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Continuous and discrete best polynomial degree reduction with Jacobi and Hahn weights

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Abstract

We show that the weighted least squares approximation of Bézier coefficients with Hahn weights provides the best polynomial degree reduction in the Jacobi L_2-norm. A discrete analogue of this result is also provided. Applications to Jacobi and Hahn orthogonal polynomials are presented.

Keywords: Degree reduction, Discrete least squares, Bézier curves, h-Bézier curves, Jacobi orthogonal polynomials, Hahn orthogonal polynomials

1. Introduction

Polynomial degree reduction refers to the process of finding the best lower degree polynomial approximation of a given polynomial with respect to a given norm. The interest in this process lies in the resulting data reduction while preserving, to a certain extent, the essential features of the original data. Polynomial degree reduction with respect to different norms attracted considerable interest for decades, e.g., L_∞-norm [7, 16], L_2-norm [12, 13, 11], L_1-norm [10], L_p-norm [4], q-norm [3].

In [12], it was shown that the least squares approximation of Bézier coefficients provides the best polynomial degree reduction in the L_2-norm. This result was generalized to include the case of the constrained polynomial degree reduction problem [1] and the case of the discrete L_2-norms polynomial degree reduction problem [2].

In the present work, we propose to find analogue results to [12] and [2] for the case of the Jacobi L_2-norm

$$||f||^2_{L_2} = \int_0^1 x^\alpha (1-x)^\beta f^2(x)dx, \quad \alpha > -1, \beta > -1. \tag{1}$$

We show that if we define the following weighted Euclidean norm of Bézier coefficients

$$||P||^2_{E_2} = \sum_{j=0}^{n} \frac{(\alpha+1)_j(\beta+1)_{n-j}}{j!(n-j)!} |p_j|^2, \tag{2}$$

where $(p_0, p_1, \ldots, p_n)$ are the Bézier coefficients of the polynomial P and $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the Pochhammer symbol with the convention that $(a)_0 := 1$, then the least squares approximation of Bézier coefficients with respect to the E_2-norm

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(2) provides the best polynomial degree reduction with respect to the L_2-norm (1). A discrete analogue of this result is also provided via replacing the Bézier coefficients in (2) by the notion of h-Bézier coefficients.

An explicit solution to the polynomial degree reduction problem with respect to the norm (1) is given in [17]. However, the proposed solution is complicated and requires explicit computation of the dual bases of discrete Bernstein bases. In contrast, our proposed solution is simple and requires only the computation of a single Moore-Penrose inverse.

We show that combining our results on the continuous and the discrete polynomial degree reduction problems leads to simple and new proofs of the connection between Jacobi orthogonal polynomials and Hahn orthogonal polynomials.

The outline of the paper is as follows: In Section 2, we give the necessary background on the theory of h-Bézier curves. In Section 3, we show that the least squares approximation of Bézier coefficients with respect to the E_2-norm (2) provides the best polynomial degree reduction with respect to the L_2-norm (1). In Section 4, we study the problem of weighted discrete polynomial degree reduction. Discrete analogues of the results of Section 3 are provided. Methods of solution to the continuous and discrete weighted polynomial degree reduction problems are presented in Section 5. Applications to Jacobi and Hahn polynomials are given in section 6. We conclude in Section 7 with possible generalizations of this work.

2. h-Bézier curves and degree elevation

In this section, we briefly review relevant preliminaries on h-Bézier curves. Denote by $\mathbb{P}_n$ the linear space of polynomials of degree at most n. The h-Bernstein basis over the unit interval $[0,1]$ of the space $\mathbb{P}_n$ is defined by [15][1]

$$B_k^n(t;h) = \binom{n}{k} \frac{\prod_{i=0}^{k-1}(t-ih)\prod_{i=0}^{n-k-1}(1-t-ih)}{\prod_{i=0}^{n-1}(1-ih)}, \quad k=0,1,\ldots,n, \tag{3}$$

where h, throughout this work, is a real number different from $1/i, i=1,2,\ldots,n-1$ and called the *sampling parameter*. The h-Bernstein basis coincides with the classical Bernstein basis for the value of the parameter $h=0$. We denote the classical Bernstein basis simply by $B_k^n, k=0,1,\ldots,n$. For $h \leq 0$, the h-Bernstein functions $B_k^n(t;h)$ are non-negative on $[0,1]$ and for any h, the basis forms a partition of unity, i.e.,

$$\sum_{k=0}^{n} B_k^n(t;h) \equiv 1 \text{ for any } t \in \mathbb{R}.$$

An h-Bézier curve is a parametric polynomial curve of the form

$$P(t) = \sum_{k=0}^{n} B_k^n(t;h)P_k, \quad P_k \in \mathbb{R}^s; \quad s \geq 2.$$

The points P_i are called the h-Bézier points or coefficients of P over the interval $[0,1]$. Many geometric design concepts related to h-Bézier curves, such as de Casteljau algorithm or subdivision, can be derived using the notion of h-blossoming introduced in [15].

