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Short-time existence of solutions for mean-field games with congestion

Diogo A. Gomes and Vardan K. Voskanyan

Abstract

We consider time-dependent mean-field games with congestion that are given by a Hamilton-Jacobi equation coupled with a Fokker-Planck equation. These models are motivated by crowd dynamics in which agents have difficulty moving in high-density areas. The congestion effects make the Hamilton-Jacobi equation singular. The uniqueness of solutions for this problem is well understood; however, the existence of classical solutions, was only known in very special cases - stationary problems with quadratic Hamiltonians and some time-dependent explicit examples. Here, we demonstrate the short-time existence of C^∞ solutions for sub-quadratic Hamiltonians.

1. *Introduction*

Here, we study time-dependent mean-field games with congestion motivated by crowd dynamics in which agents have difficulty moving in high-density areas. These games are given by the system

$$\begin{cases} -u_t - \Delta u + m^\alpha H_0\left(x, \frac{Du}{m^\alpha}\right) + b\cdot Du = V(x, m(x,t)),\\ m_t - \Delta m - \operatorname{div}(D_pH_0\left(x, \frac{Du}{m^\alpha}\right) m) - \operatorname{div}(bm) = 0,\\ u(x,T) = \Psi(x),\ m(x,0) = m_0(x). \end{cases} \tag{1.1}$$

Because we work in the spatially periodic setting, the variable x takes values on the d-dimensional torus, $\mathbb{T}^d$. The unknowns in (1.1) are the functions $u\colon \mathbb{T}^d\times[0,T]\to\mathbb{R}$ and $m\colon \mathbb{T}^d\times[0,T]\to\mathbb{R}^+$. The functions $H_0 : \mathbb{T}^d\times\mathbb{R}^d\to\mathbb{R}$, $V : \mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$, and $b, \Psi, m_0\colon \mathbb{T}^d\to\mathbb{R}$ are given C^∞ functions with $m_0 > 0$. Moreover, $V(x,m)$ is increasing in m. Detailed hypotheses on H_0, b, V, Ψ and m_0 are presented in Section 2. A concrete Hamiltonian, H_0, to which our results apply is the following: for $\gamma\in(1,2)$, set $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$. Consider the Lagrangian:

$$L_0(x,v) = a(x)(1+|v|^2)^{\frac{\gamma'}{2}}, \tag{1.2}$$

where $a\in\mathcal{C}^\infty(\mathbb{T}^d),\ a>0$. Define H_0 as the Legendre transform of L_0:

$$H_0(x,p) = \sup_v[-v\cdot p - L_0(x,v)]. \tag{1.3}$$

The uniqueness of solutions to (1.1) was proven in [**31**] (see also [**24**]) under Assumptions 10-12 of Section 2. Here, we prove the existence of smooth solutions for small terminal time, T, and sub-quadratic Hamiltonians:

Theorem 1. *Under Assumptions 1-12 (cf. Section 2), there exists a time, $T_0>0$, such that for any terminal time, $T\le T_0$, there exists a C^∞ solution (u,m) to (1.1), with $m>0$.*

Mean-field games have become an important research topic since the seminal works of J-M. Lasry and P-L. Lions [**28**], [**29**], [**30**] and M. Huang, P. Caines and R. Malhamé [**26**], [**25**]. Diverse

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questions have been studied intensively, including stationary mean-field games [**18**], [**17**], [**12**], classical and weak solutions for time-dependent problems (see, respectively, [**16**], [**15**], [**13**], [**14**] and [**33**], [**34**], [**2**]), finite state models [**9**], [**10**], [**7**], [**6**], [**21**], [**20**], extended mean-field games [**22**], and obstacle problems [**11**]. For a recent survey, see [**19**]. Congestion problems were addressed initially by P-L. Lions [**31**], who proved the uniqueness of smooth enough solutions. Two alternative approaches to congestion problems are density constraints, introduced in [**35**], [**32**], and nonlinear mobilities, see [**1**]. A recent result (see [**8**]) focuses on the stationary congestion problem with quadratic costs. The existence of smooth solutions in the time-dependent setting has not been established previously. The existence of weak solutions for small terminal time T was studied recently in [**23**]. The results in the present paper can be regarded as complementary to those in that reference. We work with subquadratic Hamiltonians, whereas in [**23**], there is a stronger constraint on the growth of H, namely, in (1.2), the exponent γ must satisfy $\gamma < 1 + \frac{1}{d+1}$. However, we require α to be small enough in contrast with [**23**] where α can be taken to be arbitrarily large.

Before proceeding, we briefly discuss the motivation for (1.1). We consider a large population of agents on $\mathbb{T}^d$, whose statistical evolution over time is encoded in an unknown probability density, $m(x,t)$. Let $(\Omega, \mathcal{F}_t, P)$ be a filtered probability space supporting a d-dimensional Brownian motion, W_t. Let $\mathbb{E}$ be the expected value operator. Consider an agent whose location at time t is x. The cost function for this agent, sometimes called a value or utility function, is

$$u(x,t) = \inf_v \mathbb{E} \int_t^T L(X_s, v_s, m(X_s,s), s)ds + \Psi(X_T),$$

where the trajectory X controlled by the dynamics,

$$dX_s = v_s ds + \sqrt{2} dW_s, \; X_0 = x,$$

and the infimum is taken over bounded $\mathcal{F}_t$-progressively measurable controls, v_s. Here, $\Psi\colon \mathbb{T}^d \to \mathbb{R}$ is the terminal cost. The Lagrangian, L, has the form

$$L(x,v,m,t) = m^\alpha L_0(x, v - b(x,t)) + V(x,m).$$

Detailed assumptions on L_0 are given in the next Section. The constant α determines the strength of the congestion effects. These are encoded in the term $m^\alpha L_0(x, v - b(x,t))$, which makes it more expensive to move in regions of high density if the drift, v, is substantially different from a reference vector field, $b\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$. Finally, the function $V\colon \mathbb{T}^d \times \mathbb{R}_0^+ \to \mathbb{R}$ accounts for other spatial preferences of the agents.

The Hamiltonian is the Legendre transform of L, given by

$$H(x,p,m,t) = \sup_v \{-v \cdot p - L(x,v,m,t)\} = m^\alpha H_0\left(x, \frac{p}{m^\alpha}\right) + b(x,t) \cdot p,$$

where H_0 is the Legendre transform of L_0. Under standard assumptions regarding rationality and symmetry, the mean-field problem that models this setup is (1.1). It comprises a second-order Hamilton-Jacobi equation for the value function, u, coupled with a Kolmogorov-Fokker-Planck equation for the density of agents, m.

We conclude this introduction by describing the structure of the paper: in Section 2, we state the main assumptions used in this manuscript. Afterwards, in Section 3, we discuss various estimates that hold for arbitrary values of the terminal time, T. Then, in Section 4, we present a new technique to address the short-time problem by controlling the growth of $\frac{1}{m}$. Next, in Section 5, we establish additional regularity for the solutions. Section 6 concludes the paper with the proof of Theorem 1.

2. *Assumptions*

Throughout this paper, we work under several assumptions that we state next. Assumptions 1 and 2 concern the smoothness of the initial and terminal data and the various functions in (1.1).

Here, we utilize C^∞ data to simplify the arguments. However, it is possible to proceed through the proofs with less regularity and to obtain the existence of solutions with C^k regularity for large enough k. Assumptions 3-7 and 10 are standard assumptions in optimal control problems, viscosity solutions, and mean-field games. They are stated explicitly for the convenience and clarity of the paper but do not result in a substantial loss of generality. A model Hamiltonian that satisfies these assumptions is (1.3). Assumptions 8 and 9 are specific to the present problem and impose, respectively, a bound on the congestion exponent and subquadratic growth for the Hamiltonian. Subquadratic Hamiltonians correspond to superquadratic Lagrangians. In (1.3), this is reflected in the condition $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$ satisfied by the exponent in (1.2). Finally, Assumptions 11 and 12 are required for uniqueness (see [**31**]).

Assumption 1. *The terminal cost, $\Psi\colon \mathbb{T}^d \to \mathbb{R}$, the reference velocity, $b\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$, and the potential, $V\colon \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$, are C^∞ functions, globally bounded with bounded derivatives of all orders.*

Assumption 2. *The initial distribution $m_0\colon \mathbb{T}^d \to \mathbb{R}$ is a C^∞ probability density: $\int_{\mathbb{T}^d} m_0(x)dx = 1$. Moreover, there exists $k_0 > 0$ such that $m_0(x) \geq k_0$ for all $x \in \mathbb{T}^d$.*

Assumption 3. *The Lagrangian $L_0\colon \mathbb{T} \times \mathbb{R}^d$ is C^∞, and the map*

$$v \mapsto L_0(x,v)$$

is strictly convex for every $x \in \mathbb{T}^d$.

Assumption 4. *L_0 is positive: $L_0(x,v) \geq 0$, $\forall (x,v) \in \mathbb{T}^d \times \mathbb{R}^d$.*

Assumption 5. *There exist conjugated powers $\gamma, \gamma' > 1$, $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$ and constants $C_i, c_i > 0$, $i=1,2$ such that*

$$C_1\frac{|v|^{\gamma'}}{\gamma'} - c_1 \leq L_0(x,v) \leq C\frac{|v|^{\gamma'}}{\gamma'} + c,\ \forall x \in \mathbb{T}^d, v \in \mathbb{R}^d.$$

Remark 1. *Let $H_0(x,p) = \sup_v \big\{ -p\cdot v - L_0(x,v)\big\}$ be the Legendre transform of L_0. The definition of the Legendre transform implies the convexity of H_0. Thus, we have*

$$H_0(x,p) - p \cdot D_pH_0(x,p) \leq H_0(x,0) = \sup_v \big\{ -L_0(x,v)\big\} \leq 0, \tag{2.1}$$

using Assumption 4.

Remark 2. *Under Assumption 3, the Hamiltonian $H_0\colon \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is C^∞.*

Remark 3. *Assumptions 3-5 imply that*

$$C_1'\frac{|p|^\gamma}{\gamma} - c_1' \leq H_0(x,p) \leq C_2'\frac{|p|^\gamma}{\gamma} + c_2', \quad \forall x \in \mathbb{T}^d, p \in \mathbb{R}^d.$$

Assumption 6. *There exist positive constants $c, C > 0$ such that*

$$p \cdot D_pH_0(x,p) - H_0(x,p) \geq c|p|^\gamma - C.$$

ASSUMPTION 7. *There exists a constant C such that*

$$|D_p H_0(x,p)| \leq C|p|^{\gamma-1} + C.$$

REMARK 4. *By combining Remark 3 with Assumption 6, we conclude that there exist positive constants c, C such that, for any $r > 1$,*

$$c|p|^\gamma \leq H_0(x,p) + rp \cdot D_p H_0(x,p) + Cr.$$

ASSUMPTION 8. *If $d \geq 4$, the exponent α in the congestion term (m^α) satisfies the inequality $0 \leq \alpha < \frac{2}{d-2}$. If $d < 4$, the exponent α satisfies $\alpha < 1$.*

ASSUMPTION 9. *H_0 has sub-quadratic growth, i.e., $\gamma < 2$.*

The next three assumptions are required for the uniqueness of the solutions.

