
\forall (u, p) \in U \times M \quad (xix)(a)
\]
\[
M_{T_u}(t_{\max}(p), x_{\max}(p)) \geq \mu_p(t_{\max}(p), x_{\max}(p))
\]
\[
\forall (m, p) \in U \times M \quad (xix)(b)
\]
\[
M_{T_u}(t_{27}(u, p), x_{27}(u, p)) \geq \mu_p(t_{27}(u, p), x_{27}(u, p))
\]
\[
\forall (u, p) \in U \times M \text{ s.t. } t_{27}(u, p) \in [t_{\min}(p), t_{\max}(p)] \quad (xix)(c)
\]
\[
M_{T_u}(t_{28}(u, p), x_{28}(u, p)) \geq \mu_p(t_{28}(u, p), x_{28}(u, p))
\]
\[
\forall (u, p) \in U \times M \text{ s.t. } t_{28}(u, p) \in [t_{\min}(p), t_{\max}(p)] \quad (xix)(d)
\]
\[
M_{T_u}(t_{29}(u, p), x_{29}(u, p)) \geq \mu_p(t_{29}(u, p), x_{29}(u, p))
\]
\[
\forall (u, p) \in U \times M \text{ s.t. } t_{29}(u, p) \in [t_{\min}(p), t_{\max}(p)] \quad (xix)(e)
\]
\[
M_{T_u}(t_{30}(u, p), x_{30}(u, p)) \geq \mu_p(t_{30}(u, p), x_{30}(u, p))
\]
\[
\forall (u, p) \in U \times M \text{ s.t. } t_{30}(u, p) \in [t_{\min}(p), t_{\max}(p)] \quad (xix)(f)
\]
\[
M_{T_u}(t_{26}(u, p), x_{26}(p, u)) \geq \mu_p(t_{26}(p, u))
\]
\[
\forall (u, p) \in U \times M \text{ s.t. } x_{26}(p, u) \in [x_{\min}(u), t_{\max}(u)] \quad (xix)(g)
\]

\[
M_{T_u}(t_{p}(p), x_{\min}(p)) \geq \Upsilon_p(t_{p}(p), x_{\min}(p))
\]
\[
\forall (u, p) \in U^2 \quad (xx)(a)
\]
\[
M_{T_u}(t_{p}(p), x_{\max}(p)) \geq \Upsilon_p(t_{p}(p), x_{\max}(p))
\]
\[
\forall (u, p) \in U^2 \quad (xx)(b)
\]
\[
M_{T_u}(t_{p}(p), x_{31}(u, p)) \geq \mu_p(t_{p}(p), x_{31}(u, p))
\]
\[
\forall (u, p) \in U^2 \text{ s.t. } x_{31}(u, p) \in [x_{\min}(p), t_{\max}(p)] \quad (xx)(c)
\]
\[
M_{T_u}(t_{p}(p), x_{32}(u, p)) \geq \mu_p(t_{p}(p), x_{32}(u, p))
\]
\[
\forall (u, p) \in U^2 \text{ s.t. } x_{32}(u, p) \in [x_{\min}(p), t_{\max}(p)] \quad (xx)(d)
\]
\[
M_{T_u}(t_{p}(p), x_{33}(u, p)) \geq \mu_p(t_{p}(p), x_{33}(u, p))
\]
\[
\forall (u, p) \in U^2 \text{ s.t. } x_{33}(u, p) \in [x_{\min}(p), t_{\max}(p)] \quad (xx)(e)
\]
\[
M_{T_u}(t_{p}(p), x_{34}(u, p)) \geq \mu_p(t_{p}(p), x_{34}(u, p))
\]
\[
\forall (u, p) \in U^2 \text{ s.t. } x_{34}(u, p) \in [x_{\min}(p), t_{\max}(p)] \quad (xx)(f)
\]
where the coefficients \( t_1(m, k), x_1(m, k), t_2(m, k), x_2(m, k), t_3(m, k), x_3(m, k), \)
\( t_4(m, k), x_4(m, k), x_5(u, k), x_6(u, k), x_7(u, k), x_8(u, k), t_9(m, n), x_9(m, n), x_{10}(u, n), \)
\( t_{11}(m, n), x_{11}(m, n), x_{12}(u, n), t_{13}(m), t_{14}(m), t_{15}(m), t_{16}(m), t_{17}(m, p), x_{17}(m, p), \)
\( t_{18}(m, p), x_{18}(m, p), t_{19}(m, p), x_{19}(m, p), t_{20}(m, p), x_{20}(m, p), t_{21}(m, p), x_{21}(m, p), \)
\( x_{22}(m, u), x_{23}(m, u), x_{24}(m, u), x_{25}(m, u), x_{26}(m, u), t_{27}(u, p), x_{27}(u, p), t_{28}(u, p), \)
\( x_{28}(u, p), x_{29}(u, p), t_{29}(u, p), t_{30}(u, p), x_{30}(u, p), x_{31}(u, p), x_{32}(u, p), x_{33}(u, p) \)
and \( x_{34}(u, p) \) are given by equations (3.34), (3.35), (3.36) and (3.37) below:

\[
\begin{align*}
t_1(m, k) &= \frac{x_{\text{min}}(m) - x_{k+1} - v_{\text{meas}}(m) t_{\text{min}}(m)}{v_f - v_{\text{meas}}(m)} \\
x_1(m, k) &= v_{\text{meas}}(m) \left( \frac{x_{\text{min}}(m) - x_{k+1} - v_{\text{meas}}(m) t_{\text{min}}(m)}{v_f - v_{\text{meas}}(m)} \right) + x_{\text{min}}(m) \\
t_2(m, k) &= \frac{x_{\text{min}}(m) - x_k - v_{\text{meas}}(m) t_{\text{min}}(m)}{u - v_{\text{meas}}(m)} \\
x_2(m, k) &= v_{\text{meas}}(m) \left( \frac{x_{\text{min}}(m) - x_k - v_{\text{meas}}(m) t_{\text{min}}(m)}{u - v_{\text{meas}}(m)} \right) + x_{\text{min}}(m) \\
t_3(m, k) &= \frac{x_{\text{min}}(m) - x_k - v_{\text{meas}}(m) t_{\text{min}}(m)}{v_f - v_{\text{meas}}(m)} \\
x_3(m, k) &= v_{\text{meas}}(m) \left( \frac{x_{\text{min}}(m) - x_k - v_{\text{meas}}(m) t_{\text{min}}(m)}{v_f - v_{\text{meas}}(m)} \right) + x_{\text{min}}(m) \\
t_4(m, k) &= \frac{x_{\text{min}}(m) - x_{k+1} - v_{\text{meas}}(m) t_{\text{min}}(m)}{u - v_{\text{meas}}(m)} \\
x_4(m, k) &= v_{\text{meas}}(m) \left( \frac{x_{\text{min}}(m) - x_{k+1} - v_{\text{meas}}(m) t_{\text{min}}(m)}{u - v_{\text{meas}}(m)} \right) + x_{\text{min}}(m) \\
x_5(u, k) &= x_{k+1} + v_f(t_r(u) - t_0) \\
x_6(u, k) &= x_{k+1} + w(t_r(u) - t_0) \\
x_7(u, k) &= x_k + v_f(t_r(u) - t_0) \\
x_8(u, k) &= x_k + w(t_r(u) - t_0)
\end{align*}
\]
\[
\begin{array}{l}
t_9(m,n) = \frac{nT v_f - v_{\text{meas}}(m)t_{\min}(m) + x_{\min}(m) - \xi}{v_f - v_{\text{meas}}(m)} - t_{\min}(m) \\
x_9(m,n) = x_{\min}(m) + \\
v_{\text{meas}}(m) \left( \frac{nT v_f - v_{\text{meas}}(m)t_{\min}(m) + x_{\min}(m) - \xi}{v_f - v_{\text{meas}}(m)} - t_{\min}(m) \right) \\
x_{10}(u,n) = \xi + v_f(t_p(u) - nT) \\
t_{11}(m,n) = \frac{nT w - v_{\text{meas}}(m)t_{\min}(m) + x_{\min}(m) - \chi}{w - v_{\text{meas}}(m)} + x_{\min}(m) + \\
v_{\text{meas}}(m) \left( \frac{nT w - v_{\text{meas}}(m)t_{\min}(m) + x_{\min}(m) - \chi}{w - v_{\text{meas}}(m)} - t_{\min}(m) \right) \\
x_{12}(u,n) = \chi + w(t_p(u) - nT) \\
t_{13}(m) = \frac{x_{\min}(m) + w t_{\min}(m)}{w} \\
t_{14}(m) = \frac{x_{\max}(m) + w t_{\max}(m)}{w} \\
t_{15}(m) = \frac{\chi - x_{\min}(m) + v_f t_{\min}(m)}{v_f} \\
t_{16}(m) = \frac{\chi - x_{\max}(m) + v_f t_{\max}(m)}{v_f}
\end{array}
\]
\[
\begin{align*}
    t_{17}(m, p) &= \frac{x_{\text{min}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{\text{meas}}(m) t_{\text{min}}(m)}{v_{\text{meas}}(p) - v_{\text{meas}}(m)} \\
    x_{17}(m, p) &= x_{\text{min}}(p) + v_{\text{meas}}(p) \left( -t_{\text{min}}(p) + x_{\text{min}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{\text{meas}}(m) t_{\text{min}}(m) \right) \\
    t_{18}(m, p) &= \frac{x_{\text{max}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{f} t_{\text{max}}(m)}{v_{\text{meas}}(p) - v_{f}} \\
    x_{18}(m, p) &= x_{\text{min}}(p) + v_{\text{meas}}(p) \left( -t_{\text{min}}(p) + x_{\text{max}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{f} t_{\text{max}}(m) \right) \\
    t_{19}(m, p) &= \frac{x_{\text{min}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{f} t_{\text{min}}(m)}{v_{\text{meas}}(p) - v_{f}} \\
    x_{19}(m, p) &= x_{\text{min}}(p) + v_{\text{meas}}(p) \left( t_{\text{min}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{f} t_{\text{max}}(m) \right) \\
    t_{20}(m, p) &= \frac{x_{\text{max}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{f} t_{\text{max}}(m)}{v_{\text{meas}}(p) - w} \\
    t_{21}(m, p) &= \frac{x_{\text{min}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{f} t_{\text{min}}(m)}{v_{\text{meas}}(p) - w} \\
    x_{21}(m, p) &= x_{\text{min}}(p) + v_{\text{meas}}(p) \left( -t_{\text{min}}(p) + x_{\text{min}}(m) - x_{\text{min}}(p) + v_{\text{meas}}(p) t_{\text{min}}(p) - v_{f} t_{\text{min}}(m) \right) \\
    x_{22}(m, u) &= x_{\text{min}}(m) + w(t_{p}(u) - t_{\text{min}}(m)) \\
    x_{23}(m, u) &= x_{\text{max}}(m) + w(t_{p}(u) - t_{\text{max}}(m)) \\
    x_{24}(m, u) &= x_{\text{min}}(m) + v_{f} (t_{p}(u) - t_{\text{min}}(m)) \\
    x_{25}(m, u) &= x_{\text{max}}(m) + v_{f} (t_{p}(u) - t_{\text{max}}(m)) \\
    x_{26}(m, u) &= x_{\text{min}}(m) + v_{\text{meas}}(m) (t_{p}(u) - t_{\text{min}}(m))
\end{align*}
\]
\[
\begin{align*}
t_{27}(u, p) &= \frac{x_{\min}(p) - x_{\max}(u) - v_{\text{meas}}(p)t_{\min}(p) + v_f t_p(u)}{v_f - v_{\text{meas}}(p)} \\
x_{27}(u, p) &= x_{\min}(p) + v_{\text{meas}}(p) \left( -t_{\min}(p) + \frac{x_{\min}(p) - x_{\max}(u) - v_{\text{meas}}(p)t_{\min}(p) + v_f t_p(u)}{v_f - v_{\text{meas}}(p)} \right) \\
t_{28}(u, p) &= \frac{x_{\min}(p) - x_{\min}(u) - v_{\text{meas}}(p)t_{\min}(p) + w t_p(u)}{w - v_{\text{meas}}(p)} \\
x_{28}(u, p) &= x_{\min}(p) + v_{\text{meas}}(p) \left( -t_{\min}(p) + \frac{x_{\min}(p) - x_{\min}(u) - v_{\text{meas}}(p)t_{\min}(p) + w t_p(u)}{w - v_{\text{meas}}(p)} \right) \\
t_{29}(u, p) &= \frac{x_{\min}(p) - x_{\min}(u) - v_{\text{meas}}(p)t_{\min}(p) + v_f t_p(u)}{v_f - v_{\text{meas}}(p)} \\
x_{29}(u, p) &= x_{\min}(p) + v_{\text{meas}}(p) \left( -t_{\min}(p) + \frac{x_{\min}(p) - x_{\min}(u) - v_{\text{meas}}(p)t_{\min}(p) + v_f t_p(u)}{v_f - v_{\text{meas}}(p)} \right) \\
t_{30}(u, p) &= \frac{x_{\min}(p) - x_{\max}(u) - v_{\text{meas}}(p)t_{\min}(p) + w t_p(u)}{w - v_{\text{meas}}(p)} \\
x_{30}(u, p) &= x_{\min}(p) + v_{\text{meas}}(p) \left( -t_{\min}(p) + \frac{x_{\min}(p) - x_{\max}(u) - v_{\text{meas}}(p)t_{\min}(p) + w t_p(u)}{w - v_{\text{meas}}(p)} \right) \\
x_{31}(u, p) &= x_{\min}(u) + v_f (t_p(p) - t_p(u)) \\
x_{32}(u, p) &= x_{\max}(u) + v_f (t_p(p) - t_p(u)) \\
x_{33}(u, p) &= x_{\min}(u) + w (t_p(p) - t_p(u)) \\
x_{34}(u, p) &= x_{\max}(u) + w (t_p(p) - t_p(u))
\end{align*}
\]

**Proof** — Note that \(\forall (k, n) \in [0, k_{\max}] \times [0, n_{\max}], \text{Dom}(M_k) \cap \text{Dom}(M_{s_n}) = \emptyset\) and that \(\forall (k, n) \in [0, k_{\max}] \times [0, n_{\max}], \text{Dom}(M_k) \cap \text{Dom}(M_{s_n}) = \emptyset\). Thus, the set of inequality constraints (3.1) can be written in the case of initial, upstream, downstream
and internal conditions as:

\[
\begin{align*}
M_{M_k}(t, \xi) &\geq \gamma_p(t, \xi) & \forall t \in [pT, (p+1)T], \forall k \in K, \forall p \in N \\
M_{M_k}(t, \chi) &\geq \beta_p(t, \chi) & \forall t \in [pT, (p+1)T], \forall k \in K, \forall p \in N \\
M_{M_k}(t, x) &\geq \mu_m(t, x) & \forall (t, x) \in \text{Dom}(\mu_m), \forall (k, m) \in K \times M \\
M_{M_k}(t, x) &\geq \Upsilon_u(t, x) & \forall (t, x) \in \text{Dom}(\Upsilon_u), \forall (k, u) \in K \times U \\
M_{\gamma_n}(t, \xi) &\geq \gamma_p(t, \xi) & \forall t \in [pT, (p+1)T], \forall (n, p) \in N^2 \\
M_{\gamma_n}(t, \chi) &\geq \beta_p(t, \chi) & \forall t \in [pT, (p+1)T], \forall (n, p) \in N^2 \\
M_{\gamma_n}(t, x) &\geq \mu_m(t, x) & \forall (t, x) \in \text{Dom}(\mu_m), \forall (n, m) \in N \times M \\
M_{\gamma_n}(t, x) &\geq \Upsilon_u(t, x) & \forall (t, x) \in \text{Dom}(\Upsilon_u), \forall (k, u) \in K \times U \\
M_{\beta_n}(t, \xi) &\geq \gamma_p(t, \xi) & \forall t \in [pT, (p+1)T], \forall (n, p) \in N^2 \\
M_{\beta_n}(t, \chi) &\geq \beta_p(t, \chi) & \forall t \in [pT, (p+1)T], \forall (n, p) \in N^2 \\
M_{\beta_n}(t, x) &\geq \mu_m(t, x) & \forall (t, x) \in \text{Dom}(\mu_m), \forall (n, m) \in N \times M \\
M_{\beta_n}(t, x) &\geq \Upsilon_u(t, x) & \forall (t, x) \in \text{Dom}(\Upsilon_u), \forall (k, u) \in K \times U \\
M_{\mu_k}(t, \xi) &\geq \gamma_p(t, \xi) & \forall t \in [pT, (p+1)T], \forall k \in M, \forall p \in N \\
M_{\mu_k}(t, \chi) &\geq \beta_p(t, \chi) & \forall t \in [pT, (p+1)T], \forall k \in M, \forall p \in N \\
M_{\mu_k}(t, x) &\geq \mu_m(t, x) & \forall (t, x) \in \text{Dom}(\mu_m), \forall (k, m) \in M^2 \\
M_{\mu_k}(t, x) &\geq \Upsilon_u(t, x) & \forall (t, x) \in \text{Dom}(\Upsilon_u), \forall k \in M, \forall u \in U \\
M_{\Upsilon_k}(t, \xi) &\geq \gamma_p(t, \xi) & \forall t \in [pT, (p+1)T], \forall k \in U, \forall p \in N \\
M_{\Upsilon_k}(t, \chi) &\geq \beta_p(t, \chi) & \forall t \in [pT, (p+1)T], \forall k \in U, \forall p \in N \\
M_{\Upsilon_k}(t, x) &\geq \mu_m(t, x) & \forall (t, x) \in \text{Dom}(\mu_m), \forall k \in U, \forall m \in M \\
M_{\Upsilon_k}(t, x) &\geq \Upsilon_u(t, x) & \forall (t, x) \in \text{Dom}(\Upsilon_u), \forall (k, u) \in U^2
\end{align*}
\]

The conditions (3.38) all involve checking that a function of \((t, x)\) is greater than another function of \((t, x)\) on a line segment of \(\mathbb{R}_+ \times [\xi, \chi]\). Yet, because of the affine structure of the initial and boundary conditions (3.3), (3.7) and (3.5) as well as the piecewise affine structure of their solutions (3.8), (3.9) and (3.10), the inequalities of the form \(\forall (t, x) \in \text{Dom}(c_i)\), \(M_{c_j}(t, x) \geq c_i(t, x)\) are equivalent to a finite number of inequalities of the form \(\forall p \in \{0, \ldots, p_{\text{max}}\}\), \(M_{c_j}(t_p, x_p) \geq c_i(t_p, x_p)\). This arises
from the fact that a piecewise affine function is positive on all points of a segment if and only if it is positive on each extremity of the segment, and on the finite number of points of the segment on which the function is not differentiable. In the present case, this property implies the equivalence of (3.38) and of (3.18), (3.19), (3.20), (3.21), (3.22), (3.23), (3.24), (3.25), (3.26), (3.27), (3.28), (3.29), (3.30), (3.31), (3.32) and (3.33). The equality constraints (3.40) arise from continuity conditions of the Moskowitz function [69].

