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Regularity of solutions in semilinear elliptic theory

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Abstract We study the semilinear Poisson equation

$$\Delta u = f(x, u) \quad \text{in} \quad B_1. \tag{1}$$

Our main results provide conditions on f which ensure that weak solutions of (1) belong to $C^{1,1}(B_{1/2})$. In some configurations, the conditions are sharp.

Keywords Semilinear elliptic theory · Partial differential equations · Regularity theory

1 Introduction

The semilinear Poisson equation (1) encodes stationary states of the nonlinear heat, wave, and Schrödinger equation. In the case when f is the Heaviside function in the u-

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variable, (1) reduces to the classical obstacle problem. For an introduction to classical semilinear theory, see [4,5].

It is well-known that weak solutions of (1) belong to the usual Sobolev space $W^{2,p}(B_{1/2})$ for any $1 \leq p < \infty$ provided $f \in L^\infty$. Recent research activity has thus focused on identifying conditions on f which ensure $W^{2,\infty}(B_{1/2})$ regularity of u.

1.1 The classical theory

There are simple examples which illustrate that continuity of $f = f(x)$ does not necessarily imply that u has bounded second derivatives: for $p \in (0, 1)$ and $x \in \mathbb{R}^2$ such that $|x| < 1$, the function

$$u(x) = x_1 x_2 (-\log |x|)^p$$

has a continuous Laplacian but is not in $C^{1,1}$ [15]. However, if f is Hölder continuous, then it is well-known that $u \in C^{2,\alpha}$; if f is Dini continuous, then $u \in C^2$ [7,11]. The sharp condition which guarantees bounded second derivatives of u is the $C^{1,1}$ regularity of $f * N$ where N is the Newtonian potential and $*$ denotes convolution; this requirement is strictly weaker than Dini continuity of f.

In the general case, the state-of-the-art is a theorem of Shahgholian [14] which states that $u \in C^{1,1}$ whenever $f = f(x, u)$ is Lipschitz in x, uniformly in u, and $\partial_u f \geq -C$ weakly for some $C \in \mathbb{R}$. In some configurations this illustrates regularity for continuous functions $f = f(u)$ which are strictly below the classical Dini-threshold in the u-variable, e.g. the odd reflection of

$$f(u) = -\frac{1}{\log(u)}$$

about the origin. Shahgholian's theorem is proved via the celebrated Alt–Caffarelli–Friedman (ACF) monotonicity formula and it seems difficult to weaken the assumptions by this method. On the other hand, Koch and Nadirashvili [10] recently constructed an example which illustrates that the continuity of f is not sufficient to deduce that weak solutions of $\Delta u = f(u)$ are in $C^{1,1}$. With all this in mind, we make the following assumption.

Assumption A Let $f = f(x, u)$ be Dini continuous in u, uniformly in x, and assume it has a $C^{1,1}$ Newtonian potential in x, uniformly in u.

One of our main results is the following statement.

Theorem 1.1 *Suppose f satisfies Assumption A. Then any solution of (1) is $C^{1,1}$ in $B_{1/2}$.*

Our assumption includes functions which fail to satisfy both conditions in Shahgholian's theorem, e.g.

$$f(x_1, x_2, t) = \frac{x_1}{\log(|x_2|)(-\log |t|)^p},$$

for $p > 1, x = (x_1, x_2) \in B_1$ and $t \in (-1, 1)$. The Newtonian potential assumption in the x-variable is essentially sharp whereas the condition in the t-variable is in general not comparable with Shahgholian's assumption.

The proof of Theorem 1.1 does not invoke monotonicity formulas and is self-contained. We consider the L^2 projection of D^2u on the space of Hessians generated by second order homogeneous harmonic polynomials on balls with radius $r > 0$ and show that the projections stay uniformly bounded as $r \to 0^+$. Although this approach has proven effective in dealing with a variety of free boundary problems [2,6,8,9], Theorem 1.1 illustrates that it is also useful in extending and refining the classical elliptic theory.

1.2 Singular case: the free boundary theory

In §4 we study the PDE (1) for functions $f = f(x, u)$ which are discontinuous in the u-variable at the origin.

If the discontinuity of f is a jump discontinuity, (1) has the structure

$$f(x, u) = g_1(x, u)\chi_{\{u>0\}} + g_2(x, u)\chi_{\{u<0\}}, \tag{2}$$

where g_1, g_2 are continuous functions such that

$$g_1(x, 0) \neq g_2(x, 0), \quad \forall x \in B_1,$$

and χ_Ω defines the indicator function of the set Ω.

Our aim is to find the most general class of coefficients g_i which generate interior $C^{1,1}$ regularity.

The classical obstacle problem is obtained by letting $g_1 = 1$, $g_2 = 0$, and it is well-known that solutions have second derivatives in L^∞ [13]. Nevertheless, by selecting $g_1 = -1$, $g_2 = 0$, one obtains the so-called unstable obstacle problem. Elliptic theory and the Sobolev embedding theorem imply that any weak solution belongs to $C^{1,\alpha}$ for any $0 < \alpha < 1$. It turns out that this is the best one can hope for: there exists a solution which fails to be in $C^{1,1}$ [3]. Hence, if there is a jump at the origin, $C^{1,1}$ regularity can hold only if the jump is positive and this gives rise to:

Assumption B $g_1(x, 0) - g_2(x, 0) \geq \sigma_0,\ x \in B_1$ for some $\sigma_0 > 0$.

The free boundary $\Gamma = \partial\{u \neq 0\}$ consists of two parts: $\Gamma^0 = \Gamma \cap \{\nabla u = 0\}$ and $\Gamma^1 = \Gamma \cap \{\nabla u \neq 0\}$. The main difficulty in proving $C^{1,1}$ regularity is the analysis of points where the gradient of the function vanishes. In this direction we establish the following result.

Theorem 1.2 *Suppose g_1, g_2 satisfy A and B. Then if u is a solution of (1), $\|u\|_{C^{1,1}(K)} < \infty$ for any $K \Subset B_{1/2}(0)\backslash\overline{\Gamma^1}$.*

At points where the gradient does not vanish, the implicit function theorem yields that the free boundary is locally a $C^{1,\alpha}$ graph for any $0 < \alpha < 1$. The solution u changes sign across the free boundary, hence it locally solves the equation $\Delta u = g_1(x,u)$ on the side where it is positive and $\Delta u = g_2(x,u)$ on the side where it is negative. If the coefficients g_i are regular enough to provide $C^{1,1}$ solutions up to the boundary—this is encoded in Assumption C—then we obtain full $C^{1,1}$ regularity.

Assumption C For any $M > 0$ there exist $\theta_0(M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ and $C_3(M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ such that for all $z \in B_{1/2}$ any solution of

$$\begin{cases} \Delta v = g_1(x,v)\chi_{\{v>0\}} + g_2(x,v)\chi_{\{v<0\}}, \ x \in B_{1/2}(z); \\ |v(x)| \le M, \ x \in B_{1/2}(z); \\ v(z) = 0, \ 0 < |\nabla v(z)| \le \theta_0; \end{cases}$$

admits a bound

$$\|D^2 v\|_{L^\infty(B_{|\nabla v(z)|/\theta_0}(z))} \le C_3.$$

Remark 1 A sufficient condition which ensures C is that g_i are Hölder continuous, see [12, Proposition 2.6] and [1, Theorem 9.3]. The idea being that at such points, the set $\{u = 0\}$ is locally $C^{1,\alpha}$ (via the implicit function theorem) and one may thereby reduce the problem to a classical PDE for which up to the boundary estimates are known.