ASSUMPTION 10. *The Hamiltonian H_0 is C^∞, and the map*

$$p \mapsto H_0(x,p)$$

is strictly convex for every $x \in \mathbb{T}^d$; that is, $D^2_{pp} H_0(x,p) > 0$, $\forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d$.

REMARK 5. *The previous Assumption implies that H_0 is uniformly convex on compacts, i.e., for any $R > 0$, there exists $\theta_R > 0$ such that $D^2_{pp} H_0(x,p) \geq \theta_R I$, $\forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d$, with $|p| \leq R$.*

ASSUMPTION 11. *For $p \neq 0$, the following inequality holds:*

$$D_p H_0(x,p) \cdot p - H_0(x,p) > \frac{\alpha}{4}\, p^t \cdot D_{pp} H_0(x,p) \cdot p.$$

ASSUMPTION 12. *The potential $V \colon \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ is strictly increasing in the second variable.*

For the Hamiltonian H_0 given by (1.3), Assumption 10 is satisfied for every $\gamma > 1$. A simple calculation shows that Assumption 11 holds if $\alpha < \frac{4(\gamma'-1)}{\gamma'} = \frac{4}{\gamma}$. A potential V for which Assumption 12 is valid is $V(x,z) = \arctan(z)$.

3. *Estimates for arbitrary terminal time*

The main result of this paper is the existence of smooth solutions to (1.1) for small values of the terminal time, T. Nevertheless, various estimates we need are valid for arbitrary T. We report those in this section.

We begin with an auxiliary Lemma

LEMMA 1. *For $0 \leq \tau \leq T$, $\phi \in C^\infty(\mathbb{T}^d)$, and $\phi \geq 0$ with $\|\phi\|_{L^1(\mathbb{T}^d)} \leq 1$, let ρ be the solution to*

$$\begin{cases} \rho_t = \Delta \rho, \\ \rho(x,\tau) = \phi(x). \end{cases} \tag{3.1}$$

Denote by 2^* the Sobolev conjugate exponent of 2, given by $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$. Then, for any q with $1 < q < \frac{2^*}{2}$, there exists a constant C_q such that

$$\|\rho\|_{L^1(L^q(dx),dt)} = \int_\tau^T \left(\int_{\mathbb{T}^d} \rho^q dx \right)^{\frac{1}{q}} dt \leq C_q.$$

Proof. By the maximum principle, $\rho \geq 0$. Furthermore, $\frac{d}{dt} \int_{\mathbb{T}^d} \rho(x,t)dx = 0$. In particular, for any $t \geq \tau$ $\|\rho(\cdot,t)\|_{L^1(\mathbb{T}^d)} = \|\phi\|_{L^1(\mathbb{T}^d)} \leq 1$. Multiplying the heat equation (3.1) by $\rho^{\delta-1}$, for $0 < \delta < 1$, and integrating by parts, we get

$$c_\delta \int_\tau^T \int_{\mathbb{T}^d} |D(\rho^{\frac{\delta}{2}})|^2 dxdt = \frac{1}{\delta} \int_{\mathbb{T}^d} (\rho^\delta(x,T) - \rho^\delta(x,\tau))dx \leq \frac{1}{\delta}, \tag{3.2}$$

for any $\varepsilon > 0$, where $c_\delta = \frac{4(1-\delta)}{\delta^2}$. Here, we used Jensen's inequality to obtain

$$0 \leq \int_{\mathbb{T}^d} \rho^\delta(x,\tau)dx, \quad \int_{\mathbb{T}^d} \rho^\delta(x,T)dx \leq 1.$$

From the Gagliardo-Nirenberg inequality,

$$\|\rho^{\frac{\delta}{2}}(\cdot,t)\|_{L^{p_\delta}(\mathbb{T}^d)} \leq C\|D(\rho^{\frac{\delta}{2}})(\cdot,t)\|^\delta_{L^2(\mathbb{T}^d)} \|\rho^{\frac{\delta}{2}}(\cdot,t)\|^{1-\delta}_{L^2(\mathbb{T}^d)},$$

where $\frac{1}{p_\delta} = (\frac{1}{2} - \frac{1}{d})\delta + \frac{1-\delta}{2}$. Since $\|\rho^{\frac{\delta}{2}}(\cdot,t)\|_{L^2(\mathbb{T}^d)} \leq \|\rho(\cdot,t)\|^{\frac{\delta}{2}}_{L^1(\mathbb{T}^d)} \leq 1$, we have

$$\|\rho(\cdot,t)\|_{L^{\frac{\delta p_\delta}{2}}(\mathbb{T}^d)} \leq C\|D(\rho^{\frac{\delta}{2}})(\cdot,t)\|^2_{L^2(\mathbb{T}^d)}.$$

Finally, integrating the previous estimate in time and using (3.2), we obtain $\|\rho\|_{L^1(L^{\frac{\delta p_\delta}{2}}(dx),dt)} \leq C_\delta$. To end the proof, we observe that $p_\delta \to 2^*$ when $\delta \to 1$. □

PROPOSITION 1. *Under Assumptions 1-4, there exists a constant $C := C(\|V\|_\infty, \|\Psi\|_\infty, T)$ such that, for any C^∞ solution (u,m) of (1.1), we have $\|m(\cdot,t)\|_{L^1(\mathbb{T}^d)} = 1,\ \forall 0 \leq t \leq T$, and, additionally, $u \geq -C$.*

Proof. By integrating the second equation, we have $\left(\int_{\mathbb{T}^d} m(x,t)dx\right)_t = 0$. Therefore, $\int_{\mathbb{T}^d} m(x,t)dx = 1$, for all $t \geq 0$. To prove the upper bound for u, we apply the nonlinear adjoint method [**5**] (for further applications, see also [**36**]). Let ζ be a solution to

$$\begin{cases} \zeta_t - \operatorname{div}(D_p H_0\left(x, \frac{Du}{m^\alpha}\right)\zeta) - \operatorname{div}(b\zeta) = \Delta\zeta \\ \zeta(x,\tau) = \phi(x), \end{cases} \tag{3.3}$$

with $\phi \in C^\infty(\mathbb{T}^d)$, $\phi \geq 0$. We multiply the first equation in (1.1) by ζ and subtract (3.3) multiplied by u. Then, we integrate by parts and gather

$$-\left[\int_{\mathbb{T}^d} u\zeta dx\right]_t + \int_{\mathbb{T}^d} m^\alpha \left[H_0\left(x, \frac{Du}{m^\alpha}\right) - \frac{Du}{m^\alpha} D_p H_0\left(x, \frac{Du}{m^\alpha}\right)\right]\zeta dx = \int_{\mathbb{T}^d} V\zeta dx.$$

Integrating on $[\tau, T]$ and using (2.1), we obtain

$$-\int_{\mathbb{T}^d} \Psi(x)\zeta(x,T)dx + \int_{\mathbb{T}^d} u(x,\tau)\phi(x)dx \geq \int_\tau^T \int_{\mathbb{T}^d} V\zeta dxdt.$$

By the maximum principle, $\zeta \geq 0$, because $\phi \geq 0$. Integrating (3.3) with respect to x, we get $\int \zeta dx = \int \phi dx$. This identity, together with the above inequality, yields

$$\int_{\mathbb{T}^d} u(x,\tau)\phi(x)dx \geq -[(T-\tau)\|V\|_\infty + \|\Psi\|_\infty]\|\phi\|_{L^1(\mathbb{T}^d)}.$$

Since this estimate holds for every C^∞ $\phi \geq 0$, we obtain $-u(x,\tau) \leq (T-\tau)\|V\|_\infty + \|\Psi\|_\infty$. □

PROPOSITION 2. *Under Assumptions 1-6, there exists a constant $C := C(\|V\|_\infty, \|\Psi\|_\infty, T)$ such that, for any C^∞ solution (u,m) to (1.1), we have*

$$\int_0^T \int_{\mathbb{T}^d} \frac{|Du|^\gamma}{m^{\bar\alpha}} dxdt \le C \tag{3.4}$$

and

$$\int_{\mathbb{T}^d} |u(x,t)|dx \le C, \ t \in [0,T], \tag{3.5}$$

where

$$\bar\alpha = (\gamma-1)\alpha < 1. \tag{3.6}$$

Proof. We integrate the first equation in (1.1) with respect to x and t. Then, we use the bounds on u from the previous proposition to get

$$\int_t^T \int_{\mathbb{T}^d} \left(m^\alpha H_0\left(x, \frac{Du}{m^\alpha}\right) + b\cdot Du\right) dxds = \int_{\mathbb{T}^d} u(x,T)dx - \int_{\mathbb{T}^d} u(x,t)dx + \int_t^T \int_{\mathbb{T}^d} V dxds \le C.$$

By Remark 3, $|p|^\gamma \le C(H_0(x,p) + b\cdot p) + C$. Accordingly,

$$\int_{\mathbb{T}^d} u(x,t)dx + \int_t^T \int_{\mathbb{T}^d} \frac{|Du|^\gamma}{m^{\bar\alpha}} dxdt$$
$$\le C \int_t^T \int_{\mathbb{T}^d} m^\alpha \left(H_0\left(x, \frac{Du}{m^\alpha}\right) + b\cdot Du\right) dxds + C\int_t^T \int_{\mathbb{T}^d} m^\alpha dxds \le C.$$

This inequality, combined with the lower bound on u of Proposition 1, yields (3.4). Moreover, since

$$\int u \le C$$

and using again the lower bound on u of Proposition 1, we obtain (3.5). □

PROPOSITION 3. *Under Assumptions 1-6, there exists a constant $C := C(\|V\|_\infty, \|\Psi\|_\infty, T)$ such that, for any C^∞ solution (u,m) to (1.1), we have*

$$\int_0^t \int_{\mathbb{T}^d} |Du|^\gamma m^{1-\bar\alpha} dxdt \le C + C\int_0^t \int_{\mathbb{T}^d} m^{1+\alpha} dxdt,$$

for all $0 \le t \le T$, where $\bar\alpha$ is given by (3.6).

