Free-Space Optical Communications with Peak and Average Constraints: High SNR Capacity Approximation

Anas Chaaban, Jean-Marie Morvan, and Mohammad Slim Alouini

Abstract—The capacity of the intensity-modulation direct-detection (IM-DD) free-space optical channel with both average and peak intensity constraints is studied. A new capacity lower bound is derived by using a truncated-Gaussian input distribution. Numerical evaluation shows that this capacity lower bound is nearly tight at high signal-to-noise ratio (SNR), while it is shown analytically that the gap to capacity upper bounds is a small constant at high SNR. In particular, the gap to the high-SNR asymptotic capacity of the channel under either a peak or an average constraint is small. This leads to a simple approximation of the high SNR capacity. Additionally, a new capacity upper bound is derived using sphere-packing arguments. This bound is tight at high SNR for a channel with a dominant peak constraint.

I. INTRODUCTION

Due to its practical simplicity, low cost, and high speed, IM-DD free-space optical communication has attracted lots of research [1]–[3]. The IM-DD channel is often modeled as a Gaussian channel with post-detection noise [2], [4]–[7]. In this model, the input signal is a positive random variable representing the intensity of the optical signal, and the output is the input plus independent Gaussian noise. In addition to the non-negativity constraint, the input signal is also restricted by a peak and an average constraint due to safety and practical considerations [8]. Although this channel is an additive Gaussian channel, the non-negativity, peak, and average constraints render it different from the intensively studied Gaussian channel with a second-moment constraint.

The fundamental limit of information transmission over this channel, in terms of bits per symbol, is given by the channel capacity. To express this capacity, one has to find the optimal input distribution. While the optimal input distribution for this channel is known to be discrete [7], a closed-form capacity expression is still to be found. Such a closed-form expression is important for extending capacity results to practical scenarios involving fading or parallel channels. Several upper and lower bounds on this capacity have been derived earlier [4], [5], and those bounds are tight in some regimes. For instance, the bounds in [4] meet at high and low SNR. The best achievable

rate to-date (not necessarily optimal) was given in [5] using a discrete input distribution with equally spaced points. This distribution achieves the low SNR capacity of the channel, which was characterized in [4].

In this paper, we derive a capacity lower bound for an IM-DD channel with both average and peak constraints, by using a truncated-Gaussian input distribution. The advantage of this distribution is that it leads to an achievable rate that can be expressed in closed-form. Furthermore, it achieves higher rates than the exponential distribution in [4]. Although it has lower rate than [5], it is easier to compute. Thus, it can be considered as a lower bound which is between the lower bounds of [4] and [5] both in terms of rate and computation complexity. Then, we show that this distribution is nearly optimal at high SNR. Namely, for a peak-to-average ratio (PAR) smaller than 2, the gap to capacity upper bounds is zero similar to [4]. For a PAR larger than 2, this gap is bounded analytically by 0.68 nats, while numerically it converges to zero as SNR increases. Based on this observation, we claim that a truncated-Gaussian input distribution is nearly optimal at high SNR.

We also provide a new capacity upper bound by interpreting the problem of capacity characterization under a peak constraint as a problem of sphere-packing in a cube. Based on this interpretation, we derive a capacity upper bound using the Steiner-Minkowski formula for polytopes [9]–[11]. The advantage of this bound is that it has a simpler expression than the one derived in [4]. As a side result, we also provide a simple alternative derivation of an upper bound given in [4], by assuming a Gaussian input with maximum variance.

The paper is organized as follows. The channel model is introduced in the Sec. II. The achievable rate by a truncated-Gaussian distribution is given in Sec. III, a new capacity upper bound is given in Sec. IV, and a numerical evaluation is given in Sec. V. The paper is concluded in Sec. VI.

Throughout the paper, we use normal-face font to denote scalars, and bold-face to denote vectors. We use $g_{\mu,\sigma}(x)$ to denote the Gaussian distribution on a random variable X with mean μ and variance σ^2, and $G_{\mu,\sigma}(x)$ to denote the corresponding cumulative distribution function. We also use $V(\cdot)$ to denote the volume of an object.