Proposition 9 [Continuity constraints] Let a set of initial, boundary and internal conditions be defined as in (3.3), and let the corresponding partial solutions be defined as \( M_{M_k}(\cdot, \cdot) \), \( M_{\gamma_n}(\cdot, \cdot) \), \( M_{\beta_n}(\cdot, \cdot) \) and \( M_{\mu_m}(\cdot, \cdot) \). Let us also assume that the model constraints (3.1) are satisfied. Let \( M_p(\cdot, \cdot) \) be defined as \( M_p(\cdot, \cdot) = \min_{k,n,m,u\mid m \neq p}(M_{M_k}(\cdot, \cdot), M_{\gamma_n}(\cdot, \cdot), M_{\beta_n}(\cdot, \cdot), M_{\mu_m}(\cdot, \cdot)) \) and \( M_o(\cdot, \cdot) \) be defined as \( M_o(\cdot, \cdot) = \min_{k,n,m,u\mid u \neq o}(M_{M_k}(\cdot, \cdot), M_{\gamma_n}(\cdot, \cdot), M_{\beta_n}(\cdot, \cdot), M_{\mu_m}(\cdot, \cdot)) \). The solution \( M(\cdot, \cdot) \) to the HJ PDE (2.5) defined by \( M(\cdot, \cdot) = \min_{k,n,m,u}(M_{M_k}(\cdot, \cdot), M_{\gamma_n}(\cdot, \cdot), M_{\beta_n}(\cdot, \cdot), M_{\mu_m}(\cdot, \cdot), M_{T_u}(\cdot, \cdot)) \) is continuous if and only if the following conditions are satisfied:

\[
\forall p \in \mathbb{M}, \quad M_p(t_{\min}(p), x_{\min}(p)) = \mu_p(t_{\min}(p), x_{\min}(p)) \quad (3.39)
\]

\[
\begin{align*}
\mu_m(t_{\min}(m), x_{\min}(m)) &= \min (M_{M_k}(t_{\min}(m), x_{\min}(m)), \\
M_{\gamma_n}(t_{\min}(m), x_{\min}(m)), M_{\beta_n}(t_{\min}(m), x_{\min}(m)), \\
M_{T_u}(t_{\min}(m), x_{\min}(m)), M_{\mu_p}(t_{\min}(m), x_{\min}(m))) \quad \forall k \in K, \\
\forall n \in \mathbb{N}, \forall u \in \mathbb{U}, \forall (m, p) \in \mathbb{M}^2
\end{align*}
\]

\[
\begin{align*}
\mu_m(t_{\max}(m), x_{\max}(m)) &= \min (M_{M_k}(t_{\max}(m), x_{\max}(m)), \\
M_{\gamma_n}(t_{\max}(m), x_{\max}(m)), M_{\beta_n}(t_{\max}(m), x_{\max}(m)), \\
M_{T_u}(t_{\max}(m), x_{\max}(m)), M_{\mu_p}(t_{\max}(m), x_{\max}(m))) \quad \forall k \in K, \\
\forall n \in \mathbb{N}, \forall u \in \mathbb{U}, \forall (m, p) \in \mathbb{M}^2
\end{align*}
\]
affine structure of the partial solutions as well as auxiliary integer variables.

mixed integer linear inequalities involving the continuous variables is more involved. It can be shown that since the piecewise affine dependency of the partial solutions with respect to the variables

These inequalities can be further rewritten as mixed integer linear inequalities using

Furthermore, the equality constraints (3.39) and (3.41) can be written as a set of mixed integer linear inequalities involving the continuous variables \( \rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}), L_1, \ldots, L_m \) and \( r_1, \ldots, r_m \), as well as auxiliary integer variables.

The proof of (3.39) is straightforward, and follows directly from the piecewise affine structure of the partial solutions \( M_{M_k} (\cdot, \cdot), M_{\gamma_n} (\cdot, \cdot), M_{\beta_n} (\cdot, \cdot) \) and \( M_{\mu_m} (\cdot, \cdot) \).

The fact that (3.39) can be written as a set of mixed integer linear inequalities is more involved. It can be shown that since \( M_{k,n,m} (\cdot, \cdot) = \min_{k,n,m} \forall p \neq p' \langle M_{M_k} (\cdot, \cdot), M_{\gamma_n} (\cdot, \cdot), M_{\beta_n} (\cdot, \cdot), M_{\mu_m} (\cdot, \cdot) \rangle \), equation (3.39) can be written as a set of inequalities involving the continuous variables \( \rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}), L_1, \ldots, L_m \) and \( r_1, \ldots, r_m \), as well as boolean variables.

An example of such derivation is shown in [82] for the case in which \( m_{\text{max}} = 1 \).

These inequalities can be further rewritten as mixed integer linear inequalities using the piecewise affine dependency of the partial solutions with respect to the variables \( \rho(1), \rho(2), \ldots, \rho(k_{\text{max}}), q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\text{max}}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\text{max}}), L_1, \ldots, L_m \) and \( r_1, \ldots, r_m \).
Proposition 10 [Data constraints] In the remainder of this dissertation, we assume that the data constraints are convex in the decision variable $v$.

Different choices of error models yield convex data constraints, such as the two examples outlined below.

Example of convex data constraints (1) — Consider a sensor measuring the boundary flows $(q_{in}(0), \ldots q_{in}(n_{max}))$ with 5% relative uncertainty, a loop detector measuring the initial density $\rho(3)$ with 10% absolute uncertainty, and no downstream sensor. In this situation, the constraints are convex inequalities (indeed, linear inequalities) in the decision variable:

$$
\begin{align*}
0.95q_{in}\text{measured}(n) & \leq q_{in}(n) \leq 1.05q_{in}\text{measured}(n) & \forall n \in [0, n_{max}] \\
\rho(3)\text{measured} - 0.1\kappa & \leq \rho(3) \leq \rho(3)\text{measured} + 0.1\kappa
\end{align*}
$$

(3.43)

Example of convex data constraints (2) — Consider two identical sensors measuring the boundary flows $(q_{in}(0), \ldots q_{in}(n_{max}))$, and $(q_{out}(1), \ldots q_{out}(n_{max}))$ which are characterized by a RMS relative error of 3%. In this situation, the constraints are convex inequalities (quadratic convex inequalities) in the decision variable:

$$
\begin{align*}
\sum_{n=0}^{n_{max}} (q_{in}(n) - q_{in}\text{measured})^2 & \leq 0.03 \sum_{n=0}^{n_{max}} (q_{in}\text{measured})^2 \\
\sum_{n=0}^{n_{max}} (q_{out}(n) - q_{out}\text{measured})^2 & \leq 0.03 \sum_{n=0}^{n_{max}} (q_{out}\text{measured})^2
\end{align*}
$$

(3.44)

In this situation the estimation problem becomes a Quadratic Program.

Proposition 11 [Travel time constraints] Travel time data can be used in the estimation process, in order to properly define this information we define the travel time as:

$$
T_{time} = t_{f_{travel}} - t_{0_{travel}}
$$

(3.45)
The travel time constraints are specified as the following equality:

\[
M_\gamma(n, t_0\text{travel}, \xi) = M_\beta(p, t_f\text{travel}, \chi) \quad \forall (n, p) \in \mathbb{N}^2
\]  

(3.46)

3.3 Numerical Example: Traffic Estimation on a single highway link

We now present an implementation of the estimation framework presented earlier on an experimental dataset. The dataset includes fixed sensor data (obtained from inductive loop detectors in the present case) travel time and mobile sensor data.

3.3.1 Experimental setup

In the following sections, the effectiveness of the method is illustrated on different traffic flow estimation problems which are formulated as Linear Programming (LP) and Mixed Integer Programming (MIP), using the Mobile Century [25, 23] dataset. The Mobile Century field experiment demonstrated the use of Nokia N-95 cellphones as mobile traffic sensors in 2008, and was a joint UC-Berkeley/Nokia project.

For the numerical applications, a spatial domain of 1.2 km is considered, located between the PeMS [71] Vehicle Detection Station (VDS) 400536 and 401529 on the Highway I - 880 N around Hayward, California. The data used in this implementation was generated on February 8th, 2008, between times 18 : 30 and 18 : 55. In our scenario, we only consider inflow and outflow data \(q_{in}^{\text{measured}}(\cdot)\) and \(q_{out}^{\text{measured}}(\cdot)\) generated by the above PeMS stations, i.e. we do not assume to know any density data. Of course the framework presented in this dissertation allows incorporation of density data, see for instance Example 1 in the previous section. The layout of the spatial domain is illustrated in Figure 4.1.
3.3.2 Initial density estimation on systems modeled by the Lighthill-Whitham-Richards PDE

In this first scenario, our objective is to estimate the initial number of vehicles on the road (which are hard to measure in practice) and compare it with the ground truth (figure 3.3).

We assumed that the data set used for the density estimation is obtained from the ground truth density map: boundary flow data is available from the PeMS sensors and GPS data is available from probe vehicles on the road. For this specific application the objective function is chosen as the initial densities, defined by \( \sum_{i=0}^{k_{max}} \rho(i) \), it is important to remember that any convex piecewise affine function of the decision variable (depending on the desired application) would be acceptable. The constraints are linear inequalities in (3.17), and comprise both model (3.18), (3.21) and (3.23) as well as data constraints. For this specific application, the data constraints are:

\[
(1 - e)q_{\text{in/out}}^{\text{measured}}(n) \leq q_{\text{in/out}}(n) \leq (1 + e)q_{\text{in/out}}^{\text{measured}}(n) \quad \forall n \in [0, n_{\text{max}}],
\]

where \( e = 0.01 = \)
Figure 3.3: Ground truth for single road density estimation. This is the initial number of vehicles that will be estimated by using flow and GPS data obtained from this image.

1\% is chosen the worst-case relative error of the sensors. The maximal densities are solutions to the following linear program:

\[
\text{Maximize } \sum_{i=0}^{k_{\text{max}}} \rho(i)
\]

\[
\begin{align*}
(1 - e)q_{\text{in}}^{\text{measured}}(n) & \leq q_{\text{in}}(n) \quad \forall n \in [0, n_{\text{max}}] \\
q_{\text{in}}(n) & \leq (1 + e)q_{\text{in}}^{\text{measured}}(n) \quad \forall n \in [0, n_{\text{max}}] \\
(1 - e)q_{\text{out}}^{\text{measured}}(n) & \leq q_{\text{out}}(n) \quad \forall n \in [0, n_{\text{max}}] \\
q_{\text{out}}(n) & \leq (1 + e)q_{\text{out}}^{\text{measured}}(n) \quad \forall n \in [0, n_{\text{max}}]
\end{align*}
\]

For this implementation, we chose 20 pieces of upstream and downstream flow data corresponding to 10 minutes of data. The parameters of the triangular flux function
are set with standard values: \( v_f = 70 \text{ mph}, w = -10 \text{ mph}, \rho_c = 30 \text{ veh/(lane.mile)} \). The optimal solution to (3.47) is illustrated in Figure 3.4. The difference between the estimated initial number of vehicles and the ground truth is 32 vehicles, which represents a relative error of 3.95%.

Figure 3.4: Single road traffic initial density estimation.
In this problem, we want to reconstruct the initial densities using two different types of data: GPS data generated by a probe vehicle and flow data. The present objective was to maximize the initial average density (worst-case average density), and the problem involves 52 variables and 1403 constraints.

Due to the low overall accuracy of the LWR scheme, the results may not be very robust in practice. In particular, the output of the estimation problem is very sensitive to model parameter changes, as illustrated in Figure 3.5.
Figure 3.5: Model parameter uncertainty effects.

In this figure, we consider the same estimation problem, but with different model parameters. For the upper subfigure, the free flow speed parameter $v_f$ of the fundamental diagram is chosen to be 65 mph. For the lower subfigure, the same parameter is chosen as 60 mph, all other parameters of the fundamental diagram remain identical. As one can see from both figures, a small change of this parameter has a considerable impact on the estimation. In the first case ($v_f = 65$ mph), the initial number of vehicles in the road is considerably lower (26% relative error with ground truth), and with $v_f = 60$ mph the relative error goes to 48%. Note that both parameters are realistic for the dataset considered, as the uncertainty on $v_f$ is on the order of 10%.
Chapter 4

Applications

This chapter was published at:

- (Section 4.1) Canepa E.S., Bayen A.M. and Claudel C.G., “Spoofing cyber attack detection in probe-based traffic monitoring systems using mixed integer linear programming”, *Networks and Heterogeneous Media*, Volume 8, Number 3, Pages 783-802, Year 2013.

- (Section 4.2) Li Y., Canepa E.S. and Claudel C.G., “Efficient robust control of first order scalar conservation laws using semi-analytical solutions”, *Discrete and Continuous Dynamical Systems - Series S*, Volume 7, Number 3, Pages 525-542, Year 2014.

4.1 Cybersecurity and privacy analysis

We now present some applications of this framework to security and privacy problems occurring in probe-based traffic information systems. Though we present only two examples for compactness, more problems could be posed as mixed integer linear programs in the same framework. Examples of such problems include:

- Real time assessment of vulnerability to attacks

- Offline assessment of worst case effects of attacks
4.1.1 Spoofing cyber attack detection as a mixed integer linear feasibility problem

Given the model, continuity and data constraints presented above, we consider the following feasibility problem:

\[
\text{Find } \bar{y} \\
\text{s.t. } \begin{cases} 
  Ay \leq b \\
  Cy \leq d 
\end{cases} 
\tag{4.1}
\]

Let us denote by \( \bar{y} \) the actual value of the decision variable corresponding to the actual traffic flow scenario. Note that in experimental situations \( \bar{y} \) cannot be measured, unless one has complete knowledge of the state of the system.

If (4.1) is infeasible, there is no set of initial, boundary and internal conditions satisfying at the same time the model and data constraints. Thus, \( \bar{y} \) is either violating the model constraints (i.e. \( A\bar{y} > b \)) or the data constraints (i.e. \( C\bar{y} > d \)), or both.

The interpretation is as follows:

- If \( \bar{y} \) violates the model constraints, then the actual traffic state function does not follow the HJ PDE \( (2.5) \), which can be caused by modeling errors of the flux function (most probable), or by phenomena that are not modeled by the HJ PDE \( (2.5) \).

- If \( \bar{y} \) violates the data constraints, our error model is incorrect. There can be three main reasons for this to happen:
  1. Incorrect error modeling, for instance caused by wrong sensor specifications
  2. Sensor faults (the error model assumes that all sensors are working according to their specifications, i.e. non faulty)
  3. Spoofing attacks
If (4.1) is feasible, there exists a set of initial, boundary and internal conditions compatible both with the traffic flow model and with the observed data. Note that this does not guarantee that no spoofing attack occurs. Indeed, a spoofing attack could occur, but the complete dataset (actual data and spoofed data) would somehow be compatible with the model and the sensor error model. In the remainder of this dissertation, we assume that a spoofing attack is detected whenever (4.1) is infeasible; though in practice this a limited approach. In order to implement a proper cyber attack detection mechanism one has to exclude sensor faults and address the limitations of the model.

4.1.2 Applications to user privacy analysis

An internal condition of the form (3.6) can be interpreted as follows. The coefficients $L_m$ and $r_m$ respectively represent the initial value of the Moskowitz function corresponding to the internal condition (at position $x_{\min}(x)$ and at time $t_{\min}(m)$), and the passing rate (number of vehicles passing the probe vehicle per unit time).

By construction, vehicles trajectories correspond to the isolines of the Moskowitz function [69], assuming that no passing occurs. In this situation, two internal conditions $\mu_m$ and $\mu_n$ are generated by the same vehicle if and only if $L_m = L_n$.

Evidently, the general assumption that vehicles are not allowed to pass each other does not hold in practice, but it is a good approximation when the flow and density of cars are averaged on a multi-lane highway. Under this approximation, the problem of reidentification [83] (i.e. do the internal conditions $\mu_m$ and $\mu_n$ originate from the same vehicle?) can be posed as:

$$\begin{align*}
\text{Minimize} & \quad |L_m - L_n| \\
\text{s. t.} & \quad Ay \leq b \\
& \quad Cy \leq d
\end{align*}$$

(4.2)
If the solution to (4.2) is zero, $\mu_m$ and $\mu_n$ can originate (though not necessarily) from the same vehicle. In the converse case, $\mu_m$ and $\mu_n$ are guaranteed to originate from two different vehicles.

An example of vehicle reidentification is shown in subsection 4.1.6.

### 4.1.3 Experimental setup

We will present some implementation examples of the spoofing attack detection and user privacy framework presented earlier; we will do this using some experimental dataset. The dataset includes fixed sensor data (obtained from inductive loop detectors in the present case) and mobile sensor data.

For the numerical applications, a spatial domain of 3.858 km is considered, located between the PeMS [71] VDS 400536 and 400284 on the Highway I - 880 N around Hayward, California. The data used in this implementation was generated on February 8th, 2008, between times 18:30 and 18:55 (local time). In our scenario, we consider inflow and outflow data $q_{\text{in/out}}^{\text{measured}}(\cdot)$ and $q_{\text{in/out}}^{\text{measured}}(\cdot)$ generated by the above PeMS stations, i.e. we do not assume to know any initial density data. We also consider internal condition data (i.e. probe vehicle data), either real (i.e. extracted from the Mobile Century dataset) or spoofed (drawn randomly according to a specific distribution). The layout of the spatial domain is illustrated in Figure 4.1.

For all subsequent applications, the data constraints are chosen are assumed to be linear: $(1-e)q_{\text{in/out}}^{\text{measured}}(n) \leq q_{\text{in/out}}(n) \leq (1+e)q_{\text{in/out}}^{\text{measured}}(n) \ \forall n \in [0,n_{\max}]$, where $e = 0.01 = 1\%$ is chosen the worst-case relative error of the flow sensors.