Theorem 1.3 *Suppose g_1, g_2 satisfy A, B and C. Let u be a solution of* (1) *and* $0 \in \Gamma^0$. *Then $u \in C^{1,1}(B_{\rho_0}(0))$, for some $\rho_0 > 0$.*

Equation (1) with right-hand side of the form (2) is a generalization of the well-studied two-phase membrane problem, where $g_i(x,u) = \lambda_i(x),\ i = 1, 2$. The $C^{1,1}$ regularity in the case when $\lambda_1 \ge 0, \lambda_2 \le 0$ are two constants satisfying B was obtained by Uraltseva [16] via the ACF monotonicity formula. Moreover, Shahgholian proved this result for Lipschitz coefficients which satisfy B [14, Example 2]. If the coefficients are Hölder continuous, the ACF method does not directly apply and under the stronger assumption that $\inf \lambda_1 > 0$ and $\inf -\lambda_2 > 0$, Edquist, Lindgren, Shahgholian [12] obtained the $C^{1,1}$ regularity via an analysis of blow-up limits and a classification of global solutions (see also [12, Remark 1.3]). Theorem 1.3 improves and extends this result.

The difficulty in the case when g_i depend also on u is that if $v := u + L$ for some linear function L, then v is no longer a solution to the same equation, so one has to get around the lack of linear invariance. Our technique exploits that linear perturbations do not affect certain L^2 projections.

The proof of Theorem 1.3 does not rely on classical monotonicity formulas or classification of global solutions. Rather, our method is based on an identity which provides monotonicity in r of the square of the L^2 norm of the projection of u onto the space of second order homogeneous harmonic polynomials on the sphere of radius r.

Theorems 1.2 and 1.3 deal with the case when f has a jump discontinuity. If f has a removable discontinuity, (1) has the structure

$$\Delta u = g(x,u)\chi_{\{u\neq 0\}}. \tag{3}$$

In this case, one may merge some observations in the proofs of the previous results with the method in [2] and prove the following theorem.

Theorem 1.4 *If g satisfies Assumption A, then every solution of* (3) *is in* $C^{1,1}(B_{1/2})$.

Theorems 1.1–1.4 provide a comprehensive theory for the general semilinear Poisson equation where the free boundary theory is encoded in the regularity assumption of f in the u-variable.

2 Technical tools

Throughout the text, the right-hand side of (1) is assumed to be bounded. Moreover, $\mathcal{P}_2$ denotes the space of second order homogeneous harmonic polynomials. A useful elementary fact is that all norms on $\mathcal{P}_2$ are equivalent.

Lemma 2.1 *The space $\mathcal{P}_2$ is a finite dimensional linear space. Consequently, all norms on $\mathcal{P}_2$ are equivalent.*

For $u \in W^{2,2}(B_1)$, $y \in B_1$ and $r \in (0, \operatorname{dist}(y, \partial B_1))$, $\Pi_y(u,r)$ is defined to be the L^2 projection operator on $\mathcal{P}_2$ given by

$$\inf_{h\in\mathcal{P}_2} \int_{B_1} \left| D^2 \frac{u(rx+y)}{r^2} - D^2 h \right|^2 dx = \int_{B_1} \left| D^2 \frac{u(rx+y)}{r^2} - D^2 \Pi_y(u,r) \right|^2 dx.$$

Calderon–Zygmund theory yields the following useful inequality for re-scalings of weak solutions of (1).

Lemma 2.2 *Let u solve* (1)*, $y \in B_{1/2}$, and $r \le 1/4$. Then for*

$$\tilde{u}_r(x) = \frac{u(rx+y) - rx\cdot\nabla u(y) - u(y)}{r^2}$$

it follows that for $1 \le p < \infty$ and $0 < \alpha < 1$,

$$\|\tilde{u}_r - \Pi_y(u,r)\|_{W^{2,p}(B_1)} \le C(n, \|f\|_{L^\infty(B_1\times\mathbb{R})}, \|u\|_{L^\infty(B_1)}, p),$$

and

$$\|\tilde{u}_r - \Pi_y(u,r)\|_{C^{1,\alpha}(B_1)} \le C(n, \|f\|_{L^\infty(B_1\times\mathbb{R})}, \|u\|_{L^\infty(B_1)}, \alpha).$$

Proof By Calderon–Zygmund theory (e.g. [2, Theorem 2.2]),

$$\|D^2 u\|_{BMO(B_{1/2})} \le C;$$

in particular,

$$\int_{B_{3/2}} |D^2\tilde{u}_r - \overline{D^2\tilde{u}_r}|^2 \le C,$$

where $\overline{D^2\tilde{u}_r}$ is the average of $D^2\tilde{u}_r$ on $B_{3/2}$. Now let

$$a = a(f, r, y) = \fint_{B_{3/2}} f(rx + y, u(rx + y))\, dx$$

and note that this quantity is uniformly controlled by $\|f\|_{L^\infty}(B_1 \times \mathbb{R})$; this fact, and the definition of Π yields (note: trace$(\overline{D^2u} - \frac{a}{n}Id) = 0$),

$$\int_{B_{3/2}} |D^2(\tilde{u}_r - \Pi_0(\tilde{u}_r, 3/2))|^2 \leq \int_{B_{3/2}} |D^2\tilde{u}_r - (\overline{D^2u} - \frac{a}{n}Id)|^2 \leq C_1.$$

Two applications of Poincaré's inequality together with the above estimate implies

$$\|\tilde{u}_r - \Pi_y(u, r) - \overline{\nabla\tilde{u}_r} \cdot x - \overline{\tilde{u}_r}\|_{W^{2,2}(B_{3/2})} \leq C_2,$$

where the averages are taken over $B_{3/2}$. Elliptic theory (e.g. [7, Theorem 9.1]) yields that for any $1 \leq p < \infty$,

$$\|\tilde{u}_r - \Pi_y(u, r) - \overline{\nabla\tilde{u}_r} \cdot x - \overline{\tilde{u}_r}\|_{W^{2,p}(B_{3/2})} \leq C_3.$$

Let $\phi := \tilde{u}_r - \overline{\nabla\tilde{u}_r} \cdot x - \overline{\tilde{u}_r}$. We have that $\phi(0) = -\overline{\tilde{u}_r}$ and $\nabla\phi(0) = -\overline{\nabla\tilde{u}_r}$; however, by the Sobolev embedding theorem, ϕ is $C^{1,\alpha}$ and thus

$$|\phi(0)| + |\nabla\phi(0)| \leq C_4$$

completing the proof of the $W^{2,p}$ estimate. The $C^{1,\alpha}$ estimate likewise follows from the Sobolev embedding theorem. □

Our analysis requires several additional simple technical lemmas involving the projection operator.

Lemma 2.3 *For any $u \in W^{2,2}(B_1)$ and $s \in [1/2, 1]$,*

$$\|\Pi_0(u, s) - \Pi_0(u, 1)\|_{L^2(B_1)} \leq C\|\Delta u\|_{L^2(B_1)},$$

and

$$\|\Pi_0(u, s) - \Pi_0(u, 1)\|_{L^\infty(B_1)} \leq C\|\Delta u\|_{L^2(B_1)},$$

for some constant $C = C(n)$.