II. THE IM-DD CHANNEL

The IM-DD channel models a scenario where light intensity is used to send information from a source to a destination. The input to the channel is a random variable $X \geq 0$ which

represents the intensity of the optical signal. This intensity is constrained due to practical and safety restrictions [8] by average and peak constraints, i.e., $\mathbb{E}[X] \leq \mathcal{E}$ and $X \leq \mathcal{A}$. Clearly, $\mathcal{E} < \mathcal{A}$.

To send a message $w \in \{1, \cdots, M\}$, the source encodes it into $\mathbf{X}(w) = (X_1(w), \cdots, X_n(w))$ for some $n \in \mathbb{N}$. Here, $X_i(w)$ are realizations of the random variable X. An intensity detector at the destination detects $\mathbf{Y} = \mathbf{X}(w) + \mathbf{Z}$ where the instances of the noise $\mathbf{Z}$ are independent and identically distributed according to $g_{0,\sigma}(z)$.[1] We say that this channel has high SNR if $\frac{\mathcal{E}}{\sigma}$ is both large.

The destination uses a decoder to recover $\hat{w} \in \{1, \cdots, M\}$ from the received signal $\mathbf{Y} = (Y_1, \cdots, Y_n)$. This encoding-decoding scheme is associated with an error probability P_e given by the probability that $\hat{w} \neq w$. The rate of this transmission $R = \frac{\log(M)}{n}$ is said to be achievable if there exist a coding scheme such that $P_e \to 0$ when $n \to \infty$. The highest achievable R is the capacity $\mathcal{C}$ of the given IM-DD channel, which is the main focus of this paper.

III. TRUNCATED-GAUSSIAN INPUT DISTRIBUTION

In this section, we derive a capacity lower bound which has a simple expression and allows simple comparison with the upper bounds. A Gaussian input distribution is good for this purpose, but it is not feasible in our case due to the non-negativity and peak constraints. To circumvent this problem, we propose the use of a truncated-Gaussian input distribution.

A. Truncated-Gaussian Input Distribution

A truncated-Gaussian distribution is described the by probability density function

$$\tilde{g}_{\mu,\nu}(x) = \begin{cases} \eta g_{\mu,\nu}(x), & x \in [0, \mathcal{A}], \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where $\eta = [G_{\mu,\nu}(\mathcal{A}) - G_{\mu,\nu}(0)]^{-1}$. The mean of this distribution is

$$\tilde{\mu} = \nu^2(\tilde{g}_{\mu,\nu}(0) - \tilde{g}_{\mu,\nu}(\mathcal{A})) + \mu. \quad (2)$$

B. Achievable Rate

The achievable rate of a truncated-Gaussian input distribution satisfying both average and peak constraints, is given in the following theorem.

Theorem 1: The capacity $\mathcal{C}$ satisfies $\mathcal{C} \geq \mathcal{R}$ where $\mathcal{R} = \mathcal{C}_0(\nu) - \Phi_1(\mu, \nu) - \Phi_2(\mu, \nu) - \Phi_3(\mu, \nu)$ for parameters μ and ν satisfying $\tilde{\mu} \leq \mathcal{E}$, with $\mathcal{C}_0(\nu) = \frac{1}{2} \log \left(1 + \frac{\nu^2}{\sigma^2}\right)$, $\Phi_1(\mu, \nu) = \log(\eta)$,

$$\Phi_2(\mu, \nu) = ((\mathcal{A} - \mu)\tilde{g}_{\mu,\nu}(\mathcal{A}) + \mu\tilde{g}_{\mu,\nu}(0)) \frac{\nu^2}{2(\nu^2 + \sigma^2)}, \quad (3)$$

$$\Phi_3(\mu, \nu) = \mathbb{E}_{X,Y} \left[\log(G_{\mu',\nu'}(\mathcal{A}) - G_{\mu',\nu'}(0))\right], \quad (4)$$

$\mu' = \frac{\mu\sigma^2 + y\nu^2}{\nu^2 + \sigma^2}$, and $\nu' = \frac{\nu\sigma}{\sqrt{\nu^2 + \sigma^2}}$, and where the expectation in $\Phi_3(\mu, \nu)$ is with respect to $f(x, y) = \tilde{g}_{\mu,\nu}(x)g_{x,\sigma}(y)$ over $x \in [0, \mathcal{A}]$ and $y \in \mathbb{R}$.