Proof. We multiply the first equation in (1.1) by m and subtract the second equation multiplied by u. Then, integration by parts yields:

$$\int_0^t \int_{\mathbb{T}^d} m^{1+\alpha} \left[\frac{Du}{m^\alpha} D_p H_0\left(x, \frac{Du}{m^\alpha}\right) - H_0\left(x, \frac{Du}{m^\alpha}\right)\right] dxdt = -\int_0^t \int_{\mathbb{T}^d} V m dx + \int_{\mathbb{T}^d} u(x,0) m_0(x) dx$$
$$- \int_{\mathbb{T}^d} u(x,t)m(x,t)dx \le C + \|m_0\|_\infty \int_{\mathbb{T}^d} |u(x,0)|dx - \inf_x u(x,t) \le C,$$

where the last inequality follows from the lower bounds on u from Proposition 1 and from the bound on $\int_{\mathbb{T}^d} |u| dx$ in the previous proposition. The claim in the statement follows from Assumption 6 by using the inequality $pD_pH_0(x,p) - H_0(x,p) \ge c|p|^\gamma - C$, for some $c, C > 0$. □

PROPOSITION 4. *Under Assumptions 1-7, there exists a constant $C := C(\|V\|_\infty, \|\Psi\|_\infty, T, \alpha)$ such that, for any C^∞ solution (u,m) of (1.1), we have*

$$\int_{\mathbb{T}^d} m^{1+\alpha}(x,t)dx + \int_0^t \int_{\mathbb{T}^d} m^{\alpha-1}(x,s)|Dm(x,s)|^2 dxds \le C.$$

Proof. We begin by multiplying the second equation by $(\alpha+1)m^\alpha$. Next, integrating by parts, we conclude that

$$\left[\int_{\mathbb{T}^d} m^{1+\alpha}(x,t)dx\right]_t = -\alpha(1+\alpha)\int_{\mathbb{T}^d} m^{\alpha-1}\left[|Dm|^2 + mD_pH_0(x,\frac{Du}{m^\alpha})\cdot Dm + mb\cdot Dm\right]dx \tag{3.7}$$

$$\leq -\alpha(1+\alpha)\int_{\mathbb{T}^d}\left[\frac{1}{2}m^{\alpha-1}|Dm|^2 - \left|D_pH_0\left(x,\frac{Du}{m^\alpha}\right)\right|^2 m^{\alpha+1} - \|b\|_\infty^2 m^{\alpha+1}\right]dx$$

$$\leq -\alpha(1+\alpha)\int_{\mathbb{T}^d}\left[\frac{1}{2}m^{\alpha-1}|Dm|^2 - C|Du|^{2(\gamma-1)}m^{\alpha+1-2\bar\alpha} - \|b\|_\infty^2 m^{\alpha+1}\right]dx$$

$$\leq -\alpha(1+\alpha)\int_{\mathbb{T}^d}\left[\frac{1}{2}m^{\alpha-1}|Dm|^2 - C|Du|^{\gamma}m^{1-\bar\alpha} - Cm^{\alpha+1}\right]dx,$$

where, in the last inequality, we have used Young's inequality:

$$|p|^{2(\gamma-1)}m^{1+\alpha-2\bar\alpha} = \left(|p|^\gamma m^{1-\bar\alpha}\right)^{\frac{2(\gamma-1)}{\gamma}}\left(m^{1+\alpha}\right)^{\frac{2-\gamma}{\gamma}} \leq \frac{2(\gamma-1)}{\gamma}|p|^\gamma m^{1-\bar\alpha} + \frac{2-\gamma}{\gamma}m^{1+\alpha}$$

and the definition of $\bar\alpha$ in (3.6). Integrating (3.7) from 0 to t and using Proposition 3, we conclude that

$$\int_{\mathbb{T}^d} m^{1+\alpha}(x,t)dx + \int_0^t\int_{\mathbb{T}^d} m^{\alpha-1}(x,s)|Dm(x,s)|^2dxds \leq C + C\int_0^t\int_{\mathbb{T}^d} m^{1+\alpha}dxdt. \tag{3.8}$$

In particular,

$$\int_{\mathbb{T}^d} m^{1+\alpha}(x,t)dx \leq C + C\int_0^t\int_{\mathbb{T}^d} m^{1+\alpha}dxdt.$$

Thus, by Gronwall's inequality, we have $\int_{\mathbb{T}^d} m^{1+\alpha}(x,t)dx \leq C$. This estimate combined with (3.8) yields

$$\int_0^t\int_{\mathbb{T}^d} m^{\alpha-1}(x,s)|Dm(x,s)|^2dxds \leq C.$$

□

COROLLARY 1. *Under Assumptions 1-7, there exists a constant $C := C(\|V\|_\infty, \|\Psi\|_\infty, T)$ such that*

$$\int_0^t\int_{\mathbb{T}^d} |Du|^\gamma m^{1-\bar\alpha}dxdt \leq C$$

Proof. The Corollary follows by combining Proposition 3 with Proposition 4. □

PROPOSITION 5. *Under Assumptions 1-8, there exists a constant $C := C(\|V\|_\infty, \|\Psi\|_\infty, T)$ such that, for any C^∞ solution (u,m) of (1.1), $u \leq C$.*

Proof. Let ρ be as in Lemma 1. Multiplying the first equation in (1.1) by ρ, subtracting the equation for ρ multiplied by u, and integrating by parts, we gather

$$-\left[\int_{\mathbb{T}^d} u\rho dx\right]_t + \int_{\mathbb{T}^d} m^\alpha\left[H_0\left(x,\frac{Du}{m^\alpha}\right) + b\cdot\frac{Du}{m^\alpha}\right]\rho dx = \int_{\mathbb{T}^d} V\rho dx.$$

Hence, integrating in time, we conclude that

$$\int_{\mathbb{T}^d} u(x,\tau)\phi(x)dx \leq \int_\tau^T\int_{\mathbb{T}^d} V\rho dxdt + \int_{\mathbb{T}^d}\Psi(x)\rho(x,T)dx + C\int_\tau^T\int_{\mathbb{T}^d} m^\alpha\rho dxdt,$$

where we used the fact that $H_0(x,p)+b\cdot p$ is bounded from below as a consequence of Remark 3. Using Holder's inequality and the bounds $\int_{\mathbb{T}^d}\rho(x,t)dx\le 1$, $\int_{\mathbb{T}^d} m^{1+\alpha}(x,t)dx\le C$, we get

$$u(x,\tau)\le (T-\tau)\|V\|_\infty+\|\Psi\|_\infty+C\|\rho\|_{L^1(L^{1+\alpha}(dx),dt)}. \tag{3.9}$$

Because Assumption 8 holds, $\alpha<\frac{2}{d-2}$. Consequently, $1+\alpha<\frac{2^*}{2}$. Therefore, we can apply Lemma 1 to prove the result. □

Corollary 2. *Under Assumptions 1-8, there exists a constant* $C:=C(\|V\|_\infty,\|\Psi\|_\infty,T-t)$ *such that, for any* C^∞ *solution* (u,m) *of* (1.1), $\|u\|_{L^\infty(\mathbb{T}^d)}\le C$.

Proof. The result follows by combining Propositions 1 and 5. □

Proposition 6. *Under Assumptions 1-9, there exist constants* $c_r, C_r=C_r(\alpha,T)>0$ *that have polynomial growth in* r*, such that, for any* C^∞ *solution* (u,m) *of* (1.1) *and* $r>1$,

$$\int_{\mathbb{T}^d}\frac{1}{m^r(x,t)}dx+c_r\int_0^t\int_{\mathbb{T}^d}\left|D\frac{1}{m^{r/2}}\right|^2dxdt+\int_0^t\int_{\mathbb{T}^d}\frac{|Du|^\gamma}{m^{r+\bar\alpha}}dxdt\le C_r+C_r\int_0^t\int_{\mathbb{T}^d}\frac{1}{m^q}dxdt,\ \forall t,$$

where $\bar\alpha$ *is given by* (3.6) *and*

$$q=r+\frac{2\bar\alpha}{2-\gamma}. \tag{3.10}$$

Proof. By adding a constant to u_0, we can assume, without loss of generality, that $u\le -1$. We fix $r>1$. We begin by multiplying the first equation in (1) by $\frac{1}{m^r}$ and adding it to the second equation multiplied by $r\frac{u}{m^{r+1}}$. After integrating by parts, we obtain

$$\begin{aligned}&-\int_{\mathbb{T}^d}\left(\frac{u}{m^r}\right)_t dx-\int_{\mathbb{T}^d}r(r+1)\frac{u|Dm|^2}{m^{r+2}}dx+\int_{\mathbb{T}^d}m^\alpha\frac{H_0+r\frac{Du}{m^\alpha}\cdot D_pH_0}{m^r}dx\\&-\int_{\mathbb{T}^d}r(r+1)\frac{uD_pH_0\cdot Dm}{m^{r+1}}dx+\int_{\mathbb{T}^d}\left[(r+1)\frac{b\cdot Du}{m^r}-r(r+1)u\frac{b\cdot Dm}{m^{r+1}}-\frac{V}{m^r}\right]dx=0.\end{aligned}$$

We integrate the inequality in t. For $m<1$, we have

$$\frac{|Du|}{m^r}\le\epsilon\frac{|Du|^\gamma}{m^{r+\bar\alpha}}+\frac{1}{m^r}+C_{r,\epsilon},$$

whereas for $m\ge1$, we have

$$\frac{|Du|}{m^r}\le|Du|\le|Du|^\gamma m^{1-\bar\alpha}+C.$$

Taking into account these estimates and the bound in Corollary 1, we get

$$\int_0^t\int_{\mathbb{T}^d}\frac{|Du|}{m^r}\le\int_0^t\int_{\mathbb{T}^d}\frac{|Du|^\gamma}{m^{r+\bar\alpha}}+\frac{1}{m^r}+\tilde C_{r,\epsilon}.$$

Then, we use the estimates:

$$c\frac{|Du|^\gamma}{m^{r+\bar\alpha}}\le m^\alpha\frac{H_0+r\frac{Du}{m^\alpha}D_pH_0}{m^r}+C\frac{r}{m^{r-\alpha}}\ \text{(see Remark 4)}$$

and

$$\frac{|Dm|}{m^{r+1}}\le\epsilon\frac{|Dm|^2}{m^{r+2}}+C_\epsilon\frac{1}{m^r}$$

to get

$$\int_{\mathbb{T}^d} \frac{1}{m^r(x,t)}dx + c_r\int_0^t\int_{\mathbb{T}^d}\left|D\frac{1}{m^{r/2}}\right|^2 dxds + c_r\int_0^t\int_{\mathbb{T}^d}\frac{|Du|^\gamma}{m^{r+\bar{\alpha}}}dxds \leq$$
$$C_r\int_0^t\int_{\mathbb{T}^d}|Du|^{\gamma-1}\frac{|Dm|}{m^{r+\bar{\alpha}+1}}dxds + C_r\int_0^t\int_{\mathbb{T}^d}\frac{1}{m^r}dxds + C_r.$$

The required estimate follows from the inequalities

$$|Du|^{\gamma-1}\frac{|Dm|}{m^{r+\bar{\alpha}+1}} \leq \epsilon\frac{|Du|^\gamma}{m^{r+\bar{\alpha}}} + \epsilon\frac{|Dm|^2}{m^{r+2}} + C_\epsilon\frac{1}{m^q},$$

and

$$\frac{1}{m^r} \leq \frac{1}{m^q} + 1,$$

where q is given by (3.10). □

4. *Short-Time Estimates*

In this section, we establish estimates for C^∞ solutions of (1.1) for small values of T. The key idea is to use the estimate in Proposition 6 to control the growth of $\frac{1}{m}$. Because $q > r$ in (3.10), we can achieve bounds only for small T. We begin with the following bound on $\frac{1}{m}$:

THEOREM 2. *Under Assumptions 1-9, there exist $r_0 > 0$, a time $t_1(r) > 0$ and constants $C = C(r,\gamma,\alpha) > 0$, $\delta > 0$, such that, for any C^∞ solution (u,m) to (1.1) and $r \geq r_0$,*

$$\int_{\mathbb{T}^d}\frac{1}{m^r(x,t)}dx \leq C\left[1 + \frac{1}{(t_1-t)^\delta}\right], \ \forall t < t_1.$$