[1] For convenience, we took a slightly different convention form [15] where one should replace h by $-h$.

To a polynomial P of degree at most n, we associate a family of parametrized polynomials by setting:

$$\mathscr{P}(t;h) := \sum_{k=0}^{n} p_k B_k^n(t;h),$$

where $(p_0, p_1, \dots, p_n)$ are the Bézier coefficients of P over $[0,1]$ with respect to the classical Bernstein basis, i.e., $\mathscr{P}(t;0) = P(t)$ for all $t \in \mathbb{R}$. The following simple lemma was proved in [2].

Lemma 1. *Let P be a polynomial of degree $n \leq N$ written as*

$$P(t) = \sum_{k=0}^{n} p_k B_k^n(t) = \sum_{k=0}^{N} q_k^{(N)} B_k^N(t). \tag{4}$$

Then for $j = 0, 1, \dots, N$

$$q_j^{(N)} = \mathscr{P}(j/N; 1/N).$$

The h-Bernstein basis (3) satisfies the identities [9]

$$B_k^n(t;h) = \frac{n+1-k}{n+1} B_k^{n+1}(t;h) + \frac{k+1}{n+1} B_{k+1}^{n+1}(t;h), \quad k = 0, 1, \dots, n \tag{5}$$

and

$$B_j^N(k/N; 1/N) = \delta_{jk}; \quad j, k = 0, 1, \dots, N. \tag{6}$$

Equation (6) shows that $B_j^N(t; 1/N) = L_j^N(t)$, where L_j^N, $j = 0, 1, \dots, N$ is the Lagrange basis functions with respect to the nodes $0, 1/N, 2/N, \dots, 1$. Therefore, $\mathscr{P}(t; 1/N)$ is the Lagrange interpolating polynomial of the degree raised Bézier coefficients to order N of the polynomial P (see [2, 8, 15]).

From now on, we adopt the following notation: For a vector $p = (p_0, ..., p_n)$, we write $\mathbb{B}^n p = \sum_{i=0}^{n} p_i B_i^n$ and $\mathbb{B}_h^n p = \sum_{i=0}^{n} p_i B_i^n(.;h)$. Since degree elevation for h-Bézier curves is independent of the parameter h (see Equation (5)), we readily obtain the following:

$$\mathbb{B}^n p \in \mathbb{P}_m \Leftrightarrow \mathbb{B}_h^n p \in \mathbb{P}_m \quad \text{for an} \quad h \in \mathbb{R}/\{1, 1/2, \dots, 1/n-1\}. \tag{7}$$

If we set $h = 1/n$ in (7), we obtain

Lemma 2. *Let $h \in \mathbb{R}/\{1, 1/2, \dots, 1/n-1\}$. If there exists a polynomial Q of degree at most m such that the h-Bézier coefficients p_k of the polynomial $\mathbb{B}_h^n p$ can be written as $p_k = Q(k)$, then the polynomial $\mathbb{B}_h^n p$ is of degree at most m.*

3. Continuous weighted polynomial degree reduction

In this section, we investigate the polynomial degree reduction problem for a specific two-parameter family of inner products and give a far-reaching generalization of the results in [12].

Let $\alpha > -1$ and $\beta > -1$ be two real numbers and define the Jacobi weighted inner product in $\mathbb{P}_n$ by

$$\langle P, Q \rangle_{L_2} = \int_0^1 x^\alpha (1-x)^\beta P(x) Q(x) dx. \tag{8}$$

Consider also the following weighted Euclidean inner product of the Bézier coefficients

$$\langle \mathbb{B}^n p, \mathbb{B}^n q \rangle_{E_2} = \sum_{j=0}^{n} \frac{(\alpha+1)_j(\beta+1)_{n-j}}{j!(n-j)!} p_j q_j, \tag{9}$$

where $(p_0, p_1, \ldots, p_n)$ and $(q_0, q_1, \ldots, q_n)$ are the Bézier coefficients of the polynomials $\mathbb{B}^n p$ and $\mathbb{B}^n q$ respectively and $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the Pochhammer symbol with the convention that $(a)_0 := 1$.