Proof: We consider a truncated-Gaussian input distribution $\tilde{g}_{\mu,\nu}(x)$ for $x \in [0, \mathcal{A}]$, for some μ and ν satisfying $\tilde{\mu} \leq \mathcal{E}$. Given this distribution, we can express the achievable rate as $R = I(X; Y)$ [14] where

$$I(X; Y) = \int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x, y) \log \left(\frac{f(y|x)}{f(y)}\right) dy dx \quad (5)$$

$$= \underbrace{\int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x, y) \log (f(y|x)) dy dx}_{I_1}$$

$$\underbrace{- \int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x, y) \log (f(y)) dy dx}_{I_2}. \quad (6)$$

Here, $f(x, y) = \tilde{g}_{\mu,\nu}(x)f(y|x)$ is the joint distribution of (X, Y), $f(y|x) = g_{x,\sigma}(y)$ is the conditional distribution of Y given X, and $f(y)$ is the marginal distribution of the channel output Y. The distribution of Y can be calculated as follows

$$f(y) = \int_0^{\mathcal{A}} f(x)f(y|x)dx = \eta g_{\mu,\sigma_y}(y)[G_{\mu',\nu'}(\mathcal{A}) - G_{\mu',\nu'}(0)]$$

where $\sigma_y = \sqrt{\nu^2 + \sigma^2}$, and μ' and ν' are as defined in the statement of the theorem. Note that $I_1 = -h(Z) = -\frac{1}{2} \log(2\pi e \sigma^2)$. By substituting $f(y)$ in I_2, we can show that $I_2 = -\frac{1}{2} \log \left(2\pi e \sigma_y^2\right) + \sum_{i=1}^{3} \Phi_i(\mu, \nu)$, where $\Phi_i(\mu, \nu)$, $i \in \{1, 2, 3\}$ are as defined in the statement of the theorem (see Appendix A). This proves the achievability of $\mathcal{R}$. ∎

The difficulty in computing $\mathcal{R}$ arises due to the term $\Phi_3(\mu, \nu)$. However, note that this term is always negative, and hence it always increases the achievable rate. This leads to the following lower bound which is rather easier to compute.

Corollary 1: The capacity $\mathcal{C}$ satisfies $\mathcal{C} \geq \mathcal{R}'$ where $\mathcal{R}' = \mathcal{C}_0(\nu) - \Phi_1(\mu, \nu) - \Phi_2(\mu, \nu)$ for parameters μ and ν satisfying $\tilde{\mu} \leq \mathcal{E}$, with $\mathcal{C}_0(\nu)$, $\Phi_1(\mu, \nu)$ and $\Phi_2(\mu, \nu)$ as defined in Theorem 1.

Proof: By Theorem 1 and the negativity of $\Phi_3(\mu, \nu)$. ∎

The lower bound $\mathcal{R}'$ is easier to compute, for a given μ, and ν, than the lower bound in [5] (denoted here $\mathcal{R}_F$), for a given number of mass points K. We denote the latter bound . The two bounds share a common difficulty which is the choice of μ and ν for $\mathcal{R}'$ and and the choice of K for $\mathcal{R}_F$. Those parameters are optimized numerically. It is worth to note that the lower bounds in [4] are the easiest to compute.

C. Simplification at High SNR

For the sake of comparison with upper bounds, the achievable rate $\mathcal{R}'$ can be further simplified at high SNR as follows

$$\mathcal{R}' \geq \tilde{\mathcal{R}} \approx \mathcal{C}_0(\nu)$$

by choosing $\tilde{\mu} = \min \left\{\frac{\mathcal{A}}{2}, \mathcal{E}\right\}$ and $\nu = \mu/3$. This choice leads to $\mu \approx \tilde{\mu}$, and therefore, $\nu \approx \frac{1}{3} \min \left\{\frac{\mathcal{A}}{2}, \mathcal{E}\right\}$. With this choice, we get $\Phi_1(\mu, \nu)$ and $\Phi_2(\mu, \nu)$ which are both negligible with respect to $\mathcal{C}_0(\nu)$ at high SNR. This leads to the following inequality at high SNR

$$\mathcal{R} \geq \mathcal{R}' \geq \tilde{\mathcal{R}} = \mathcal{C}_0 \left(\min \left\{\frac{\mathcal{A}}{6}, \frac{\mathcal{E}}{3}\right\}\right). \quad (7)$$

---

[1] Note that we consider a discrete-time channel model similar to [4], [5] rather than the continuous time model of [12], [13].