Proof. We choose a sufficiently large r_0 such that $\frac{2^*}{2}r = \frac{dr}{d-2} > q = r + \frac{2\bar{\alpha}}{2-\gamma}$ for $r \geq r_0$. Let $\lambda > 0$ be such that $\frac{2^*}{2}r\lambda + r(1-\lambda) = q$; that is, $\lambda = \frac{\bar{\alpha}(d-2)}{(2-\gamma)r} < 1$ for $r \geq r_0$. We set $\bar{\lambda} = \frac{2^*}{2}\lambda = \frac{\bar{\alpha}d}{(2-\gamma)r}$. Provided r_0 is large enough, $\bar{\lambda} < 1$ and $\beta = \frac{1-\bar{\lambda}}{1-\lambda} > 1$ for all $r \geq r_0$. Then, using Hölder's and Young's inequalities, we obtain

$$\int_{\mathbb{T}^d}\frac{1}{m^q}dx \leq \left(\int_{\mathbb{T}^d}\frac{1}{m^{\frac{2^*}{2}r}}dx\right)^\lambda\left(\int_{\mathbb{T}^d}\frac{1}{m^r}dx\right)^{1-\lambda} = \left[\left(\int_{\mathbb{T}^d}\frac{1}{m^{\frac{2^*}{2}r}}dx\right)^{2/2^*}\right]^{\bar{\lambda}}\left[\left(\int_{\mathbb{T}^d}\frac{1}{m^r}dx\right)^\beta\right]^{1-\bar{\lambda}} \leq$$
$$\varepsilon\bar{\lambda}\left(\int_{\mathbb{T}^d}\frac{1}{m^{\frac{2^*}{2}r}}dx\right)^{2/2^*} + \frac{1}{\varepsilon^\tau}(1-\bar{\lambda})\left(\int_{\mathbb{T}^d}\frac{1}{m^r}dx\right)^\beta,$$

for any $\varepsilon > 0$ and some exponent $\tau > 0$. From Sobolev's inequality,

$$\int_{\mathbb{T}^d}\frac{|Dm|^2}{m^{r+2}}dx = \frac{4}{r^2}\int_{\mathbb{T}^d}\left|D\left(\frac{1}{m^{r/2}}\right)\right|^2 dx \geq c\frac{4}{r^2}\left(\int_{\mathbb{T}^d}\frac{1}{m^{\frac{2^*}{2}r}}\right)^{2/2^*}dx - \frac{4}{r^2}\int_{\mathbb{T}^d}\frac{1}{m^r}dx.$$

By combining Proposition 6 and the above inequalities with the estimate

$$\int_{\mathbb{T}^d}\frac{1}{m^r}dx \leq \varepsilon\left(\int_{\mathbb{T}^d}\frac{1}{m^r}dx\right)^\beta + C_\varepsilon, \ \forall\varepsilon > 0,$$

we obtain

$$\int_{\mathbb{T}^d}\frac{1}{m^r(x,t)}dx \leq C + C\int_0^t\left(\int_{\mathbb{T}^d}\frac{1}{m^r}dx\right)^\beta dt, \ \forall t \in [0,T]. \tag{4.1}$$

Let $h(t) = \int_{\mathbb{T}^d}\frac{1}{m(x,t)^r}dx$ and $H(t) = \int_0^t h^\beta(s)ds$. Then, the previous inequality reads

$$h(t) \leq C_{r,\gamma,T} + C_{r,\gamma,T}H(t).$$

Thus,

$$\dot{H}(t) = h^{\beta}(t) \leq C_{r,\alpha,\gamma,T}(1+H(t))^{\beta}. \tag{4.2}$$

Integrating (4.2) and taking into account that $H(0)=0$, we get

$$(1+H(t))^{1-\beta} \geq 1-(\beta-1)C_{r,\gamma,T}t.$$

Accordingly,

$$H(t) \leq \frac{1}{[1-(\beta-1)C_{r,\gamma,T}t]^{\frac{1}{\beta-1}}} \qquad \text{for all } t < t_1(r) := \frac{1}{(\beta-1)C_{r,\gamma,T}}.$$

Consequently,

$$\int_{\mathbb{T}^d} \frac{1}{m(x,t)^r}dx = h(t) \leq C_{r,\gamma,T} + C_{r,\gamma,T}H(t) \leq C + \frac{C}{(t_1-t)^{\frac{1}{\beta-1}}},\ t<t_1.$$

□

Corollary 3. *Suppose that Assumptions 1-9 hold. Let r_0 and $t_1(r)$ be as in Theorem 2. For $r>r_0$, let $t \leq t_1(r) \equiv t_1$. Then, there exist constants C_r and δ_r such that, for any C^∞ solution (u,m) of* (1.1),

$$\int_0^t \int_{\mathbb{T}^d} \left| D\frac{1}{m^{r/2}} \right|^2 dxdt \leq C_r + \frac{C_r}{(t_1-t)^{\delta_r}}, \quad \forall\, t<t_1. \tag{4.3}$$

Iterating the estimates from Proposition 6, we prove next uniform bounds in r.

Proposition 7. *Under Assumptions 1-9, there exist $r_1>0$ and constants $C=C(r,\gamma,\alpha)>0$, $\beta_r>1$, such that, for any C^∞ solution (u,m) to* (1.1) *and $r \geq r_1$,*

$$\left\| \frac{1}{m} \right\|_{L^\infty([0,t]\times\mathbb{T}^d)} \leq C_t \left(1 + \left\| \frac{1}{m} \right\|^{\beta_r}_{L^\infty([0,t],L^r(\mathbb{T}^d))} \right).$$

Proof. For $r>1$, choose $\theta_n>0$ such that

$$r^{n+1}+\delta = (1-\theta_n)r^n + \theta_n \frac{2^*}{2} r^{n+1},$$

where $\delta = \frac{2\bar{\alpha}}{2-\gamma}$; that is, $\theta_n = \frac{1-\frac{1}{r}+\frac{\delta}{r^{n+1}}}{\frac{2^*}{2}-\frac{1}{r}} > 0$. Set $\lambda_n = \frac{2^*}{2}\theta_n$ and $\beta_n = \frac{1-\theta_n}{1-\lambda_n}$. Then, there exists $r_1>1$ such that, for any $r \geq r_1$ and any $n \geq 1$, we have $\lambda_n<1$. We fix a time t. As in the previous proposition, using a weighted Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{T}^d} \frac{1}{m^{r^{n+1}+\delta}}dx &\leq \left[\left(\int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2}r^{n+1}}}dx\right)^{2/2^*}\right]^{\lambda_n} \left[\left(\int_{\mathbb{T}^d} \frac{1}{m^{r^n}}dx\right)^{\beta_n}\right]^{1-\lambda_n} \\ &\leq \varepsilon\lambda_n \left(\int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2}r^{n+1}}}dx\right)^{2/2^*} + \frac{1}{\varepsilon^\tau}(1-\lambda_n)\left(\int_{\mathbb{T}^d} \frac{1}{m^{r^n}}dx\right)^{\beta_n},\end{aligned}$$

where $\varepsilon>0$ and $\tau>0$ is a suitable exponent. On the other hand, Proposition 6 and Sobolev's inequality imply that

$$\int_{\mathbb{T}^d} \frac{1}{m^{r^{n+1}}(x,t)}dx + \int_0^t \left(\int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2}r^{n+1}}(x,s)}dx\right)^{2/2^*} ds \leq C_{r^{n+1}} + C_{r^{n+1}} \int_0^t\int_{\mathbb{T}^d} \frac{1}{m^{r^{n+1}+\delta}(x,s)}dxds.$$

From these two inequalities, we conclude that

$$\int_{\mathbb{T}^d} \frac{1}{m^{r^{n+1}}(x,t)}dx \leq C_{r^{n+1}} + C_{r^{n+1}} \int_0^t \left(\int_{\mathbb{T}^d} \frac{1}{m^{r^n}(x,s)}dx\right)^{\beta_n} ds.$$

Define $A_n(t) = \max_{[0,t]} \int_{\mathbb{T}^d} \frac{1}{m^{r^n}(x,\cdot)} dx$. From the above estimate,

$$1 + A_{n+1}(t) \leq \max\{1, t\} C_n (1 + A_n(t))^{\beta_n},$$

where $C_n = O(r^{nk})$ for some $k > 1$. Proceeding inductively, we get

$$(1 + A_{n+1}(t))^{\frac{1}{\beta_1 \cdot \ldots \cdot \beta_n}} \leq C_t^{\sum_{i=1}^n \frac{1}{\beta_1 \cdot \ldots \cdot \beta_i}} r^{\sum_{i=1}^n \frac{ik}{\beta_1 \cdot \ldots \cdot \beta_i}} (1 + A_1).$$

Since

$$\beta_n = \frac{1 - \theta_n}{1 - \lambda_n} = r \left(1 + \frac{(\frac{2^*}{2} r - 1)\delta}{r^{n+1}(\frac{2^*}{2} - 1 - \frac{2^*}{2} \frac{\delta}{r^n})} \right) := r(1 + q_n),$$

where $q_n = O(r^{-n}) > 0$, the series $\sum_{i=1}^{\infty} \frac{ik}{\beta_1 \cdot \ldots \cdot \beta_i}$, $\sum_{i=1}^{\infty} \frac{1}{\beta_1 \cdot \ldots \cdot \beta_i}$, and the infinite product $\prod_{i=1}^{\infty}(1 + q_i)$ converge. From this, we obtain

$$\left\| \frac{1}{m} \right\|_{L^\infty([0,t], L^{r^{n+1}}(\mathbb{T}^d))} \leq C_t \left(1 + \left\| \frac{1}{m} \right\|_{L^\infty([0,t], L^r(\mathbb{T}^d))}^{\beta_r} \right),$$

for some constants $C_t > 0$ and $\beta_r = \prod_{i=1}^{\infty}(1 + q_i) > 1$ that do not depend on the solution. By sending $n \to \infty$, we obtain the result. □

The results of Theorem 2, Proposition 7 and Corollary 3 prove the following:

THEOREM 3. *Under Assumptions 1-9, there exist a time* $T_0 > 0$ *and constants* $C = C(\gamma, \alpha) > 0$ *such that, for any* C^∞ *solution* (u, m) *to* (1.1),

$$\left\| \frac{1}{m} \right\|_{L^\infty([0,T_0] \times \mathbb{T}^d)} \leq C.$$

5. *Short-time regularity of the value function*

Building upon the results in the previous section, we prove next the extended regularity of the solutions of (1.1).

LEMMA 2. *Let* $w : \mathbb{T}^d \times [0, T] \to \mathbb{R}$ *be a non-negative solution of the Fokker-Planck equation:*

$$w_t(x, t) - \Delta w - \operatorname{div}(g(x, t) w(x, t)) = 0, \tag{5.1}$$

with $w(x, 0) = m_0(x)$. *Assume that, for some* $p_0 > d$, *every* $r > 1$ *and some constant* $C_r > 0$, *the drift* g *satisfies* $\|g\|_{L^r([0,T_0], L^{p_0}(\mathbb{T}^d))} \leq C_r$. *Then, there exist constants* $C_{q,r}$ *such that* $\|w\|_{L^\infty([0,T_0], L^q(\mathbb{T}^d))} \leq C_{q,r}$, *for all* $q > 1$.