Proof Let $f = \Delta u$ and v be the Newtonian potential of f, i.e.

$$v(x) = \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)\chi_{B_1}(y)}{|x - y|^{n-2}} dx,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Since $u - v$ is harmonic,

$$\Pi_0(u - v, s) = \Pi_0(u - v, 1);$$

therefore

$$\Pi_0(u, s) - \Pi_0(u, 1) = \Pi_0(v, s) - \Pi_0(v, 1).$$

Invoking bounds on the projection (e.g. [2, Lemma 3.2]) and Calderon–Zygmund theory (e.g. [2, Theorem 2.2]), it follows that

$$\begin{aligned} \|\Pi_0(u, s) - \Pi_0(u, 1)\|_{L^2(B_1)} &= \|\Pi_0(v, s) - \Pi_0(v, 1)\|_{L^2(B_1)} \\ &\le C\|\Delta v\|_{L^2(B_1)} = C\|\Delta u\|_{L^2(B_1)}. \end{aligned}$$

The L^∞ bound follows from the equivalence of the norms in the space $\mathcal{P}_2$. □

Lemma 2.4 *Let u solve (1). Then for all $0 < r \le 1/4$, $s \in [1/2, 1]$ and $y \in B_{1/2}$,*

$$\sup_{B_1} |\Pi_y(u, rs) - \Pi_y(u, r)| \le C,$$

and

$$\sup_{B_1} |\Pi_y(u, r)| \le C \log(1/r),$$

for some constant $C = C(n, \|f\|_{L^\infty(B_1\times\mathbb{R})}, \|u\|_{L^\infty(B_1)})$.

Proof Note that

$$\Pi_y(u, rs) - \Pi_y(u, r) = \Pi_0(\tilde{u}_r, s) - \Pi_0(\tilde{u}_r, 1),$$

where

$$\tilde{u}_r(x) = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2}$$

as before. From Lemma 2.3 we have that

$$\|\Pi_0(\tilde{u}_r, s) - \Pi_0(\tilde{u}_r, 1)\|_{L^\infty(B_1\times\mathbb{R})} \le C\|\Delta\tilde{u}_r\|_{L^2(B_1)} \le C\|f\|_{L^\infty(B_1)}.$$

As for the second inequality in the statement of the lemma let $r_0 = 1/4$ and $s \in [1/2, 1]$. Then we have that

$$\sup_{B_1} |\Pi_y(u, sr_0/2^j)| \leq \sup_{B_1} |\Pi_y(u, sr_0/2^j) - \Pi_y(u, r_0/2^j)| + \sum_{k=0}^{j-1} \sup_{B_1} |\Pi_y(u, r/2^{k+1}) - \Pi_y(u, r/2^k)| + \sup_{B_1} |\Pi_y(u, r_0)| \leq Cj \leq C \log\left(\frac{2^j}{sr_0}\right),$$

for all □ .

The previous tools imply a growth estimate on weak solutions solution of (1).

Lemma 2.5 *Let u solve (1). Then for $y \in B_{1/2}$ and $r > 0$ small enough,*

$$\sup_{B_r(y)} |u(x) - u(y) - (x - y)\nabla u(y)| \leq Cr^2 \log(1/r).$$

Proof Let

$$\tilde{u}_r = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2}.$$

The assertion of the Lemma is equivalent to the estimate

$$\|\tilde{u}_r\|_{L^\infty(B_1)} \leq C \log(1/r),$$

for r small enough. Lemma 2.4 and the $C^{1,\alpha}$ estimates of Lemma 2.2 imply

$$\|\tilde{u}_r\|_{L^\infty(B_1)} \leq \|\tilde{u}_r - \Pi_y(u, r)\|_{L^\infty(B_1)} + \|\Pi_y(u, r)\|_{L^\infty(B_1)} \leq C + C \log(1/r) \leq C \log(1/r),$$

provided r is small enough. □

Next lemma relates the boundedness of the projection operator and the boundedness of second derivatives of weak solutions of (1).

Lemma 2.6 *Let u be a solution to (1). If for each $y \in B_{1/2}$ there is a sequence $r_j(y) \to 0^+$ as $j \to \infty$ such that*

$$M := \sup_{y \in B_{1/2}} \sup_{j \in \mathbb{N}} \|D^2 \Pi_y(u, r_j(y))\|_{L^\infty(B_{1/2})} < \infty,$$

then

$$|D^2 u| \leq C \text{ a.e. in } B_{1/2},$$

for some constant $C = C(M, n, \|f\|_{L^\infty(B_1 \times \mathbb{R})}, \|u\|_{L^\infty(B_1)}) > 0$.

Proof Let $y \in B_{1/2}$ be a Lebesgue point for D^2u and $r_j = r_j(y) \to 0^+$ as $j \to \infty$. Then by utilizing Lemma 2.2,

$$\begin{aligned}|D^2u(y)| &= \lim_{j\to\infty} \fint_{B_{r_j}(y)} |D^2u(z)|dz \\ &\le \limsup_{j\to\infty} \fint_{B_{r_j}(y)} |D^2u(z) - D^2\Pi_y(u, r_j)|dz + M \\ &\le C.\end{aligned}$$

Since a.e. $z \in B_{1/2}$ is a Lebesgue point for D^2u, the proof is complete. □

Next, we introduce another projection that we need for our analysis. Define $Q_y(u, r)$ to be the minimizer of

$$\inf_{q\in\mathcal{P}_2} \int_{\partial B_1} \left| \frac{u(rx + y)}{r^2} - q(x)\right|^2 d\mathcal{H}^{n-1}.$$

The following lemma records the basic properties enjoyed by this projection, cf. [2, Lemma 3.2].

Lemma 2.7 (i) *$Q_y(\cdot, r)$ is linear;*
(ii) *if u is harmonic $Q_y(u, s) = Q_y(u, r)$ for all $s < r$;*
(iii) *if u is a linear function then $Q_y(u, r) = 0$;*
(iv) *if u is a second order homogeneous polynomial then $Q_y(u, r) = u$;*
(v) $\|Q_0(u, s) - Q_0(u, 1)\|_{L^2(\partial B_1)} \le C_s\|\Delta u\|_{L^2(B_1)}$, *for* $0 < s < 1$;
(vi) $\|Q_0(u, 1)\|_{L^2(\partial B_1)} \le \|u\|_{L^2(\partial B_1)}$.

Proof (i) This is evident.
(ii) It suffices to prove $Q_y(u, r) = Q_y(u, 1)$ for $r < 1$. Let

$$\sigma_2 = \frac{Q_y(u, 1)}{\|Q_y(u, 1)\|_{L^2(\partial B_1)}}$$

and for $i \ne 2$, let σ_i be an ith degree harmonic polynomial. Then there exist coefficients a_i such that

$$u(x + y) = \sum_{i=0}^{\infty} a_i\sigma_i(x), \quad x \in \partial B_1;$$

in particular, $a_2 = \|Q_y(u, 1)\|_{L^2(\partial B_1)}$. Let

$$v(x) = \sum_{i=0}^{\infty} a_i\sigma_i(x), \quad x \in B_1.$$

Then v is a harmonic and $u(x+y)=v(x)$ for $x\in\partial B_1$. Hence, we have that $u(x+y)=v(x)$ for $x\in B_1$ and in particular

$$u(x+y)=\sum_{i=0}^{\infty}a_i\sigma_i(x),\quad x\in B_1.$$

Therefore

$$\frac{u(rx+y)}{r^2}=\sum_{i=0}^{\infty}a_i\frac{\sigma_i(rx)}{r^2}=\sum_{i=0}^{\infty}a_ir^{i-2}\sigma_i(x),\quad x\in B_1,$$

so $Q_y(u,r)=a_2\sigma_2(x)=Q_y(u,1)$.

