Efficient Outage Probability Evaluation of Diversity Receivers Over Generalized Gamma Channels

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Abstract

In this paper, we are interested in determining the cumulative distribution function of the sum of generalized Gamma in the setting of rare event simulations. To this end, we present an efficient importance sampling estimator. The main result of this work is the bounded relative property of the proposed estimator. This result is used to accurately estimate the outage probability of multibranch maximum ratio combining and equal gain combining diversity receivers over generalized Gamma fading channels. Selected numerical simulations are discussed to show the robustness of our estimator compared to naive Monte Carlo.

Index Terms

Generalized Gamma, importance sampling, Monte Carlo, bounded relative error, outage probability, diversity techniques.

I. Introduction

The generalized Gamma (GG) distribution [1] is a flexible distribution that includes many popular fading models such as Weibull or Nakagami-m as special cases. It has the same functional form as the well-known $\alpha-\mu$ distribution [2, 3]. The main difference between the two distributions resides in the number of parameters. While the $\alpha-\mu$ probability density function (PDF) is written in terms of the physical fading parameters α and μ, the GG distribution has three parameters. According to [2, 3], the GG distribution can be used to model the small-scale variation of the fading signal in non-line-of sight wireless radio-frequency channels. The reader is referred to [1] for the mathematical formulation of the GG distribution and to [2, 3] for the physical interpretation of the $\alpha-\mu$ fading model.

Diversity techniques are often used to reduce the fading caused by multipath transmission channel [4, Chap. 9]. They rely on receiving multiple transmitted signal replicas affected by independent fadings. When some of these techniques are considered, one of the main challenges in evaluating the system performance is that the sum of the fading envelopes or powers is involved. Although the study of diversity receivers for many important fading channels received a great deal of attention, only few works investigated the performance of diversity receivers over GG fading channels. This amounts to finding the distribution of the sum of GG variates. Stacy managed in his seminal work [1] to derive an infinite series representation of the cumulative distribution function (CDF) of the sum of GG variates. However, the author points out that such representation presents certain computational challenges. Most of the other attempts fail to provide a closed-form expression for this cumbersome problem and only propose to tackle it by deriving approximate solutions that usually involve a truncation error. In [5], Piboongungon *et al.* investigated the the average symbol error rate (SER) performance of both maximum ratio combining (MRC) and equal gain combining (EGC) receivers over GG fading channels. The authors used the moment generation function (MGF)-based approach [4] in the case of MRC receivers and the characteristic function (CHF)-based approach [6] for the EGC receivers. In [7], Sagias *et al.* proposed an upper bound for the sum of GG random variables (RVs) for the purpose of studying the performance of EGC receivers. First, they derived the expression of the CDF of the product of GG variates. Then, by exploiting the well-known inequality between the arithmetic and geometric means, the authors were able to provide an upper bound for the problem of the sum. Later on, da Costa *et al.* presented a highly accurate closed-form approximations to the PDF and CDF of the sum of independent identically distributed (i.i.d.) $\alpha-\mu$ variates [8] by approximating the sum of i.i.d $\alpha-\mu$ RVs by one $\alpha-\mu$ RV. To determine the parameters α and μ of the approximate distribution, a moment-based method was introduced. This result was used to approximate the outage probability as well as the average bit error probability for EGC and MRC diversity techniques. Although numerical simulations show a good agreement between the approximate and the exact solution, the proposed approach is restricted to the i.i.d case. Recently, closed-form expressions for the SER of EGC and MRC receivers over $\alpha - \mu$ fading were derived in [9]. Using the Mellin transform, El Ayadi *et al.* expressed the SER in terms of the Fox H-functions.

In summary, determining an exact expression for the outage probability is a challenging task in many cases, and to the best of our knowledge, no closed-form results for the outage probability of multi-branch diversity receivers operating over GG fading channels were derived in the literature. In this case, the only way to study the system performance is by means of numerical simulations, e.g. using Monte Carlo (MC) method. Due to the simplicity of its implementation, MC can be seen as a powerful technique when the problem is too complex and difficult to be solved analytically. However, it proves its inefficiency to estimate a rare event probability, i.e. probability lower than 10^{-8}, since too many samples are needed to guarantee a good quality estimator. To solve this problem, many accelerated simulation methods have been proposed in the literature. Recently, some efforts have been made to propose efficient estimators to evaluate the outage probability with diversity techniques for wireless communication systems. For instance, the authors in [10] have addressed this problem using two unified importance sampling (IS) schemes. However, the work was limited to the i.i.d. Later on, the authors proposed in [11, 12] a mean-shift IS scheme that evaluates the outage probability of multibranch MRC diversity receivers over Gamma-Gamma fading channels in both the i.i.d and independent and not necessarily identically distributed (i.n.i.d) cases. The proposed approach requires much less samples comparing to conventional MC schemes for achieving a given accuracy. In this paper, we chose to propose an efficient and a robust IS estimator is introduced to estimate the left tail probability of the sum of i.n.i.d GG variates.