We are now in a position to state our first main result.

Theorem 1. *The orthogonal complements of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the inner product (8) and the Euclidean inner product (9) are equal.*

Proof. Denote by $\mathbb{P}_{m,n}$ the orthogonal complement of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the Euclidean inner product $\langle \cdot, \cdot \rangle_{E_2}$. Let $\mathbb{B}^n q$ be an element of $\mathbb{P}_{m,n}$. Then $\langle \mathbb{B}^n q, \mathbb{B}^n p \rangle_{E_2} = 0$ for any element $\mathbb{B}^n p \in \mathbb{P}_m$. Let s be an integer less than or equal to m. Then

$$\langle \mathbb{B}^n q, x^s \rangle_{L_2} = \int_0^1 x^{\alpha+s}(1-x)^\beta \mathbb{B}^n q \, dx = \sum_{k=0}^{n} q_k \int_0^1 x^{\alpha+s}(1-x)^\beta B_k^n(x) \, dx.$$

Therefore,

$$\langle \mathbb{B}^n q, x^s \rangle_{L_2} = \langle \mathbb{B}^n q, \mathbb{B}^n \phi \rangle_{E_2}, \tag{10}$$

where $\phi = (\phi_0, \phi_1, \ldots, \phi_n)$ with

$$\phi_k = \frac{k!(n-k)!}{(\alpha+1)_k(\beta+1)_{n-k}} \int_0^1 x^{\alpha+s}(1-x)^\beta B_k^n(x) \, dx.$$

To show that the inner product (10) vanishes for any $s \leq m$, we proceed as follows:

$$\int_0^1 x^{\alpha+s}(1-x)^\beta B_k^n(x) \, dx = \binom{n}{k} \frac{\Gamma(\alpha+s+k+1)\Gamma(n+\beta-k+1)}{\Gamma(\alpha+\beta+2+s+n)},$$

where Γ is the Gamma function. Thus

$$\int_0^1 x^{\alpha+s}(1-x)^\beta B_k^n(x) \, dx = \binom{n}{k} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{(\alpha+1)_{s+k}(\beta+1)_{n-k}}{(\alpha+\beta+2)_{s+n}}.$$

Therefore,

$$\phi_k = n! \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{(\alpha+k+1)(\alpha+k+2)\ldots(\alpha+k+s)}{(\alpha+\beta+2)_{s+n}}.$$

This clearly shows that $\phi(k) := \phi_k$ is a polynomial of degree at most $s \leq m$ in the variable k and consequently by Lemma 2, the inner product (10) vanishes for any $s \leq m$. Therefore, $\mathbb{P}_{m,n}$ is contained in the orthogonal complement of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the weighted Euclidean inner product $\langle \cdot, \cdot \rangle_{E_2}$. We conclude the proof by invoking the equality of the dimension of the two orthogonal complements. $\square$

A direct consequence of Theorem 2 is the following:

Corollary 1. *Given a polynomial P of degree n, the approximation problem*

$$\min_{Q \in \mathbb{P}_m} ||P - Q|| \tag{11}$$

has the same minimizer for the norm induced either by the inner product (8) or by the inner product (9).

We have the following factorization of weighted polynomial degree reduction: Denote by $\mathscr{P}_{m,n}$ the linear operator that maps polynomials of degree n to their best weighted L_2-approximation in degree m. Then $\mathscr{P}_{m,n} = \mathscr{P}_{m,l}\mathscr{P}_{l,n}$ with $m \le l \le n$.

Remark 1. In the case $\alpha = 0, \beta = 0$, Corollary 1 leads to the results given in [12].

4. Discrete weighted polynomial degree reduction

In this section, we give a discrete analogue of Corollary 1. The discrete analogue has important applications to the problem of the best polynomial degree reduction at different sampling of the original curve. It also gives a method for understanding the relations between Jacobi orthogonal polynomials and Hahn orthogonal polynomials.