We divided the spatial domain into four segments of equal distance $X=965$ m. We also set $T=30$ s as the aggregation time for the flow data ($T$ is determined by the granularity of PeMS data). All MILPs have been implemented using IBM Ilog Cplex working on a Macbook operating MacOS X. The problems described in this dissertation are tractable: they typically involve hundreds of variables and thousands
Figure 4.1: Spatial domain considered for the numerical implementation. The upstream and downstream PeMS stations are delimiting a 3.858 km spatial domain, outlined by a solid line. The direction of traffic flow is represented by an arrow.

of constraints, and are solved in a few tens seconds.

4.1.4 Cyber attack detection example

Our objective is to show the effects of a spoofing attack on traffic flow estimates using mixed boundary flow and probe vehicle data. For this specific application the objective function is chosen as the total number of vehicles at initial time, defined by:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=0}^{F_{\text{max}}} \rho(i) \\
\text{s. t.} & \quad Ay \leq b \\
& \quad Cy \leq d
\end{align*}
\]  

(4.3)

We are mostly interested in the feasibility of the model and data constraints, which will act as a proxy for detecting fake data injection. We consider 20 blocks of upstream (3.7) and downstream (3.5) boundary conditions as well as 6 blocks of (real) internal (3.6) conditions.

As no spoofed data is injected, (4.3) is feasible, and the traffic density maps
corresponding to the minimum and maximum values of the objective function are shown in Figure 4.2 below.

Figure 4.2: Scenarios corresponding to the minimum and maximum number of vehicles (no spoofing attack).

In all subfigures, we compute the scenario for which the initial number of vehicles is the largest (or the smallest), given the boundary data as well as probe data (dashed segments) Top: minimized number of vehicles. Bottom: maximized number of vehicles.

We now simulate the effects of a spoofing attack on the traffic estimates, as well as its detection by the proposed scheme. For this, we progressively incorporate spoofed probe data in the problem and check the solution to (4.3). An attack is detected when (4.3) becomes infeasible.

For our specific scenario, infeasibility occurred after we added only 3 fake internal conditions when the former were drawn from a normal distribution centered around 15 mph, with a standard deviation of 1.25 mph. Note that average speed reported
by the vehicles was around 40 mph, with a standard deviation of 10 mph. Since
the traffic speed was higher than the velocity of the fake internal conditions, these
additions tend to increase the estimated minimal possible density of vehicles on the
highway, as illustrated in Figure 4.3. For the scenario illustrated in Figure 4.3 six
fake internal conditions were randomly chosen with a mean value of 35 mph and a
standard deviation of 1.25 mph. These fake internal conditions have a stronger effect
on the estimation of the minimal number of vehicles than in the previous case, and
go undetected since they do not alter the velocity profile too significantly. Hence, the
best strategy for an attacker is not necessarily to inject fake data that is too far away
from the current traffic condition. Based on these results, this framework has the
potential to detect an attacker manipulation taking place at the low level of a traffic
monitoring system. Since the detection procedure relies on the model, a potential
attacker must be aware of the limitations inherent to the model to plan a proper
manipulation that goes undetected.

The mixed integer linear constraints arising from the model and from the data
define a feasible set in which the real coefficients of the initial, boundary and internal
conditions lie. Adding spoofing data will generate additional inequality constraints,
which will reduce the size of the feasible set. Hence, spoofing cyber attacks can narrow
down the estimates to a specific point of the feasible set, giving artificial confidence
in the (wrong) estimates generated by the system.

4.1.5 Effects of average speed and vehicle distribution on cyber
attack detection

We now illustrate on a specific example the ability of the algorithm to detect a
cyber attack resulting from a faked internal condition that would result in a model
violation. In this specific example, we consider 20 blocks of upstream (3.7) and
downstream (3.5) boundary conditions as well as 6 blocks of (real) internal (3.6)
Figure 4.3: Scenarios corresponding to the computed minimum number of vehicles with spoofing attacks.

The scenario is identical to 4.2 with the addition of spoofed data. Whenever spoofed data is considered, we consider the scenario perturbed by the spoofed data for which an additional piece of spoofed data leads to the detection of the attack by the proposed scheme. Top: minimized number of vehicles with no fake internal condition. Center: minimized number of vehicles with two additional fake internal conditions chosen randomly with a mean value of 15 mph and a standard deviation of 1.25 mph. Bottom: minimized number of vehicles with six additional fake internal conditions chosen randomly with a mean value of 35 mph and a standard deviation of 1.25 mph.
conditions. We add a single internal condition associated with a very low speed (compared to the speed estimated by the model in the area of influence of this internal condition). As expected, problem (4.1) becomes infeasible with this new internal condition. This example is illustrated in Figure 4.4.

Figure 4.4: Example of cyber attack detection. This scenario shows how a single internal condition associated with an unreasonable velocity (compared with the model prediction) can result in an infeasibility of problem (4.1). Top: maximized number of vehicles with no faked internal condition (corresponding to lowest possible average velocity). Bottom: configuration of the internal conditions resulting in an infeasibility of (4.1). The faked internal condition is highlighted in black, and is corresponding to a speed that is much slower than the worst-case speed forecasted by the model in this area.

We now illustrate the effects of the average velocity associated with the faked internal conditions on cyber attack detection. As before, we consider 20 blocks of upstream (3.7) and downstream (3.5) boundary conditions as well as 6 blocks of (real) internal (3.6) conditions. We then generate 12 different sets of internal conditions
Table 4.1: Number of internal conditions required for infeasibility of \((4.1)\) on different scenarios.

We consider the same problem as previously, with 6 blocks of real internal conditions, and randomly generate sets of internal conditions associated with some speed range. The faked internal conditions from the corresponding sets are then added in order into \((4.1)\), until the problem becomes infeasible. Low numbers mean that an attack is detected almost immediately, while large numbers indicate that some amount of faked data can go unnoticed by the model/data consistency check.

Associated with different speed ranges of internal conditions (3 sets per speed range), and progressively add these internal conditions until problem \((4.1)\) becomes infeasible. The results are summarized in table 4.1.

In this specific scenario, the vehicle speed range on the highway was between 30-50 mph. Hence, very low speeds are very likely to cause an infeasibility of the model and data constraints, and indeed an attacker sending such faked internal conditions in the system would be detected immediately. Of course, faked internal conditions corresponding to speeds that are consistent with the average speed of traffic are more likely to go undetected (as in the speed range 30–40 mph in table 4.1), but not always, as illustrated by the first line of this table. In this specific cases, the spatio-temporal locations of the internal conditions caused an infeasibility of \((4.1)\).

4.1.6 Vehicle reidentification example

Our objective is now to illustrate the performance of the framework described above on a non trivial case of vehicle reidentification. We consider the same physical setup as previously, with 20 blocks of upstream \((3.7)\) and downstream \((3.5)\) boundary conditions. We also consider 3 blocks of (real) internal conditions \((3.6)\), extracted from the mobile century dataset. Among these 3 blocks, two originate from the same Mobile
Century test vehicle, and one originates from another mobile century vehicle. The layout is illustrated in Figure 4.5.

![Figure 4.5: Vehicle reidentification problem layout.](image)

In this problem, we consider one block of internal condition (left) generated by a given probe vehicle. We also have two additional blocks of internal condition, generated after the first one. Among these two blocks, one comes from the same vehicle that generated block #1.

Vehicle reidentification problems are at the core of user privacy analysis for probe-based traffic information systems. Indeed, the average distance to confusion [83, 84] is an important metric to evaluate user privacy. However, typical algorithms such as the one used in [83] do not take into account the effects of the flow model. For instance, the reidentification model used in [83] assumes that the velocity of vehicles is more or less constant, and looks for the best candidate within a region of the space-time domain satisfying this constraint. If we apply this procedure to the problem described in Figure 4.5, it is easy to visually check that GPS track #3 is the most probable successor of GPS track #1, since it is in the alignment of track #1. However, in this specific case GPS track #2 is actually the successor to GPS track #1, and GPS track #3 has been generated by another probe vehicle. The model-based reidentification scheme (4.2) is able to capture this fact: minimizing $|L_1 - L_2|$ gives 0, while minimizing $|L_1 - L_3|$ gives 41. Thus, the nonzero optimal value of (4.2) rules out GPS track #3 as a possible successor to GPS track #1, a result that does not seem obvious at all.
by looking at the configuration in Figure 4.5. The density maps corresponding to the computations of (4.2) are illustrated in Figure 4.6.

![Figure 4.6: Example of reidentification.](image)

The corresponding scenario is described in Figure 4.5. Top: GPS data from vehicle to be reidentified. Bottom: The reidentification example with two options that will be evaluated for a match ($|L_1 - L_2|$ and $|L_1 - L_3|$). A nonzero label difference means that both tracks cannot be generated by the same vehicle, according to both the model and the available data.

This result suggests that the framework can help in the vehicle reidentification problem, which is importance for privacy analysis. Indeed, it is very likely that if an attacker gains access to some private probe vehicle data, he or she can also gain access to additional traffic flow measurements from sensors, which are sometimes even public (for instance the PeMS system operating in California, see [71]). As we will show in
next section adding additional traffic information to the estimation framework can benefit the vehicle reidentification in some cases.

4.1.7 Benefits of additional traffic information for vehicle reidentification

In the example below, we illustrate the benefits of additional knowledge for the vehicle reidentification example previously described. As mentioned before, we consider two possible candidates for a reidentification problem, and show that adding some elements of data can improve the results in some cases.

![Figure 4.7: Effects of additional traffic data on reidentification.](image)

In these figures, we illustrate how additional data can remove the ambiguity between two possible successors. The initial configuration is shown in the two upper subfigures for which both possible successors are possible. In the upper left subfigure, we show an example of configuration compatible with IC 2 being the successor of IC 1, while the upper right subfigure shows an example of configuration compatible with IC 3 being the successor of IC 1. In the lower subfigures, we show that adding two additional internal conditions (IC 4 and IC 5) lead with IC 2 being the only possible successor of IC 1. Indeed, the lower left subfigure shows an example of configuration compatible with IC 2 being the successor of IC 1. However, in the lower right subfigure, the minimal possible label difference between IC 3 and IC 1 (given the additional data) jumps to 65.
4.1.8 Model-based reidentification scheme

The reidentification scheme described in [83] attempts to match vehicles from both subsets by assuming a constant velocity of the transmitting vehicle, as illustrated in Figure 4.8 below. This scheme could be improved by checking model feasibility constraints beforehand, which we illustrate in the same figure.

In these figures, our objective is to determine if IC 2, IC 3, IC 4 or IC 5 are originating from the same vehicle as IC 1. The dotted line represents the actual trajectory of the vehicle that generated IC 1, which ends in IC 2 in this figure. The upper subfigure shows the non model-based reidentification scheme described in [83], in which the velocity of the vehicle is assumed to be constant, and the successor of IC 1 is chosen as the closest to the predicted position of the vehicle (circle), assuming a constant velocity (dashed line). The lower subfigure represents a model-based reidentification scheme based on the same assumption, though it removes the infeasible successor points (based on the output of (4.2): a nonzero value indicates that the chosen successor is infeasible). In the lower subfigure example, IC 1 and IC 5 are infeasible, which yields only two candidates: IC 2 and IC 3. Since IC 2 is the closest to the expected future position of the vehicle, it is selected as the likely successor of IC 1.

In preliminary testing with classical traffic flow parameters (obtained from the Highway Capacity Manual [85]) on 300 reidentification problems drawn randomly (in each problem, the actual internal condition is to be chosen among 4 candidates) lead to the following results:

As one can see from the table, the proposed algorithm outperforms the naive reidentification scheme on real data by 8.3%. This proves that, in congested conditions, the model based scheme brings additional correct answers, specially when the GPS
Naive algorithm | Model based check and naive selection
---|---
Correct match | Correct match | Unambiguous match | Wrong match
86 | 111 | 21 | 189
Wrong match | 214 | 21

Table 4.2: Comparison of the reidentification techniques
Naive and model based reidentification results over 300 examples. The model based reidentification outperforms the naive by 25 correct matches.

Data used is sparse like in the examples used for this application. One of the biggest issues when dealing with model-based reidentification of vehicles in practice is model parameter uncertainty, which affects the robustness of the results, as well as prevent the full use of all available data. Indeed, using all available data (in large datasets) would lead to model infeasibility (since the model is assumed to be perfect). In the above table, unambiguous matches refer to matches in which the minimal difference between labels was exactly 0. In 90 other cases (the majority), none of the possible successors could be matched using the model (i.e. the solutions to (4.2) for all possible successors was nonzero). In the latter cases, we used the minimal computed difference between labels (i.e. the solution to (4.2)) as a proxy to select the most likely successor, namely the internal condition for which the difference is minimal. This scheme, as expected, performs similar to the naive scheme (90 matches out of 300), so the zero-label difference framework brings an additional 8.3% correct matches.

Since the Lighthill-Whitham-Richards traffic flow model is not a perfect description of traffic flow propagation, the benefits of the method depends on its relative accuracy on the considered dataset. Handling model uncertainty (which is not done in the current framework) is thus critical to use all available traffic data without making the problem infeasible, and will probably lead to better matching performance.

We illustrate once again the problem of model parameter uncertainty in Figure 3.5 below, in which we show that slightly modifying the model parameters (using a small
number of internal conditions) can lead to dramatically different reidentification results.

4.2 Traffic control on a single highway link

In the previous section, we showed that traffic flow estimation problems could be formulated as optimization programs with linear constraints. In this section, we briefly extend this formulation to traffic flow control problems.

Model constraints describe the physical limitations that originate from the physics of traffic flow propagation, and thus remain identical in this control application. Measurement data constraints $C_{\text{data}}, d_{\text{data}}$ also remain identical, though for the sake of boundary control we sometimes have to remove these constraints to simulate a given input.

4.2.1 Data constraints

In the subsequent problems, we assume that initial conditions are given and cannot be controlled. For instance, in the examples below, the initial condition is fixed by setting prescribed measurements $\rho(m)^{\text{meas}}$, resulting in linear equality constraints in the decision variable described above.

Conversely, we assume that the inflows and outflows ($q_{\text{in}}(t)$ and $q_{\text{out}}(t)$) on the highway link can be controlled for all times; this means that value conditions are created along the upstream and downstream position and their explicit solutions are constraining the outcome (density map). It is also possible to control an arbitrary number of these and set the remainder to prescribed values.

4.2.2 Objective function

Several objective functions can be considered when solving the optimal control problem. Possible functions include the total amount of downstream flow in given simu-
lation time \( t_{max} \), the total time needed to clear congestion or the minimization of the difference between the final state and a prescribed traffic density profile.

In this example, we choose for instance to maximize downstream flow, leading to the following linear program:

\[
\text{Minimize } - \sum_{i=1}^{n_{max}} q_{out}(i) \\
\text{s. t. } \begin{cases} 
A_{model} y \leq b_{model} \\
\rho(k) = \rho(k)_{\text{meas}} 
\end{cases}
\] (4.4)

### 4.2.3 Implementation

In this implementation, we consider a spatial domain of 3.858 km, located between the PeMS VDS 400536 and 400284 on Highway I-880 N around Hayward, California.

The link is divided into 6 segments of equal length \( X = 643 \, m \). The initial density measurements on 6 segments are defined as 6 piece-wise affine constants in the range \([0.5\rho_c, 3\rho_c]\) that represent a partly congested link of a highway. We simulate 15 boundary conditions with a granularity of \( T = 30 \, s \).

We solve the LP (4.4) using IBM Ilog Cplex on a Macbook operating MacOS X. As illustrated in Fig. 4.9, the optimal control of the upstream and downstream flows alleviates traffic congestion by limiting its upstream incoming flow and eventually achieves a free flow condition in less than 7 minutes (simulation time). This particular optimization problem involves 36 variables and 600 constraints, and is solved in 0.02 s.

### 4.2.4 Computational complexity

To the best of our knowledge, previously reported highway control methods require the discretization of the highway link into cells. Let \( n \) represent the number of cells, and \( m \) represent the number of time steps. The number of decision variables required to solve such problems is at least \( n \times m \), since the effects of the model are
In this example, the initial condition is fixed. We control the upstream and downstream boundary flows to maximize the cumulated downstream flow in the interval \([0, t_{\text{max}}]\).
propagated through all cells, at all times. Our approach requires less variables: only $2 \times m$ variables are required for each link, assuming that the initial densities are fixed. Thus the proposed algorithm is expected to significantly outperform (in terms of computational time) other methods whenever the links considered have much more than two cells in average.

4.3 The LWR model with Bounded Acceleration phase

The LWR model (2.2) has been proved to be very robust, simple and tractable for many applications. However, it is unable to capture some traffic features that are commonly observed such as traffic instabilities, stop-and-go waves or kinematic constraints of real vehicles. For instance, the LWR model can produce unrealistic vehicles acceleration or deceleration at the downstream of bottlenecks or traffic signals.

Historically, two macroscopic models were designed starting from the original LWR model to take into account the boundedness of traffic acceleration: while Lebacque has developed a two-phase flow model allowing for general concave fundamental diagrams but depending on a complex mathematical sound basis [86, 87], Leclercq has proposed a model based on a relevant partitioning of the time-space domain that rely on the assumption of a triangular flow-density FD [88, 75]. When a piecewise linear (triangular) fundamental diagram is assumed, it is noteworthy that both models are equivalent to

\[
\frac{\partial k(t, x)}{\partial t} + \frac{\partial (k(t, x)v(t, x))}{\partial x} = 0 \quad \text{with} \quad \frac{\partial v(t, x)}{\partial t} + v(t, x) \frac{\partial v(t, x)}{\partial x} = \begin{cases} 
< a, & \text{if } v(t, x) = V_e(k(t, x)) \\
a, & \text{if } v(t, x) < V_e(k(t, x))
\end{cases}
\]

(4.5)

where $a$ is the maximal acceleration assumed to be identical for all the vehicles.
$V_e : k \mapsto V_e(k)$ denotes the equilibrium speed-density FD such that $\psi(k) = kV_e(k)$ for any $k \in [0, \kappa]$. It is noteworthy that the vehicle trajectories in the bounded acceleration (BA) areas, say when $v \neq V_e(k)$, can be explicitly computed as parabolas (shown below). This BA phase constitutes the single difference with the original LWR model (2.2).

By considering the Moskowitz function $M$ defined previously in (2.3), the LWR-BA model (4.5) can be expressed as a Hamilton-Jacobi PDE as follows

$$\frac{\partial M(t, x)}{\partial t} - \psi \left( - \frac{\partial M(t, x)}{\partial x} \right) = 0, \quad \text{if } v(t, x) = V_e(k(t, x)), \quad (4.6a)$$

$$\begin{cases} 
\frac{\partial M(t, x)}{\partial t} + v(t, x) \frac{\partial M(t, x)}{\partial x} = 0, \\
\frac{\partial v(t, x)}{\partial t} + v(t, x) \frac{\partial v(t, x)}{\partial x} = a, 
\end{cases} \quad \text{if } v(t, x) < V_e(k(t, x)). \quad (4.6b)$$

For additional material on the bounded acceleration LWR model, the interested reader is referred to [89]. It should be notice that an extension of the methodology provided in [79] for a family of second order models recast in Lagrangian coordinates is available in [90].