Proof. Multiplying (5.1) by $q w^{q-1}$ and integrating by parts, we get

$$\int_{\mathbb{T}^d} w^q(x, t) dx - \int_{\mathbb{T}^d} w^q(x, 0) dx + q(q - 1) \int_0^t \int_{\mathbb{T}^d} w^{q-2}(x, s) |Dw(x, s)|^2 dx ds$$
$$= q(q - 1) \int_0^t \int_{\mathbb{T}^d} w^{q-1} g \cdot Dw dx ds.$$

From this, using the Cauchy inequality, we have the estimate

$$\int_{\mathbb{T}^d} w^q(x, t) dx + \int_0^t \int_{\mathbb{T}^d} |D w^{\frac{q}{2}}(x, s)|^2 dx ds \leq C_q + C_q \int_0^t \int_{\mathbb{T}^d} |g|^2 w^q dx ds. \tag{5.2}$$

The previous bound together with Sobolev's inequality implies

$$\int_0^t \|w^q(\cdot,s)\|_{L^{\frac{2^*}{2}}(\mathbb{T}^d)} ds \leq C_q \int_0^t \int_{\mathbb{T}^d} (1+|g|^2) w^q dx ds$$
$$\leq \tilde{C}_q \left(1 + \int_0^t \|1+|g(\cdot,s)|^2\|^r_{L^{p_1}(\mathbb{T}^d)} ds + \int_0^t \|w^q(\cdot,s)\|^{r'}_{L^{p_1'}(\mathbb{T}^d)} ds\right),$$

for any $r>1$, where $p_1 = \frac{p_0}{2}$ and the conjugate powers r' and p_1' satisfy $\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{p_1}+\frac{1}{p_1'}=1$. Recall that $\|w(\cdot,t)\|_{L^1(\mathbb{T}^d)}=1$. Moreover, because $p_1>\frac{d}{2}$, we have $p_1' < \frac{2^*}{2}$. Therefore, by interpolation,

$$\|w(\cdot,s)\|_{L^{qp_1'}(\mathbb{T}^d)} \leq \|w(\cdot,s)\|^{\theta}_{L^{q\frac{2^*}{2}}(\mathbb{T}^d)} \|w(\cdot,s)\|^{1-\theta}_{L^1(\mathbb{T}^d)} = \|w(\cdot,s)\|^{\theta}_{L^{q\frac{2^*}{2}}(\mathbb{T}^d)},$$

for some $\theta<1$. Hence, $\|w^q(\cdot,s)\|_{L^{p_1'}(\mathbb{T}^d)} \leq \|w^q(\cdot,s)\|^{\theta}_{L^{\frac{2^*}{2}}(\mathbb{T}^d)}$. By combining these bounds, we have the estimate

$$\int_0^t \|w^q(\cdot,s)\|_{L^{\frac{2^*}{2}}(\mathbb{T}^d)} ds \leq C_q \left(\int_0^t \|1+|g(\cdot,s)|^2\|^r_{L^{p_1}(\mathbb{T}^d)} ds + \int_0^t \|w^q(\cdot,s)\|^{r'\theta}_{L^{\frac{2^*}{2}}(\mathbb{T}^d)} ds\right).$$

Finally, by choosing a large enough r such that $r'\theta<1$, we obtain $\int_0^t \|w^q(\cdot,s)\|_{L^{\frac{2^*}{2}}(\mathbb{T}^d)} ds \leq C_q$. To end the proof, we observe that, from (5.2), it follows that, for any $q>1$, there exists C_q such that $\int_{\mathbb{T}^d} w^q(x,t)dx \leq C_q$ for any $t\in[0,T]$. □

The next Lemma uses the Gagliardo Niremberg theorem to obtain additional regularity. This is a critical point where we use the assumption that H is subquadratic.

LEMMA 3. *Under Assumptions 1-9, there exist a time $T_0>0$ and constants $C_p, C_{r,p} = C(\gamma,\alpha,r,p)>0$ such that, for any C^∞ solution (u,m) to (1.1),*

$$\|u_t\|_{L^r(0,T_0;L^p(\mathbb{T}^d))} + \left\|D^2u\right\|_{L^r(0,T_0;L^p(\mathbb{T}^d))} \leq C_{r,p}, \qquad \forall 1<r,p<+\infty.$$

$$\|m\|_{L^\infty(0,T_0;L^p(\mathbb{T}^d))} \leq C_p,$$

Furthermore,

$$\|Du\|_{L^{p\gamma}(\mathbb{T}^d\times[0,T_0])} \leq C, \qquad \forall p>1.$$

Proof. We choose T_0 as in Theorem 3. By the Gagliardo-Nirenberg interpolation inequality, Corollary 2, and by taking into account that $\gamma<2$, we get

$$\|Du(\cdot,t)\|_{L^{p\gamma}(\mathbb{T}^d)} \leq c_{p,d}\|D^2u(\cdot,t)\|^{\frac{1}{2}}_{L^p(\mathbb{T}^d)} \|u(\cdot,t)\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^d)} \leq C\|D^2u(\cdot,t)\|^{\frac{1}{2}}_{L^p(\mathbb{T}^d)}.$$

For this reason, we have the bound

$$\||Du|^\gamma\|_{L^r(0,T_0;L^p(\mathbb{T}^d))} = \left(\int_0^{T_0} \|Du(\cdot,t)\|^{r\gamma}_{L^{p\gamma}(\mathbb{T}^d)} dt\right)^{\frac{1}{r}} \leq$$
$$C\left(\int_0^{T_0} \|D^2u(\cdot,t)\|^{\frac{r\gamma}{2}}_{L^p(\mathbb{T}^d)} dt\right)^{\frac{1}{r}} \leq C\left\|D^2u\right\|^{\frac{\gamma}{2}}_{L^r(0,T_0;L^p(\mathbb{T}^d))},$$

where we used that $\frac{\gamma}{2}<1$ again. Then, from Theorem 3 and standard regularity results for the heat equation (see, for instance, [**27**]), we have

$$\|u_t\|_{L^r(0,T_0;L^p(\mathbb{T}^d))} + \left\|D^2u\right\|_{L^r(0,T_0;L^p(\mathbb{T}^d))} \leq C\,\||Du|^\gamma\|_{L^r(0,T_0;L^p(\mathbb{T}^d))} + C\,\|m^\alpha\|_{L^r(0,T_0;L^p(\mathbb{T}^d))}$$
$$\leq C\left\|D^2u\right\|^{\frac{\gamma}{2}}_{L^r(0,T_0;L^p(\mathbb{T}^d))} + C\,\|m^\alpha\|_{L^r(0,T_0;L^p(\mathbb{T}^d))}.$$

Since $\frac{\gamma}{2} < 1$, we obtain

$$\|u_t\|_{L^r(0,T_0;L^p(\mathbb{T}^d))} C \left\|D^2 u\right\|_{L^r(0,T_0;L^p(\mathbb{T}^d))} \leq C \|m^\alpha\|_{L^r(0,T_0;L^p(\mathbb{T}^d))}.$$

The above arguments also imply that

$$\|Du\|_{L^{r\gamma}(0,T_0;L^{p\gamma}(\mathbb{T}^d))} \leq C \left\|D^2 u\right\|^{\frac{1}{2}}_{L^r(0,T_0;L^p(\mathbb{T}^d))} \leq C + C \|m^\alpha\|^{\frac{1}{2}}_{L^r(0,T_0;L^p(\mathbb{T}^d))} \leq C_{r,p}$$

for all $r, p > 1$.

We recall that $\alpha < \frac{d-2}{d}$. By Proposition 4, we have $\left\|m^{\alpha+1}\right\|_{L^\infty(0,T_0;L^1(\mathbb{T}^d))} \leq C$. Hence, for some $p_0 > \frac{d}{2}$, $\|m^\alpha\|_{L^\infty(0,T_0;L^{p_0}(\mathbb{T}^d))} \leq C$. Consequently, $\|Du\|_{L^r(0,T_0;L^{p_0\gamma}(\mathbb{T}^d))} \leq C_r$ for any $r > 1$. Note that $p_0\gamma/(\gamma-1) > \frac{d}{2}\gamma/(\gamma-1) > d$. Therefore, D_pH is bounded in $L^r(0,T_0;L^p(\mathbb{T}^d))$ for some $p > d$ and any $r > 1$. Finally, Lemma 2 implies that m is bounded in $L^\infty(0,T_0;L^q(\mathbb{T}^d))$ for any $q > 1$. □

LEMMA 4. *Under Assumptions 1-9, there exist a time $T_0 > 0$ and a constant $C > 0$ such that, for any C^∞ solution (u,m) to (1.1), we have $\|Du\|_{L^\infty(\mathbb{T}^d\times[0,T_0])} \leq C$.*

Proof. From Theorem 3 and Lemma 3, it follows that the equation for u can be written as

$$\begin{cases} u_t + \Delta u = f, \\ u(x,T_0) = \Psi(x), \end{cases}$$

where $f \in L^r(\mathbb{T}^d \times [0,T_0])$ for every $r > 1$. From this and by the same reasoning as in [**16**], we obtain

$$\|Du\|_{L^\infty(\mathbb{T}^d\times[0,T_0])} \leq C.$$

□

LEMMA 5. *Under Assumptions 1-9, there exist a time $T_0 > 0$ and a constant $C > 0$ such that, for any C^∞ solution (u,m) to (1.1), we have $\|m\|_{L^\infty(\mathbb{T}^d\times[0,T_0])}, \|Dm\|_{L^\infty(\mathbb{T}^d\times[0,T_0])} \leq C$.*

Proof. From the estimates in Theorem 3 and Lemmas 3 and 4, it follows that for suitable functions a and c, bounded in $L^p(\mathbb{T}^d \times [0,T_0])$ for every $p > 1$, the equation for m can be written as

$$\begin{cases} -m_t(x,t) + \Delta m(x,t) = a(x,t) \cdot Dm(x,t) + c(x,t)m(x,t), \\ m(x,0) = m_0(x). \end{cases}$$

Let $w = \ln m$. Then

$$\begin{cases} -w_t(x,t) + \Delta w(x,t) + |Dw|^2 - a(x,t) \cdot Dw(x,t) = c(x,t), \\ w(x,0) = \ln m_0(x). \end{cases}$$

The adjoint method, applied as in [**12**], yields $\|w\|_{L^\infty(\mathbb{T}^d\times[0,T_0])}, \|Dw\|_{L^\infty(\mathbb{T}^d\times[0,T_0])} \leq C$. These estimates imply the result. □

THEOREM 4. *Under Assumptions 1-9, there exist a time $T_0 > 0$ and constants $C_{k,l,p} > 0$, for $k, l \in \mathbb{N}$, $p > 1$ such that, for any C^∞ solution (u,m) to (1.1), we have $\|D_t^k D_x^l u\|_{L^p(\mathbb{T}^d\times[0,T_0])} \leq C_{k,l,p}$.*

Proof. The result follows from a simple bootstrapping argument. As a starting point, we use the regularity given by Theorem 3 and Lemmas 3, 4, and 5. Then, the Theorem is proved by repeatedly using the parabolic regularity on the equations for u, m, and their derivatives. □

6. *Existence of solutions*

To establish the existence of solutions, we use the continuation method. For that, we introduce the problem

$$\begin{cases} -u_t - \Delta u + m^\alpha H_\lambda\left(x, \frac{Du}{m^\alpha}\right) + b_\lambda \cdot Du = V_\lambda(x, m(x,t)), \\ m_t - \Delta m - \operatorname{div}(D_p H_\lambda\left(x, \frac{Du}{m^\alpha}\right) m) - \operatorname{div}(b_\lambda m) = 0, \\ u(x,T) = \Psi_\lambda(x),\ m(x,0) = m_\lambda(x), \end{cases} \tag{6.1}$$

where $0 \leq \lambda \leq 1$, $H_\lambda(x,p) = (1-\lambda)H_0 + \lambda(1+|p|^2)^{\frac{\gamma}{2}}$, $b_\lambda = (1-\lambda)b$, $V_\lambda = (1-\lambda)V + \lambda \arctan(m)$, $\Psi_\lambda = (1-\lambda)\Psi$, $m_\lambda = (1-\lambda)m_0 + \lambda$. The terminal time T satisfies $T \in \mathcal{T} = [0, T_0]$, where T_0 is as in Theorem 4.