(iii) and (iv) These are evident.

(v) Similar to Lemma 2.3.

(vi) This follows from the fact that $Q_0(u,1)$ is the L^2 projection of u.

□

We also employ following simple observation in the subsequent analysis.

Remark 2 If H is a Hilbert space and $E\subset H$ a subspace, then for any $x\in H$ and $e\in E$, $\langle x,e\rangle=\langle Proj_E(x),e\rangle$ (since we may write $x=Proj_E(x)+y$, where $y\in E^{\perp}$).

Next we prove some technical results for $Q_y(u,r)$ and establish a precise connection between $\Pi_y(u,r)$ and $Q_y(u,r)$ by showing that the difference is uniformly bounded in r.

Lemma 2.8 *For $u\in W^{2,p}(B_1(y))$ with $p>n$ and $r\in(0,1]$,*

$$\frac{d}{dr}Q_y(u,r)=\frac{1}{r}Q_0(x\cdot\nabla u(x+y)-2u(x+y),r).$$

Proof Firstly,

$$Q_y(u,r)=Q_0\left(\frac{u(rx+y)}{r^2},1\right).$$

Since u is $C^{1,\alpha}$ if $p>n$ and Q is linear bounded operator, it follows that

$$\begin{aligned}\frac{d}{dr}Q_y(u,r)&=Q_0\left(\frac{d}{dr}\frac{u(rx+y)}{r^2},1\right)=Q_0\left(\frac{rx\cdot\nabla u(rx+y)-2u(rx+y)}{r^3},1\right)\\&=\frac{1}{r}Q_0(x\cdot\nabla u(x+y)-2u(x+y),r).\end{aligned}$$

□

Lemma 2.9 *Let $u \in W^{2,p}(B_1(y))$ with $p > n$ and $q \in \mathcal{P}_2$. Then*

$$\int_{B_1} q(x)\Delta u(x+y)dx = \int_{\partial B_1} q(x)\left(x \cdot \nabla u(x+y) - 2u(x+y)\right) d\mathcal{H}^{n-1}. \quad (4)$$

Proof Integration by parts implies

$$\int_{B_1} q(x)\Delta u(x+y)dx = \int_{B_1} \Delta q(x)u(x+y)dx + \int_{\partial B_1} q(x)\frac{\partial u(x+y)}{\partial n} - u(x+y)\frac{\partial q(x)}{\partial n} d\mathcal{H}^{n-1}.$$

By taking into account that q is a second order homogeneous polynomial it follows that

$$\frac{\partial q(x)}{\partial n} = 2q(x), \quad x \in \partial B_1.$$

Moreover,

$$\frac{\partial u(x+y)}{\partial n} = x \cdot \nabla u(x+y), \quad x \in \partial B_1.$$

Combining these equations yields (4). □

Lemma 2.10 *Let $u \in W^{2,p}(B_1(y))$ with $p > n$ and $0 < r \leq 1$. Then for every $q \in \mathcal{P}_2$,*

$$\int_{\partial B_1} q(x)\frac{d}{dr}Q_y(u,r)(x)d\mathcal{H}^{n-1} = \frac{1}{r}\int_{B_1} q(x)\Delta u(rx+y)dx.$$

Proof Let $\tilde{u}_r(x) = u(rx+y)/r^2$. From Lemmas 2.8, 2.9, and the fact that $Q_0(\cdot, 1)$ is the projection onto the space of homogeneous harmonic polynomials of degree two, we obtain

$$\begin{aligned}
&\int_{\partial B_1} q(x)\frac{d}{dr}Q_y(u,r)(x)d\mathcal{H}^{n-1} \\
&= \frac{1}{r}\int_{\partial B_1} q(x)Q_0\left(\frac{rx \cdot \nabla u(rx+y) - 2u(rx+y)}{r^2}, 1\right) d\mathcal{H}^{n-1} \\
&= \frac{1}{r}\int_{\partial B_1} q(x)Q_0\left(x \cdot \nabla \tilde{u}_r(x) - 2\tilde{u}_r(x), 1\right) d\mathcal{H}^{n-1}
\end{aligned}$$

$$= \frac{1}{r} \int_{\partial B_1} q(x)\,(x \cdot \nabla \tilde{u}_r(x) - 2\tilde{u}_r(x))\, d\mathcal{H}^{n-1}$$
$$= \frac{1}{r} \int_{B_1} q(x) \Delta \tilde{u}_r(x) dx = \frac{1}{r} \int_{B_1} q(x) \Delta u(rx + y) dx,$$

(the third equality follows from Remark 2). □

Lemma 2.11 *For $u \in W^{2,p}(B_1(y))$ with $p > n$ and $0 < r \le 1$,*

$$\frac{d}{dr} \int_{\partial B_1} Q_y^2(u, r) d\mathcal{H}^{n-1} = \frac{2}{r} \int_{B_1} Q_y(u, r) \Delta u(rx + y) dx.$$

Proof By Lemmas 2.8, 2.10 we get

$$\frac{d}{dr} \int_{\partial B_1} Q_y^2(u, r) d\mathcal{H}^{n-1} = 2 \int_{\partial B_1} Q_y(u, r) \frac{d}{dr} Q_y(u, r) d\mathcal{H}^{n-1}$$
$$= \frac{2}{r} \int_{B_1} Q_y(u, r) \Delta u(rx + y) dx.$$

□

Lemma 2.12 *Let $f \in L^\infty(B_1)$, u be a solution of (1) and $y \in B_{1/2}$. For $0 < r < 1/2$ consider*

$$u_r(x) := \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - \Pi_y(u, r),$$
$$v_r(x) := \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - Q_y(u, r).$$

Then

(i) *$u_r - v_r$ is bounded in C^∞, uniformly in r;*
(ii) *the family $\{v_r\}$ is bounded in $C^{1,\alpha}(B_1) \cap W^{2,p}(B_1)$, for every $0 < \alpha < 1$ and $p > 1$.*

Proof (i) For each r, the difference $u_r - v_r = Q_y(u, r) - \Pi_y(u, r)$ is a second order harmonic polynomial. Therefore, it suffices to show that the L^∞ norm of that difference admits a bound independent of r. Note that

$$u_r - v_r = Q_y(u, r) - \Pi_y(u, r)$$
$$= Q_0\left(\frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - \Pi_y(u, r), 1\right) = Q_0(u_r, 1).$$

Hence, by Lemma 2.2,

$$\sup_r \sup_{B_1} |Q_0(u_r, 1)| \le C \sup_r \sup_{B_1} |u_r| < \infty.$$

(ii) Lemma 2.2 implies that $\{u_r\}_{r>0}$ is bounded in $C^{1,\alpha}(B_1) \cap W^{2,p}(B_1)$ for every $\alpha < 1$ and $p > 1$. Hence, the result follows from (i). □

3 $C^{1,1}$ regularity: general case

In this section we utilize the previous technical tools and prove $C^{1,1}$ regularity provided that $f = f(x,t)$ satisfies Assumption A:

Assumption A (i)

$$|f(x,t_2) - f(x,t_1)| \le \omega(|t_2 - t_1|),$$

and

$$\int_0^\epsilon \frac{\omega(t)}{t} dt < \infty,$$

for some $\epsilon > 0$;