The rest of this paper is organized as follows. First, we describe the problem setting in Section II. Then, we give a brief description of the IS method in Section III. In section IV, we present our approach to estimate the outage probability for diversity receivers over GG fading channels as well as the main theorem proving the efficiency of the proposed method in the i.n.i.d case. Prior to concluding, we show, in Section V, some selected simulation results related to the evaluation of the outage probability of multibranch diversity receivers over GG fading channels. We also compare the computational efficiency of our approach with naive MC.

II. System Model

The instantaneous signal-to-noise ratio (SNR) expression at the diversity receiver, is given by [13]

$$\gamma_{end} = \frac{E_s}{N_0\sqrt{L^{1-p+q}}} \left(\sum_{\ell=1}^{L} X_\ell^p\right)^q, \tag{1}$$

where $(p,q) = (1,2)$ for the EGC case and $(p,q) = (2,1)$ for the MRC case. The ratio $\frac{E_s}{N_0}$ is the SNR per symbol at the transmitter, L is the number of diversity branches, and $\{X_\ell\}_{\ell=1}^{L}$ are the channel envelopes which are modeled as i.n.i.d GG RVs with parameters $(\beta_\ell, m_\ell, \Omega_\ell)$, $\ell = 1,\ldots,L$, whose PDFs are given by [1, Eq. (1)]

$$f_{X_\ell}(x) = \frac{\beta_\ell m_\ell^{m_\ell} x^{\beta_\ell m_\ell - 1}}{\Omega_\ell{}^{m_\ell}\Gamma(m_\ell)} \exp\left(-\frac{m_\ell}{\Omega_\ell}x^{\beta_\ell}\right), x \geq 0,$$
$$\ell = 1,\ldots,L, \tag{2}$$

where β_ℓ and m_ℓ are two positive real numbers that represent the distribution shape parameters, $\Gamma(\cdot)$ is the Gamma function [14, Sec. (8.31)], and Ω_ℓ is linked to the mean of X_ℓ as

$$\mathbb{E}\left[X_\ell\right] = \left(\frac{\Omega_\ell}{m_\ell}\right)^{\frac{1}{\beta_\ell}} \frac{\Gamma\left(m_\ell + \frac{1}{\beta_\ell}\right)}{\Gamma(m_\ell)},\ \ell = 1,\ldots,L. \tag{3}$$

The GG(β, m, Ω) distribution was introduced by Stacy [1] as a generic model that covers Weibull, Gamma and other distributions as special cases. Table I summarizes these special cases when one or more of the GG distribution parameters is set equal to specific values.

TABLE I

Special cases of the GG distribution.

Distribution	m	β	Ω
Chi-squared	$\in \mathbb{N}\setminus\{0\}$	1	$2m$
Erlang	$\in \mathbb{N}\setminus\{0\}$	1	–
Exponential	1	1	–
Rayleigh	1	2	–
One-sided Gaussian	$\frac{1}{2}$	2	–
Gamma	–	1	–
Nakagami-m	–	2	–
Weibull	1	–	–

The quality of a communication system can be evaluated by computing the outage probability. This metric is a function of the transmission technique used, but also the channel on which the signal is transmitted. More specifically, for a given threshold γ_{th}, the outage probability P is defined as the probability that the instantaneous SNR drops below γ_{th}, i.e.

$$P = \mathbb{P}(\gamma_{end} \leq \gamma_{th}) = \mathbb{P}\left(\sum_{\ell=1}^{L} X_\ell^p \leq \left(\frac{N_0}{E_s}\sqrt{L^{1-p+q}}\gamma_{th}\right)^{\frac{1}{2}}\right). \quad (4)$$

In the remainder of the paper, we will focus on the EGC case and we will show how the approach can be easily used for the MRC case. Thereby, unless stated otherwise, P will be given by

$$P = \mathbb{P}\left(S_L = \sum_{\ell=1}^{L} X_\ell \leq \gamma_0 = \left(\frac{N_0 L}{E_s}\gamma_{th}\right)^{\frac{1}{2}}\right). \quad (5)$$

We can see that our goal of estimating the outage probability reduces to find the CDF of the sum of GG RVs. In particular, we are interested in the case where the outage probability requirements are very low, i.e. in the range of 10^{-6} to 10^{-10}. For instance, this is common in areas such as wireless backhauling using Free Space optics (FSO) [15, 16] and millimeter wave [17, 18].

III. Importance Sampling

This section gives a brief introduction to the main ideas behind the IS method. Later on, we shall discuss in depth the proposed IS scheme for the purpose of estimating the outage probability of L-branch EGC diversity receivers over i.n.i.d GG fading channels.

The outage probability is given by $P = \mathbb{E}\left[\mathbb{1}_{(S_L \leq \gamma_0)}\right]$, where $\mathbb{E}[\cdot]$ is the expectation with respect to (w.r.t) the probability measure under which the PDF of X_ℓ is $f_{X_\ell}(\cdot)$, $\ell = 1, 2, \ldots, L$. The naive MC estimator of (5) is thus

$$\hat{P}_{MC} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{(S_L(\omega_i) \leq \gamma_0)}, \quad (6)$$

where N is the number of MC samples, $\mathbb{1}_{(\cdot)}$ is the indicator function, and $\{S_L(\omega_i)\}_{i=1}^N$ are i.i.d. realizations of the RV S_L. The sequence $\{X_\ell(\omega_i)\}_{\ell=1}^L$ is sampled independently according to the PDFs (2) for each realization of S_L.