Fix a positive integer $n \ge 1$ and two real numbers $\alpha > -1$ and $\beta > -1$. For each integer $N \ge n$, we define the equidistant partition $X_N = (x_0, x_1, ..., x_N)$ of the interval $[0,1]$ by requiring $x_j = j/N, j = 0,1,\dots,N$. Denote by h the sampling parameter $h = 1/N$. The discrete weighted L_2^h inner product $\langle \cdot,\cdot \rangle_{L_2^h}$ over $\mathbb{P}_n$ is defined as

$$\langle P, Q \rangle_{L_2^h} := \sum_{j=0}^{N} \frac{(\alpha+1)_j(\beta+1)_{N-j}}{j!(N-j)!} P(x_j)Q(x_j). \tag{12}$$

Moreover, we define the weighted h-Euclidean inner product $\langle \cdot,\cdot \rangle_{E_2^h}$ of the h-Bézier coefficients over $\mathbb{P}_n$ by

$$\langle \mathbb{B}_h^n p, \mathbb{B}_h^n q \rangle_{E_2^h} := \sum_{i=0}^{n} \frac{(\alpha+1)_i(\beta+1)_{n-i}}{i!(n-i)!} p_i q_i, \tag{13}$$

where $(p_0, p_1, \dots, p_n)$ and $(q_0, q_1, \dots, q_n)$ are the h-Bézier coefficients of the polynomials $\mathbb{B}_h^n p$ and $\mathbb{B}_h^n q$ respectively with $h = 1/N$. To state the main result of this section, we need the following proposition.

Proposition 1. *Given an integer $n \ge 1$. Let R be a polynomial of degree at most $s \le n$ and denote by $x_j = j/N, j = 0, ..., N$ with $N \ge n$. The function*

$$\psi(k) = \sum_{j=0}^{N} R(j) \frac{(\alpha+1)_j(\beta+1)_{N-j}}{j!(N-j)!} \frac{k!(n-k)!}{(\alpha+1)_k(\beta+1)_{n-k}} B_k^n(x_j; 1/N)$$

is a polynomial in k of degree at most s.

Proof. We proceed by induction on N. If $N = n$ then invoking (6) we obtain $\psi(k) = R(k)$ which is a polynomial of degree at most s in k. Now let us prove the proposition for $N+1$ assuming its validity for any $l \le N$. Consider the function

$$\psi(k) = \sum_{j=0}^{N+1} R(j) \frac{(\alpha+1)_j(\beta+1)_{N+1-j}}{j!(N+1-j)!} \frac{k!(n-k)!}{(\alpha+1)_k(\beta+1)_{n-k}} B_k^n\left(\tilde{x}_j; 1/(N+1)\right), \tag{14}$$

where $\tilde{x}_j = j/(N+1),\ j = 0, ..., N+1$. If we write

$$B_k^n(t) = \sum_{i=0}^{N} q_i^{(N)} B_i^N(t) = \sum_{i=0}^{N+1} q_i^{(N+1)} B_i^{N+1}(t)$$

then by Lemma 1 and Equation (5), for $j = 0, 1, \dots, N+1$

$$B_k^n(\tilde{x}_j; 1/(N+1)) = q_j^{(N+1)} = \frac{N+1-j}{N+1} q_j^{(N)} + \frac{j}{N+1} q_{j-1}^{(N)},$$

with the convention that $q_{-1}^{(N)} = q_{N+1}^{(N)} = 0$. Therefore,

$$B_k^n(\tilde{x}_j; 1/(N+1)) = \frac{N+1-j}{N+1} B_k^n(x_j; 1/N) + \frac{j}{N+1} B_k^n\left(x_{j-1}; 1/N\right). \qquad (15)$$

Inserting (15) into (14) gives

$$\psi(k) = \frac{1}{N+1} \sum_{j=0}^{N} \tilde{R}(j) \frac{(\alpha+1)_j(\beta+1)_{N-j}}{j!(N-j)!} \frac{k!(n-k)!}{(\alpha+1)_k(\beta+1)_{n-k}} B_n^k(x_j; 1/N),$$

where $\tilde{R}(j) = (N+\beta+1-j)R(j) + (\alpha+1+j)R(j+1)$. Since the polynomial R is of degree at most s, the polynomial $\tilde{R}(x) = (N+\beta+1-x)R(x) + (\alpha+1+x)R(x+1)$ is also of degree at most s. Thus by the induction hypothesis $\psi(k)$ is a polynomial in k of degree at most s. □

Now, we are in a position to prove the following.