The simple achievable rate $\tilde{\mathcal{R}}$ is suitable for comparison with capacity upper bounds. In [4], it was shown that the high-SNR capacity of a peak-constrained channel is upper bounded by $\frac{1}{2}\log\left(\frac{\mathcal{A}^2}{2\pi e\sigma^2}\right)$. In the same paper, it was shown that the high-SNR capacity of an average-constrained channel is upper bounded by $\frac{1}{2}\log\left(\frac{e\mathcal{E}^2}{2\pi\sigma^2}\right)$.[2] By combining these two bounds, we have the high-SNR capacity upper bound for the average- and peak-constrained channel $\mathcal{C}_{\text{High SNR}} \leq \overline{\mathcal{C}}_{\text{High SNR}}$ where

$$\overline{\mathcal{C}}_{\text{High SNR}} = \min\left\{\frac{1}{2}\log\left(\frac{\mathcal{A}^2}{2\pi e\sigma^2}\right), \frac{1}{2}\log\left(\frac{e\mathcal{E}^2}{2\pi\sigma^2}\right)\right\}. \quad (8)$$

Using (7), we conclude

$$\overline{\mathcal{C}}_{\text{High SNR}} - \mathcal{R} \leq \overline{\mathcal{C}}_{\text{High SNR}} - \tilde{\mathcal{R}} \leq \frac{1}{2}\log\left(\frac{9e}{2\pi}\right),$$

if $\mathcal{E} \leq \frac{\mathcal{A}}{2}$, i.e., the achievable rate $\mathcal{R}$ is within $\frac{1}{2}\log\left(\frac{9e}{2\pi}\right) < 0.68$ nats of the upper bound. This gap can be reduced to ≈ 0 by numerically optimizing with respect to μ and ν as we shall see in Section V. On the other hand, for $\mathcal{E} \geq \frac{\mathcal{A}}{2}$ we have

$$\overline{\mathcal{C}}_{\text{High SNR}} - \mathcal{R} \leq \overline{\mathcal{C}}_{\text{High SNR}} - \tilde{\mathcal{R}} \leq \frac{1}{2}\log\left(\frac{36}{2\pi e}\right),$$

i.e., $\mathcal{R}$ is within $\frac{1}{2}\log\left(\frac{36}{2\pi e}\right) < 0.38$ nats of $\overline{\mathcal{C}}_{\text{High SNR}}$ in this case. This gap can be reduced to zero by choosing $\mu = \frac{\mathcal{A}}{2}$ and $\nu = \mathcal{A}$ and evaluating $\mathcal{R}'$ as given in Corollary 1.

IV. CAPACITY UPPER BOUNDS

Here, we derive an upper bound for a peak-constrained channel using a sphere-packing approach. The resulting bound is close to the bound given in [4, (20)] but has a simpler expression, and is derived using an approach similar to [6].

A. Peak-constrained Channel

The capacity of a peak-constrained channel can be interpreted as a problem of packing spheres in a cube using similar arguments as [16]. Since a codeword $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ satisfies the constraint $0 \leq X_i \leq \mathcal{A} \ \forall i = 1, \cdots, n$, this codeword is confined to an n-dimensional cube with side-length $\mathcal{A}$. We call this n-cube $\mathcal{W}_{\mathcal{A}}^n$. On the other hand, the noise $\mathbf{Z} = (Z_1, \cdots, Z_n)$ satisfies $E[Z_i^2] = \sigma^2$. By the law of large numbers, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for $n \geq N$, $\left|\frac{1}{n}\sum_{i=1}^{n} Z_i^2 - \mathbb{E}[Z^2]\right| < \epsilon$. Therefore, $\mathbf{Z}$ has a Euclidean norm which approaches

$$\lambda \triangleq \sqrt{n\sigma^2}$$

for large n. This confines the noise to the shell of an n-dimensional ball of radius λ, denoted $\mathcal{B}_{\lambda}^n$.