Rare events are events with a very small probability of occurrence but have a significant contribution to the MC estimation. The IS method [19] is one of the most used approaches in the evaluation of rare events probabilities. The basic idea behind is to modify the dynamics of the simulation so that the rare event happens more frequently. This can be accomplished by changing the underlying PDF of the RV in question. The goal is thereby to find a clever change in probability that, given a certain confidence interval, reduces the number of required simulation runs. It may happen that IS does not improve the estimation when a poor choice of the new PDF, known as the biased PDF, is introduced. In fact, a bad choice may lead to a large likelihood ratio and thus we end up with an estimator having a larger variance than the MC estimator. The effectiveness of this method relies on the good change of the underlying PDF. The reader is directed to [20] for a succinct review of the use of IS in communication systems.

By introducing new biased densities $\{f^*_{X_\ell}(\cdot)\}_{\ell=1}^{L}$, we can re-write $P = \mathbb{E}^*\left[\mathbb{1}_{(S_L \leq \gamma_0)} \mathcal{L}(X_1, \ldots, X_L)\right]$ where $\mathbb{E}^*[\cdot]$ denotes the expectation w.r.t the probability measure under which the PDF of X_ℓ is $f^*_{X_\ell}(\cdot)$, $\ell = 1, 2, \ldots, L$.

The likelihood ratio $\mathcal{L}(X_1, \ldots, X_L)$ is defined as

$$\mathcal{L}(X_1, \ldots, X_L) = \prod_{\ell=1}^{L} \frac{f_{X_\ell}(X_\ell)}{f^*_{X_\ell}(X_\ell)}. \tag{7}$$

The sole purpose of IS is to "encourage" the sampling from the importance region $\{S_L \leq \gamma_0\}$ and thereby to make such event more likely to happen under the new distribution. In this case, the IS estimator of (5) is

$$\hat{P}_{IS} = \frac{1}{N^*} \sum_{i=1}^{N^*} \mathbb{1}_{(S_L(\omega_i) \leq \gamma_0)} \mathcal{L}(X_1(\omega_i), \ldots, X_L(\omega_i)), \tag{8}$$

where for each realization $i = 1, \ldots, N$, the sequence $\{X_\ell(\omega_i)\}_{\ell=1}^{L}$ are sampled independently according to the biased PDFs $\{f^*_\ell(\cdot)\}_{\ell=1}^{L}$.

IS consists in running the simulations by exhibiting a change in the original distribution, for which the studied event is no longer rare. To correct the bias introduced by this manipulation, the simulations are weighted by the likelihood ratio which is the Radon-Nikodym density of the original distribution w.r.t the biased one. The goal of a IS simulation is therefore to provide biased PDFs $\{f^*_{X_\ell}(\cdot)\}_{\ell=1}^{L}$ in order to reduce the variance of the estimator. The versatility of the IS method is quite large due to the different possible choices of the biased PDFs and thereby the challenge is to make the right choice of such PDFs. An inadequate choice can produce a large likelihood ratio resulting in potentially more time-consuming computations than naive MC simulation. To assess the goodness of an IS approach, many criteria has been introduced in previous works (see, for instance, [21] and references therein) among them we find the bounded relative error, one of the desirable property in the field of rare events algorithms.

IV. Proposed Approach

In this section, a clever choice of the biased PDF is introduced. In fact, we propose to introduce the parameter $\Omega_\ell^* = \Omega_\ell - \Omega_{0,\ell}$ in the new biased PDF where $\Omega_{0,\ell}$ satisfies $0 \leq \Omega_{0,\ell} < \Omega_\ell$ and as $\gamma_0 \to 0$, it approaches Ω_ℓ, $\ell = 1, \ldots, L$. This choice is justified by two reasons: (i) the biased PDF belongs to the same family as the original PDF so the sampling from it should be simple and no extra effort will be dedicated to this issue and (ii) this choice will lead to a variance scaling (see Theorem 1).

In this case, the biased PDF is

$$f^*_{X_\ell}(x) = \frac{\beta_\ell m_\ell^{m_\ell} x^{\beta_\ell m_\ell - 1}}{(\Omega_\ell - \Omega_{0,\ell})^{m_\ell}\Gamma(m_\ell)} \exp\left(-\frac{m_\ell x^{\beta_\ell}}{\Omega_\ell - \Omega_{0,\ell}}\right), x \geq 0,$$
$$\ell = 1, \ldots, L. \tag{9}$$

For the selection of the parameters $\{\Omega_{0,\ell}\}_{\ell=1}^L$, we require that the equation $\mathbb{E}^*[S_L] = \gamma_0$ holds. This equation has infinitely many solutions. We choose a particular solution of the following form

$$\Omega_{0,\ell} = \Omega_\ell - \alpha_\ell \gamma_0^{\beta_\ell}, \forall \ell = 1, \ldots L, \tag{10}$$

where $\alpha_\ell = \left[\frac{m_\ell \Gamma(m_\ell)}{\Gamma(m_\ell + \frac{1}{\beta_\ell})L}\right]^{\beta_\ell}$.