(ii) The Newtonian potential of $x \mapsto f(x,t)$ is $C^{1,1}$ locally uniformly in t: for $v_t := f(\cdot, t) * N$ where N is the Newtonian potential,

$$\sup_{a \le t \le b} \|D^2 v_t\|_{L^\infty(B_1)} < \infty, \quad \text{for all} \quad a, b \in \mathbb{R}.$$

Proof of Theorem 1.1 Let $y \in B_{1/2}$ and $v = v_{u(y)} = f(x, u(y)) * N$. Note that if

$$u_r(x) = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - \Pi_y(u, r),$$

then

$$\Pi_y(u, r/2) - \Pi_y(u, r) = \Pi_y(u_r, 1/2) - \Pi_y(u_r, 1) = \Pi_y(u_r, 1/2).$$

Using this identity, Lemmas 2.3 and 2.5

$$\begin{aligned}
&\|\Pi_y(u, r/2) - \Pi_y(u, r) - \Pi_y(v, r/2) + \Pi_y(v, r)\|_{L^\infty(B_1)} \\
&= \|\Pi_y(u_r, 1/2) - \Pi_y(v_r, 1/2) - \Pi_y(u_r, 1) + \Pi_y(v_r, 1)\|_{L^\infty(B_1)} \\
&= \|\Pi_y(u_r - v_r, 1/2) - \Pi_y(u_r - v_r, 1)\|_{L^\infty(B_1)} \\
&\le C\|\Delta u_r - \Delta v_r\|_{L^2(B_1)}
\end{aligned}$$

$$= \|f(rx+y,u(rx+y)) - f(rx+y,u(y))\|_{L^2(B_1)}$$
$$\leq C\omega\left(\sup_{B_r(y)} |u(x)-u(y)|\right) \leq C\omega\left(c(r+r^2\log\frac{1}{r})\right) \leq C\omega(cr),$$

for $r > 0$ sufficiently small ($|\nabla u(y)|$ is controlled by $\|u\|_{W^{2,p}(B_1)}$). Hence, for $r_0 > 0$ small enough and $y \in B_{1/2}$ we have

$$\|\Pi_y(u, r_0/2^j) - \Pi_y(u, r_0)\|_{L^\infty(B_1)}$$
$$\leq \left\| \sum_{k=1}^{j} \Pi_y(v, r_0/2^k) - \Pi_y(v, r_0/2^{k-1}) \right\|_{L^\infty(B_1)}$$
$$+ \sum_{k=1}^{j} \left\| \Pi_y(u, r_0/2^k) - \Pi_y(u, r_0/2^{k-1}) - \Pi_y(v, r_0/2^k) + \Pi_y(v, r_0/2^{k-1}) \right\|_{L^\infty(B_1)}$$
$$\leq C\|D^2 v_{u(y)}\|_{L^\infty(B_1)} + C\sum_{k=1}^{\infty} \omega\left(\frac{cr}{2^{k-1}}\right) \leq \tilde{C}(\|D^2 v_{u(y)}\|_{L^\infty(B_1)} + 1)$$
$$\leq \tilde{C}\left(\sup_{|s|\leq \sup|u|} \|D^2 v_s\|_{L^\infty(B_1)} + 1\right).$$

Thus

$$\|\Pi_y(u, r_0/2^j)\|_{L^\infty(B_1)} \leq \|\Pi_y(u, r_0)\|_{L^\infty(B_1)} + \tilde{C}(\|D^2 v_{u(y)}\|_{L^\infty(B_1)} + 1). \tag{5}$$

We conclude via Lemmas 2.4 and 2.6. □

Remark 3 To generate examples, consider $f(x,t) = \phi(x)\psi(t)$. If $\phi \in L^\infty$ and ψ is Dini, then f satisfies condition (i). If $\phi * N$ is $C^{1,1}$ and ψ is locally bounded, then f satisfies (ii). Thus if $\phi * N$ is $C^{1,1}$ and ψ is Dini, then f satisfies both conditions. In particular, f may be strictly weaker than Dini in the x-variable.

Remark 4 The projection Q_y has similar properties to Π_y. Consequently, if f satisfies Assumption A, (5) holds for Π_y replaced by Q_y,

$$\|Q_y(u, r_0/2^j)\|_{L^\infty(B_1)} \leq \|Q_y(u, r_0)\|_{L^\infty(B_1)} + \tilde{C}(\|D^2 v_{u(y)}\|_{L^\infty(B_1)} + 1). \tag{6}$$

4 $C^{1,1}$ regularity: discontinuous case

The goal of this section is to investigate the optimal regularity for solutions of (1) with f having a jump discontinuity in the t-variable. This case may be viewed as a free boundary problem. The idea is to employ again an L^2 projection operator.

4.1 Two-phase obstacle problem

Suppose $f = f(x, u)$ has the form

$$f(x, u) = g_1(x, u)\chi_{\{u>0\}} + g_2(x, u)\chi_{\{u<0\}},$$

where g_1, g_2 are continuous. We recall from the introduction that if f has a jump in u at the origin, then we assume it to be a positive jump:

Assumption B $g_1(x, 0) - g_2(x, 0) \geq \sigma_0,\ x \in B_1$ for some $\sigma_0 > 0$.

Remark 5 In the unstable obstacle problem, i.e. $g_1 = -1$, $g_2 = 0$, there exists a solution which is $C^{1,\alpha}$ for any $\alpha \in (0, 1)$ but not $C^{1,1}$.

Let $\Gamma^0 := \Gamma \cap \{|\nabla u| = u = 0\}$ and $\Gamma^1 := \Gamma \cap \{|\nabla u| \neq 0\}$. Our main result provides optimal growth away from points with sufficiently small gradients.

Theorem 4.1 *Suppose $g_1, g_2 \in C^0$ satisfy B. Then for all constants $\theta, M > 0$ there exist $r_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ and $C_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ such that for any solution of (1) with $\|u\|_{L^\infty(B_1)} \leq M$*

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C_0, \tag{7}$$

for all y, r such that $r \leq r_0$ and $y \in B_{1/2} \cap \Gamma \cap \{|\nabla u(y)| < \theta r\}$. Consequently, for the same choice of r and y we have that

$$\sup_{x \in B_r} |u(x + y) - x \cdot \nabla u(y)| \leq C_1 r^2, \tag{8}$$

for some constant $C_1(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$.

The proof of the theorem is carried out in several steps. A crucial ingredient is the following monotonicity result.