When $\lambda = 1$, (6.1) has a unique solution, namely $u \equiv (1 - \frac{\pi}{4})t$, $m \equiv 1$. We prove that the set Λ of values $0 \leq \lambda \leq 1$ for which (6.1) admits a solution is relatively open and closed. Therefore, $\Lambda = [0,1]$ and, in particular, (1) admits a solution.

For $k \geq -1$, we set $F^k(\mathcal{T};\mathbb{T}^d) = \cap_{2k_1+k_2=k} H^{k_1}(\mathcal{T}; H^{k_2}(\mathbb{T}^d))$, where the intersection is taken over all integers, $k_1 \geq 0, k_2 \geq -1$. The space $F^k(\mathcal{T};\mathbb{T}^d)$ is a Banach space endowed with the norm

$$\|f\|_{F^k(\mathcal{T};\mathbb{T}^d)} = \sum_{2k_1+k_2=k} \|f\|_{H^{k_1}(\mathcal{T};H^{k_2}(\mathbb{T}^d))}.$$

Moreover, there exists $\tilde{k}_d$, depending only on the dimension d, such that, for $k \geq \tilde{k}_d$, the space F^{k-2} is an algebra. Let $k \geq \tilde{k}_d$ and consider the operator

$$\mathcal{M}_\lambda \colon F^k(\mathcal{T};\mathbb{T}^d) \times F^k(\mathcal{T};\mathbb{T}^d) \to F^{k-2}(\mathcal{T};\mathbb{T}^d) \times F^{k-2}(\mathcal{T};\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d)$$

given by

$$\mathcal{M}_\lambda \begin{bmatrix} u \\ m \end{bmatrix} = \begin{bmatrix} m_t - \Delta m - \operatorname{div}(D_p H_\lambda\left(x, \frac{Du}{m^\alpha}\right) m) + \operatorname{div}(b_\lambda m) \\ u_t + \Delta u - m^\alpha H_\lambda\left(x, \frac{Du}{m^\alpha}\right) - b_\lambda \cdot Du + V_\lambda(x,m) \\ m(x,0) - m_\lambda(x) \\ u(x,T) - \Psi_\lambda(x) \end{bmatrix}.$$

Then, (6.1) is equivalent to

$$\mathcal{M}_\lambda \begin{bmatrix} u \\ m \end{bmatrix} = 0, \tag{6.2}$$

and (1) then reads as $\mathcal{M}_0 \begin{bmatrix} u \\ m \end{bmatrix} = 0$. Moreover, as we remarked before, $\mathcal{M}_1 \begin{bmatrix} u \\ m \end{bmatrix} = 0$ has only the trivial solution $u \equiv (1 - \frac{\pi}{4})t$, $m \equiv 1$. We consider the linearized operator $\mathcal{L}$:

$$\mathcal{L}_\lambda \begin{bmatrix} v \\ f \end{bmatrix} = \lim_{\varepsilon \to 0} \frac{\mathcal{M}_\lambda \begin{bmatrix} u + \varepsilon v \\ m + \varepsilon f \end{bmatrix} - \mathcal{M}_\lambda \begin{bmatrix} u \\ m \end{bmatrix}}{\varepsilon} =$$

$$\begin{bmatrix} f_t - \Delta f - \operatorname{div}\left[D_p H_\lambda\left(x, \frac{Du}{m^\alpha}\right) f + m^{1-\alpha} D^2_{pp} H_\lambda\left(x, \frac{Du}{m^\alpha}\right) \cdot Dv - \alpha f D^2_{pp} H_\lambda\left(x, \frac{Du}{m^\alpha}\right) \cdot \frac{Du}{m^\alpha} + b_\lambda f\right] \\ v_t + \Delta v - \alpha m^{\alpha-1} f\left(H_\lambda\left(x, \frac{Du}{m^\alpha}\right) - \frac{Du}{m^\alpha} D_p H_\lambda\left(x, \frac{Du}{m^\alpha}\right)\right) - D_p H_\lambda\left(x, \frac{Du}{m^\alpha}\right) Dv - b_\lambda \cdot Dv + D_z V_\lambda f \\ f(x,0) \\ v(x,T) \end{bmatrix}.$$

Note that $\mathcal{L}_\lambda : F^k(\mathcal{T};\mathbb{T}^d) \times F^k(\mathcal{T};\mathbb{T}^d) \to F^{k-2}(\mathcal{T};\mathbb{T}^d) \times F^{k-2}(\mathcal{T};\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d)$, for all large enough k. However, if u and m are C^∞ solutions to (6.1), then $\mathcal{L}_\lambda$ admits a unique extension as a bounded linear operator $\mathcal{L}_\lambda : F^k(\mathcal{T};\mathbb{T}^d) \times F^k(\mathcal{T};\mathbb{T}^d) \to F^{k-2}(\mathcal{T};\mathbb{T}^d) \times F^{k-2}(\mathcal{T};\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d)$, for all $k \geq 1$.

The form $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(\mathbb{T}^d)$. To apply the inverse function theorem, we need to prove that the linear operator $\mathcal{L}_\lambda$ is invertible. For this, we begin by showing that the equation $\mathcal{L}_\lambda w = W$ has a unique weak solution in the sense of the following definition:

Definition 1. *For $h, g \in L^2(0, T_0; L^2(\mathbb{T}^d)), A, B \in L^2(\mathbb{T}^d)$, set*

$$W(x,t) = \begin{bmatrix} h(x,t) \\ g(x,t) \\ A(x) \\ B(x) \end{bmatrix}.$$

A function $w = \begin{bmatrix} v \\ f \end{bmatrix}$, with

$$v, f \in L^2(0, T_0; H^1(\mathbb{T}^d)) \text{ and } v_t, f_t \in L^2(0, T_0; H^{-1}(\mathbb{T}^d)), \text{ that is } v, f \in F^1(\mathcal{T}; \mathbb{T}^d), \tag{6.3}$$

is a weak solution of $\mathcal{L}_\lambda w = W$ if:
(i). *for any $\bar v, \bar f \in H^1(\mathbb{T}^d)$ and for a.e. t, $0 \le t \le T_0$, we have*

$$\begin{cases} \langle f_t, \bar f\rangle + \left\langle Df + D_pH_\lambda f + m^{1-\alpha} D^2_{pp}H_\lambda \cdot Dv - \alpha f D^2_{pp} H_\lambda \cdot Q + f b_\lambda, D\bar f\right\rangle = \langle h, \bar f\rangle \\ \langle v_t, \bar v\rangle - \langle Dv, D\bar v\rangle - \left\langle \alpha m^{\alpha-1} f\left(H_\lambda - Q\cdot D_pH_\lambda\right) + D_pH_\lambda \cdot Dv + b_\lambda \cdot Dv - D_z V_\lambda f, \bar v\right\rangle = \langle g, \bar v\rangle. \end{cases} \tag{6.4}$$

Here $Q = \frac{Du}{m^\alpha}$ and the Hamiltonian H_λ and its derivatives are evaluated at the point (x, Q).

(ii). *$f(x,0) = A(x), v(x, T_0) = B(x)$.*

Remark 6. *Note that (6.3) implies $v, f \in C(\mathcal{T}; L^2(\mathbb{T}^d))$ (see e.g., [**4**], Section 5.9.2, Theorem 3). Therefore, the traces $f(x,0)$ and $v(x,T_0)$ are well defined.*

Theorem 5 Uniqueness of weak solutions. *Let (u_λ, m_λ) be a C^∞ solution to (6.1), and let T_0 be as in Theorem 4. Then, under Assumptions 1–12, there exists at most one weak solution to the equation $\mathcal{L}_\lambda w = W$ in the sense of Definition 1.*

Proof. Since the equation $\mathcal{L}_\lambda w = W$ is linear, it is enough to prove that $\mathcal{L}_\lambda w = 0$ has only the trivial solution $w = 0$. For this, we take $\bar f = v, \bar v = f$ in (6.4). Adding both equations and integrating in time, we obtain

$$\begin{aligned} 0 = &\int_0^{T_0}\int_{\mathbb{T}^d} \Big[\alpha m^{\alpha-1} f^2 \left(Q\cdot D_pH_\lambda - H_\lambda\right) + m^{1-\alpha} Dv \cdot D^2_{pp} H_\lambda \cdot Dv \\ &- \alpha f Q \cdot D^2_{pp} H_\lambda \cdot Dv + D_z V_\lambda f^2\Big] dxdt = \\ &\int_0^{T_0}\int_{\mathbb{T}^d} \Big[\alpha m^{\alpha-1} f^2 \left(Q\cdot D_pH_\lambda - H_\lambda - \frac{\alpha}{4} Q\cdot D^2_{pp}H_\lambda \cdot Q\right) \\ &+ m^{\alpha-1}\left(m^{1-\alpha}Dv - \frac{\alpha}{2} fQ\right)^t \cdot D^2_{pp}H_\lambda \cdot \left(m^{1-\alpha}Dv - \frac{\alpha}{2}fQ\right) + D_zV_\lambda f^2\Big] dxdt, \end{aligned}$$

where we set $Q = \frac{Du}{m^\alpha}$. Using the estimates from Theorem 3, Lemma 4, Remark 4, and Assumption 12, we conclude that at the solution (u_λ, m_λ) to (6.1), there exist constants $\theta_1, \theta_2 > 0$ that do not depend on the solution and λ, such that the above expression is bounded from below by

$$\theta_1 \int_0^{T_0}\int_{\mathbb{T}^d} m^{\alpha-1}\left|m^{1-\alpha}Dv - \frac{\alpha}{2}fQ\right|^2 + \theta_2 |f|^2\, dxdt.$$

Thus, we get $f = 0, Dv = 0$. Consequently, $v \equiv v(t)$. Next, by looking at the second equation in (6.4), for $\bar{v} = v(t)$ and $\bar{f} = 0$, we obtain

$$\frac{d}{dt}\langle v, v\rangle = 0.$$

Using the boundary conditions for v, we conclude that $v = 0$. Therefore, $w = 0$. □