Theorem 2. *The orthogonal complements of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the discrete L_2^h inner product (12) and the E_2^h-Euclidean inner product (13) are equal.*

Proof. Denote by $\mathbb{P}_{m,n}$ the orthogonal complement of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the Euclidean inner product $\langle \cdot, \cdot \rangle_{E_2^h}$. Let $\mathbb{B}_h^n q$ be an element of $\mathbb{P}_{m,n}$. Thus we have $\langle \mathbb{B}_h^n q, \mathbb{B}_h^n p \rangle_{E_2^h} = 0$ for any element $\mathbb{B}_h^n p \in \mathbb{P}_m$. Let s be an integer less than or equal to m. Then

$$\langle \mathbb{B}_h^n q, x^s \rangle_{L_2^h} = \sum_{j=0}^{N} \left(\frac{j}{N}\right)^s \frac{(\alpha+1)_j(\beta+1)_{N-j}}{j!(N-j)!} \mathbb{B}_h^n q(j/N) = \langle \mathbb{B}_h^n q, \mathbb{B}_h^n \phi \rangle_{E_2^h}, \qquad (16)$$

where $\phi = (\phi_0, \phi_1, \dots, \phi_n)$ with

$$\phi_k = \sum_{j=0}^{N} \left(\frac{j}{N}\right)^s \frac{(\alpha+1)_j(\beta+1)_{N-j}}{j!(N-j)!} \frac{k!(n-k)!}{(\alpha+1)_k(\beta+1)_{n-k}} B_k^n(j/N; 1/N).$$

Since $\mathbb{B}_h^n q$ is an element of $\mathbb{P}_{m,n}$ then according to Lemma 2, the inner product (16) vanishes if and only if $B_h^n \phi$ is an element of $\mathbb{P}_m$ i.e.; $\phi(k) := \phi_k$ is a polynomial in k of degree at most m. This is true by invoking Proposition 1. Therefore, $\mathbb{P}_{m,n}$ is contained in the orthogonal complement of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the h-Euclidean inner product $\langle \cdot, \cdot \rangle_{L_2^h}$. We conclude the proof by invoking the equality of the dimension of the two orthogonal complements. □

A direct consequence of Theorem 2 is the following.

Corollary 2. *Given a polynomial P of degree n, the approximation problem*

$$\min_{Q \in \mathbb{P}_m} ||P - Q|| \qquad (17)$$

has the same minimizer for the norm induced by either the inner product (12) or the inner product (13).

5. Solution methods

We shall now use degree raising to solve the minimization problems (11) and (17). We begin by fixing notation. Denote by $A_{n,m}$ the degree raising matrix that maps the Bézier coefficients of a polynomial P of degree m to the Bézier coefficients of P when raised to degree n. The matrix $A_{n,m}$ is of order $(n+1)\times(m+1)$ and can be decomposed into a product of elementary degree raising matrices as $A_{n,m} = A_{n,n-1}A_{n-1,n-2}\ldots A_{m+1,m}$ where

$$A_{k,k-1}(i,j) = \begin{cases} i/k & \text{if} \quad j = i-1, \\ 1-i/k & \text{if} \quad j = i, \\ 0 & \text{else.} \end{cases}$$

Let P be a polynomial of degree n with h-Bézier coefficients $p^h = (p_0^h, p_1^h, \ldots, p_n^h)$. From now on, we assume that the parameter h can attain the value 0 and in this case p^h should be interpreted as the classical Bézier coefficients of the polynomial P. Denote by W (resp. $\sqrt{W}$) the diagonal $(n+1)\times(n+1)$ weight matrix with diagonal elements ω_j (resp. $\sqrt{\omega_j}$); $j = 0,1,\ldots,n$, where

$$\omega_j = \frac{(\alpha+1)_j(\beta+1)_{n-j}}{j!(n-j)!}.$$

The h-Bézier coefficients q^h of the polynomial Q that solve the minimization problem (11) for $h = 0$, and the minimization problem (17) for $h \neq 0$ are solutions to the least squares problem

$$\min_{q^h \in \mathbb{R}^{m+1}} ||\sqrt{W}A_{n,m}q^h - \sqrt{W}p^h||_2. \tag{18}$$

Using the pseudo-inverse, the solution to (18) is given by

$$q^h = \left(\sqrt{W}A_{n,m}\right)^{\dagger}\sqrt{W}p^h = \left(A_{n,m}^t W A_{n,m}\right)^{-1} A_{n,m}^t W p^h. \tag{19}$$