Lemma 4.2 *Suppose $g_1, g_2 \in C^0$ satisfy B. Then for all constants $\theta, M > 0$ there exist $\kappa_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ and $r_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ such that for any solution u of (1) with $\|u\|_{L^\infty(B_1)} \leq M$ if*

$$\|Q_y(u, r)\|_{L^2(\partial B_1)} \geq \kappa_0,$$

for some $0 < r < r_0$ and $y \in B_{1/2} \cap \Gamma \cap \{|\nabla u(y)| < \theta r\}$, then

$$\frac{d}{dr} \int_{\partial B_1} Q_y^2(u, r)\, d\mathcal{H}^{n-1} > 0.$$

Proof If the conclusion is not true, then there exist radii $r_k \to 0$, solutions u_k and points $y_k \in B_{1/2} \cap \Gamma_k \cap \{|\nabla u_k(y_k)| < \theta r_k\}$ such that $\|u_k\|_{L^\infty(B_1)} \le M$, and $\|Q_{y_k}(u_k, r_k)\|_{L^2(\partial B_1)} \to \infty$, and

$$\frac{d}{dr} \int_{\partial B_1} Q^2_{y_k}(u_k, r) d\mathcal{H}^{n-1} \bigg|_{r=r_k} \le 0.$$

Let

$$T_k := \|Q_{y_k}(u_k, r_k)\|_{L^2(\partial B_1)},$$

and consider the sequence

$$v_k(x) = \frac{u_k(r_k x + y_k) - r_k x \cdot \nabla u_k(y_k)}{r_k^2} - Q_{y_k}(u_k, r_k).$$

Without loss of generality we can assume that $y_k \to y_0$ for some $y_0 \in B_{1/2}$. Lemma 2.2 implies the existence of a function v such that up to a subsequence

$$v_k(x) = \frac{u_k(r_k x + y_k) - r_k x \cdot \nabla u_k(y_k)}{r_k^2} - Q_{y_k}(u_k, r_k) \to v, \text{ in } C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^n).$$

By $C^{1,\alpha}$ convergence, $v(0) = |\nabla v(0)| = 0$. Moreover, for $q_k(x) := Q_{y_k}(u_k, r_k)/T_k$, we can assume that up to a further subsequence, $q_k \to q$ in C^∞ for some $q \in \mathcal{P}_2$. Note that

$$\begin{aligned} \Delta v_k(x) &= g_1(r_k x + y_k, u_k(r_k x + y_k))\chi_{\{u_k(r_k x + y_k) > 0\}} \\ &\quad + g_2(r_k x + y_k, u_k(r_k x + y_k))\chi_{\{u_k(r_k x + y_k) < 0\}} \end{aligned}$$

hence

$$\Delta v_k \to \Delta v = g_1(y_0, 0)\chi_{\{q(x) > 0\}} + g_2(y_0, 0)\chi_{\{q(x) < 0\}}.$$

By Lemma 2.11,

$$\begin{aligned} 0 \ge \frac{d}{dr} \int_{\partial B_1} Q^2_{y_k}(u_k, r) d\mathcal{H}^{n-1} \bigg|_{r=r_k} &= \frac{2}{r_k} \int_{B_1} Q_{y_k}(u_k, r_k) \Delta u_k(r_k x + y_k) dx \\ &= \frac{2T_k}{r_k} \int_{B_1} q_k(x) \Delta v_k(x) dx. \end{aligned}$$

Therefore

$$\int_{B_1} q_k(x) \Delta v_k(x) dx \le 0.$$

On the other hand

$$\lim_{k\to\infty}\int_{B_1} q_k(x)\Delta v_k(x)dx = \int_{B_1} q(x)\left(g_1(0,y_0)\chi_{\{q(x)>0\}} + g_2(0,y_0)\chi_{\{q(x)<0\}}\right)dx$$
$$= (g_1(0,y_0) - g_2(0,y_0)) \int_{\{q(x)>0\}} q(x)dx > 0,$$

a contradiction. □

Proof of Theorem 4.1 Let κ_0 and r_0 be the constants from Lemma 4.2. Without loss of generality we can assume that $r_0 \le 1/4$. From Lemmas 2.4 and 2.12 we have that

$$\|Q_y(u, r_0)\|_{L^2(\partial B_1)} \le C\log\frac{1}{r_0}, \tag{9}$$

for all $y \in B_{1/2}$, where $C = C(M, \|g_1\|_\infty, \|g_2\|_\infty, n)$ is a constant. Let

$$C_0 = \max\left(\kappa_0, 2C\log\frac{1}{r_0}\right). \tag{10}$$

We claim that

$$\|Q_y(u, r)\|_{L^2(\partial B_1)} \le C_0,$$

for $r \le r_0$ and $y \in B_{1/2} \cap \Gamma \cap \{|\nabla u(y)| < \theta r\}$. Let us fix y such that $|\nabla u(y)| \le \theta r_0$ and consider

$$T_y(r) := \|Q_y(u, r)\|_{L^2(\partial B_1)}$$

as a function of r on the interval $|\nabla u(y)|/\theta \le r \le r_0$. Let

$$e := \inf\{r \text{ s.t. } T_y(r) \le C_0\}. \tag{11}$$

By (9) and (10), we have that $T_y(r_0) \le C_0$, so $|\nabla u(y)|/\theta \le e \le r_0$. If $e > |\nabla u(y)|/\theta$ then $T_y(e) = C_0$ and by Lemma 4.2 we have that $T_y'(e) > 0$, so $T_y(r) < C_0$ for $e - \varepsilon < r < e$ which contradicts (11).

Therefore, $e = |\nabla u(y)|/\theta$ and $T_y(r) \le C_0$ for all $|\nabla u(y)|/\theta \le r \le r_0$ which proves (7).

Inequality (8) follows from Lemmas 2.2 and 2.12. □

Theorem 4.1 implies $C^{1,1}$ regularity away from Γ^1 in the case the coefficients g_i are regular enough to provide $C^{1,1}$ solutions away from the free boundary, i.e. Theorem 1.2.

Remark 6 Note that A is the condition given in Theorem 1.1. If g_i only depend on x, then this reduces to the assumption that the Newtonian potential of g_i is $C^{1,1}$, which is sharp.

Proof of Theorem 1.2 Suppose A and B hold. We show that for every $\delta > 0$ there exists $C_\delta > 0$ such that for all $y \in B_{1/2}(0)$ such that $\text{dist}(y, \Gamma^1) \geq \delta$, there exists $r_y > 0$ such that

$$\|Q_y(u,r)\|_{L^2(\partial B_1(0))} \leq C_\delta, \tag{12}$$

for $r \leq r_y$.

Consequently,

$$|u(x) - u(y) - \nabla u(y)(x-y)| \leq \tilde{C}_\delta |x-y|^2 \tag{13}$$

for $|x-y| \leq r_y$, $y \in B_{1/2}(0)$ and $\text{dist}(y, \Gamma^1) \geq \delta$; this readily yields the desired result.

Note that (13) follows from (12) via Lemmas 2.2 and 2.12.

Without loss of generality assume that $\delta \leq r_0$, where $r_0 > 0$ is the constant from Theorem 4.1. For every $y \in B_{1/2}(0)$ consider the ball $B_{\delta/2}(y)$. Then there are two possibilities.

(i) $B_{\delta/2}(y) \cap \Gamma^0 = \emptyset$.
In this case $B_{\delta/2} \cap \Gamma = \emptyset$, hence u satisfies the equation

$$\Delta u = g_i(x,u)$$

in $B_{\delta/2}(y)$ for $i = 1$ or $i = 2$. Inequality (6) in Remark 4 and Assumption A yield

$$\|Q_y(u,r)\|_{L^2(\partial B_1(0))} \leq C \log \frac{4}{\delta} + C(\|D^2 v^i_{u(y)}\|_{L^\infty(B_1)} + 1) \leq C_\delta,$$

for $r \leq \delta/4$, where $v^i_{u(y)}$ is defined as the solution to $\Delta w(x) = g_i(x, u(y))$.