Now to guarantee a vanishing probability of error, the codewords $\mathbf{X}$ have to be distributed within $\mathcal{W}_{\mathcal{A}}^n$ in such a way that the noise balls centered at those codewords do not intersect. Thus, we need to upper bound the number of $\mathcal{B}_{\lambda}^n$ that can be packed with their centers inside $\mathcal{W}_{\mathcal{A}}^n$. The limit of this upper bound $\overline{M}_n$ as $n \to \infty$ will be an upper bound on the number of codewords that can be sent over our IM-DD channel, and a capacity upper bound can be obtained as
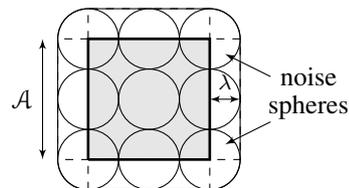
Fig. 1: Packing spheres with centers inside a cube. The convex envelope around the cube is its λ-neighborhood.

$\mathcal{C}_{\mathcal{A}} \leq \lim_{n\to\infty} \frac{\log(\overline{M}_n)}{n}$. Here, we use the sub-script $\mathcal{A}$ to indicate that this is the capacity of the channel with only a peak intensity constraint $\mathcal{A}$.

The noise balls extend the cube to the so-called λ-neighborhood of $\mathcal{W}_{\mathcal{A}}^n$, which we denote by $\mathcal{W}_{\mathcal{A}}^n(\lambda)$ (see Fig. 1 for a 2-dimensional example). An upper bound $\overline{M}_n$ can be derived by calculating the volume $V(\mathcal{W}_{\mathcal{A}}^n(\lambda))$ and then dividing by $V(\mathcal{B}_{\lambda}^n)$.

The volume $V(\mathcal{W}_{\mathcal{A}}^n(\lambda))$ can be calculated using the Steiner-Minkowski theorem for polytopes [9, Proposition 12.3.6]. According to this theorem, the volume $V(\mathcal{W}_{\mathcal{A}}^n(\lambda))$ can be written as a polynomial of degree n in λ, given by

$$V(\mathcal{W}_{\mathcal{A}}^n(\lambda)) = \sum_{i=0}^{n} L_i(\mathcal{W}_{\mathcal{A}}^n)\lambda^i, \quad (9)$$

for some coefficients $L_i(\mathcal{W}_{\mathcal{A}}^n)$ depending on the geometry of $\mathcal{W}_{\mathcal{A}}^n$. Thus, for evaluating $V(\mathcal{W}_{\mathcal{A}}^n(\lambda))$, it remains to determine the coefficients $L_i(\mathcal{W}_{\mathcal{A}}^n)$. These coefficients were given in [10] for any convex body $\mathcal{T}$ as

$$L_i(\mathcal{T}) = \sum_{\mathcal{T}^{n-i} \in \partial\mathcal{T}} V(\mathcal{T}^{n-i})V(\mathcal{B}_1^i)\theta_{\mathcal{T}^{n-i},\mathcal{T}}, \quad (10)$$

where $\mathcal{T}^{n-i}$ is a generic $n-i$ dimensional face of $\partial\mathcal{T}$ the boundary of $\mathcal{T}$, and $\theta_{\mathcal{T}^{n-i},\mathcal{T}}$ is the normalized dihedral external angle of $\mathcal{T}^{n-i}$ in $\mathcal{T}$ (see [10] for the definition of this angle).

Consider Fig. 1 which shows $\mathcal{W}_{\mathcal{A}}^2(\lambda)$ for example. The extension of the square by λ extends each corner to a quarter of $\mathcal{B}_{\lambda}^2$, and each edge to a half of a 2-dimensional cylinder (rectangle) of length $\mathcal{A}$ and radius λ. From this point of view, we can write

$$V(\mathcal{W}_{\mathcal{A}}^2(\lambda)) = \mathcal{A}^2 + 4(2^{-1})\mathcal{A}^1(2\lambda) + 4(2^{-2})\mathcal{A}^0(\pi\lambda^2)$$

$$= \sum_{i=0}^{n} 2^i\binom{n}{n-i}2^{-i}\mathcal{A}^{n-i}V(\mathcal{B}_{\lambda}^i),$$