As an extension of [79], [89] give the explicit solutions for the LWR model with bounded acceleration (4.5) or (4.6a)-(4.6b) for its HJ PDE expression, under the value conditions (3.1).

Unlike the classical solutions for the LWR model, a bounded acceleration phase is introduced (see more precisely (4.6b)) which takes into account the time laps that is needed for a vehicle to go from an initial velocity $v(\tilde{t}, \tilde{x})$ to the free flow speed $v_f$, with a constant acceleration $a$. This duration is thus given by

$$\tau = \frac{v_f - v(\tilde{t}, \tilde{x})}{a}.$$
During this BA phase, the vehicles have a parabolic trajectory described by

\[
\begin{align*}
\ddot{x}(t) &= a, \\
\dot{x}(t) &= a(t - \tilde{t}) + v(\tilde{t}, \tilde{x}), \quad \text{for all} \quad 0 \leq t - \tilde{t} \leq \tau, \\
x(t) &= \frac{a}{2}(t - \tilde{t})^2 + v(\tilde{t}, \tilde{x})(t - \tilde{t}) + \tilde{x},
\end{align*}
\]

After the acceleration phase, we assume that the vehicles move with the free flow speed \( v_f \).

### 4.3.1 Setting for the solutions of LWR with Bounded Acceleration phase

Recall the previously prescribed uni-directional stretch of arterial road \( A := [\xi, \chi] \subset \mathbb{R} \) where \( \xi \) and \( \chi \) denote respectively the upstream and downstream boundaries of the link. Again, the time domain is defined as before by \([0, t_f] \) with a given \( 0 < t_f < +\infty \).

The following assumptions are used for the explicit solutions:

(A1) The spatio-temporal domain is discretized into \( n \) uniform spatial segments \([x_i, x_{i+1}] \) with \( i \in \left[ 0, n - 1 \right] \) and and and \( m \) discrete time segments \([t_j, t_{j+1}] \) with \( j \in \left[ 0, m - 1 \right] \) such that

\[
\xi := x_0 < x_1 < \cdots < x_n := \chi \quad \text{and} \quad 0 =: t_0 < t_1 < \cdots < t_m := t_f.
\]

As mentioned earlier, we do not need the discrete time and spatial steps to be uniform.

(A2) The initial, upstream and downstream and internal boundary conditions denoted respectively by \( M_{\text{ini}}, M_{\text{up}}, M_{\text{down}} \) and \( M_{\text{intern}}^{(l)} \) are piecewise affine on the discrete space and time segments \( \left([x_i, x_{i+1}] \right)_i \) and \( \left([t_j, t_{j+1}] \right)_j \).

We define by \( (k_i)_{0 \leq i \leq n-1} \in \mathbb{R}^n_+ \) the set of initial densities, by \( (q_j)_{0 \leq j \leq m-1} \in \mathbb{R}^m_+ \)
the set of upstream flows, by \((p_j)_{0 \leq j \leq m-1} \in \mathbb{R}_+^m\) the set of downstream flows and by \(\left(M^{(l)}, q_{\text{intern}}^{(l)}\right)_{0 \leq l \leq o-1} \in (\mathbb{R} \times \mathbb{R}_+)^o\) the set of internal boundary conditions. Since these values are constant but they are not known exactly it is important to recall that the objective of our work is to determine these constants thanks to an optimization problem that boils down to a Mixed Integer-Linear Program (MILP).

4.3.2 Explicit solutions

Below, we give the explicit expressions of the partial solutions \(M_{c_{\text{ini}}}^{(i)}, M_{c_{\text{up}}}^{(j)}, M_{c_{\text{down}}}^{(j)}\) and \(M_{c_{\text{intern}}}^{(l)}\) under the assumptions (A0), (A1) and (A2).

Initial condition (free flow)

If \(0 \leq \rho_i \leq \rho_c\), the solution component associated with the affine initial condition (2.16) (see Figure 4.11) is expressed by:

\[
M_{c_{\text{ini}}}^{(i)}(t, x) = \begin{cases} 
(i) & \rho_c(tv_f - x) + b_i + x_i(\rho_c - \rho_i) : x_i + tw \leq x \leq x_i + tv_f \\
(ii) & \rho_i(tv_f - x) + b_i : x_i + tv_f \leq x \leq x_{i+1} + tv_f \\
(iii) & -x_{i+1}\rho_i + b_i : x \geq x_{i+1} + tv_f 
\end{cases}
\]

With \(b_i = \rho_ix_i - \sum_{m=0}^{i-1} (x_{m+1} - x_m)\kappa\)

Initial condition (congested flow)

If \(\rho_c \leq \rho_i \leq \kappa\), the solution component associated with the affine initial condition (2.16) (see Figure 4.12) is expressed by:
\[
\mathbf{M}_{\text{ini}}^{(t)}(t, x) = \begin{cases} 
(i) & \rho_i(tw - x) - \kappa tw + b_i \\
& : x_i + tw \leq x \leq x_i + 1 + tw \\
(ii) & \rho_i(tw - x) - \kappa tw + b_i + \frac{1}{2}\rho_i a T_4^2 \\
& : x_i + 1 + tw \leq x \leq x_a + (t - \tau)w \text{ when } t \geq \tau \text{ and} \\
& \quad x_i + 1 + tw \leq x \leq x_i + 1 + v_{\text{ini}}^{(i)} t + \frac{a}{2} t^2 \text{ when } t \leq \tau \\
(iii) & \frac{1}{v_{\text{ini}}^{(i)}} \left[ x_{i+1} + \tau v_{\text{ini}}^{(i)} + \frac{1}{2} a \tau^2 + (t - \tau) v_f - x \right] \left( \rho_i w - \rho_i v_f - \kappa w \right) + \\
& \quad \rho_i (\tau v_{\text{ini}}^{(i)} + \frac{1}{2} a \tau^2 + (t - \tau) v_f - x) + b_i \\
& : x_a + (t - \tau) w \leq x \leq x_a + (t - \tau) v_f \text{ when } t \geq \tau \\
(iv) & -x_{i+1} \rho_i + b_i \\
& : x_a + (t - \tau) v_f \text{ when } t \geq \tau \text{ or} \\
& \quad x \geq x_{i+1} + v_{\text{ini}}^{(i)} t + \frac{a}{2} t^2 \text{ when } t \leq \tau
\end{cases}
\] (4.8)

where the auxiliary variables are given by

\[
\begin{align*}
\dot{t} &= 0, \quad \bar{x} = x_{i+1}, \quad v(\bar{t}, \bar{x}) = v_{\text{ini}}^{(i)}, \quad \tau = \frac{v_f - v_{\text{ini}}^{(i)}}{a} \\
x_a &= x_{i+1} + v_{\text{ini}}^{(i)} \tau + \frac{a}{2} \tau^2 \\
T_4 &:= \frac{(w - v_{\text{ini}}^{(i)}) + \sqrt{(w - v)^2 - 2a (x_{i+1} - x + tw)}}{a}.
\end{align*}
\]

**Upstream boundary condition (free flow)**

We assume that \(\rho_{\text{up}}^{(j)} \leq \rho_c\), for a given \(j\), say that the upstream boundary \((x = x_0)\) is in free-flow on \([t_j, t_{j+1}]\). Then the solution component associated with the affine upstream boundary condition (2.18) (see Figure 4.13) is expressed by:
\[ M_{c_{up}}(t, x) = \begin{cases} 
(i) & d_j + q_j \left( t - \frac{x - x_0}{v_f} \right) \\
& : x_0 + v_f(t - t_{j+1}) \leq x \leq x_0 + v_f(t - t_j) \\
(ii) & d_j + q_j t_{j+1} + \rho_c((t - t_{j+1})v_f - (x - x_0)) \\
& : x_0 \leq x \leq x_0 + v_f(t - t_{j+1}) \\
(iii) & d_j + q_j t_j \\
& : x \geq x_0 + v_f(t - t_j) 
\end{cases} \]

With \( d_j = -q_j t_j + \sum_{m=0}^{j-1} (t_{m+1} - t_m) q_m \).

**Upstream boundary condition (congested)**

We assume that \( \rho_{up}^{(j)} > \rho_c \), for a given \( j \), say that the upstream boundary \((x = x_0)\) is congested on \([t_j, t_{j+1}]\). This case has not been studied in [89]. It can be seen as a special case of an internal boundary condition for a fixed bottleneck \( (V_{intern} = 0) \) located at \( x = x_0 \).

For any \( j \), let us define

\[
\begin{align*}
\rho_{up}^{(j)} &= \frac{q_j}{\rho_{up}^{(j)}} = \frac{q_j}{q_j + w^\kappa} w \\
\tau &= \frac{v_f - v_{up}^{(j)}}{a} \\
x_a &= x_0 + v_{up}^{(j)} \tau + \frac{a}{2} \tau^2
\end{align*}
\]

Then the solution component associated with the affine upstream boundary condition [2.18] (see Figure 4.14) is computed as follows:
\[ M_c^{(j)}(t, x) = \begin{cases} 
(i) & d_j + q_j t_{j+1} - \rho_c (x - x_0 - v_f(t - t_{j+1})) \\
& : x \leq x_0 + \frac{v_{up}^{(j)}}{a} (t - t_{j+1}) + \frac{a}{2} (t - t_{j+1})^2 \text{ and} \\
& \quad x \leq x_a + w(t - t_{j+1} - \tau) \text{ and } t \geq t_{j+1} \\
(ii) & d_j + q_j t_{j+1} - \rho_c (x - x_0 - v_f(t - t_{j+1}) - \frac{a}{2} \tau^2) \\
& : x \leq x_a + v_f(t - t_{j+1} - \tau) \text{ and} \\
& \quad x \geq x_a + w(t - t_{j+1} - \tau) \text{ and } t \geq t_{j+1} + \tau \\
(iii) & d_j + q_j \left[ t + \frac{1}{a} \left( \frac{v_{up}^{(j)}}{a} - \sqrt{\left( \frac{v_{up}^{(j)}}{a} \right)^2 + 2a(x - x_0)} \right) \right] \\
& : x \leq x_0 + \frac{v_{up}^{(j)}}{a} (t - t_j) + \frac{a}{2} (t - t_j)^2 \text{ and} \\
& \quad x \geq x_0 + \frac{v_{up}^{(j)}}{a} (t - t_{j+1}) + \frac{a}{2} (t - t_{j+1})^2 \text{ when } t \geq t_{j+1} \text{ and} \\
& \quad x_0 \leq x \leq x_a \text{ and } t_j \leq t \leq t_{j+1} + \tau \\
(iv) & d_j + q_j \left[ t - \frac{x - x_a}{v_f} - \tau \right] \\
& : x \leq x_a + v_f(t - t_j - \tau) \text{ and} \\
& \quad x \geq x_a + v_f(t - t_{j+1} - \tau) \text{ and} \\
& \quad x \geq x_a \text{ and } t \geq t_j + \tau \\
(v) & d_j + q_j t_j \\
& : x \leq x_0 + v_f(t - t_j) \text{ and} \\
& \quad x \geq x_0 + \frac{v_{up}^{(j)}}{a} (t - t_j) + \frac{a}{2} (t - t_j)^2 \text{ when } t \leq t_j + \tau \\
& \quad \text{or} \\
& \quad x \geq x_a + v_f(t - t_j - \tau) \text{ when } t \geq t_j + \tau 
\end{cases} \]

**Downstream boundary condition**

As previously mentioned, we only consider congested downstream boundary conditions that have a domain of influence that belongs to our computation domain \([t_0, T] \times [x_0, x_n]\).
We define

\[
\begin{aligned}
&v_{\text{down}}^{(j)} = \frac{p_j}{p_j + w\kappa} w \\
&\tau = \frac{v_f - v_{\text{down}}^{(j)}}{a} \\
x_a = x_n + v_{\text{down}}^{(j)}\tau + \frac{a}{2}\tau^2 \\
t_a = t_{j+1} + \tau
\end{aligned}
\]

If \(v_{\text{down}}^{(j)} < v_f\), then the solution component associated with the affine downstream boundary condition (2.20) (see Figure 4.15) is expressed by:

\[
M_{c_{\text{down}}}^{(j)}(t, x) =
\begin{cases}
(i) & b_j + p_j t + \left(\frac{p_j}{w} + \kappa\right)(x_n - x) \\
& : x_n + w(t - t_j) \leq x \leq x_n + w(t - t_{j+1}) \\
(ii) & b_j + p_j t_{j+1} + \left(\frac{w - v_{\text{down}}^{(j)}}{t - t_{j+1}} - \frac{\sqrt{(w - v_{\text{down}}^{(j)})^2 + 2a(w(t_{j+1} - t) + x - x_a)}}{a}\right)w \\
& : x_n + w(t - t_{j+1}) \leq x \leq x_n + v_{\text{down}}^{(j)}(t - t_{j+1}) + \frac{a}{2}(t - t_{j+1})^2 \\
& \text{when } 0 \leq (t - t_{j+1}) \leq \tau \text{ and } x_n + w(t - t_{j+1}) \leq x \leq x_a + (t - t_a)v_f \text{ when } (t - t_{j+1}) \geq \tau \\
(iii) & b_j + \left(p_j t_{j+1} + \frac{w\kappa}{v_f - w}\right)(x - x_a + v_f(t_a - t)) \\
& : x_a + (t - t_a)w \leq x \leq x_a + (t - t_a)v_f \text{ and } (t - t_{j+1}) \geq \tau \\
(iv) & b_j + \left(\frac{p_j}{w} + \kappa\right)wt_{j+1} \\
& : x \geq x_a + (t - t_a)v_f \text{ when } (t - t_{j+1}) \geq \tau \text{ and } x \geq x_n + v_{\text{down}}^{(j)}(t - t_{j+1}) + \frac{a}{2}(t - t_{j+1})^2 \text{ when } 0 \leq (t - t_{j+1}) \leq \tau
\end{cases}
\]
With \( b_j = M_{ini}(x_n) - p_j t_j + \sum_{m=0}^{j-1} (t_{m+1} - t_m)p_m \).

Notice that (iv) in (4.11) is outside of the computational domain.

**Internal boundary condition**

We define the following auxiliary variables:

\[
\begin{align*}
  x_{max}^{(l)} &= x_{min}^{(l)} + V_{intern}^{(l)} (t_{max}^{(l)} - t_{min}^{(l)}) \\
  \tau &= \frac{v_f - V_{intern}^{(l)}}{a} \\
  x_{a, min} &= x_{min}^{(l)} + V_{intern}^{(l)} \tau + \frac{a}{2} \tau^2 \\
  t_{a, min} &= t_{min}^{(l)} + \tau \\
  x_{a, max} &= x_{max}^{(l)} + V_{intern}^{(l)} \tau + \frac{a}{2} \tau^2 \\
  t_{a, max} &= t_{max}^{(l)} + \tau 
\end{align*}
\]

The solution component associated with the affine internal boundary condition (2.22) (see Figure 4.16) is expressed by:
\[ M_{\text{internal}}^{(l)}(t, x) = \begin{cases} 
(i) & M^{(l)} - v_2 t_{\min}^{(l)} + ((x_{\min}^{(l)} - x) + tv_2) \rho_2 \\
& \quad : x \leq x_{\min}^{(l)} + V(t - t_{\min}^{(l)}) \text{ and} \\
& \quad x \geq x_{\min}^{(l)} + w(t - t_{\min}^{(l)}) \text{ and} \\
& \quad x \leq x_{\max}^{(l)} + w(t - t_{\max}^{(l)}) \text{ and} \\
& \quad t \geq t_{\min}^{(l)} \\
(ii) & M^{(l)} - v_2 t_{\min}^{(l)} + ((x_{\min}^{(l)} - x + tv_2) \rho_2 + \frac{\rho_2 w^2}{2} T_2^2 \\
& \quad : x \geq x_{\max}^{(l)} + w(t - t_{\max}^{(l)}) \text{ and} \\
& \quad x \leq x_{a,\max}^{(l)} + w(t - t_{a,\max}^{(l)}) \text{ and} \\
& \quad x \leq x_{\max}^{(l)} + V(t - t_{a,\max}^{(l)}) + \frac{a}{2} (t - t_{\max}^{(l)})^2 \text{ and} \\
& \quad t \geq t_{\min}^{(l)} \\
(iii) & M^{(l)} - v_2 t_{\min}^{(l)} + ((x_{\min}^{(l)} - x + tv_2) \rho_2 + \frac{\rho_2 w^2}{2} T_2^2 \\
& \quad : x \leq x_{\min}^{(l)} + V(t - t_{\min}^{(l)}) + \frac{a}{2} (t - t_{\min}^{(l)})^2 \text{ and} \\
& \quad x \geq x_{a,\min}^{(l)} + V(t - t_{a,\min}^{(l)}) \text{ and} \\
& \quad x \geq x_{\max}^{(l)} + V(t - t_{a,\max}^{(l)}) + \frac{a}{2} (t - t_{\max}^{(l)})^2 \text{ and} \\
& \quad x \geq x_{\min}^{(l)} + V(t - t_{a,\min}^{(l)}) \text{ and} \\
& \quad t \geq t_{\min}^{(l)} \\
(iv) & M^{(l)} + v_2 (t_1 - t_{\min}^{(l)}) - v_f t_1 + ((x_{\min}^{(l)} - x + \tau v + \frac{ax^2}{2} + (t - \tau) v_f) \rho_2 \\
& \quad : x \leq x_{a,\min}^{(l)} + v_f (t - t_{a,\min}^{(l)}) \text{ and} \\
& \quad x \geq x_{a,\max}^{(l)} + v_f (t - t_{a,\max}^{(l)}) \text{ and} \\
& \quad x \geq x_{a,\min}^{(l)} + V(t - t_{a,\min}^{(l)}) \text{ and} \\
& \quad t \geq t_{\min}^{(l)} + \tau \\
(v) & M^{(l)} + v_2 (t_2 - t_{\min}^{(l)}) - v_f t_2 + ((x_{\min}^{(l)} - x + \tau v + \frac{ax^2}{2} + (t - \tau) v_f) \rho_2 \\
& \quad : x \geq x_{a,\max}^{(l)} + v_f (t - t_{a,\max}^{(l)}) \text{ and} \\
& \quad x \leq x_{a,\max}^{(l)} + w(t - t_{a,\max}^{(l)}) \text{ and} \\
& \quad t \geq t_{\max}^{(l)} + \tau \\
(vi) & M^{(l)} \\
& \quad : \text{elsewhere} \end{cases} \]
where $\rho_2$ is the upper solution of $Q(\rho) - \rho V_{\text{intern}}^{(l)} = q_{\text{intern}}^{(l)}$ such that $\rho_c \leq \rho_2 \leq \kappa$, say

$$\rho_2 = \frac{q_{\text{intern}}^{(l)} + w \kappa}{w - V_{\text{intern}}^{(l)}} \quad \text{and} \quad v = \frac{Q(\rho_2)}{\rho_2} = w \left(1 - \frac{\kappa}{\rho_2}\right)$$

and

$$T_2 = \frac{(V_{\text{intern}}^{(l)} - v) + \sqrt{(V_{\text{intern}}^{(l)} - v)^2 + 2a(x - x_{\text{min}}^{(l)} - V_{\text{intern}}^{(l)}(t - t_{\text{min}}^{(l)}))}}{a}$$

$$T_4 = \frac{(w - v) + \sqrt{(w - v)^2 + 2a(x - x_{\text{max}}^{(l)} - w(t - t_{\text{max}}^{(l)}))}}{a}$$

$$t_1 = \frac{1}{v_f - V_{\text{intern}}^{(l)}} \left( x_{\text{min}}^{(l)} - V_{\text{intern}}^{(l)} - x + \tau v + \frac{a \tau^2}{2} + (t - \tau) v_f \right)$$

$$t_2 = \frac{1}{v_f - w} \left( x_{\text{max}}^{(l)} - w t_{\text{max}}^{(l)} - x + \tau v + \frac{a \tau^2}{2} + (t - \tau) v_f \right)$$