(ii) $B_{\delta/2}(y) \cap \Gamma^0 \neq \emptyset$.
Let $w \in \Gamma^0$ be such that $d := |y - w| = \text{dist}(y, \Gamma_0)$. We have that $d \leq \delta/2$. As in the previous step, (6) and Assumption A yield

$$\|Q_y(u,r)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u,d/2)\|_{L^2(\partial B_1(0))} + C(\|D^2 v^i_{u(y)}\|_\infty + 1),$$

for $r \leq d/2$. From Theorem 4.1 we have that

$$\left| u\left(y + \frac{d}{2}z\right)\right| \leq C\left|y + \frac{d}{2}z - w\right|^2 \leq Cd^2,$$

for all $|z| \leq 1$ because $d \leq \delta/2 \leq r_0$. On the other hand, by definition,

$$Q_y(u,d/2) = \text{Proj}_{\mathcal{P}_2}\left(\frac{u\left(y + \frac{d}{2}z\right)}{d^2/4}\right),$$

where $\mathrm{Proj}_{\mathcal{P}_2}$ is the $L^2(\partial B_1(0))$ projection on the space $\mathcal{P}_2$. Hence, by Lemma 2.7 vi,

$$\|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} \le \left\| \frac{u\left(y + \frac{d}{2}z\right)}{d^2/4} \right\|_{L^2(\partial B_1(0))} \le C,$$

which yields

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \le C + C(\|D^2 v^i_{u(y)}\|_\infty + 1),$$

for $r \le d/2$.

The proof is now complete. □

Lastly we point out that if the coefficients g_i are regular enough to provide $C^{1,1}$ solutions at points where the gradient does not vanish, then we obtain full interior $C^{1,1}$ regularity. Recall from the introduction the following assumption:

Assumption C For any $M > 0$ there exist $\theta_0(M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ and $C_3(M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ such that for all $z \in B_{1/2}$ any solution of

$$\begin{cases} \Delta v = g_1(x, v)\chi_{\{v>0\}} + g_2(x, v)\chi_{\{v<0\}}, \ x \in B_{1/2}(z); \\ |v(x)| \le M, \ x \in B_{1/2}(z); \\ v(z) = 0, \ 0 < |\nabla v(z)| \le \theta_0; \end{cases}$$

admits a bound

$$\|D^2 v\|_{L^\infty(B_{|\nabla v(z)|/\theta_0}(z))} \le C_3.$$

Theorem 4.1 and C imply Theorem 1.3.

Proof of Theorem 1.3 Our strategy is to consider several cases. The main idea is to note that by Lemmas 2.12 and 2.6 the assertion follows if we show that there exist $\rho_0, C > 0$ such that for every $y \in B_{\rho_0}(0)$ there exists $r_y > 0$ such that

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \le C \tag{14}$$

for $0 < r \le r_y$.

Let ρ_0 be such that $|\nabla u(y)| \le \theta_0$ for $y \in B_{\rho_0}(0)$, where θ_0 is the constant from Assumption C (we can do this because u is $C^{1,\alpha}$ and $0 \in \Gamma^0$). For $y \in B_{\rho_0}(0)$ let $d := \mathrm{dist}(y, \Gamma)$ and let $w \in \Gamma$ be such that $d = |y - w|$.

From Theorem 1.2 we can assume that $2d < r_0$ where r_0 is the constant in Theorem 4.1. One of the following cases is possible.

(i) $d = 0$, $y \in \Gamma^0$.
In this case we have that (14) holds for $r \le r_0$ by Theorem 4.1 (note that $\nabla u(y) = 0$).

(ii) $d = 0$, $y \in \Gamma^1$.
Here, the $C^{1,1}$ bound follows directly from Assumption C.

(iii) $d > 0$, $w \in \Gamma^0$. Here, we repeat the argument in case (ii) of the proof of Theorem 1.2 line by line and obtain that (14) is valid for $r \leq d/2$.

(iv) $d > 0$, $w \in \Gamma^1$.
From Theorem 4.1 we have that

$$|u(\tilde{z} + w) - \tilde{z} \cdot \nabla u(w)| \leq C_1 |\tilde{z}|^2 \tag{15}$$

for $|\nabla u(w)|/\theta_0 \leq |\tilde{z}| \leq r_0$. On the other hand by Assumption C we obtain that (15) holds for $|\tilde{z}| \leq |\nabla u(w)|/\theta_0$. Hence, (15) holds for all $\tilde{z}$ such that $|\tilde{z}| \leq r_0$. Thanks to Assumption A, we have via Remark 4 that

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} + C(\|D^2 v^i_{u(y)}\|_{L^\infty(B_1)} + 1), \tag{16}$$

for $r \leq d/2$.
Furthermore, by the projection's invariance of affine additions

$$\begin{aligned} Q_y(u, d/2) &= \mathrm{Proj}_{\mathcal{P}_2}\left(\frac{u\left(y + \frac{d}{2}z\right) - \frac{d}{2}z \cdot \nabla u(y) - u(y)}{d^2/4} \right) \\ &= \mathrm{Proj}_{\mathcal{P}_2}\left(\frac{u\left((y + \frac{d}{2}z\right) - \left(y + \frac{d}{2}z - w\right) \cdot \nabla u(w)}{d^2/4} \right). \end{aligned}$$

Hence by applying (15) with $\tilde{z} = y + \frac{d}{2}z - w$ we have that

$$\begin{aligned} \|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} &\leq \left\| \frac{u\left(w + (y + \frac{d}{2}z - w)\right) - \left(y + \frac{d}{2}z - w\right) \cdot \nabla u(w)}{d^2/4} \right\|_{L^2(\partial B_1(0))} \\ &\leq C, \end{aligned}$$

which combined with (16) yields

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C + C(\|D^2 v^i_{u(y)}\|_{L^\infty(B_1)} + 1),$$

for $r \leq d/2$. Note that $|\tilde{z}| \leq r_0$ is guaranteed because we have chosen $d < \frac{r_0}{2}$.

□

The previous analysis applies to the following example.

Example Let $g_i(x, u) = \lambda_i(x)$ for $i = 1, 2$, where λ_i are such that

(i) $\lambda_1(x) - \lambda_2(x) \geq \sigma_0 > 0$ for all $x \in B_1$;
(ii) $\lambda_1(x), \lambda_2(x)$ are Hölder continuous.

We recall from the introduction that under the stronger assumption $\inf_{B_1} \lambda_1 > 0$, $\inf_{B_1} -\lambda_2 > 0$, this problem is studied in [12] and the optimal interior $C^{1,1}$ regularity is established. The authors use a different approach based on monotonicity formulas and an analysis of global solutions via a blow-up procedure.

4.2 No-sign obstacle problem

Here we observe that Assumption A implies that the solutions of (3) are in $C^{1,1}(B_{1/2})$. This theorem was proven in [2] (Theorem 1.2) for the case when $g(x,t)$ depends only on x. Under Assumption A, appropriate modifications of the proof in [2] work also for the general case; since the arguments are similar, we provide only a sketch of the proof and highlight the differences.

Sketch of the proof of Theorem 1.4 Let $\tilde{\Gamma} := \{y \text{ s.t. } u(y) = |\nabla u(y)| = 0\}$. For $r > 0$ let $\Lambda_r := \{x \in B_1 \text{ s.t. } u(rx) = 0\}$ and $\lambda_r := |\Lambda_r|$.

The proof of Theorem 1.2 in [2] consists of the following ingredients.