Theorem 1. The choice of the biased PDF as in (9) where $\Omega_{0,\ell}$ satisfies $0 \leq \Omega_{0,\ell} < \Omega_\ell$ and $\lim\limits_{\gamma_0 \to 0} \Omega_{0,\ell} = \Omega_\ell$, $\ell = 1, \ldots, L$ leads to a variance scaling of the RV X_ℓ under the new PDF, i.e.

$$\mathbb{V}^*[X_\ell] = c \times \mathbb{V}[X_\ell], \tag{11}$$

where c is a positive constant such that $\lim\limits_{\gamma_0 \to 0} c = 0$ and $\mathbb{V}[\cdot]$ (receptively $\mathbb{V}^*[\cdot]$) is the variance of X_ℓ w.r.t $f_{X_\ell}(\cdot)$ (respectively $f^*_{X_\ell}(\cdot)$).

Proof: See Appendix A. ■

The following theorem characterizes the efficiency of our proposed IS estimator.

Theorem 2. Let $\{X_\ell\}_{\ell=1}^{L}$ be a sequence of i.n.i.d GG RVs and $f^*_{X_\ell}(\cdot)$ be defined as in (9) where $\Omega_{0,\ell}$ is given by (10). Then, the IS estimator (8) has a relative bounded error, i.e.

$$\limsup_{\gamma_0 \to 0} \frac{\mathbb{E}^*\left[\mathbb{1}_{(S_L \leq \gamma_0)} \mathcal{L}^2(X_1, \ldots, X_L)\right]}{P^2} < +\infty. \tag{12}$$

provided that $\min_{1 \leq \ell \leq L} \beta_\ell > 1$.

Proof: See Appendix B. ■

Remark 1. To accurately estimate the probability P, the naive MC simulation requires a number of samples of the order of $\mathcal{O}(P^{-1})$. However, for the same accuracy requirement and when the IS estimator is endowed with the relative bounded error property, the number of simulation runs N required remains bounded independently of how small the outage probability P is.

Remark 2. In this remark, we show briefly how our approach is easily extendable to the MRC case. First, we recall the expression of the outage probability in this case

$$\mathbb{P}\left(\sum_{\ell=1}^{L} Z_\ell \leq \eta_0 = \frac{N_0}{E_s}\gamma_{th}\right). \tag{13}$$

where $Z_\ell = X_\ell^2$ is also a GG RV with parameters $(\frac{\beta_\ell}{2}, m_\ell, \Omega_\ell)$.
As we can see from (13), the problem is again reduced to finding the CDF of the sum of GG variates. Using the approach described in this section, we can easily evaluate the outage probability given by (13) for the MRC scenario.

A pseudo-code detailing how to estimate the outage probability P using our proposed IS approach is as follows

Algorithm 1 Our IS Algorithm to compute the outage probability (5) (EGC case)

Input: L, N^*, γ_0, $\{(m_\ell, \beta_\ell, \Omega_\ell)\}_{\ell=1}^{L}$.

for $\ell = 1$ to L **do**
  Evaluate $\mathbf{\Omega_0} = \{\Omega_{0,\ell}\}_{\ell=1}^{L}$ according to Eq. (10).
end for
for $i = 1$ to N^* **do**
  - Generate a realization of $\mathbf{X} = \{X_\ell(\omega_i)\}_{\ell=1}^{L}$ from
  the PDFs $= \{f^*_{X_\ell}(\cdot)\}_{\ell=1}^{L}$ given by (9).
  - Compute $\mathbb{1}_{(S_L(\omega_i) \leq \gamma_0)} \mathcal{L}(X_1(\omega_i), \ldots, X_L(\omega_i))$.
end for

Output: Evaluate IS estimator using Eq. (8).

V. Simulation Results

This section presents the numerical simulations regarding the estimation of the outage probability using both naive MC and our proposed IS method. The accuracy as well as the efficiency of both methods is analyzed.

To compare the efficiency of IS to naive MC, we need to compare the number of simulation runs required by each method to achieve the same accuracy requirement ε. To this end, we introduce the relative error of naive MC simulation

$$\varepsilon = \frac{\alpha}{P} \sqrt{\frac{P(1-P)}{N}}, \tag{14}$$

where $\alpha = 1.96$ which corresponds to a 95% confidence interval.