To assess the quality of the solution to the minimization problem (17), we need algorithms to compute the h-Bézier coefficients of the polynomial P. One method for computing the h-Bézier coefficients of P is to invoke the notion of h-blossoming introduced in [15]. Namely, to find the unique multi-affine symmetric function $f(u_1,u_2,\ldots,u_n)$ such that

$$f(u, u+h, \ldots, u+(n-1)h) = P(u) \text{ for any } u \in \mathbb{R}.$$

The h-Bézier representation $P = \mathbb{B}_h^n p^h$ can be derived as:

$$p_k^h = f(kh, (k+1)h, \ldots, (n-1)h, 1, 1+h, \ldots, 1+(k-1)h), \quad k = 0,1,\ldots,n.$$

An alternative and more efficient method is to use the conversion formula between the h-Bernstein basis and the discrete Legendre polynomials as described in [2]. However, the method we adopt here, is as follows: Let P be a polynomial of degree n with h-Bézier coefficients $p^h = (p_0^h, p_1^h, \ldots, p_n^h)$ i.e.;

$$P(t) = \sum_{j=0}^{n} p_j^h B_j^n(t;h) = \sum_{j=0}^{N} p_j^{(N)} B_j^N(t;h).$$

Denote by $V_h = (P(0), P(h), P(2h), \ldots, P(1))^T$. Then by (6), we have

$$A_{N,n}(p_0^h, p_1^h, \ldots, p_n^h)^T = (p_0^{(N)}, p_1^{(N)}, \ldots, p_N^{(N)})^T = V_h.$$

Therefore, we can compute the h-Bézier coefficients of P using the pseudo-inverse as

$$p^h = A^{\dagger}_{N,n} V_h. \tag{20}$$

Hence, according to (19), the solution to the problem (17) as a function of the parameter $h \neq 0$ is given by

$$q^h = \left(A^t_{n,m} W A_{n,m}\right)^{-1} A^t_{n,m} W A^{\dagger}_{N,n} V_h.$$

Summarizing,

Theorem 3. *The Bézier coefficients q of the solution to the minimization problem (11) are given by*

$$q = \left(A^t_{n,m} W A_{n,m}\right)^{-1} A^t_{n,m} W p.$$

The h-Bézier coefficients q^h of the solution to the minimization problem (17) are given by

$$q^h = \left(A^t_{n,m} W A_{n,m}\right)^{-1} A^t_{n,m} W p^h = \left(A^t_{n,m} W A_{n,m}\right)^{-1} A^t_{n,m} W A^{\dagger}_{N,n} V_h,$$

where $V_h = (P(0), P(h), P(2h), \ldots, P(1))$ and $A_{i,j}$ denote the degree raising matrices.

Example 1. *Consider a planar Bézier curve P of degree nine with control points* $(70,2), (-20,-8), (-10,-57), (-85,-9), (-3,10), (27,146), (11,16), (113,107), (86,52), (31,-29)$. *Figure 1 shows an application of Theorem 3 to the continuous and discrete polynomial degree reduction from the degree nine polynomial P to a degree five polynomial Q_5, for fixed parameters $\alpha = -1/2, \beta = -1/2$ and different sampling parameter h. In each case we give the uniform error reduction*

$$E_\infty = \max_{t \in X_N} ||P(t) - Q_5(t)|| \sim \max_{t \in [0,1]} ||P(t) - Q_5(t)||,$$

where $X_N = \{0, 1/N, 2/N, \ldots, N-1/N, 1\}$ and $N = 1000$.

6. Applications to Jacobi and Hahn polynomials

In this section we derive relations between orthogonal polynomials with respect to the inner products (8) and (12). These relations are derived using two ingredients. The first is that degree elevation of h-Bézier curves does not depend on the parameter h and the second is the following simple observation [5]: Let $\langle \cdot, \cdot \rangle_E$ be a given inner product on $\mathbb{P}_n$ and $||.||_E$ be the induced norm. Denote by $S_0, S_1, \ldots, S_n$ a set of orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_E$ such that the exact degree of S_i is equal i for $i = 0, 1, \ldots, n$. Then, for a given polynomial P of degree n, the solution to the minimization problem

$$\min_{Q \in \mathbb{P}_{n-1}} ||P - Q||_E \tag{21}$$

is given by

$$Q = P - \alpha S_n, \quad \text{where} \quad \alpha = \frac{a}{b}, \tag{22}$$

with a (resp. b) the leading coefficient of P (resp. S_n) i.e.; α is chosen such that the degree of Q is $n-1$.