Remark 4 It is noteworthy that letting $V_{\text{intern}}^{(l)}$ going to zero (for any $0 \leq l \leq o - 1$) such that $x_{\text{min}}^{(l)} = x_{\text{max}}^{(l)} = x_0$ (or respectively $= x_n$) and

$$q_{\text{intern}}^{(l)} = v \rho_2 = w(\rho_2 - \kappa)$$

and assuming that there exists a $j \in [0, m - 1]$ such that $t_{\text{min}}^{(l)} = t_j$ and $t_{\text{max}}^{(l)} = t_{j+1}$ ($q_{\text{intern}}^{(l)} = q_j$, $M^{(l)} = d_j + t_j q_j$, $\rho_2 = \rho_{\text{up}}^{(j)}$ and $v = v_{\text{up}}^{(j)}$ or resp. $q_{\text{intern}}^{(l)} = p_j$, $M^{(l)} = b_j + t_j p_j$ and $v = v_{\text{down}}^{(j)}$), then passing to the limit in (4.12), one can recover the expressions of the partial solution for congested upstream boundary conditions (4.10) (resp. for downstream boundary conditions (4.11)).

### 4.4 The optimization-based queue length estimation method

The explicit solutions for the LWR-BA model under piecewise affine boundary conditions have been stated in the previous section. We are now ready to introduce our optimization problem for the estimation of the queue length. The decision variable
associated with the value conditions (2.16), (2.18), (2.20) and (2.22) is defined as follows

\[ y := \left( \ldots, \rho_i, \ldots, q_j, \ldots, p_j, \ldots, M^{(l)}, q_{\text{intern}}^{(l)}, \ldots \right). \quad (4.14) \]

This decision variable will allow to determine the set of the boundary values that optimize a prescribed cost function under some constraints. The objective function should recapture the most likely link dynamics with a general reconstruction of the averaged queuing behavior on the arterial. The constraints are presented below. We distinguish the model constraints that appear from the physical limitations due to the considered traffic flow model and the data constraints that are triggered by the measurements we have on the current traffic state, under some uncertainties due to sensor inaccuracies. These constraints are stated as convex inequality constraints in terms of the decision variable \( y \).

### 4.4.1 Model constraints

**Preliminaries**

Following [31], we have the following property:

**Proposition 12** [Compatibility conditions] Let us consider a family of value conditions \( c_j \) that are lower semi-continuous functions defined on subsets of \([0, T] \times [\xi, \chi]\) and let us define their minimum

\[ c(t, x) := \min_{j \in \mathcal{J}} c_j(t, x). \]
Then, the solution $M$ of (4.6a)-(4.6b) and (3.1) verifies

$$M(t, x) = c(t, x), \text{ for any } (t, x) \in \text{Dom}(c),$$

if and only if the following set of compatibility conditions

$$M_{c_i}(t, x) \geq c_j(t, x), \text{ for all } i, j \in \mathbb{J}, \text{ and } (t, x) \in \text{Dom}(c_j)$$

are satisfied.

**Proof —** See Proposition 3.6 in [31].

In the next Subsections, we give the semi-analytical expressions of the compatibility conditions for the LWR-BA model under the assumptions (A0), (A1) and (A2). Notice that the full derivation of the set of inequalities for the LWR model are provided in [91].

Before, we need to introduce the following notations:

1. The parabola parameters

   $$
   \begin{align*}
   t_a(\tilde{t}, \tilde{x}) & := \tilde{t} + \tau = \tilde{t} + \frac{v_f - v(\tilde{t}, \tilde{x})}{a}, \\
   x_a(\tilde{t}, \tilde{x}) & := \frac{a}{2} \tau^2 + v(\tilde{t}, \tilde{x}) \tau + \tilde{x} \left( = x \left( t_a(\tilde{t}, \tilde{x}) \right) \right)
   \end{align*}
   \tag{4.15}
   $$

2. Consider the intersection between two straight lines

   $$y_0 = x_0 + (t - t_0)v_0 \quad \text{and} \quad y_1 = x_1 + (t - t_1)v_1.$$
Whenever $v_0 \neq v_1$, it occurs at time

$$
\tilde{T} : (x_0, t_0, v_0, x_1, t_1, v_1) \mapsto \frac{x_1 - x_0 + v_0 t_0 - v_1 t_1}{v_0 - v_1} \quad (4.16)
$$

3. Consider the intersection point(s) between a straight line

$$
y_0 = x_0 + (t - t_0)v_0
$$

and a parabola

$$
y_1 = x_1 + (t - t_1)v_1 + \frac{a}{2}(t - t_1)^2
$$

that is given by the solution(s) of the following polynomial of degree 2

$$
\frac{a}{2} t^2 + (v_1 - v_0 - at_1) t + \left( x_1 - x_0 + t_0v_0 - t_1v_1 + \frac{a}{2}(t_1)^2 \right) = 0
$$

Let us set

$$
\Delta = B^2 - 4AC
$$

$$
= (v_1 - v_0)^2 + 2a [(t_1 - t_0)v_0 - (x_1 - x_0)]
$$

Then we have to distinguish the following cases

- If $\Delta < 0$, there is no real solution.
- If $\Delta = 0$, then there is a single solution $t := -\frac{B}{2A} = t_1 + \frac{(v_0 - v_1)}{a}$
- If $\Delta > 0$, then there exist two solutions $t_+, t_-$ (with $t_+ > t_-$) that are computed thanks to $t_\pm := -\frac{B \pm \sqrt{\Delta}}{2A} = t_1 + \frac{(v_0 - v_1)}{a}

\pm \sqrt{\left( \frac{v_1 - v_0}{a} \right)^2 + \frac{2}{a} [(t_1 - t_0)v_0 - (x_1 - x_0)]}$.

As we are looking at the intersection points for checking compatibility conditions, we implicitly assume that in all cases we have $\Delta \geq 0$. We thus set the following function that gives the time(s) for which the line and the parabola
intersect

\[ \hat{T}_\pm : (x_0, t_0, v_0, x_1, t_1, v_1) \mapsto t_1 + \frac{(v_0 - v_1)}{a} \]

\[ \pm \sqrt{\left( \frac{v_1 - v_0}{a} \right)^2 + \frac{2}{a} \left[ (t_1 - t_0)v_0 - (x_1 - x_0) \right]} \].

(4.17)

Initial condition (free flow)

See Figure 4.11

\[ \begin{cases} 
    M_{c_{ini}}^{(i)} \left( t_0 + \frac{x_n - x_i}{w}, x_0 \right) \geq c^{(j)}_{up} \left( t_0 + \frac{x_n - x_i}{w}, x_0 \right), \\
    \text{for any } i, j \text{ such that } t_0 + \frac{x_n - x_i}{w} \in [t_j, t_{j+1}], \\
    M_{c_{ini}}^{(i)} \left( t_0 + \frac{x_n - x_i}{v_f}, x_n \right) \geq c^{(j)}_{down} \left( t_0 + \frac{x_n - x_i}{v_f}, x_n \right), \\
    \text{for any } i, j \text{ such that } t_0 + \frac{x_n - x_i}{v_f} \in [t_j, t_{j+1}], \\
    M_{c_{ini}}^{(i)} \left( t^{(i,l)}_1, x^{(i,l)}_1 \right) \geq c^{(l)}_{intern} \left( t^{(i,l)}_1, x^{(i,l)}_1 \right), \\
    \text{for any } i, l \text{ such that } t^{(l)}_{min} \leq t^{(i,l)}_1 \leq t^{(l)}_{max}, \\
    M_{c_{ini}}^{(i)} \left( t^{(i,l)}_2, x^{(i,l)}_2 \right) \geq c^{(l)}_{intern} \left( t^{(i,l)}_2, x^{(i,l)}_2 \right), \\
    \text{for any } i, l \text{ such that } t^{(l)}_{min} \leq t^{(i,l)}_2 \leq t^{(l)}_{max}, 
\end{cases} \]

(4.18)

with

\[ \begin{align*}
    t^{(i,l)}_1 & := \hat{T} \left( x_i, t_0, v_f, x^{(i,l)}_1, t^{(l)}_{min}, V^{(l)}_{intern} \right), \\
    x^{(i,l)}_1 & := x^{(l)}_{min} + V^{(l)}_{intern} \left( t^{(i,l)}_1 - t^{(l)}_{min} \right), \\
    t^{(i,l)}_2 & := \hat{T} \left( x_i, t_0, w, x^{(i,l)}_2, t^{(l)}_{min}, V^{(l)}_{intern} \right), \\
    x^{(i,l)}_2 & := x^{(l)}_{min} + V^{(l)}_{intern} \left( t^{(i,l)}_2 - t^{(l)}_{min} \right). 
\end{align*} \]
Initial condition (congested flow)

See Figure 4.12.

\[
\begin{align*}
M_{c_{\text{ini}}}^{(i)} \left( t_0 + \frac{x_0 - x_i}{w} , x_0 \right) & \geq c_{\text{up}}^{(j)} \left( t_0 + \frac{x_0 - x_i}{w} , x_0 \right), \\
& \text{for any } i, j \text{ such that } t_0 + \frac{x_0 - x_i}{w} \in [t_j, t_{j+1}], \\
M_{c_{\text{ini}}}^{(i)} \left( t_a + \frac{x_a - x_i}{w} , x_0 \right) & \geq c_{\text{up}}^{(j)} \left( t_a + \frac{x_a - x_i}{w} , x_0 \right), \\
& \text{for any } i, j \text{ such that } t_a + \frac{x_a - x_i}{w} \in [t_j, t_{j+1}], \\
M_{c_{\text{ini}}}^{(i)} \left( t_a + \frac{x_n - x_a}{w} , x_n \right) & \geq c_{\text{down}}^{(j)} \left( t_a + \frac{x_n - x_a}{w} , x_n \right), \\
& \text{for any } i, j \text{ such that } t_a + \frac{x_n - x_a}{w} \in [t_j, t_{j+1}], \\
M_{c_{\text{ini}}}^{(i)} \left( t_a + \frac{x_n - x_i}{v_f} , x_n \right) & \geq c_{\text{down}}^{(j)} \left( t_a + \frac{x_n - x_i}{v_f} , x_n \right), \\
& \text{for any } i, j \text{ such that } t_a + \frac{x_n - x_i}{v_f} \in [t_j, t_{j+1}], \\
M_{c_{\text{ini}}}^{(i)} \left( t_0^{(i,j)} , x_n \right) & \geq c_{\text{down}}^{(j)} \left( t_0^{(i,j)} , x_n \right), \\
& \text{for any } i, j \text{ such that } t_0^{(i,j)} \in [t_j, t_{j+1}], \\
M_{c_{\text{ini}}}^{(i)} \left( t_1^{(i,l)} , x_1^{(i,l)} \right) & \geq c_{\text{intern}}^{(l)} \left( t_1^{(i,l)} , x_1^{(i,l)} \right), \\
& \text{for any } i, l \text{ such that } t_1^{(i,l)} \leq t_1^{(i,l)} \leq t_{\text{max}}, \\
M_{c_{\text{ini}}}^{(i)} \left( t_2^{(i,l)} , x_2^{(i,l)} \right) & \geq c_{\text{intern}}^{(l)} \left( t_2^{(i,l)} , x_2^{(i,l)} \right), \\
& \text{for any } i, l \text{ such that } t_2^{(i,l)} \leq t_2^{(i,l)} \leq t_{\text{max}}, \\
M_{c_{\text{ini}}}^{(i)} \left( t_3^{(i,l)} , x_3^{(i,l)} \right) & \geq c_{\text{intern}}^{(l)} \left( t_3^{(i,l)} , x_3^{(i,l)} \right), \\
& \text{for any } i, l \text{ such that } t_3^{(i,l)} \leq t_3^{(i,l)} \leq t_{\text{max}}, \\
M_{c_{\text{ini}}}^{(i)} \left( t_4^{(i,l)} , x_4^{(i,l)} \right) & \geq c_{\text{intern}}^{(l)} \left( t_4^{(i,l)} , x_4^{(i,l)} \right), \\
& \text{for any } i, l \text{ such that } t_4^{(i,l)} \leq t_4^{(i,l)} \leq t_{\text{max}},
\end{align*}
\]
\[
\begin{align*}
  t_0^{(i,j)} &= \hat{T}_+ \left( x_n, t_j, 0, x_{i+1}, t_0, v^{(i)} \right), \\
  t_1^{(i,l)} &= \hat{T}_- \left( x_{i+1}, t_a, v_f, x_{\min}, t_{\min}, V(l)_{\text{intern}} \right), \\
  x_1^{(i,l)} &= x^{(i,l)}_{\min} + V(l)_{\text{intern}} \left( t_1^{(i,l)} - t^{(l)}_{\min} \right), \\
  t_2^{(i,l)} &= \hat{T}_- \left( x_{i+1}, t_a, v_f, x^{(l)}_{\min}, t_{\min}, V(l)_{\text{intern}} \right), \\
  x_2^{(i,l)} &= x^{(l)}_{\min} + V(l)_{\text{intern}} \left( t_2^{(l)} - t^{(l)}_{\min} \right), \\
  t_3^{(i,l)} &= \hat{T}_- \left( x_{i+1}, t_0, w, x^{(l)}_{\min}, t^{(l)}_{\min}, V(l)_{\text{intern}} \right), \\
  x_3^{(i,l)} &= x^{(l)}_{\min} + V(l)_{\text{intern}} \left( t_3^{(l)} - t^{(l)}_{\min} \right), \\
  t_4^{(i,l)} &= \hat{T}_- \left( x^{(l)}_{\min}, t_{\min}, V(l)_{\text{intern}}, x_{i+1}, t_0, v^{(i)} \right), \\
  x_4^{(i,l)} &= x^{(l)}_{\min} + V(l)_{\text{intern}} \left( t_4^{(l)} - t^{(l)}_{\min} \right).
\end{align*}
\]

See Figure 4.13.
See Figure 4.14.

Upstream boundary condition (congested)

\[
\begin{align*}
M_{c_{\text{up}}}^{(j,t)} & \left( t_a + \frac{x_0 - x_a}{w}, x_0 \right) \geq c_{\text{up}}^{(j')} \left( t_a + \frac{x_0 - x_a}{w}, x_0 \right), \\
\quad \text{for any } j, j' \text{ such that } t_a + \frac{x_0 - x_a}{w} \in [t_{j'}, t_{j'} + 1], \\
M_{c_{\text{up}}}^{(j,t)} & \left( t_j + \frac{x_n - x_0}{v_f}, x_n \right) \geq c_{\text{down}}^{(j')} \left( t_j + \frac{x_n - x_0}{v_f}, x_n \right), \\
\quad \text{for any } j, j' \text{ such that } t_j + \frac{x_n - x_0}{v_f} \in [t_{j'}, t_{j'} + 1], \\
M_{c_{\text{up}}}^{(j,t)} & \left( t_a + \frac{x_n - x_a}{w}, x_n \right) \geq c_{\text{down}}^{(j')} \left( t_a + \frac{x_n - x_a}{w}, x_n \right), \\
\quad \text{for any } j, j' \text{ such that } t_a + \frac{x_n - x_a}{w} \in [t_{j'}, t_{j'} + 1], \\
M_{c_{\text{up}}}^{(j,t)} & \left( t_0^{(j,j')}, x_n \right) \geq c_{\text{down}}^{(j')} \left( t_0^{(j,j')}, x_n \right), \
\quad \text{for any } j, j' \text{ such that } t_0^{(j,j')} \in [t_{j'}, t_{j'} + 1], \\
M_{c_{\text{up}}}^{(j,t)} & \left( t_1^{(j,l)}, x_1^{(j,l)} \right) \geq c_{\text{intern}}^{(l)} \left( t_1^{(j,l)}, x_1^{(j,l)} \right), \\
\quad \text{for any } j, l \text{ such that } t_1^{(j,l)} \leq t_{\text{max}}^{(l)}, \\
M_{c_{\text{up}}}^{(j,t)} & \left( t_2^{(j,l)}, x_2^{(j,l)} \right) \geq c_{\text{intern}}^{(l)} \left( t_2^{(j,l)}, x_2^{(j,l)} \right), \\
\quad \text{for any } j, l \text{ such that } t_1^{(j,l)} \leq t_{\text{max}}^{(l)}, \\
M_{c_{\text{up}}}^{(j,t)} & \left( t_3^{(j,l)}, x_3^{(j,l)} \right) \geq c_{\text{intern}}^{(l)} \left( t_3^{(j,l)}, x_3^{(j,l)} \right), \\
\quad \text{for any } j, l \text{ such that } t_1^{(j,l)} \leq t_{\text{max}}^{(l)}, \\
M_{c_{\text{up}}}^{(j,t)} & \left( t_4^{(j,l)}, x_4^{(j,l)} \right) \geq c_{\text{intern}}^{(l)} \left( t_4^{(j,l)}, x_4^{(j,l)} \right), \\
\quad \text{for any } j, l \text{ such that } t_1^{(j,l)} \leq t_{\text{max}}^{(l)},
\end{align*}
\]