Similarly, we define the relative error of the IS method as

$$\varepsilon^* = \frac{\alpha}{P} \sqrt{\frac{\mathbb{V}^*[\mathbb{1}_{(S_L \leq \gamma_0)} \mathcal{L}(X_1, \ldots, X_L)]}{N}}. \tag{15}$$

Let ϵ_0 be a fixed accuracy requirement. Using Eqs. (14) and (15), we can determine the number of samples needed by naive MC and IS simulations respectively

$$N = P(1-P) \left(\frac{\alpha}{P\epsilon_0} \right)^2, \tag{16}$$

$$N^* = \mathbb{V}^*[\mathbb{1}_{(S_L \leq \gamma_0)} \mathcal{L}(X_1, \ldots, X_L)] \left(\frac{\alpha}{P\epsilon_0} \right)^2. \tag{17}$$

Table II details the fading parameters (m_ℓ, β_ℓ) used in this section for two scenarios $L = 4$ and $L = 6$. In Fig. 1, we plot the outage probability P against the SNR threshold γ_{th} using naive MC (blue curve) and our proposed IS approach (red curve). The solid line represents the EGC scenario while the dashed line is for the MRC case. Similar conclusions can be drawn for both type of diversity. In fact, we notice that for the range of probabilities between 10^{-1} and 10^{-4}, both methods match for the two cases. However, as the probability becomes smaller, naive MC fails to estimate the outage probability with the same accuracy as our method. In fact, we can see that for $L = 6$, naive MC with $N = 10^7$ samples is unable to estimate accurately the outage probabilities below 10^{-6} while the proposed IS scheme can estimate P even with a small number of samples $N^* = 10^4$.

TABLE II

FADING PARAMETERS USED TO SIMULATE THE OUTAGE PROBABILITY OF L-BRANCH MRC DIVERSITY RECEIVERS OVER I.N.I.D GG FADING MODEL.

L	Fading Parameters (m_ℓ, β_ℓ)
4	(2, 1.5), (2.5, 2), (3.2, 2.2), (3.5, 1.5)
6	(3, 1.5), (2.5, 1.8), (2.8, 2.5), (3, 2.2), (3.25, 1.5), (2.5, 1.2)

The behavior of the number of required simulations runs by both methods for a fixed accuracy requirement $\varepsilon = \varepsilon^* = 5\%$ is depicted in Fig. 2 for the EGC case. We can observe that, for high outage probabilities, naive MC is sufficient for the estimation of P since the number of simulation runs for both methods is quite the same. Our method outperforms naive MC in the region of rare events, i.e $P < 10^{-6}$. In this region, the number of samples N needed by MC to estimate P up to 95% accuracy grows rapidly whereas N^* remains almost constant. This goes hand in hand with the bounded relative property of our IS estimator. To illustrate this idea, we can look at the number of samples N^* required by IS, for $L = 4$ is approximately 2.45×10^3 (respectively 1.2×10^5) times less than the number of samples used in MC simulations for γ_{th} = 18 dB (respectively γ_{th} = 16 dB). Similar conclusions can be drawn for the MRC scenario.

In Fig. 3, we plot the relative errors ε and ε^* of both naive MC and IS for the case when $L = 4$ using $N = 10^8$ and $N^* = 10^4$ samples, respectively. In this plot, we show the result for the EGC case as similar conclusions are valid for the MRC scenario. Although the relative error of naive MC is relatively smaller than the one of IS for relatively high probabilities, it tends

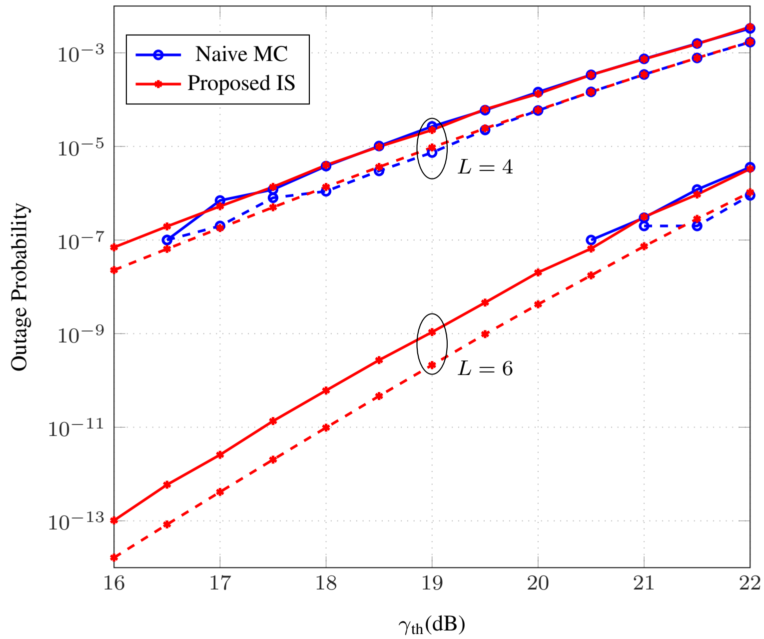

Fig. 1. Outage probability of L-branch diversity receivers over i.n.i.d GG fading model with $E_s/N_0 = 10$ dB and $\Omega = 10$ dB. Number of samples $N = 10^7$ and $N^* = 10^4$. EGC case: solid line and MRC case: dashed line.

to grow as the outage probability becomes smaller unlike the IS relative error which remains almost constant no matter how small the probability P is. This shows how our proposed approach outperforms naive MC.

VI. Conclusion

In this paper, we presented a novel approach for the efficient estimation of the left tail of the sum of GG variates. Our approach is based on a variance scale type of IS scheme. We showed that the proposed estimator is endowed with the bounded relative error criterion. Capitalizing on this result, we were able to efficiently evaluate the outage probability of L-branch diversity receivers over GG fading channels. Simulation results show the accuracy as well as the efficiency of our proposed IS estimator compared to the naive MC estimator.