To prove the existence of weak solutions, we apply the Galerkin approximation method (see e.g., [**4**]). We consider a sequence of C^∞ functions $e_k = e_k(x),\ k \in \mathbb{N}$ such that $\{\ e_k\ \}_{k=1}^{\infty}$ is an orthogonal basis of $H^1(\mathbb{T}^d)$ and an orthonormal basis of $L^2(\mathbb{T}^d)$. We construct a sequence of finite dimensional approximations to weak solutions of (6.1) as follows: let $v_N, f_N \colon [0, T_0] \to H^1(\mathbb{T}^d)$, where

$$f_N(t) = \sum_{k=1}^{N} A_N^k(t) e_k, \quad v_N(t) = \sum_{k=1}^{N} B_N^k(t) e_k.$$

We show that we can select the coefficients A_n^k, B_N^k such that

$$\begin{cases} \langle f_N', e_k\rangle + \left\langle Df_N + f_N D_pH_\lambda + m^{1-\alpha} D_{pp}^2 H_\lambda \cdot Dv_N - \alpha f_N D_{pp}^2 H_\lambda \cdot Q + f_N b_\lambda, De_k\right\rangle \\ \qquad = \langle h, e_k\rangle, \\ \\ \langle v_N', e_k\rangle - \langle Dv_N, De_k\rangle \\ \qquad - \left\langle \alpha m^{\alpha-1} f_N \left(H_\lambda - Q\cdot D_pH_\lambda\right) + D_pH_\lambda \cdot Dv_N + b_\lambda \cdot Dv_N - D_z V_\lambda f_N, e_k\right\rangle = \langle g, e_k\rangle \end{cases} \tag{6.5}$$

and

$$A_N^k(0) = \langle A, e_k\rangle,\ B_N^k(T_0) = \langle B, e_k\rangle, \qquad k = 1, 2, \dots, N. \tag{6.6}$$

This system (6.5) is equivalent to:

$$\begin{cases} \dot{A}_N^k + \sum_{l=1}^N \left\langle De_l + e_l D_pH_\lambda - \alpha e_l D_{pp}^2 H_\lambda \cdot Q + e_l b_\lambda, De_k\right\rangle A_N^l \\ \qquad + \sum_{l=1}^N \langle m^{1-\alpha} D_{pp}^2 H_\lambda \cdot De_l, De_k\rangle B_N^l = \langle h, e_k\rangle, \\ \\ \dot{B}_N^k - \sum_{l=1}^N \langle De_l + D_pH_\lambda \cdot De_l + b_\lambda \cdot De_l, De_k\rangle B_N^l \\ \qquad - \sum_{l=1}^N \left\langle \alpha m^{\alpha-1} e_l \left(H_\lambda - Q\cdot D_pH_\lambda\right) - e_l D_z V_\lambda, e_k\right\rangle A_N^l = \langle g, e_k\rangle. \end{cases} \tag{6.7}$$

Because (6.7) is a linear system of ordinary differential equations (ODEs), the only difficulty in proving the existence of solutions concerns the boundary conditions (6.6). Existence is not immediate because half of the boundary conditions are given at the initial time, whereas the other half are given at the terminal time. From standard theory of ordinary differential equations, the initial value problem for (6.7) (that is, with $A_N^k(0)$ and $B_N^k(0)$ prescribed) has a unique solution. Hence, to prove the existence of solutions to (6.7), it is enough to show the existence of solutions for the corresponding homogeneous problem:

$$\begin{cases} \dot{\tilde{A}}_n^k + \sum_{l=1}^N \left\langle De_l + e_l D_pH_\lambda - \alpha e_l D_{pp}^2 H_\lambda \cdot Q + e_l b_\lambda, De_k\right\rangle \tilde{A}_N^l \\ \qquad + \sum_{l=1}^N \langle m^{1-\alpha} D_{pp}^2 H_\lambda \cdot De_l, De_k\rangle \tilde{B}_N^l = 0 \\ \\ \dot{\tilde{B}}_N^k - \sum_{l=1}^N \langle De_l + D_pH_\lambda \cdot De_l + b_\lambda \cdot De_l, De_k\rangle \tilde{B}_N^l \\ \qquad - \sum_{l=1}^N \left\langle \alpha m^{\alpha-1} e_l \left(H_\lambda - Q\cdot D_pH_\lambda\right) - e_l D_z V_\lambda, e_k\right\rangle \tilde{A}_N^l = 0, \end{cases} \tag{6.8}$$

with arbitrary $\tilde{A}_N^k(0)$ and $\tilde{B}_N^k(T_0)$, $1 \le k \le N$. Indeed, any solution to (6.7)-(6.6), (A, B) can be written as a sum of a particular solution to (6.7), $(\bar{A}, \bar{B})$, for instance with

$$\bar{A}_N^k(0) = 0,\ \bar{B}_N^k(0) = 0,\ k = 1, 2, \dots, N,$$

and a solution, $(\tilde{A}, \tilde{B})$, to (6.8) with suitable initial and terminal conditions such that (6.6) holds for $(A, B) = (\bar{A} + \tilde{A}, \bar{B} + \tilde{B})$.

Next, we regard the solution of the initial value problem for the homogeneous system corresponding to (6.7) as a linear operator on $\mathbb{R}^{2N}$:

$$(A_N(0), B_N(0)) \mapsto (A_N(0), B_N(T_0)). \tag{6.9}$$

We need to prove that this mapping is surjective. Since (6.9) is a linear mapping from $\mathbb{R}^{2N}$ to $\mathbb{R}^{2N}$, surjectivity is equivalent to injectivity. Therefore, it suffices to prove that the homogeneous system of ODEs corresponding to (6.7) subject to initial-terminal conditions $A_N(0) = B_N(T_0) = 0$ has only the trivial solution $A_N = B_N \equiv 0$. Let f_N, v_N solve (6.5) with $h = g \equiv 0, A = B \equiv 0$. From (6.5), we obtain (6.4) for $f = \bar{v} = f_N, v = \bar{f} = v_N$. Using the same argument as in Theorem 5, we conclude that $f_N = v_N \equiv 0$.

Next, we provide energy estimates for these approximations to ensure the weak convergence of approximate solutions through some subsequence.

THEOREM 6. *Suppose that Assumptions 1–12 hold. Then, for small enough T_0, there exists a constant C such that, for any C^∞ solution (u_λ, m_λ) to (6.1), we have*

$$\begin{aligned}\max_{0\leq t\leq T_0} \|(f_N, v_N)\|_{(L^2(\mathbb{T}^d))^2} + \|(f_N, v_N)\|_{(L^2(0,T_0;H^1(\mathbb{T}^d)))^2} + \|(f_N', v_N')\|_{(L^2(0,T_0;H^{-1}(\mathbb{T}^d)))^2} \\ \leq C\left(\|h\|_{L^2(0,T_0;L^2(\mathbb{T}^d))} + \|g\|_{L^2(0,T_0;L^2(\mathbb{T}^d))} + \|A\|_{L^2(\mathbb{T}^d)} + \|B\|_{L^2(\mathbb{T}^d)}\right).\end{aligned}$$

Proof. We assume that T_0 is small enough so that Theorem 4 holds. Using the linearity of (6.5), we observe that (6.4) holds for $f = \bar{f} = f_N, v_N = \bar{v}_N = v_N$. Then, using Hölder's inequality and the estimates from Theorem 4, we obtain the following system of inequalities:

$$\begin{cases} \left(\|f_N\|^2_{L^2(\mathbb{T}^d)}\right)_t + \|Df_N\|^2_{L^2(\mathbb{T}^d)} \leq C\left(\|h\|^2_{L^2(\mathbb{T}^d)} + \|Dv_N\|^2_{L^2(\mathbb{T}^d)} + \|f_N\|^2_{L^2(\mathbb{T}^d)}\right), \\ \left(\|v_N\|^2_{L^2(\mathbb{T}^d)}\right)_t - \|Dv_N\|^2_{L^2(\mathbb{T}^d)} \geq -C\left(\|g\|^2_{L^2(\mathbb{T}^d)} + \|v_N\|^2_{L^2(\mathbb{T}^d)} + \|f_N\|^2_{L^2(\mathbb{T}^d)}\right). \end{cases} \tag{6.10}$$

From the second inequality and using Gronwall's inequality, we get

$$\|v_N(\cdot, t)\|^2_{L^2(\mathbb{T}^d)} \leq C\int_t^{T_0} \left(\|g(\cdot, s)\|^2_{L^2(\mathbb{T}^d)} + \|f_N(\cdot, s)\|^2_{L^2(\mathbb{T}^d)}\right) ds + C\|B\|^2_{L^2(\mathbb{T}^d)},$$

and further

$$\int_0^{T_0} \|Dv_N(\cdot, s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C\int_0^{T_0} \left(\|g(\cdot, s)\|^2_{L^2(\mathbb{T}^d)} + \|f_N(\cdot, s)\|^2_{L^2(\mathbb{T}^d)}\right) ds + C\|B\|^2_{L^2(\mathbb{T}^d)}.$$

From this, combined with the first inequality in (6.10), and using Gronwall's inequality once more, we have

$$\begin{aligned}\|f_N(\cdot, t)\|^2_{L^2(\mathbb{T}^d)} \leq & C\int_0^{T_0} \left(\|g(\cdot, s)\|^2_{L^2(\mathbb{T}^d)} + \|h(\cdot, s)\|^2_{L^2(\mathbb{T}^d)} + \|f_N(\cdot, s)\|^2_{L^2(\mathbb{T}^d)}\right) ds \\ & + C\left(\|A\|^2_{L^2(\mathbb{T}^d)} + \|B\|^2_{L^2(\mathbb{T}^d)}\right).\end{aligned}$$

Thus, for small enough T_0, we get

$$\sup_{t\in[0,T_0]} \|f_N(\cdot, t)\|^2_{L^2(\mathbb{T}^d)} \leq C\int_0^{T_0} \left(\|g(\cdot, s)\|^2_{L^2(\mathbb{T}^d)} + \|h(\cdot, s)\|^2_{L^2(\mathbb{T}^d)}\right) ds + C\left(\|A\|^2_{L^2(\mathbb{T}^d)} + \|B\|^2_{L^2(\mathbb{T}^d)}\right).$$

Consequently,

$$\int_0^{T_0} \|Df_N(\cdot, t)\|^2_{L^2(\mathbb{T}^d)} \leq C\int_0^{T_0} \left(\|g(\cdot, s)\|^2_{L^2(\mathbb{T}^d)} + \|h(\cdot, s)\|^2_{L^2(\mathbb{T}^d)}\right) ds + C\left(\|A\|^2_{L^2(\mathbb{T}^d)} + \|B\|^2_{L^2(\mathbb{T}^d)}\right).$$

Thus, we have

$$\max_{0\le t\le T_0}\|(f_N,v_N)\|_{(L^2(\mathbb{T}^d))^2}+\|(f_N,v_N)\|_{(L^2(0,T_0;H^1(\mathbb{T}^d)))}$$
$$\le C\left(\|h\|_{L^2(0,T_0;L^2(\mathbb{T}^d))}+\|g\|_{L^2(0,T_0;L^2(\mathbb{T}^d))}+\|A\|_{L^2(\mathbb{T}^d)}+\|B\|_{L^2(\mathbb{T}^d)}\right).$$