- Interior $C^{1,1}$ estimate
- Quadratic growth away from the free boundary
- [2, Proposition 5.1]

Let us recall that the interior $C^{1,1}$ estimate is the inequality

$$\|u\|_{C^{1,1}(B_{d/2})} \leq C\left(\|D^2 v\|_{L^\infty(B_d)} + \frac{\|u\|_{L^\infty(B_d)}}{d^2}\right), \tag{17}$$

where $\Delta u(x) = g(x)$ for $x \in B_d$ and v, the Newtonian potential of g, is $C^{1,1}$. This estimate is purely a consequence of g having a $C^{1,1}$ Newtonian potential.

Quadratic growth away from the free boundary is a bound

$$|u(x)| \leq C \operatorname{dist}(x, \tilde{\Gamma})^2. \tag{18}$$

The first observation in [2] is that if $g(x,t) = g(x)$ has a $C^{1,1}$ Newtonian potential, then (17) and (18) yield $C^{1,1}$ regularity for the solution. Indeed, "far" from the free boundary, the solution u solves the equation $\Delta u = g(x)$ and is locally $C^{1,1}$ by assumption. For points close to the free boundary, u solves the same equation but now on a small ball centered at the point of interest and touching the free boundary. At this point one invokes (18) and by (17) obtains that the $C^{1,1}$ bound does not blow up close to the free boundary (see Lemma 4.1 in [2]).

To prove (18), the authors prove in Proposition 5.1 [2] that if the projection $\Pi_y(u,r)$ (for some $y \in \tilde{\Gamma}$) is large enough then the density λ_r of the coincidence set diminishes at an exponential rate. On the other hand, if λ_r diminishes at an exponential rate, $\Pi_y(u,r)$ has to be bounded. Consequently, by invoking Lemma 2.2 one obtains (18).

Now let g satisfy A.

- Interior $C^{1,1}$ estimate
  In the general case, (17) is replaced by

$$\|Q_y(u,s)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u,r)\|_{L^2(\partial B_1(0))} + C(\|D^2 v_{u(y)}\|_\infty + 1), \tag{19}$$

  where $0 < s < r < d$, $\Delta v_{u(y)} = g(x, u(y))$ and $\Delta u = f(x,u)$ in $B_d(y)$. Estimate (19) is purely a consequence of Assumption A via Remark 4.

- [2, Proposition 5.1]
  In this proposition, it is shown that there exists C such that if $\Pi_y(u,r) \geq C$ then

$$\lambda_{r/2}^{1/2} \leq \frac{\tilde{C}}{\|\Pi_y(u,r)\|_{L^\infty(B_1)}} \lambda_r^{1/2} \tag{20}$$

for some $\tilde{C} > 0$. The inequality is obtained by the decomposition

$$\frac{u(rx+y)}{r^2} = \Pi_y(u,r) + h_r + w_r,$$

where h_r, w_r are such that

$$\begin{cases} \Delta h_r = -g(rx+y)\chi_{\Lambda_r} & \text{in } B_1, \\ h_r = 0 & \text{on } \partial B_1, \end{cases}$$

and

$$\begin{cases} \Delta w_r = g(rx+y) & \text{in } B_1, \\ w_r = \frac{u(rx+y)}{r^2} - \Pi_y(u,r) & \text{on } \partial B_1. \end{cases}$$

The authors show that

$$\begin{aligned} \|D^2 h_r\|_{L^2(B_{1/2})} &\leq C\|g\|_{L^\infty}\|\chi_{\Lambda_r}\|_{L^2(B_1)}, \\ \|D^2 w_r\|_{L^\infty(B_{1/2})} &\leq C\left(\|g\|_{L^\infty} + \|u\|_{L^\infty(B_1)}\right). \end{aligned} \tag{21}$$

In the general case one may consider the decomposition

$$\frac{u(rx+y)}{r^2} = Q_y(u,r) + h_r + w_r + z_r,$$

where h_r, w_r, z_r are such that

$$\begin{cases} \Delta h_r = -g(rx+y,0)\chi_{\Lambda_r} & \text{in } B_1, \\ h_r = 0 & \text{on } \partial B_1, \end{cases}$$

and

$$\begin{cases} \Delta w_r = g(rx+y,0) & \text{in } B_1, \\ w_r = \frac{u(rx+y)}{r^2} - Q_y(u,r) & \text{on } \partial B_1, \end{cases}$$

and

$$\begin{cases} \Delta z_r = \left(g(rx+y,u(rx+y)) - g(rx+y,0)\right)\chi_{B_1\setminus\Lambda_r} & \text{in } B_1, \\ z_r = 0 & \text{on } \partial B_1. \end{cases}$$

Evidently, estimates (21) are still valid. Additionally, we have

$$\|D^2 z_r\|_{L^2(B_{1/2})} \le C\|\Delta z_r\|_{L^2(B_1)} \le C\omega(r^2 \log \frac{1}{r}), \tag{22}$$

since $g(x, t)$ is uniformly Dini in t.

Combining (21) and (22) and arguing as in [2] one obtains the existence of $C > 0$ such that

$$\lambda_{r/2}^{1/2} \le \frac{\tilde{C}}{\|Q_y(u, r)\|_{L^2(\partial B_1)}} \lambda_r^{1/2} + \omega\left(r^2 \log \frac{1}{r}\right), \tag{23}$$

whenever $\|Q_y(u, r)\|_{L^2(\partial B_1)} \ge C$.

- Quadratic growth away from the free boundary
  In [2], the norms of $\Pi_y(u, r/2^k)$, $k \ge 1$ are estimated in terms of the sum $\sum_{j=0}^{\infty} \lambda_{r/2^j}$. If the norms of projections are unbounded, one obtain estimate (20) which implies convergence of the previous sum and hence boundedness of the projections. This is a contradiction. Similarly, in the general case the norms of $Q_y(u, r/2^k)$, $k \ge 1$ can be estimated by

$$\sum_{j=0}^{\infty} \lambda_{r/2^j} + \sum_{j=0}^{\infty} \omega\left(\left(\frac{r}{2^k}\right)^2 \log \frac{2^k}{r^2}\right).$$

Inequality (23) and Dini continuity imply

$$\sum_{j=0}^{\infty} \omega\left(\left(\frac{r}{2^k}\right)^2 \log \frac{2^k}{r^2}\right), \sum_{j=0}^{\infty} \lambda_{r/2^j} < \infty,$$

if the norms of projections are unbounded. Furthermore, one completes the proof of the quadratic growth as in [2].

To verify that the above ingredients imply $C^{1,1}$ regularity, we split the analysis into two cases. If we are "far" from the free boundary, u locally solves $\Delta u = g(x, u)$ so by Theorem 1.1 u is $C^{1,1}$. If we are close to the free boundary then u solves $\Delta u = g(x, u)$ in a small ball $B_d(y)$ that touches the free boundary. We invoke (19) for $0 < s < r = d/2$ and the quadratic growth to obtain

$$\begin{aligned}
\|Q_y(u, s)\|_{L^2(\partial B_1(0))} &\le \|Q_y(u, d/2)\|_{L^2(\partial B_1)} + C(\|D^2 v_{u(y)}\|_\infty + 1) \\
&\le C\left\|\frac{u(y + d/2x)}{d^2/4}\right\|_{L^2(\partial B_1)} + C(\|D^2 v_{u(y)}\|_\infty + 1) \\
&\le C + C(\|D^2 v_{u(y)}\|_\infty + 1).
\end{aligned}$$

for $s \le d/2$.

So there exists a constant C such that for all $y \in B_{1/2}$ there exist radii $r_j(y) \to 0$ such that

$$Q_y(u, r_j(y)) \leq C.$$

We conclude via Lemma 2.6. □

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