with $n = 2$. In this expression, $2^i\binom{n}{n-i}$ is the number of $n-i$ dimensional faces of $\mathcal{W}_{\mathcal{A}}^n$ [9], 2^{-i} is the normalized dihedral external angle of the $n-i$ dimensional face in $\mathcal{W}_{\mathcal{A}}^n$, and $\mathcal{A}^{n-i}V(\mathcal{B}_{\lambda}^i)$ is the volume of the cylinder formed by the orthogonal product of the $n-i$ dimensional face (which is $\mathcal{W}_{\mathcal{A}}^{n-i}$ [9]) and $\mathcal{B}_{\lambda}^i$. By extending this argument to higher dimensions, we obtain

$$L_i(\mathcal{W}_{\mathcal{A}}^n)\lambda^i = \binom{n}{n-i}\mathcal{A}^{n-i}V(\mathcal{B}_{\lambda}^i), \quad (11)$$

as a special case of (10) for a cube. Based on this, we obtain the following upper bound.

Theorem 2: The capacity $\mathcal{C}$ satisfies $\mathcal{C} \leq \mathcal{C}_\mathcal{A} \leq \overline{\mathcal{C}}_\mathcal{A}$ where $\overline{\mathcal{C}}_\mathcal{A} = \sup_{\alpha \in [0,1]} B_\mathcal{A}(\alpha)$, and

$$B_\mathcal{A}(\alpha) = (1-\alpha) \log\left(\frac{\mathcal{A}}{\sqrt{2\pi e}\sigma}\right) - \log\left((1-\alpha)^{1-\alpha}\alpha^{\frac{3\alpha}{2}}\right).$$

Proof: The upper bound $\mathcal{C} \leq \mathcal{C}_\mathcal{A}$ clearly follows since dropping a constraint does not reduce capacity. The upper bound $\mathcal{C}_\mathcal{A} \leq \overline{\mathcal{C}}_\mathcal{A}$ is obtained as follows. First, we replace $V(\mathcal{B}^i_\lambda)$ in (11) by $(\sqrt{\pi}\lambda)^i/\Gamma\left(1 + \frac{i}{2}\right)$ ($\Gamma(\cdot)$ is the Gamma function), and we replace λ by $\sqrt{n\sigma^2}$. Then we substitute $L_i(\mathcal{W}^n_\mathcal{A})\lambda^i$ from (11) in the expression of $V(\mathcal{W}^n_\mathcal{A}(\lambda))$ (9), and divide by $V(\mathcal{B}^n_\lambda)$. This leads to the upper bound $\overline{M}_n = \sum_{i=0}^n N_i$, where $N_i = \binom{n}{n-i}\left(\frac{\mathcal{A}}{\sqrt{\pi}\lambda}\right)^{n-i}\frac{\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(1+\frac{i}{2}\right)}$. The rest of the proof is based on $\mathcal{C}_\mathcal{A} \leq \lim_{n\to\infty}\frac{1}{n}\log(\overline{M}_n)$. This limit is equal to $\sup_{\alpha \in [0,1]} B_\mathcal{A}(\alpha)$, which can be proved along the same steps as in [6]. ∎

B. Average and Peak Constraints

Here, we provide an alternative derivation of one of the bounds in [4]. This upper bound is given as follows.

Theorem 3 ([4, (11) & (19)]): The capacity $\mathcal{C}$ satisfies $\mathcal{C} \leq \overline{\mathcal{C}}$ where $\overline{\mathcal{C}} = \frac{1}{2}\log\left(1 + \frac{\mathcal{A}^2}{4\sigma^2}\right)$ if $\mathcal{E} \geq \frac{\mathcal{A}}{2}$ and $\overline{\mathcal{C}} = \frac{1}{2}\log\left(1 + \frac{\mathcal{E}(\mathcal{A}-\mathcal{E})}{\sigma^2}\right)$ otherwise.

This upper bound was derived in [4] using a duality-bounding approach. It can be derived alternatively as follows. The capacity is given by $\mathcal{C} = \max_{f(x)} I(X;Y)$ [14], where $f(x)$ denotes the probability distribution of X, and the maximization is over all $f(x)$ satisfying $X \in [0,\mathcal{A}]$ and $\mathbb{E}[X] \leq \mathcal{E}$. By relaxing the first constraint to $X \in \mathbb{R}$ and assuming that X satisfies $\mathrm{Var}(X) \leq P$ for some $P > 0$, the capacity maximizing input distribution is Gaussian $g_{0,\sqrt{P}}(x)$, and we have $\mathcal{C} \leq \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$. It remains to show that the variance of the input is bounded. The maximum variance of the distribution of $X \in [0,\mathcal{A}]$ that satisfies $\mathbb{E}[X] = \mu \leq \mathcal{E}$ is achieved when X follows a Bernoulli distribution $f_\mu(\mathcal{A}) = \frac{\mu}{\mathcal{A}}$ and $f_\mu(0) = 1 - \frac{\mu}{\mathcal{A}}$. In particular, for any distribution $f(x)$ of $X \in [0,\mathcal{A}]$ with mean μ, the variance satisfies