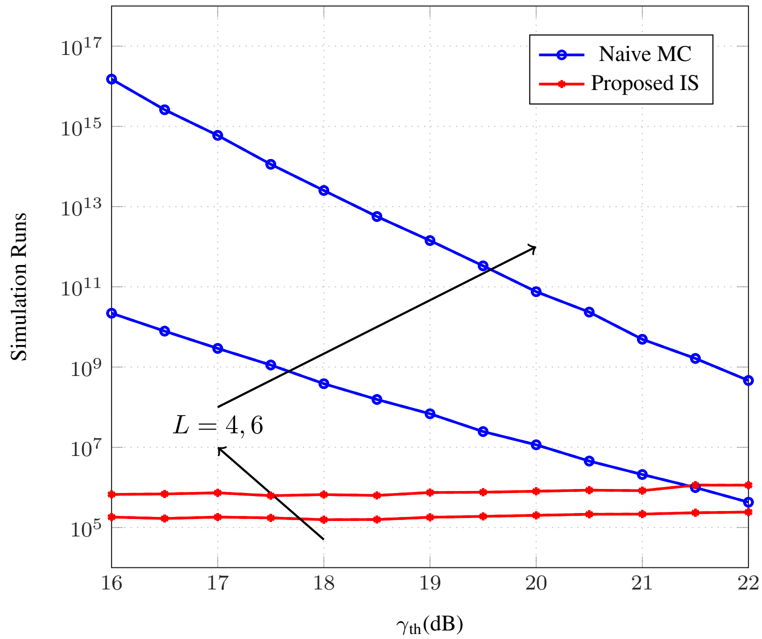

Fig. 2. Number of required simulation runs for 5% relative error for L-branch EGC diversity receivers over i.n.i.d GG fading model with $E_s/N_0 = 10$ dB and $\Omega = 10$ dB.

Appendix A
Proof of Theorem 1

Proof: We start by recalling the expression of the variance of X_ℓ under $f_{X_\ell}(\cdot)$

$$\mathbb{V}[X_\ell] = \left(\frac{\Omega_\ell}{m_\ell}\right)^{\frac{2}{\beta_\ell}} \left[\frac{\Gamma\left(m_\ell + \frac{2}{\beta_\ell}\right)}{\Gamma\left(m_\ell\right)} - \left(\frac{\Gamma\left(m_\ell + \frac{1}{\beta_\ell}\right)}{\Gamma\left(m_\ell\right)}\right)^2\right]. \tag{A.1}$$

Let $b = \left[\frac{\Gamma\left(m_\ell + \frac{2}{\beta_\ell}\right)}{\Gamma(m_\ell)} - \left(\frac{\Gamma\left(m_\ell + \frac{1}{\beta_\ell}\right)}{\Gamma(m_\ell)}\right)^2\right]$, so that the variance writes $\mathbb{V}[X_\ell] = b \times \left(\frac{\Omega_\ell}{m_\ell}\right)^{\frac{2}{\beta_\ell}}$. Similarly, we can write the the variance of X_ℓ under $f^*_{X_\ell}(\cdot)$ as

$$\begin{aligned}\mathbb{V}[X_\ell] &= b \times \left(\frac{\Omega_\ell - \Omega_{0,\ell}}{m_\ell}\right)^{\frac{2}{\beta_\ell}} \\ &= b \times \left(\frac{\Omega_\ell}{m_\ell}\right)^{\frac{2}{\beta_\ell}} \left(1 - \frac{\Omega_{0,\ell}}{\Omega_\ell}\right)^{\frac{2}{\beta_\ell}} \\ &= \left(1 - \frac{\Omega_{0,\ell}}{\Omega_\ell}\right)^{\frac{2}{\beta_\ell}} \mathbb{V}[X_\ell].\end{aligned} \tag{A.2}$$

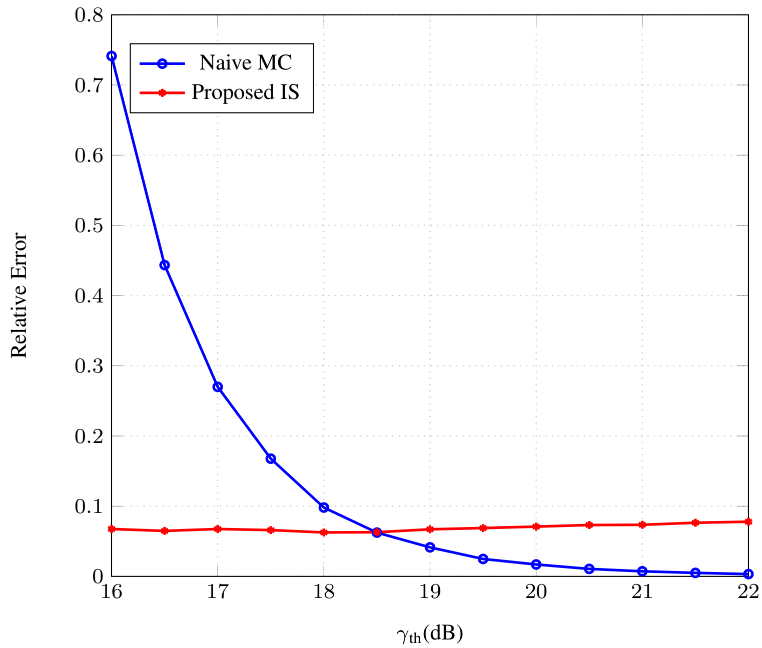

Fig. 3. Relative error of both methods for $L = 4$ with number of samples $N = 10^8$ and $N^* = 10^4$.