(4.21)
with

\[
\begin{align*}
t_a &= t_a(t_{j+1}, x_0), \\
x_a &= x_a(t_{j+1}, x_0), \\
\end{align*}
\]

\[
\begin{align*}
t_0^{(j,j')} &= \hat{T}_0 \left( x_n, t_{j'}, 0, x_0, t_{j+1}, v_{up}^{(j)} \right), \\
t_1^{(j,l)} &= \hat{T}_1 \left( x_0, t_j, v_f, x_{\min}^{(l)}, t_{\min}^{(l)}, V_{\text{intern}}^{(l)} \right), \\
x_1^{(j,l)} &= x_{\min}^{(l)} + V_{\text{intern}}^{(l)}(t_1^{(j,l)} - t_{\min}^{(l)}), \\
t_2^{(j,l)} &= \hat{T}_2 \left( x_a, t_a, v_f, x_{\min}^{(l)}, t_{\min}^{(l)}, V_{\text{intern}}^{(l)} \right), \\
x_2^{(j,l)} &= x_{\min}^{(l)} + V_{\text{intern}}^{(l)}(t_2^{(j,l)} - t_{\min}^{(l)}), \\
t_3^{(j,l)} &= \hat{T}_3 \left( x_a, t_a, w, x_{\min}^{(l)}, t_{\min}^{(l)}, V_{\text{intern}}^{(l)} \right), \\
x_3^{(j,l)} &= x_{\min}^{(l)} + V_{\text{intern}}^{(l)}(t_3^{(j,l)} - t_{\min}^{(l)}), \\
t_4^{(j,l)} &= \hat{T}_4 \left( x_0, t_{j+1}, v_{up}^{(j)}, x_{\min}^{(l)}, t_{\min}^{(l)}, V_{\text{intern}}^{(l)} \right), \\
x_4^{(j,l)} &= x_{\min}^{(l)} + V_{\text{intern}}^{(l)}(t_4^{(j,l)} - t_{\min}^{(l)}).
\end{align*}
\]
Downstream boundary condition

See Figure [4.15]

\[
\begin{align*}
M_{c_{\text{down}}}^{(j)} \left( t_j + \frac{x_0 - x_n}{w}, x_0 \right) & \geq c_{\text{up}}^{(j')} \left( t_j + \frac{x_0 - x_n}{w}, x_0 \right), \\
& \text{for any } j, j' \text{ such that } t_j + \frac{x_0 - x_n}{w} \in [t_{j'}, t_{j'+1}],
\end{align*}
\]

\[
\begin{align*}
M_{c_{\text{down}}}^{(j)} \left( t_a + \frac{x_0 - x_a}{w}, x_0 \right) & \geq c_{\text{up}}^{(j')} \left( t_a + \frac{x_0 - x_a}{w}, x_0 \right), \\
& \text{for any } j, j' \text{ such that } t_a + \frac{x_0 - x_a}{w} \in [t_{j'}, t_{j'+1}],
\end{align*}
\]

\[
\begin{align*}
M_{c_{\text{down}}}^{(j)} \left( t_{(j,l)}^{(1)}, x_{(j,l)}^{(1)} \right) & \geq c_{\text{intern}}^{(l)} \left( t_{(j,l)}^{(1)}, x_{(j,l)}^{(1)} \right), \\
& \text{for any } j, l \text{ such that } t_{l_{\text{min}}} \leq t_{(j,l)}^{(1)} \leq t_{l_{\text{max}}},
\end{align*}
\]

\[
\begin{align*}
M_{c_{\text{down}}}^{(j)} \left( t_{(j,l)}^{(2)}, x_{(j,l)}^{(2)} \right) & \geq c_{\text{intern}}^{(l)} \left( t_{(j,l)}^{(2)}, x_{(j,l)}^{(2)} \right), \\
& \text{for any } j, l \text{ such that } t_{l_{\text{min}}} \leq t_{(j,l)}^{(2)} \leq t_{l_{\text{max}}},
\end{align*}
\]

with

\[
\begin{align*}
t_a & = t_a \left( t_j + 1, x_n \right), \\
x_a & = x_a \left( t_j + 1, x_n \right), \\
\end{align*}
\]

\[
\begin{align*}
t_{(j,l)}^{(1)} & = \hat{T} \left( x_n, t_j + 1, w, x_{(j,l)}^{(1)} \min, V_{(j,l)}^{(1)} \min \right), \\
x_{(j,l)}^{(1)} & = x_{(j,l)}^{(1)} + V_{(j,l)}^{(1)} \min \left( t_{(j,l)}^{(1)} - t_{(j,l)}^{(1)} \min \right), \\
\end{align*}
\]

\[
\begin{align*}
t_{(j,l)}^{(2)} & = \hat{T} \left( x_a, t_a, w, x_{(j,l)}^{(2)} \min, V_{(j,l)}^{(2)} \min \right), \\
x_{(j,l)}^{(2)} & = x_{(j,l)}^{(2)} + V_{(j,l)}^{(2)} \min \left( t_{(j,l)}^{(2)} - t_{(j,l)}^{(2)} \min \right).
\end{align*}
\]
See Figure 4.16

\begin{align}
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{\min}^{(l)} + \frac{x_0 - x_{\min}^{(l)}}{w}, x_0 \right) & \geq \mathbf{c}_{\text{up}}^{(j)} \left( t_{\min}^{(l)} + \frac{x_0 - x_{\min}^{(l)}}{w}, x_0 \right), \\
\text{for any } l, j \text{ such that } & t_{\min}^{(l)} + x_0 - x_{\min}^{(l)} \in [t_j, t_{j+1}], \\
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{\max}^{(l)} + \frac{x_0 - x_{\max}^{(l)}}{w}, x_0 \right) & \geq \mathbf{c}_{\text{up}}^{(j)} \left( t_{\max}^{(l)} + \frac{x_0 - x_{\max}^{(l)}}{w}, x_0 \right), \\
\text{for any } l, j \text{ such that } & t_{\max}^{(l)} + x_0 - x_{\max}^{(l)} \in [t_j, t_{j+1}], \\
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{\max}^{(l)} + \frac{x_n - x_{\max}^{(l)}}{w}, x_n \right) & \geq \mathbf{c}_{\text{down}}^{(j)} \left( t_{\max}^{(l)} + \frac{x_n - x_{\max}^{(l)}}{w}, x_n \right), \\
\text{for any } l, j \text{ such that } & t_{\max}^{(l)} + x_n - x_{\max}^{(l)} \in [t_j, t_{j+1}], \\
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{\max}^{(l)} + \frac{x_{\max}^{(l)} - w}{v_f}, x_n \right) & \geq \mathbf{c}_{\text{down}}^{(j)} \left( t_{\max}^{(l)} + \frac{x_{\max}^{(l)} - w}{v_f}, x_n \right), \\
\text{for any } l, j \text{ such that } & t_{\max}^{(l)} + x_{\max}^{(l)} - w \in [t_j, t_{j+1}], \\
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{\min}^{(l)} + \frac{x_n - x_{\min}^{(l)}}{v_f}, x_n \right) & \geq \mathbf{c}_{\text{down}}^{(j)} \left( t_{\min}^{(l)} + \frac{x_n - x_{\min}^{(l)}}{v_f}, x_n \right), \\
\text{for any } l, j \text{ such that } & t_{\min}^{(l)} + x_n - x_{\min}^{(l)} \in [t_j, t_{j+1}], \\
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{\min}^{(l)} + \frac{x_0 - x_{\min}^{(l)}}{v_f}, x_0 \right) & \geq \mathbf{c}_{\text{down}}^{(j)} \left( t_{\min}^{(l)} + \frac{x_0 - x_{\min}^{(l)}}{v_f}, x_0 \right), \\
\text{for any } l, j \text{ such that } & t_{\min}^{(l)} + x_0 - x_{\min}^{(l)} \in [t_j, t_{j+1}], \\
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{0}^{(l)}, x_0^{(l)} \right) & \geq \mathbf{c}_{\text{down}}^{(j)} \left( t_{0}^{(l)}, x_0^{(l)} \right), \\
\text{for any } l, j \text{ such that } & t_{0}^{(l)} \in [t_j, t_{j+1}], \\
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{1}^{(l)}, x_1^{(l)} \right) & \geq \mathbf{c}_{\text{down}}^{(j)} \left( t_{1}^{(l)}, x_1^{(l)} \right), \\
\text{for any } l, j \text{ such that } & t_{1}^{(l)} \in [t_j, t_{j+1}], \\
\mathbf{M}_{c_{\text{intern}}}^{(l)} \left( t_{\min}^{(l)} + \frac{x_n - x_{\min}^{(l)}}{V_l^{(l)}_{\text{intern}}}, x_n \right) & \geq \mathbf{c}_{\text{down}}^{(j)} \left( t_{\min}^{(l)} + \frac{x_n - x_{\min}^{(l)}}{V_l^{(l)}_{\text{intern}}}, x_n \right), \\
\text{for any } l, j \text{ such that } & t_{\min}^{(l)} + x_n - x_{\min}^{(l)} \in [t_j, t_{j+1}] \text{ and } V_l^{(l)}_{\text{intern}} \neq 0,
\end{align}
and

\[
\begin{align*}
\text{M}_{c_{\text{intern}}}^{(l)}(t_1^{(l,l')}, x_1^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_1^{(l,l')}, x_1^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_1^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\text{M}_{c_{\text{intern}}}^{(l)}(t_2^{(l,l')}, x_2^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_2^{(l,l')}, x_2^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_2^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\text{M}_{c_{\text{intern}}}^{(l)}(t_3^{(l,l')}, x_3^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_3^{(l,l')}, x_3^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_3^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\text{M}_{c_{\text{intern}}}^{(l)}(t_4^{(l,l')}, x_4^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_4^{(l,l')}, x_4^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_4^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\text{M}_{c_{\text{intern}}}^{(l)}(t_5^{(l,l')}, x_5^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_5^{(l,l')}, x_5^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_5^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\text{M}_{c_{\text{intern}}}^{(l)}(t_6^{(l,l')}, x_6^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_6^{(l,l')}, x_6^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_6^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\text{M}_{c_{\text{intern}}}^{(l)}(t_7^{(l,l')}, x_7^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_7^{(l,l')}, x_7^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_7^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\text{M}_{c_{\text{intern}}}^{(l)}(t_8^{(l,l')}, x_8^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_8^{(l,l')}, x_8^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_8^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\text{M}_{c_{\text{intern}}}^{(l)}(t_9^{(l,l')}, x_9^{(l,l')}) & \geq c_{\text{intern}}^{(l')} (t_9^{(l,l')}, x_9^{(l,l')}), \\
\text{for any } l, l' \text{ such that } t_{\text{min}}^{(l')} \leq t_9^{(l,l')} \leq t_{\text{max}}^{(l')} , \\
\end{align*}
\]

(4.24)
with

\[
\begin{align*}
\text{with} & \\
\begin{align*}
t_{\text{in}} &= t_a(t_{\text{in}}, x_{\text{in}}), \\
x_{\text{in}} &= x_a(t_{\text{in}}, x_{\text{in}}), \\
t_{\text{max}} &= t_a(t_{\text{max}}, x_{\text{max}}), \\
x_{\text{max}} &= x_a(t_{\text{max}}, x_{\text{max}}), \\
t_{0}^{(l,j)} &= \hat{T}_+(x_n, t_{j}, 0, x_{\text{in}}, t_{\text{in}}, V_{\text{intern}}), \\
t_{1}^{(l,j)} &= \hat{T}_+(x_n, t_{j}, 0, x_{\text{max}}, t_{\text{max}}, V_{\text{intern}}), \\
t_{1}^{(l,l')} &= \hat{T}(x_{\text{in}}, t_{\text{in}}, w, x_{\text{in}}, t_{\text{min}}, V_{\text{intern}}), \\
x_{1}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{1}^{(l,l')} - t_{\text{min}}), \\
t_{2}^{(l,l')} &= \hat{T}(x_{\text{in}}, t_{\text{min}}, v_f, x_{\text{in}}, t_{\text{min}}, V_{\text{intern}}), \\
x_{2}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{2}^{(l,l')} - t_{\text{min}}), \\
t_{3}^{(l,l')} &= \hat{T}(x_{\text{in}}, t_{\text{max}}, x_{\text{in}}, t_{\text{min}}, V_{\text{intern}}), \\
x_{3}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{3}^{(l,l')} - t_{\text{min}}), \\
t_{4}^{(l,l')} &= \hat{T}(x_{\text{in}}, t_{\text{min}}, v_f, x_{\text{in}}, t_{\text{min}}, V_{\text{intern}}), \\
x_{4}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{4}^{(l,l')} - t_{\text{min}}), \\
t_{5}^{(l,l')} &= \hat{T}(x_{\text{in}}, t_{\text{in}}, V_{\text{intern}}, x_{\text{in}}, t_{\text{min}}, V_{\text{intern}}), \\
x_{5}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{5}^{(l,l')} - t_{\text{min}}), \\
t_{6}^{(l,l')} &= \hat{T}(x_{\text{in}}, t_{\text{min}}, v_f, x_{\text{in}}, t_{\text{min}}, V_{\text{intern}}), \\
x_{6}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{6}^{(l,l')} - t_{\text{min}}), \\
t_{7}^{(l,l')} &= \hat{T}(x_{\text{in}}, t_{\text{in}}, w, x_{\text{in}}, t_{\text{min}}, V_{\text{intern}}), \\
x_{7}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{7}^{(l,l')} - t_{\text{min}}), \\
t_{8}^{(l,l')} &= \hat{T}_+(x_{\text{in}}, t_{\text{in}}, V_{\text{intern}}, x_{\text{in}}, t_{\text{min}}, V_{\text{intern}}), \\
x_{8}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{8}^{(l,l')} - t_{\text{min}}), \\
t_{9}^{(l,l')} &= \hat{T}_+(x_{\text{in}}, t_{\text{in}}, V_{\text{intern}}, x_{\text{in}}, t_{\text{max}}, V_{\text{intern}}), \\
x_{9}^{(l,l')} &= x_{\text{in}} + V_{\text{intern}}(t_{9}^{(l,l')} - t_{\text{min}}).
\end{align*}
\]
4.4.2 Data constraints

In addition to the model constraints, we consider a separate set of data constraints that arise from known measurements on traffic densities, boundary flows or travel time samples obtained thanks to probe vehicles.

We consider the additional assumption

(A3) The data constraints are linear with respect to the decision variable $y$ defined in (4.14) such that they can be represented in matrix form as follows

$$ C_{\text{data}} y \leq d_{\text{data}}. $$

The data constraints that are considered in this paper encompass

1. Downstream boundary outflow estimation due to the red light located at $x = \chi$

   $$ p_j = 0, \text{ for any } j \text{ s.t. } \Omega_{\text{red}} \cap [t_j, t_{j+1}] \neq \emptyset, $$

   where the set of traffic signal timings $\Omega_{\text{red}}$ is assumed to known.

2. Flow measurements $q_{\text{meas}}$ coming from fixed sensors with known errors $e_{\text{flow}}$

   and located at the upstream boundary $x = \xi$

   $$ (1 - e_{\text{flow}})q_{\text{meas}}(t) \leq q_j \leq (1 + e_{\text{flow}})q_{\text{meas}}(t), \text{ for any } t \in [t_j, t_{j+1}]. $$

3. Travel times estimates $d_{\text{travel}}$ provided by mobile sensors with known errors bound $e_{\text{time}}$

   $$ M(t_{\text{exit}} - d_{\text{travel}} - e_{\text{time}}, \xi) \leq M(t_{\text{exit}}, \chi) \leq M(t_{\text{exit}} - d_{\text{travel}} + e_{\text{time}}, \xi). $$

The probe data are sampled from 5% to 15% of vehicles crossing the arterial
link $A = [\xi, \chi]$.

Figure 4.10: Example of Traffic estimation with the LWR-BA model. This is a traffic estimation on a single highway using the LWR-BA model constraints and the queue estimation application.

4.4.3 Setting of the optimization problem

It is noteworthy that the solutions described in equations (4.7)-(4.12) associated with the adequate vale conditions (2.16), (2.18), (2.20) and (2.22) are all linear in the decision variable $y$ and all these constraints described by (4.18)-(4.24) are also linear in $y$. Therefore, we can represent the model constraints in the matrix form as follows

$$A_{\text{model}} y \leq b_{\text{model}}.$$

These model constraints encode the limitations due to the physics of traffic flows.

We are now ready to set the optimal control problem that we will use for the queue length estimation. We follow [92]. Notice that the optimization framework
was previously proposed in [31, 91]. It reads as follows

\begin{equation}
\text{Maximize } g(y) \\
\text{subject to } \begin{cases} A_{\text{model}} y \leq b_{\text{model}}, & \text{(model constraints)} \\ C_{\text{data}} y \leq d_{\text{data}}, & \text{(data constraints)} \end{cases} \tag{4.25}
\end{equation}

The objective function \( g : y \mapsto g(y) \) can be any convex piecewise affine function of the decision variable. In our case, it is defined as follows

\[ g(y) = (0_{\mathbb{R}^n}, 0_{\mathbb{R}^m}, 1_{\mathbb{R}^m}, 0_{\mathbb{R}^m \times \mathbb{R}^m}) \cdot y^T = \sum_{j=0}^{m-1} p_j. \]

Solving the optimization problem (4.25) leads to an optimal solution denoted by

\[ y^* := [\rho_1^*, \ldots, \rho_n^*, q_1^*, \ldots, q_m^*, p_1^*, \ldots, p_m^*, (M^{(1)})^*, (q_{\text{inter}}^{(1)})^*, \ldots, (M^{(o)})^*, (q_{\text{inter}}^{(o)})^*] \]

\[ = \arg \max_y g(y) \]

that can be used to compute the traffic states \( M \) and \( \rho = -\frac{\partial M}{\partial x} \) thanks to the explicit solutions (4.7)-(4.12). The queue lengths are then deduced by computing at each time step the extremal points of the set

\[ Q_\varepsilon(t) := \left\{ (\alpha, \beta) \bigg| \begin{array}{l} \xi \leq \alpha < \beta \leq \chi, \\ |\rho(t, z) - \kappa| \leq \varepsilon, \quad \forall z \in [\alpha, \beta] \end{array} \right\} \]

where \( \varepsilon > 0 \) is a prescribed sensitivity parameter.
Figure 4.11: Domain of influence of the initial condition $c^{(i)}_{ini}$ when $0 \leq \rho^{(i)}_{ini} \leq \rho_c$.