From equation (6.4), for any $\bar{f},\bar{v}\in span\{e_k\}_{k=1}^N$ with $\|\bar{f}\|_{L^2(0,T_0;H^1(\mathbb{T}^d))}\le 1$, $\|\bar{v}\|_{L^2(0,T_0;H^1(\mathbb{T}^d))}\le 1$ we get

$$\begin{cases}\int_0^{T_0}\langle f_N(s),\bar{f}\rangle ds\le C\left(\|h\|^2_{L^2(0,T_0;L^2(\mathbb{T}^d))}+\|g\|^2_{L^2(0,T_0;L^2(\mathbb{T}^d))}+\|A\|^2_{L^2(\mathbb{T}^d)}+\|B\|^2_{L^2(\mathbb{T}^d)}\right),\\ \int_0^{T_0}\langle v_N(s),\bar{v}\rangle ds\le C\left(\|h\|^2_{L^2(0,T_0;L^2(\mathbb{T}^d))}+\|g\|^2_{L^2(0,T_0;L^2(\mathbb{T}^d))}+\|A\|^2_{L^2(\mathbb{T}^d)}+\|B\|^2_{L^2(\mathbb{T}^d)}\right),\end{cases}$$

because $f_N,v_N\in span\{e_k\}_{k=1}^N$ as well. These inequalities imply the required estimates. □

THEOREM 7 Existence of weak solutions. *Let (u_λ,m_λ) be a C^∞ solution to (6.1) and let T_0 be as in Theorem 4. Then, under Assumptions 1–12, there exists a weak solution to the equation $\mathcal{L}_\lambda w=W$ in the sense of (6.4).*

Proof. According to the energy estimates, there exist subsequences of v_N,f_N and functions $v,f\in L^2(0,T_0;H^1(\mathbb{T}^d))$, with $v'=v_t, f'=f_t\in L^2(0,T_0;H^{-1}(\mathbb{T}^d))$, such that

$$\begin{cases}v_N\rightharpoonup v,\ f_N\rightharpoonup f,\ \text{weakly in }\ L^2(0,T_0;H^1(\mathbb{T}^d))\\ v_N'\rightharpoonup v',\ f_N'\rightharpoonup f',\ \text{weakly in }\ L^2(0,T_0;H^{-1}(\mathbb{T}^d)).\end{cases}$$

For fixed N_0, let $\bar{v},\bar{f}\in span\{e_k\}_{k=1}^{N_0}$ with $\|\bar{v}\|_{L^2(0,T_0;H^1(\mathbb{T}^d))},\|\bar{f}\|_{L^2(0,T_0;H^1(\mathbb{T}^d))}\le 1$. According to the definition of v_N,f_N, we have that (6.4) holds for every $N\ge N_0$. Weak convergence then implies (6.4) for v,f and any $\bar{v},\bar{f}\in span\{e_k\}_{k=1}^{N_0}$. The above convergence implies that $v_N\rightharpoonup v,\ f_N\rightharpoonup f$ also in $C(0,T_0;L^2(\mathbb{T}^d))$. Therefore, the initial and terminal conditions on f,v hold as well. Since $\cup_{N\ge 1}span\{e_k\}_{k=1}^N$ is dense in $L^2(0,T_0;H^1(\mathbb{T}^d))$, the proof is complete. □

THEOREM 8 Higher Regularity. *Let (u_λ,m_λ) be a C^∞ solution of (6.1) and let T_0 be as in Theorem 4. Assume that $A,B\in H^{k+1}(\mathbb{T}^d),h,g\in F^{2k}(\mathcal{T};\mathbb{T}^d)$ and let $W=[h,g,A,B]^t$. Then, under Assumptions 1–12, for any weak solution $w=[f,v]^t$ of $\mathcal{L}_\lambda w=W$, we have $v,f\in F^{2k+2}(\mathcal{T};\mathbb{T}^d)$.*

The proof draws on the regularizing properties of the heat equation and a bootstrap argument. We use the following result:

LEMMA 6. *Let $\tilde{h}\in H^{k_1}(\mathcal{T};H^{k_2}(\mathbb{T}^d))$, $\tilde{g}\in H^{2k_1+k_2+1}(\mathbb{T}^d)$ for some $k_1,k_2\ge 0$, and let $\tilde{u}\in F^1(\mathcal{T};\mathbb{T}^d)$ be a weak solution of the heat equation*

$$\begin{cases}\tilde{u}_t-\Delta\tilde{u}=\tilde{h}\\ \tilde{u}(x,0)=\tilde{g}(x).\end{cases}$$

Then $\tilde{u}\in H^{k_1}(\mathcal{T};H^{k_2+2}(\mathbb{T}^d))\cap H^{k_1+1}(\mathcal{T};H^{k_2}(\mathbb{T}^d))$.

Proof. The Lemma is proved easily using induction. The base case $k_1=k_2=0$ is a standard regularity result for the heat equation. □

From the second equation of (6.4), we have that v is a weak solution to

$$\begin{cases} v_t + \Delta v = g \\ \quad +\alpha m^{\alpha-1} f \left(H_\lambda\left(x, \frac{Du}{m^\alpha}\right) + \frac{Du}{m^\alpha} D_p H_\lambda\left(x, \frac{Du}{m^\alpha}\right)\right) + D_p H_\lambda\left(x, \frac{Du}{m^\alpha}\right) Dv + b_\lambda \cdot Dv + D_z V_\lambda f \\ \\ v(x, T_0) = B(x). \end{cases} \tag{6.11}$$

Since the right-hand side of the previous PDE belongs to $L^2(0, T_0, L^2(\mathbb{T}^d))$, using Lemma 6, we conclude that $v \in L^2(\mathcal{T}; H^2(\mathbb{T}^d)) \cap H^1(\mathcal{T}; L^2(\mathbb{T}^d))$.

Next, the first equation of (6.4) implies that f is a weak solution to

$$\begin{cases} f_t - \Delta f = h \\ \quad + \operatorname{div}\left[D_p H_\lambda\left(x, \frac{Du}{m^\alpha}\right) f - \alpha f D^2_{pp} H_\lambda\left(x, \frac{Du}{m^\alpha}\right) \cdot \frac{Du}{m^\alpha} + m^{1-\alpha} D^2_{pp} H_\lambda\left(x, \frac{Du}{m^\alpha}\right) \cdot Dv + b_\lambda f\right] \\ \\ f(x, 0) = A(x). \end{cases} \tag{6.12}$$

From the regularity of v obtained above, we conclude that the right-hand side of this equation is also in $L^2(0, T_0, L^2(\mathbb{T}^d))$. For that reason, according to Lemma 6, $f \in L^2(\mathcal{T}; H^2(\mathbb{T}^d)) \cap H^1(\mathcal{T}; L^2(\mathbb{T}^d))$.

Now, we assume that $v, f \in F^{2i}(\mathcal{T}; \mathbb{T}^d)$ for some $i \leq k$. We prove that $v, f \in F^{2i+2}(\mathcal{T}; \mathbb{T}^d)$. First, note that since $v, f \in H^{k_1}(\mathcal{T}; H^{k_2}(\mathbb{T}^d))$ for every k_1, k_2 with $2k_1 + k_2 = 2i$, the expression on the right-hand side of (6.11) is in $H^{k_1}(\mathcal{T}; H^{k_2-1}(\mathbb{T}^d))$. Thus, using Lemma 6, we get $v \in H^{k_1}(\mathcal{T}; H^{k_2+1}(\mathbb{T}^d))$. We know now that the right-hand side of (6.11) is in $H^{k_1}(\mathcal{T}; H^{k_2}(\mathbb{T}^d))$. Using Lemma 6 the second time, we conclude that $v \in H^{k_1}(\mathcal{T}; H^{k_2+2}(\mathbb{T}^d)) \cap H^{k_1+1}(\mathcal{T}; H^{k_2}(\mathbb{T}^d))$.

Now, we have that the right-hand side of (6.12) is in $H^{k_1}(\mathcal{T}; H^{k_2-1}(\mathbb{T}^d))$. Thus, using Lemma 6 twice as above, we get $f \in H^{k_1}(\mathcal{T}; H^{k_2+2}(\mathbb{T}^d)) \cap H^{k_1+1}(\mathcal{T}; H^{k_2}(\mathbb{T}^d))$. From what we have proved, it follows that $v, f \in H^{\tilde{k}_1}(\mathcal{T}; H^{\tilde{k}_2}(\mathbb{T}^d))$, for every $\tilde{k}_1, \tilde{k}_2$ with $2\tilde{k}_1 + \tilde{k}_2 = 2i + 2$. Consequently, $v, f \in F^{i+2}(\mathcal{T}; \mathbb{T}^d)$.

Proof Proof of Theorem 1.

Theorem 4 and the Arzela-Ascoli Theorem imply that the set Λ is a closed subset of the interval $[0, 1]$. We prove that it is also open. Let $\lambda_0 \in \Lambda$. Using Theorem 4, we see that the operator

$$\mathcal{L}_{\lambda_0} \colon F^{2k}(\mathcal{T}; \mathbb{T}^d) \times F^{2k}(\mathcal{T}; \mathbb{T}^d) \to F^{2k-2}(\mathcal{T}; \mathbb{T}^d) \times F^{2k-2}(\mathcal{T}; \mathbb{T}^d) \times H^{2k-1}(\mathbb{T}^d) \times H^{2k-1}(\mathbb{T}^d)$$

is bounded for every $k \geq 1$. Using Theorems 5, 7, and 8, we conclude that $\mathcal{L}_{\lambda_0}$ is bijective. It is thus also invertible. We choose a large enough k such that $H^l(\mathcal{T}; H^l(\mathbb{T}^d))$, where $l = \lfloor \frac{2k}{3} \rfloor$, is an algebra. By the inverse function theorem ([**3**]), there is a neighborhood U of λ_0 where the equation $\mathcal{M}_\lambda \begin{bmatrix} u \\ m \end{bmatrix} = 0$ has a unique solution (u_λ, m_λ) in $F^{2k}(\mathcal{T}; \mathbb{T}^d) \times F^{2k}(\mathcal{T}; \mathbb{T}^d)$. Then, $u_\lambda, m_\lambda \in H^l(\mathcal{T}; H^l(\mathbb{T}^d))$. The inverse function theorem implies that the mapping $\lambda \mapsto (u_\lambda, m_\lambda)$ is continuous. Hence, we can assume that in the neighborhood U, m_λ is bounded away from zero. This observation, together with the fact that $H^l(\mathcal{T}; H^l(\mathbb{T}^d))$ is an algebra allows us to use the regularity theory and a bootstrap argument to conclude that (u_λ, m_λ) are C^∞. Accordingly, $U \subset \Lambda$. Consequently, we have proved that Λ is an open set in $[0, 1]$. Because $1 \in \Lambda$, we know that $\Lambda \neq \emptyset$. Therefore, $\Lambda = [0, 1]$. In particular, $0 \in \Lambda$. □

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King Abdullah University of Science and Technology (KAUST), CEMSE Division, Thuwal 23955-6900. Saudi Arabia, and KAUST SRI, Center for Uncertainty Quantification in Computational Science and Engineering.

diogo.gomes@kaust.edu.sa
vardan.voskanyan@kaust.edu.sa