Thus, $c = \left(1 - \frac{\Omega_{0,\ell}}{\Omega_\ell}\right)^{\frac{2}{\beta_\ell}}$. Clearly c is a positive constant such that $\lim\limits_{\gamma_0 \to 0} c = 0$ since $0 \leq \Omega_{0,\ell} < \Omega_\ell$ and $\lim\limits_{\gamma_0 \to 0} \Omega_{0,\ell} = \Omega_\ell$. ■

Appendix B
Proof of Theorem 2

Proof: The likelihood ratio is given by

$$\begin{aligned} \mathcal{L}(X_1, \ldots, X_L) &= \prod_{\ell=1}^{L} \frac{f_{X_\ell}(X_\ell)}{f^*_{X_\ell}(X_\ell)} \\ &= \prod_{\ell=1}^{L} \left(\frac{\Omega_\ell - \Omega_{0,\ell}}{\Omega_\ell}\right)^{m_\ell} \prod_{\ell=1}^{L} \exp\left(m_\ell \left[\frac{1}{\Omega_\ell - \Omega_{0,\ell}} - \frac{1}{\Omega_\ell}\right] x_\ell^{\beta_\ell}\right). \end{aligned} \tag{B.1}$$

Replacing the expression of $\Omega_{0,\ell}$ in (9), we get

$$\begin{aligned} \mathcal{L}(X_1, \ldots, X_L) = &\prod_{\ell=1}^{L} \left(\frac{\alpha_\ell \gamma_0^{\beta_\ell}}{\Omega_\ell}\right)^{m_\ell} \\ &\times \exp\left(\sum_{\ell=1}^{L} m_\ell \left[\frac{1}{\alpha_\ell \gamma_0^{\beta_\ell}} - \frac{1}{\Omega_\ell}\right] x_\ell^{\beta_\ell}\right). \end{aligned} \tag{B.2}$$

Thereby, the likelihood can be bounded by

$$\mathcal{L}(X_1,\ldots,X_L) \leq \prod_{\ell=1}^{L}\left(\frac{\alpha_\ell}{\Omega_\ell}\right)^{m_\ell}\gamma_0^{\sum\limits_{\ell=1}^{L} m_\ell\beta_\ell}$$
$$\times\exp\left(\sum_{\ell=1}^{L}\frac{m_\ell}{\alpha_\ell}\left(\frac{x_\ell}{\gamma_0}\right)^{\beta_\ell}\right). \tag{B.3}$$

Let $\eta_0 = \max\limits_{1\leq\ell\leq L}\frac{m_\ell}{\alpha_\ell}$, $\beta_{max} = \max\limits_{1\leq\ell\leq L}\beta_\ell$, and $\beta_{min} = \min\limits_{1\leq\ell\leq L}\beta_\ell$. We define the two sets

$$I \triangleq \left\{\frac{x_\ell}{\gamma_0} : \frac{x_\ell}{\gamma_0} < 1\right\}, \tag{B.4}$$

$$J \triangleq \left\{\frac{x_\ell}{\gamma_0} : \frac{x_\ell}{\gamma_0} \geq 1\right\}. \tag{B.5}$$

We can write

$$\sum_{\ell=1}^{L}\left(\frac{x_\ell}{\gamma_0}\right)^{\beta_\ell} = \sum_{\ell\in I}\left(\frac{x_\ell}{\gamma_0}\right)^{\beta_\ell} + \sum_{\ell\in J}\left(\frac{x_\ell}{\gamma_0}\right)^{\beta_\ell}$$
$$\leq \sum_{\ell\in I}\left(\frac{x_\ell}{\gamma_0}\right)^{\beta_{min}} + \sum_{\ell\in J}\left(\frac{x_\ell}{\gamma_0}\right)^{\beta_{max}}. \tag{B.6}$$

Using the embedding inequality of L^1 into L^α for $\alpha \geq 1$ [22], we can write, for any positive real numbers $\{v_\ell\}_{\ell=1}^{L}$

$$\sum_{\ell=1}^{L} v_\ell^\alpha \leq \left(\sum_{\ell=1}^{L} v_\ell\right)^\alpha. \tag{B.7}$$

Since $\min\limits_{1\leq\ell\leq L}\beta_\ell > 1$, we can use this inequality for $v_\ell = \frac{x_\ell}{\gamma_0}$, $\ell = 1,\ldots,L$ to get

$$\sum_{\ell=1}^{L}\left(\frac{x_\ell}{\gamma_0}\right)^{\beta_\ell} \leq \left(\sum_{\ell\in I}\frac{x_\ell}{\gamma_0}\right)^{\beta_{min}} + \left(\sum_{\ell\in J}\frac{x_\ell}{\gamma_0}\right)^{\beta_{max}}$$
$$\leq \left(\sum_{\ell=1}^{L}\frac{x_\ell}{\gamma_0}\right)^{\beta_{min}} + \left(\sum_{\ell=1}^{L}\frac{x_\ell}{\gamma_0}\right)^{\beta_{max}}. \tag{B.8}$$