Figure 4.12: Domain of influence of the initial condition $c^{(i)}_{ini}$ when $\rho_c < \rho^{(i)}_{ini} \leq \kappa$. 
Figure 4.13: Domain of influence of the upstream boundary condition $c_{up}^{(j)}$ when $0 \leq \rho_{up}^{(j)} \leq \rho_c$.

Figure 4.14: Domain of influence of the upstream boundary condition $c_{up}^{(j)}$ when $\rho_{up}^{(j)} \geq \rho_c$. 
Figure 4.15: Domain of influence of the downstream boundary condition $c_{\text{down}}^{(j)}$.

Figure 4.16: Domain of influence of the internal boundary condition $c_{\text{intern}}^{(l)}$. 
Figure 4.17: Queue estimation on a single link. Queue estimation example involving the Bounded Acceleration model and real traffic data.
Chapter 5

Extension to Highway networks

This chapter was published at:

• (Section 5.2) Li Y., Canepa E.S. and Claudel C.G., “Optimal Control of Scalar Conservation Laws Using Linear/Quadratic Programming: Application to Transportation Networks”, *IEEE Transactions on Control of Network Systems*, Volume 1, Number 1, Pages 28-39, Year 2014.

5.1 Junction models

In this section, we generalize the above framework to traffic state estimation on road networks. For this, we first need to derive the boundary flows occurring at the junctions. This process is done through a junction model, which we now outline.

**Proposition 13** [Conservation of vehicles] We consider a general junction (without storage capacity) integrating on-ramps, off-ramps and incoming as well as outgoing links, and illustrated in Figure 5.1.

The conservation of vehicles at the junction imposes the following equality constraint.

\[
\sum q_{out_i} + \sum q_{ramp}^{in} = \sum q_{in_j} + \sum q_{ramp}^{out}\quad (5.1)
\]

The quantities \(\sum q_{out_i}\) are the flows leaving their respective road (to cross the junction) and \(\sum q_{in_j}\) represent the flows entering the roads (after crossing the junc-


Figure 5.1: Junction convention and flow conservation.

...tion). $\sum q_{in_k}^{ramp}$ are the external flows entering the network through the intersection, conversely, $\sum q_{out_n}^{ramp}$ are the flows exiting the network.

Note that the above equality constraint is insufficient in practice to derive the actual incoming and outgoing flows. Following [93, 94], we assume that the flow at the junctions is the solution to a LP. This LP maximizes the total inflow through the junction under the constraints of an allocation matrix (which splits the flow according to the driver preferences), and physical constraints of demand and supply occurring at the boundaries of the incoming and outgoing links respectively.

The equation of conservation of flows (5.1) is linear in the decision variable. We assume that the volume of traffic entering the junction ($U_i's$) through $\sum q_{out_i}$ and $\sum q_{in_k}^{ramp}$ is distributed among the exit options ($O_j's$) according to an allocation parameter $\alpha_{Outs,Ins} \geq 0$. Since the junction has no storage capacity the sum of all the allocation parameters from a fixed incoming option among all the output options must be equal to one:

$$\sum_j \alpha_{O_j,U_i} = 1 \quad (5.2)$$

The relation of the incoming-outgoing flows in terms of the allocation parameters...
is encoded by the allocation matrix:

\[
\begin{bmatrix}
\sum_j O_j
\end{bmatrix} = A \begin{bmatrix}
\sum_i U_i
\end{bmatrix}
\]  \hspace{1cm} (5.3)

The well-posedness constraints imposed by the LWR model on the junctions can be written as

\[
\begin{align*}
q_{\text{out}_i} & \leq d_i \\
q_{\text{ramp}_{\text{in}_k}} & \leq d_{u,k} \\
q_{\text{in}_j} & \leq s_j \\
q_{\text{ramp}_{\text{out}_n}} & \leq s_{d,n}
\end{align*}
\]  \hspace{1cm} (5.4)

where \(d_i\) correspond to the demand of link \(i\), \(d_{u,k}\) corresponds to the demand of the on-ramp \(k\), \(s_j\) correspond to the supply of link \(j\) and \(s_{d,n}\) correspond to the supply of off-ramp \(n\). Since the demand and supplies of incoming and outgoing links is a mixed integer linear function of the decision variable \(\text{[95]}\) (while the flows are linear in the decision variable), these constraints are mixed integer linear. Finally, the maximization of the sum of the flows through the junction imposes that at least one of the inequalities in (5.4) becomes an equality, adding additional integer variables to the problem.

Hence, by combining all of the above constraints the junction constraints can be written as mixed integer linear inequalities in the decision variable, while the model constraints (on each links) are also mixed integer linear. If the objective to be minimized (as part of the estimation) is a linear function of the decision variable (which is typically the case), then the problem of estimating the state of traffic on a general highway network remains a MILP.

Hence, the estimation of the state of traffic on a general network can be posed as a mixed integer linear program, in which the integer variables are a consequence of the junction models and of internal data (either internal boundary conditions or
5.1.1 Implementation

We illustrate the above results by implementing a MILP to solve two travel time estimation problems (with distinct network structures) involving different segments of the I-880 highway located in the San Francisco bay area.

The first problem will involve three roads, two junctions with a ramp on and off respectively as can be seen in Figure 5.2. For this problem, we consider data generated by the PeMS [71] VDS 400674 and 400640 for the upstream and downstream position respectively (Figure 5.2). The stations are located on the Highway I - 880 N around Hayward, California, the space domain of the entire network is 6 miles. We assumed that we don’t have data in the junctions except for the allocation matrix which is predefined by the user, we are modeling the junctions with the approach described earlier. The traffic flow allocation matrix for both junctions in this example is the following:

\[
\begin{bmatrix}
q_{in} \\
ramp_{out_1}
\end{bmatrix} =
\begin{bmatrix}
0.9 & 1 \\
0.1 & 0
\end{bmatrix}
\begin{bmatrix}
q_{out_1} \\
ramp_{in_1}
\end{bmatrix}
\] (5.5)

An example of the travel time estimation of the structure mentioned above is show in Figure 5.3 below, the result obtained by the estimation toolbox is compared with the ground truth, that is, GPS data obtained during the Mobile Century experiment. This particular example has 447 variables and 3127 constraints and took 4.57 seconds to solve on an iMac with a processor Intel Core i5-2400 @2.5GHz.

A series of estimations with different traffic conditions were analyzed, comparing the estimated travel time with the ground truth obtained from GPS data. After 30 estimations the RMS error obtained is 11 seconds.

The second structure is a merge and will involve three roads and one junction
In this problem we want to estimate the travel time of the network from the upstream to downstream position; the network consists of three roads connected through two junctions with a ramp on and off respectively, the latter is used to consider the flow leaving and entering the highway on the intersections.

with a ramp on and off as can be seen in Figure 5.4. For this problem, we consider data generated by the PeMS VDS stations 400490 and 400685 for the upstream and downstream position respectively (Figure 5.2). The stations are located on the Highway I - 880 N around Hayward, California, the space domain of the entire network is 10.1 miles. Also, travel time data obtained on the Mobile Century experiment was added as data constraints to the problem. The road merging to the I-880 is Decoto Road, from which there is no data available; a random generator of reasonable flows in the upstream position of Decoto Rd. was included. We assumed that we don’t have data in the junction except for the allocation matrix which is again predefined by the user. The traffic flow allocation matrix in this problem is the following:

\[
\begin{bmatrix}
q_{\text{in}_1} \\
ramp_{\text{out}_1}
\end{bmatrix} =
\begin{bmatrix}
0.9 & 1 & 1 \\
0.1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
q_{\text{out}_1} \\
ramp_{\text{out}_2} \\
q_{\text{in}_1}
\end{bmatrix}
\] 

(5.6)

Two different objective functions were selected to estimate the travel time. The
Figure 5.3: Traffic time estimation example structure 1.
In all subfigures, we compute the density map for which the initial number of vehicles is the lowest, given the boundary data or the junction model, the sample car path is highlighted in red. For this example the estimated travel time is 443 seconds and the ground truth is 445 seconds. Top: Density map of the last highway segment, this segment has data in the downstream end. Center: Density map of the middle highway segment, this segment has a junction in both ends. Bottom: Density map of the initial segment of the highway, this segment has data on the upstream end.
We want to estimate the travel time of the network from the upstream to downstream position; the network consists of three roads connected through one junction with a ramp on and off, the latter is used to consider the flow leaving and entering the highway on the intersection.

The first objective function is the minimization of the initial densities, an example is shown in Figure 5.5. The second objective function is the minimization of the L1 norm between the decision variables in order to obtain a more uniform density map (Figure 5.6). The results obtained by the estimation toolbox are compared with the ground truth, that is, travel time data validated during the Mobile Century experiment. The example in Figure 5.5 has 546 variables and 4336 constraints and took 5.12 seconds to solve on an iMac with a processor Intel Core i5-2400 @2.5GHz.

The travel time estimation was evaluated using different traffic conditions, a comparison of the results obtained by the estimator with both objective functions can be observed in Figure 5.7. From the figure we can conclude that with the L1 norm minimization the travel time is always overestimated, however it can capture better a sudden increase in the travel time, like the one happening at 13:00 hours of the evaluated day.
Figure 5.5: Traffic time estimation example 1 merge structure.

In these subfigures we compute the density map for which the initial number of vehicles is the lowest, given the boundary data or the junction model, the sample car path is highlighted in red. For this example the estimated travel time is 767 seconds and the ground truth is 769 seconds. Top: Density map of the last highway segment, this segment has data in the downstream end and a travel time constraint. Bottom: Density map of the initial segment of the highway, this segment has data on the upstream end and a travel time constraint.
Figure 5.6: Traffic time estimation example 2 merge structure.
For these subfigures we compute the density map for which the L1 norm of the decision variables is the minimum, creating a more uniform density map. Top: Density map of the last highway segment, this segment has data in the downstream end and a travel time constraint. Bottom: Density map of the initial segment of the highway, this segment has data on the upstream end and a travel time constraint.
5.2 Control on Highway Networks

In this section, we generalize the control framework from a single link to a highway network. In the remainder of this dissertation, to avoid confusion with the terms $q_{in}, q_{out}$ when modeling the junctions, we replace them with $q^{us}, q^{ds}$ to denote upstream and downstream flows on links.

5.2.1 Network model

The highway network is modeled as a directed graph consisting of vertices $v \in \mathcal{V}$ and edges $e \in \mathcal{E}$. Each edge represents a link on the mainstream highway with a length $L_e$ and a set of physical parameters (e.g. the speed limit $v_{max}$, the number of lanes $W_e \in \mathbb{N}^+$). On-ramps $on \in \mathcal{ON}$ are defined as a special edge with length $L_{on} \rightarrow 0$, and a set of parameters (e.g. $v_{max}$, a number of lanes $W_{on} \in \{0\} \cup \mathbb{N}^+$), and a direction to a vertex $v$. Off-ramp $off \in \mathcal{OFF}$ are defined similarly with $W_{off} \in \{0\} \cup \mathbb{N}^+$ and a direction out from a vertex $v$. A junction $j \in \mathcal{J}$ is defined as $J_j := (v_j, \mathcal{I}_j, \mathcal{O}_j, on_j, off_j)$ and consists of a vertex $v_j \in \mathcal{V}$, a set of incoming edges $e_{in} \in \mathcal{I}_j$, a set of outgoing edges $e_{out} \in \mathcal{O}_j$, an on-ramp $on_j \in \mathcal{ON}$ and an off-ramp $off_j \in \mathcal{OFF}$. Note that by setting the number of lanes on an on-ramp or an off-ramp to zero, we can eliminate the corresponding on-ramp or off-ramp in this junction $J_j$.

A network example is illustrated in Fig. 5.8.

To generalize the control framework across junctions, we make the following assumptions:

1. Junctions have no vehicle storage capacity.

2. The up-/downstream and on-ramp flows can all be completely controlled. Nowadays, only on-ramp flows can be controlled using ramp metering or traffic lights; up-/downstream flows on highway links are usually uncontrolled, though they
Figure 5.7: Travel times Comparison.
The ground truth are the travel times experienced on February 8th, 2008 during the Mobile Century experiment between Mowry and Winton Avenue (10.1 miles) compared with the results obtained by the travel time estimation using the merge structure and two different objective functions.

Figure 5.8: Network graph definition
Several highway links are connected by junctions. Each junction can have one on-ramp, one off-ramp, both, or none.
are controlled on some secondary roads. It is however reasonable to assume that in the future all highway links will become controlled.

3. At each junction $J_j$, the incoming flows from $I_j$ and $on_j$ are routed to outgoing edges $O_j$ and $off_j$ according to a preference matrix.

4. The traffic flow from on-ramp $on_j$ is not routed to the off-ramp $off_j$ at the same junction.

With these assumptions, at a junction $j \in J$ with incoming edges $e_in \in I_j$, outgoing edges $e_out \in O_j$, one on-ramp $on_j \in ON_j$, and one off-ramp $off_j \in OFF_j$, the variables of interest are related by equation (5.7) below:

$$\begin{bmatrix} A_j & A_{on}^m \\ B_j & 0 \end{bmatrix} \begin{bmatrix} q^{ds} \\ q^{on}_j \end{bmatrix} = \begin{bmatrix} q^{us} \\ q^{off}_j \end{bmatrix} \tag{5.7}$$

Column vectors $q^{us}$ and $q^{ds}$ contain $q^{us}_{e_out}(t, \xi_{e_out})$, $q^{ds}_{e_in}(t, \chi_{e_in})$ denoting the upstream and downstream flows on respective links. The variables $q^{on}_j(t, \cdot)$ and $q^{off}_j(t, \cdot)$ are scalar values representing flows on on-ramps $on$ and off-ramps $off$.

$A_j$ is a $|O_j| \times |I_j|$ matrix, where each element $\alpha^j_{mn}(t)$ denoting the percentage of the incoming flow $q^{ds}_{e_in}(t, \chi_{e_in})$ at this junction routed to an outgoing link $e_out \in O_j$.

$A_{on}^m$ is a $|O_j| \times 1$ matrix, where each element $\alpha^{on}_{mn}(t)$ denotes the percentage of the on-ramp flow $q^{on}_j(t, \cdot)$ routed to an outgoing link $e_out \in O_j$.

$B_j$ is a $1 \times |I_j|$ matrix, where each element $\beta^{off}_n(t)$ denotes the percentage of the incoming flow $q^{ds}_{e_in}(t, \chi_{e_in})$ routed to the off-ramp $off_j \in OFF_j$.

From the conservation of vehicles across the junction, the total outflow from the junction is equal to the total inflow. Therefore, the parameters in Equation (5.7) satisfy:

$$\sum_{e_out \in O_j} \alpha^j_{e_out, e_in}(t) = 1 - \beta^{off}_n(t) \quad \forall j \in J \tag{5.8}$$
5.2.2 Formulation of the network control problem

The model and data constraints for the single link traffic control problem (4.4) still apply on each link of the highway network. The model for junctions in Equation (5.7) (5.8) can be incorporated into the linear program as equality constraints.

Let us consider a highway network with links $E$ and junctions $J$. We define a new decision variable $y$ as

$$
y := [\rho_{1i}(0,\cdot), q_{1w}^s(t,\xi_1), \ldots, q_{1d}^s(t,\chi_1), \ldots, \rho_{ni}(0,\cdot), \ldots, q_{on}^m(t,\cdot), \ldots, q_{on}^m(t,\cdot)]$$

The variables $q_{1o}^m(t,\cdot), \ldots, q_{on}^m(t,\cdot)$ can be regarded as control inputs. By defining a certain objective function $f(\cdot)$, the traffic network control can be posed as an optimization problem in which the constraints of the model and of the data are linear.

$$\begin{align*}
\text{Minimize} & \quad f(y) \\
\text{s. t.} & \quad A_{\text{model}}y \leq b_{\text{model}} \\
& \quad C_{\text{data}}y = d_{\text{data}} \\
& \quad E_{\text{conj}}y = f_{\text{conj}}
\end{align*}
$$

(5.9)

5.2.3 Highway definition

In the remainder of this dissertation, we consider a section of highway I210 East from vehicle detection station (VDS) 716585, Pasadena, CA to 718290, Monrovia, CA as illustrated in Fig. 5.9. This highway section consists of 17 links and 16 junctions. Note that some links that are smaller than 200 m are regarded as a part of the preceding link for simplicity (this does not affect results in practice, as the traffic flow model does not capture the physics of traffic at this scale). All on-ramps have one lane while the number of lanes on off-ramps ranges from 1 to 3. The numbers of lanes on the mainstream highway are respectively 6 for links 1 \sim 2, 5 for links 3 \sim 10, 4 for links
11 ∼ 17.

5.2.4 Network congestion alleviation using on-ramp flow control (ramp metering)

Since the available information from public traffic feeds is not sufficient for us to compute all coefficients in problem (5.9), we make the following assumptions on some of the unknown variables:

- We assume that the initial densities $\rho_{e}^{ini}(0, \cdot)$ and the mainstream incoming flow to the first link $q_{1}^{in}(\cdot, \xi_{e1})$ are known in advance for the simulation time domain $(0, t_{max})$. This data can be obtained based on past density and flow measurements. For this application, we use the data collected by the PeMS (Performance Measurement System), which provides flow measurements at different points of the highway network.

- We assume that the portion $\beta_j$ of the traffic flow on link $e_j$ leaving the highway through the off-ramp on junction $J_j$ is time invariant and only depends on the
number of lanes on the off-ramp:

\[ \beta_j(t) := 0.1 \cdot W_{off,j} \]

- We assume that the control input variables \( q_{on}^e(t) \) can be controlled at each time step (that is, every 30 seconds)

\[ q_{on}^e(t, \cdot) := q_{on}^e([t/T], \cdot) \in \{0 \cup \mathbb{R}^+\} \]

**Decision variable**

As discussed previously, the variables \( \rho_{e}^{ini}(0, x), q_{us}^e(t, \xi_e), q_{ds}^e(t, \chi_e) \) describe the optimization problem on link \( e \). Since the control framework on one single link is still valid for the each link on the highway network, the set of variables on each link is part of the decision variable in the global network optimization problem.

The on-ramp flow \( q_{on}^e \) at the junction \( J \) plays the role of a control input, and is thus part of the decision variable. Note that the boundary flows \( q_{us}, q_{ds} \) that were used as control inputs in the single link example are now constrained by \( q_{on} \), and therefore are not control inputs anymore.