Shifted Jacobi polynomials $J_n^{\alpha,\beta}, n = 0, 1, \ldots$ are orthogonal polynomials with respect to the inner product (8) and are given in terms of the generalized hypergeometric functions as

$$J_n^{\alpha,\beta}(x) = {}_2F_1\left(\begin{matrix} -n & n+\alpha+\beta+1 \\ & \alpha+1 \end{matrix}; x\right),$$

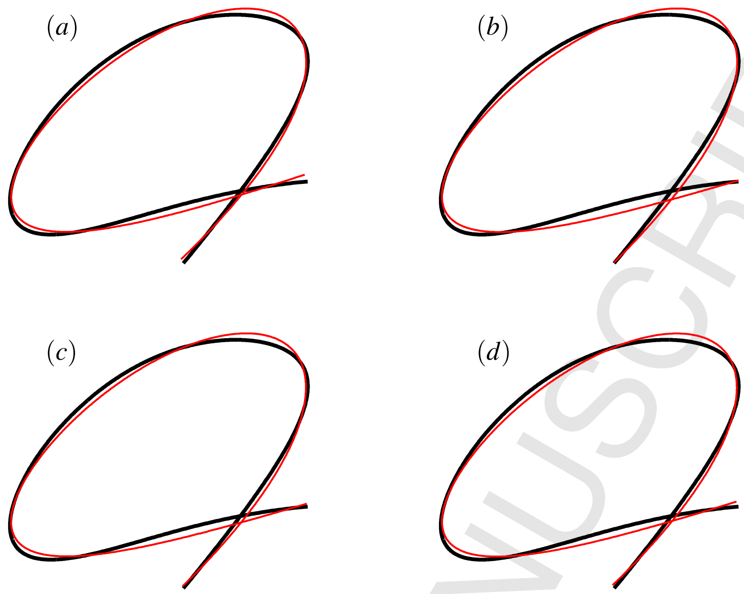

Figure 1: Degree reduction of the Bézier curve of degree 9 (black curve) to degree 5 (red curves) for fixed parameters $\alpha = -1/2, \beta = -1/2$ and different sampling parameter h. (a) continuous degree reduction i.e.; $h = 0$; $E_\infty = 2.7842$. (b) degree reduction with $h = 0.1$; $E_\infty = 3.5854$. (c) degree reduction with $h = 0.05$; $E_\infty = 2.9582$. (d) degree reduction with $h = 0.02$; $E_\infty = 2.8834$.

where

$$ {}_nF_m\left(\begin{matrix} a_1\ a_2\ \dots\ a_n \\ b_1\ b_2\ \dots\ b_m \end{matrix};x\right) = \sum_{j=0}^{\infty} \frac{(a_1)_j(a_2)_j\dots(a_n)_j}{(b_1)_j(b_2)_j\dots(b_m)_j}\frac{x^j}{j!}. \tag{23}$$

The leading coefficient, ρ_n, of the shifted Jacobi polynomial $J_n^{\alpha,\beta}$ is given by

$$\rho_n = (-1)^n \frac{(\alpha+\beta+n+1)_n}{(\alpha+1)_n}. \tag{24}$$

Hahn polynomials are orthogonal polynomials with respect to the weighted inner product (9). Hahn polynomials can be expressed using the generalized hypergeometric series as:[2]

$$Q_n(x;\alpha,\beta,h) = {}_3F_2\left(\begin{matrix} -n & -\frac{x}{h} & n+\alpha+\beta+1 \\ \alpha+1 & -\frac{1}{h} & \end{matrix};1\right). \tag{25}$$

The leading coefficient, l_n, of the Hahn polynomial $Q_n(x;\alpha,\beta,h)$ is given by

$$l_n = \frac{(\alpha+\beta+n+1)_n N^n}{(\alpha+1)_n(-N)_n} \quad \text{with} \quad N = \frac{1}{h}. \tag{26}$$

Using Corollaries 1 and 2 we prove the following.

[2]For convenience, we normalize Hahn polynomials to be orthogonal with respect to the partition $\{0, 1/N, 2/N, \dots, 1\}$ instead of the usual $\{0, 1, \dots, N\}$.