Thus, we can write

$$\mathcal{L}(X_1,\ldots,X_L) \leq \prod_{\ell=1}^{L}\left(\frac{\alpha_\ell}{\Omega_\ell}\right)^{m_\ell}\gamma_0^{\sum\limits_{\ell=1}^{L} m_\ell\beta_\ell}\times$$
$$\exp\left(\eta_0\left[\frac{1}{\gamma_0^{\beta_{min}}}\left(\sum_{\ell=1}^{L}x_\ell\right)^{\beta_{min}} + \frac{1}{\gamma_0^{\beta_{max}}}\left(\sum_{\ell=1}^{L}x_\ell\right)^{\beta_{max}}\right]\right). \tag{B.9}$$

Therefore, we obtain the following upper bound

$$\mathbb{E}^*\left[\mathbb{1}_{\{S_L\leq\gamma_0\}}\mathcal{L}^2(X_1,\ldots,X_L)\right]\leq\prod_{\ell=1}^{L}\left(\frac{\alpha_\ell}{\Omega_\ell}\right)^{2m_\ell}\gamma_0^{2\sum\limits_{\ell=1}^{L}m_\ell\beta_\ell}$$
$$\times\exp(4\eta_0). \tag{B.10}$$

On the other hand, we have that

$$\bigcap_{\ell=1}^{L}\{X_\ell\leq\frac{\gamma_0}{L}\}\subset\{\sum_{\ell=1}^{L}X_\ell\leq\gamma_0\}. \tag{B.11}$$

In the i.n.i.d scenario, this leads to

$$P\geq\prod_{\ell=1}^{L}\mathbb{P}\left(X_\ell\leq\frac{\gamma_0}{L}\right). \tag{B.12}$$

We recall that the CDF of a GG RV is given by [1]

$$F_{X_\ell}(x)=\frac{\gamma\left(m_\ell,\frac{m_\ell}{\Omega_\ell}x^{\beta_\ell}\right)}{\Gamma(m_\ell)}, \tag{B.13}$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete Gamma defined in [14, Eq. (8.350.1)]. Since we have $\gamma(s,z)\underset{z\to0}{\sim}\frac{z^s}{s}$ [23, Eq. (2.272)], then we can write

$$\gamma\left(m_\ell,\frac{m_\ell}{\Omega_\ell}\left(\frac{\gamma_0}{L}\right)^{\beta_\ell}\right)\underset{\gamma_0\to0}{\sim}\frac{m_\ell^{m_\ell-1}\gamma_0^{\beta_\ell m_\ell}}{\Omega_\ell^{m_\ell}L^{\beta_\ell m_\ell}} \tag{B.14}$$

Thereby, the CDF has the following asymptotic expansion

$$F_{X_\ell}\left(\frac{\gamma_0}{L}\right)\underset{\gamma_0\to0}{\sim}\frac{m_\ell^{m_\ell-1}\gamma_0^{\beta_\ell m_\ell}}{\Omega_\ell^{m_\ell}L^{\beta_\ell m_\ell}\Gamma(m_\ell)} \tag{B.15}$$

A lower bound on P is given by

$$P\geq\prod_{\ell=1}^{L}F_{X_\ell}\left(\frac{\gamma_0}{L}\right)\underset{\gamma_0\to0}{\sim}\prod_{\ell=1}^{L}\frac{m_\ell^{m_\ell-1}\gamma_0^{\beta_\ell m_\ell}}{\Omega_\ell^{m_\ell}L^{\beta_\ell m_\ell}\Gamma(m_\ell)}. \tag{B.16}$$

Thus, we get as $\gamma_0\to0$

$$\frac{1}{P^2}\leq\prod_{\ell=1}^{L}\left[\frac{\Omega_\ell^{m_\ell}L^{\beta_\ell m_\ell}\Gamma(m_\ell)}{m_\ell^{m_\ell-1}}\right]^2\gamma_0^{-2\sum\limits_{\ell=1}^{L}\beta_\ell m_\ell}. \tag{B.17}$$

Combining (B.10) and (B.17), we obtain

$$\limsup_{\gamma_0\to0}\frac{\mathbb{E}^*\left[\mathbb{1}_{\{S_L\leq\gamma_0\}}\mathcal{L}^2(X_1,\ldots,X_L)\right]}{P^2}$$
$$\leq\prod_{\ell=1}^{L}\left[\frac{[\Gamma(m_\ell)]^{1+m_\ell\beta_\ell}}{m_\ell^{m_\ell-1}[\Gamma(m_\ell+\frac{1}{\beta_\ell})]^{m_\ell\beta_\ell}}\right]^2\exp\left(4\eta_0\right). \tag{B.18}$$

and hence the proof is concluded. ■

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