We thus define the decision variable for the global network traffic control problem as:

\[
y := [\rho_1^{ini}(0,1), \ldots, \rho_1^{ini}(0,k_m), q_1^{us}(1,\xi_1), \ldots q_1^{us}(n_m,\xi_{e_1}), q_1^{ds}(1,\chi_1), \ldots, q_1^{ds}(n_m,\chi_{e_1})]
\]

\[
\ldots
\]

\[
q_{on}(1,\cdot), \ldots q_{on}(n_m,\cdot), q_{on}^m(1,\cdot), \ldots
\]

In the above definition, \( k_m \) denotes the number of segments used to describe the density profile on links and \( n_m \) denotes the number of boundary conditions in the time domain. For simplicity we used the same number of segments to describe the
density function on each link, and we used a constant time step, though the proposed framework is valid for an arbitrary spatial and temporal sampling.

Constraints for links and junctions

The inequality constraints in LP (4.4) are a function of the number of lanes in each link. Data constraints are set for the initial conditions on all links with a relative error. All data constraints for flow variables $q^{us}, q^{ds}, q^{on}$ are relaxed (to allow the control action to operate) except for the upstream flow $q^{us}_1$ on the first link. The upstream flow $q^{us}_1$ is fed to the control problem using the measured data from PeMS, but allowing a relative measurement error level of 1%.

Since there is no intersection between two highway sections in the present case, the vehicle flow continuity equations (5.7) and (5.8) across the junction can be expressed in a simpler form.

Consider two links $e_j, e_{j+1}$ joining in $J_j$, and their corresponding upstream and downstream flows $q^{us}_{j/j+1}, q^{ds}_{j/j+1}$. Vehicle conservation across the junction $J_j$ can be formulated as equation (5.10) below.

\[
(1 - \beta_j) q^{ds}_j(t, \chi_{e_j}) + q^{on}_j(t, \cdot) = q^{us}_j(t, \xi_{e_{j+1}}) \quad (5.10)
\]

Objective function

Several important objectives motivated from highway transportation can be defined explicitly in the decision variable. Examples include the maximal cumulated number of vehicles exiting the highway portion at given time $t_{max}$, the least number of congested segments on the highway, the smallest time required for alleviating traffic congestion on a highway network. Other objectives (for instance minimizing total fuel consumption) could also be formulated in this framework.

In this example, we define our objective function as maximizing the total down-
stream flow from all links with a weight function $w(t)$ in the given time interval $(0, t_{\text{max}})$:

$$f(y) = -\sum_{i=1}^{n} \sum_{t=1}^{n_m} w(t) \cdot q_{is}^d(t, \chi_{ei})$$

Note that this objective function represents a combination of the objectives used in the single link control problem investigated earlier. It corresponds to the total flow on the highway with higher weight assigned to upstream links. This enables us to maximize the total flow exiting this portion of the highway while not unrealistically block the upstream links. In addition, the weight function $w(t)$ is chosen as a decreasing exponential in the interval $(0, t_{\text{max}})$, which forces the optimization solver to alleviate highway congestion in a shortest time. For this specific example, we chose $w(t) = e^5 \cdot (t_{\text{max}} - t) / t_{\text{max}}$.

Other choices of objective functions are also possible with special formations, for instance the minimization of congested areas at $t_{\text{max}}$ requires additional boolean variables and turns the problem into a MILP, or the minimal switched control effort requires a quadratic formation in the decision variables and turns the problem into a MIQP which will be investigated in the next example.

**Implementation**

The highway section is defined as uniformly (mildly) congested at $t = 0$ with the initial density on each link of the highway section constrained as:

$$\rho_e^{ini}(0, k) = 1.2 \rho_c(e) \quad \forall k \in \{1, 2 \ldots k_m\}$$

where $\rho_c$ is the critical density that varies across links.

The upstream flow on the first link $q_{is}^d(t, \xi_{e1})$ for simulation is extracted from the data set collected by VDS 7176733 from 23:25 to 23:50 on 2/25/2013. Due to the lack of density data from PeMS, we simply defined the initial density as mildly congested.
to show how the congestion would be dissolved. And the purpose of using real flow
data is to demonstrate the framework’s applicability on dealing with continuously
changing flows. However, the defined initial density could be incompatible with the
real flow data extracted from PeMS, for instance it is impossible to have an upstream
flow $q^{us} = q_{max}$ when the link is congested.

We solve the linear program (5.9) defined in subsection 5.2.2 using IBM Ilog
Cplex on a Macbook Pro operating MacOS X. This particular optimization problem
has 677 variables and 7713 constraints (with 240 equality constraints), and is solved
in 0.01 seconds. Setting up the matrices (with Matlab) takes about 17 seconds.

Figure 4 illustrates the evolution of traffic density, and shows how the congestion is
alleviated by controlling on-ramps along the highway. With a simulation time $t_{max} = 450s$, the highway alleviated the congestion and achieved the free flow condition in 5
minutes (simulation time) and then utilized the maximal capacity of the highway by
allowing the maximal possible flow from each on-ramp.

5.2.5 Highway congestion alleviation using switched control
(traffic lights)

We now show the applicability of the control framework presented earlier on a hybrid
control problem, in which the on-ramps are actuated by traffic signals.

Two commonly used actuators for controlling traffic flow on highways are traffic
lights and varying speed limits. The following example compares the performance
of traffic light actuation to continuous ramp metering (investigated earlier). For
simplicity, we assume that there are always vehicles waiting on on-ramps (though in
practice this may not be the true). Thus the on-ramp flows can be defined as:

$$
q_{j}^{on}(t, \cdot) = q_{max} \cdot b_{j}^{on}(t, \cdot)
$$
Figure 5.10: Evolution of traffic densities with on-ramp control
The horizontal axis corresponds to the total postmile from the upstream of the highway portion in m; the vertical axis corresponds to the density in veh/m. The three steps horizontal lines (green, black) are respectively the maximal density and the critical density on each link (these variables depend on the number of lanes). Vertical lines mark the position of junctions with arrows denoting on- or off-ramps. Red line plots the density evolution on the highway portion over time.
where $q_{\text{max}}$ is the maximal flow allowed on one lane, and $b^\text{on}_j(t, \cdot)$ is a boolean variable describing state of the traffic light. The traffic light can be actuated at each time step:

$$b^\text{on}_j(t, \cdot) = \begin{cases} 1 & \text{Green} \\ 0 & \text{Red} \end{cases}$$

Accordingly, the decision variable is now defined as

$$y := \left[ \rho^\text{ini}_1(0, 1), \ldots, \rho^\text{ini}_1(0, k_m), q^\text{us}_1(1, \xi_{e_1}), \ldots, q^\text{us}_1(n_m, \xi_{e_1}), \right.$$  

$$\left. q^\text{ds}_1(1, \chi_{e_1}), \ldots, q^\text{ds}_1(n_m, \chi_{e_1}), \rho^\text{ini}_n(0, 1), \ldots, q^\text{ds}_n(n_m, \chi_{e_n}), \right.$$  

$$\left. b^\text{on}_1(1, \cdot), \ldots, b^\text{on}_1(n_m, \cdot), \ldots, b^\text{on}_m(1, \cdot), \ldots \right]$$

With the introduction of boolean variables, the optimization problem is now a Mixed Integer Linear Program (MILP). We solve this MILP with the same objective function, constraints, and initial conditions as in the previous example. Similarly, we used IBM Ilog Cplex on a Macbook Pro operating MacOS X. This MILP has 677 variables (among which 150 boolean variables) and 7713 constraints (with 240 equality constraints), and is solved (using branch and bound methods) in 0.22 seconds. Setting up the matrices (with Matlab) takes about 17 seconds.

The simulated evolution of traffic density involving switched control is comparable to the density evolution in Fig. 4 for most of the time, though some congestion appears at the downstream boundary of some links because of the switched control action.

The optimized value of the objective function in this case is $f(y) = -17647.5$ which is greater (less optimal) than the optimal value corresponding to the previously investigated continuous control scenario $f(y) = -17728.3$. This is expected, since switched controls are a subcategory of the general control functions investigated earlier. Physically, traffic lights control will cause a rush of vehicles during some periods of time and might create some additional congestion.
Figure 5.11: Highway network I10-I605, CA.
This network can be decomposed into 67 links (red triangles) due to either an on- or off-ramp, or due to a change in the number of lanes. Blue arrows at the beginning of the highway denote the upstream flow. Each arrow along the highway denotes an on- or off-ramp. The total length of this network is approximately 40 km.

Nevertheless, a completely uncongested traffic state is reached after approximately 8 minutes (simulation time) using switched control.

5.2.6 Multi-highway network

As the result in the previous example on Highway I210 shows, one major advantage of this control framework is its relatively high computational performance, yet with a solution that is guaranteed to be globally optimal. In this section, we further demonstrate its efficiency on a larger highway network, and use this example to introduce the companion MATLAB toolbox.

Highway network definition

We consider a section of highway I605 North from VDS 762640, Industry, Los Angeles, CA to 773760, Baldwin Park, Los Angeles, CA, which intersects a section of highway I10 East from VDS 716092, Alhambra, CA to 717232, West Covina, CA
as illustrated in Figure 5.11. These two sections of highway consist of 67 links, 37 on-ramps and one junction between I605 and I10 mainlines. The total length of the two highway sections is ca. 40 km.

Objective functions

As previously, we keep the total downstream flow as one of our objectives:

\[ f_{ds}(y) := - \sum_{i=1}^{n} \sum_{t=1}^{n_m} w(t) \cdot q_{d,i}^{ds}(t, \chi_{e_i}) \]

However, the portion of upstream flows that does not exit the highway link by the end of the simulation time will not contribute to this objective, since it is solely defined by the total downstream flow. As a result, the CPLEX solver tends to give a solution with zero incoming upstream flow for the final portion of the simulation time as illustrated in Figure 4.9. Therefore, we similarly define another objective as the total upstream flow.

\[ f_{us}(y) := - \sum_{i=1}^{n} \sum_{t=1}^{n_m} w(t) \cdot q_{u,i}^{us}(t, \chi_{e_i}) \]

The traffic flow model used in this study, while robust, is not perfect. In particular, it predicts unrealistically large accelerations [86, 75] whenever the inflow is discontinuous. Thus, for practical applications, one must use relatively smooth control inputs, that is, a smooth change of the ramp metering or a low number of switches of the traffic lights, or risk suboptimal control in practice since the capacity of the road will be reduced by accelerating vehicles. We define this additional objective as a quadratic function, which therefore transforms the MILP developed previously into a MIQP.

\[
f_{\text{control}}(y) := \begin{cases} 
\sum_{o \in \mathcal{O}} \sum_{t=1}^{n_m} (q_{on}(t) - q_{on}(t-1))^2 & \text{Ramp metering} \\
\sum_{o \in \mathcal{O}} \sum_{t=1}^{n_m} (b_{on}(t) - b_{on}(t-1))^2 & \text{Switched control}
\end{cases}
\]
We consider a weighted sum of the objective components defined above as our objective function.

\[ f_{\text{comb}} := w_{ds} \cdot f_{ds} + w_{us} \cdot f_{us} + w_{\text{control}} \cdot f_{\text{control}} \]

**Simulation results**

Following the same procedure in the example on highway I210, we first constrain the network to be slightly congested by imposing an uniform initial density \( \rho_{e}^{\text{meas}}(0, k) = 1.2 \rho_c \) on each link. We do not impose additional constraints on any flow variables (up/downstream, on-ramps and off-ramps), except on the boundaries of the network on which we input flow data obtained from the PeMS system. We simulate 10 steps with a step time of \( T = 30s \), with ramp metering control.

We choose the combined objective function with 3 weight configurations as fol-
\[ f_{\text{comb}} = 1 \cdot f_{\text{ds}} + 1 \cdot f_{\text{us}} + w_{\text{control}} \cdot f_{\text{control}} \]

where \( w_{\text{control}} = \{0, 0.5, 5\} \).

This MIQP has 1777 variables, 16124 inequality constraints, 650 equality constraints, and is solved on a Macbook Pro in 0.02s, 0.31s, and 0.32s respectively for \( w_{\text{control}} = \{0, 0.5, 5\} \).

Figure 5.12 illustrates the control inputs under different levels of emphasis on the smoothness. With \( w_{\text{control}} = 0 \), the control input at on-ramps has very fast fluctuations, and thus may not be applicable to ramp metering control as it would require very high variations in input. Conversely, with \( w_{\text{control}} = 5 \), the control input does not follow the evolution of the state of traffic fast enough, which results in less efficient traffic control.

The companion Matlab toolbox allows the user to tune \( w_{\text{control}} \) to obtain an efficient and simultaneously slow varying control input to the transportation network. Of course other choices of objective functions are possible to achieve this.

### 5.3 Queue estimation on networks

We now present the implementation of our estimation framework to the case of an arterial road by considering it as a particular network. More precisely, we will consider the road not as a set of isolated single links as it has been done in [96, 97] for a general setting, or in [92] for queue estimation on arterials; but as an oriented graph made of a series of links (the edges) linked by a junction (the vertex). For a presentation of the general framework, the interested reader is kindly referred to [53]. Without loss of generality, we will restrain ourselves to the case of two links with one intermediary junction. The junction is assumed to be point-wise, say it has no storage capacity.

It is worth mentioning that this approach is also equivalent to considering the
junction as a fixed bottleneck and thus to add an internal condition with the appropriate flow conditions.

Moreover, to take into account the lateral inflows and outflows at the junctions due to the turning movements to and from other roads (see Figure 5.13) we assume that there are one on-ramp and one off-ramp at each junction.

Figure 5.13: Schematic representation of turning movements at an intersection as a one-by-one junction with on- and off-ramps.

5.3.1 Numerical examples

NGSIM dataset on Lankershim Boulevard, Los Angeles, CA, USA

The sample was collected on an arterial street, namely the Lankershim Boulevard in Los Angeles, CA, encompassing 4 intersections equipped with traffic signals (see Figure 5.14). The detailed trajectory data of 2,442 vehicles (time resolution of 10 samples per second) originate from five high-definition cameras, monitoring a 1,600-foot stretch of road from 08:28 a.m. to 09:00 a.m. on Thursday June 16, 2005.

The section is composed as follows (see Table 5.1):

The signals are located as follows:
Figure 5.14: Schematic representation of the Lankershim arterial

- Signal 86 downstream Link 1
- Signal 87 between Link 1 and Link 2
- Signal 88 between Link 2 and Link 3
- Signal 89 between Link 3 and Link 4
- Signal 90 between Link 4 and Link 5

Results

The results of the estimation procedure in [92] are not totally satisfying since the method only gives an insight on the averaged dynamics of traffic flow on the multilanes section. Moreover, the variability due to the presence of specific left or right-turn lanes (or left or right-turn pockets) is difficult to handle. However, data
<table>
<thead>
<tr>
<th>Link number</th>
<th>NorthBound (NB)</th>
<th>SouthBound (SB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 lanes with no possible turn movements</td>
<td>3 lanes with 1 permissive left-turn lane 1 right-turn lane 1 left-turn lane</td>
</tr>
<tr>
<td>2</td>
<td>4 lanes to 5 lanes (downstream) 1 designated left-turn lane 1 dedicated right-turn lane 1 permissive right-turn lane</td>
<td>3 lanes with no possible turn movements</td>
</tr>
<tr>
<td>3</td>
<td>3 lanes to 5 lanes (downstream) 1 dedicated left-turn lane</td>
<td>3 lanes to 6 lanes (downstream) 2 dedicated left-turn lanes 1 right-turn pocket 3 through-lanes</td>
</tr>
<tr>
<td>4</td>
<td>4 lanes to 5 lanes (downstream) intermediate entry-exit point small left-turn pocket (downstream)</td>
<td>4 lanes to 5 lanes (downstream) 1 dedicated left-turn lane 1 dedicated right-turn lane</td>
</tr>
<tr>
<td>5</td>
<td>5 lanes 1 dedicated left-turn lane</td>
<td>4 lanes</td>
</tr>
</tbody>
</table>

Table 5.1: Description of the Lankershim Boulevard geometry

constraints coming from at least 5% travel times data enable to make more precise estimation of queue lengths.
Figure 5.15: Queue estimate comparisons on link 2, NorthBound

Figure 5.16: Queue estimate comparisons on link 2, SouthBound
Figure 5.17: Queue estimate comparisons on link 3, SouthBound

Figure 5.18: Queue estimate comparisons on link 4, NorthBound
Chapter 6

Concluding Remarks

6.1 Summary

This dissertation presented a new traffic estimation and control framework based on mixed integer linear programming optimization in which the state of the system is modeled by the Lighthill-Whitham-Richards PDE. Using a Lax-Hopf formula, we show that the constraints arising from the model, as well as the measured data result in linear inequality constraints for a specific decision variable, the objective function depends on the application in which the framework will be used. We also show that the method can be extended to highway networks, at the expense of increasing the size of the decision variable, hence affecting the computational time. Numerical implementations of the estimation and control both on single roads and on highways were presented. Real traffic data was used in the implementations presented on this dissertation, the data is coming from the Performance Measurement Systems (PeMS) and the Mobile Century experiment in California. The inclusion of different type of traffic data illustrates the ability of the framework to use any traffic information available on a city. To the best of my knowledge the bounded acceleration phase on a queue estimation application has never been included before.

This framework has the advantage of being exact and efficient for small-scale networks and gives the ability for the user to select any objective function and explore the possible state estimates associated with a given dataset. The nature of this framework allows the implementation of traffic estimation on a distributed manner
for large urban areas, using smart city hardware the framework presented on this dissertation could be computed locally by microcontrollers places around a city.

6.2 Future Research Work

On this dissertation, the traffic model was assumed to be noiseless, future work will involve investigating the model uncertainty by evaluating the model parameters effect on the result. Also, the implementation of the framework on a real smart city operation will be evaluated.
REFERENCES


APPENDICES
A Publications


- Li Y., Canepa E. and Claudel C., “Exact solutions to robust control problems involving scalar hyperbolic conservation laws using Mixed Integer Linear Programming"

• Canepa E. S. and Claudel C. G., “Spoofing cyber attack detection in probe-based traffic monitoring systems using mixed integer linear programming”, Computing, Networking and Communications (ICNC), 2013 International Conference on, Pages 327-333, Year 2013. doi=10.1109/ICNC.2013.6504104


• Costeseque G., Canepa E. S. and Claudel C. G., “Optimization framework for the LWR model with bounded acceleration: application to queue estimation on arterials using mobile data”, In preparation