Proposition 2. *The coefficients of the Bernstein representation of shifted Jacobi polynomials are equal to the coefficients of the h-Bernstein representation of Hahn polynomials for any parameter h, i.e.; if*

$$J_n^{\alpha,\beta}(x) = \sum_{i=0}^{M} p_i B_i^M(x), \quad M \geq n$$

then

$$Q_n(x;\alpha,\beta,h) = \sum_{i=0}^{M} p_i B_i^M(x;h).$$

Proof. Let P be the polynomial of exact degree n with h-Bézier coefficients $p^h = (a,0,...,0)$ over the interval $[0,1]$, where a is given by

$$\frac{(\alpha+\beta+n+1)_n}{(\alpha+1)_n}.$$

Denote by Q the polynomial solution of degree $n-1$ to the minimization problem (17). Noting that the leading coefficient of the polynomial P is equal to the leading coefficient (26) of the Hahn polynomial $Q_n(x;\alpha,\beta,h)$, we deduce by (21) and (22) that

$$Q_n(x;\alpha,\beta,h) = P(x) - Q(x).$$

Denote by q^h the h-Bézier coefficients of the polynomials Q over the interval $[0,1]$. Using (19) we obtain

$$\begin{aligned} Q_n(x;\alpha,\beta,h) = P(x) - Q(x) &= \mathbb{B}_h^n(p^h - A_{n,n-1}q^h) \\ &= \mathbb{B}_h^n\left(I - A_{n,n-1}\left(A_{n,n-1}^t W A_{n,n-1}\right)^{-1} A_{n,n-1}^t W\right) p^h. \end{aligned} \tag{27}$$

Similarly, let $\tilde{P}$ be the polynomial of exact degree n with Bézier coefficients $\tilde{p} = (a,0,\ldots,0)$ and denote by $\tilde{Q}$ the polynomial solution of degree $n-1$ to the minimization problem (11). Then noting that the leading coefficient of the polynomial $\tilde{P}$ is equal to the leading coefficient (24) of the shifted Jacobi polynomial $J_n^{\alpha,\beta}$, we obtain

$$J_n^{\alpha,\beta}(x) = \tilde{P}(x) - \tilde{Q}(x) = \mathbb{B}^n\left(I - A_{n,n-1}\left(A_{n,n-1}^t W A_{n,n-1}\right)^{-1} A_{n,n-1}^t W\right)\tilde{p}. \tag{28}$$

Comparing (27) and (28), while noting that the matrix

$$I - A_{n,n-1}(A_{n,n-1}^t W A_{n,n-1})^{-1} A_{n,n-1}^t W$$

is independent of the parameter h, we conclude that the coefficients of the Bernstein representation to order n of the shifted Jacobi polynomials are equal to the coefficients of the h-Bernstein representation of Hahn polynomials to order n for any parameter h, i.e.; the validity of the proposition for $M = n$. We complete the proof by invoking the fact that degree elevation of h-Bézier curves is independent of the parameter h. □

Given a Bernstein representation of shifted Jacobi polynomial as

$$J_n^{\alpha,\beta}(x) = \sum_{i=0}^{M} p_i B_i^M(x), \quad M \geq n,$$

then by Proposition 2 and Equation (6) with $h = 1/M$, we have

$$p_i = Q_n\left(\frac{i}{M};\alpha,\beta,\frac{1}{M}\right), \quad i = 0,1,\ldots,M.$$

Therefore, we recover the following result of Ciesielski in [6] which was proved using a different approach.

Corollary 3. *Shifted Jacobi polynomials $J_n^{\alpha,\beta}$ can be expressed in terms of the Bernstein basis of order $M \geq n$ as*

$$J_n^{\alpha,\beta}(x) = \sum_{j=0}^{M} {}_3F_2\left(\begin{matrix}-n & -j & n+\alpha+\beta+1\\ & \alpha+1 & -M\end{matrix};1\right) B_j^M(x). \tag{29}$$

Combining Corollary 3 and Proposition 2, we also get that for any $N, M \geq n$

$${}_3F_2\left(\begin{matrix}-n & -xN & n+\alpha+\beta+1\\ & \alpha+1 & -N\end{matrix};1\right) = \sum_{j=0}^{M} {}_3F_2\left(\begin{matrix}-k & -j & n+\alpha+\beta+1\\ & \alpha+1 & -M\end{matrix};1\right) B_j^M(x;1/N).$$

The last expression was derived in [14] using the Olinde-Rodrigues formula for Hahn polynomials.

7. Conclusion

In this work we generalized the results of [2, 12] for Jacobi L_2-norms and Hahn discrete L_2-norms. We presented new interpretations of the connection between Jacobi orthogonal polynomials and Hahn orthogonal polynomials. A question that naturally arises from this work is to characterize all inner products for which an analogue result to Corollary 1 can be derived. We believe that Jacobi inner products are the only inner products with such property but such a claim remains unsolved. Moreover, the most important research direction in our agenda is to find a continuous and discrete counterparts to the results of [1] for the constrained polynomial degree reduction.

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