$$\mathrm{Var}(X) = \int_0^\mathcal{A} (x-\mu)^2 f(x)dx \tag{12}$$

$$\leq \int_0^\mathcal{A} \left[\frac{x}{\mathcal{A}}(\mathcal{A}-\mu)^2 + \left(1 - \frac{x}{\mathcal{A}}\right)\mu^2\right] f(x)dx \tag{13}$$

$$= \mu(\mathcal{A}-\mu), \tag{14}$$

with equality if X is distributed according to $f_\mu(x)$. Now since $\mu \leq \mathcal{E}$, then the variance of the distribution $f_\mu(x)$ is maximized if $\mu = \frac{\mathcal{A}}{2}$ when $\mathcal{E} \geq \frac{\mathcal{A}}{2}$ and $\mu = \mathcal{E}$ otherwise. This leads to the same bound derived in [4] as stated in Theorem 3.

V. NUMERICAL EVALUATION

Bounds on $\mathcal{C}$ are plotted in Fig. 2 and 3 for $\mathcal{E} = \frac{\mathcal{A}}{10}$ and $\mathcal{E} = \frac{\mathcal{A}}{2}$. It can be seen from these figures that the achievable rate $\mathcal{R}$ is higher than $\mathcal{R}_{L1}$ given in [4, (10)]. Furthermore, $\mathcal{R}$ approaches capacity at high SNR. In fact, the lower bound $\mathcal{R}'$ also approaches the high-SNR capacity as we shall see next.
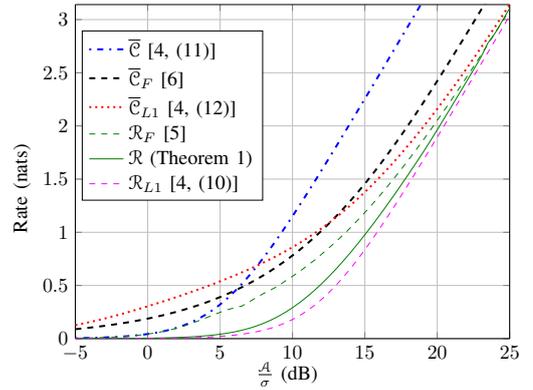

Fig. 2: Capacity upper and lower bounds for $\mathcal{E} = \frac{\mathcal{A}}{10}$.
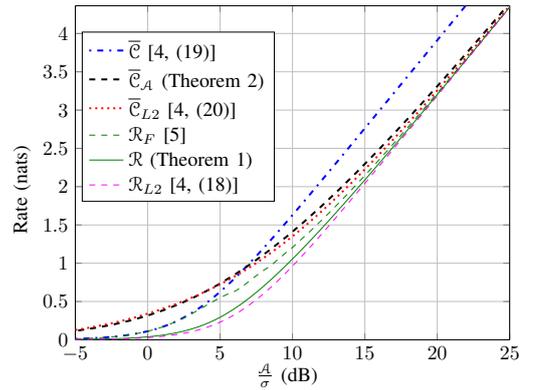

Fig. 3: Capacity upper and lower bounds for $\mathcal{E} = \frac{\mathcal{A}}{2}$.

Note that the lower bound $\mathcal{R}_F$ given in [5] is the tightest. However, $\mathcal{R}_F$ which is achievable by using a discrete input distribution does not have a closed-form expression. From this point of view, $\mathcal{R}'$ has two advantages: being nearly tight at high SNR, and being easier to compute and express than $\mathcal{R}_F$.

It can also be seen that $\overline{\mathcal{C}}_\mathcal{A}$ is close to $\overline{\mathcal{C}}_{L2}$ given in [4, (20)] for $\mathcal{E} \geq \frac{\mathcal{A}}{2}$. However, $\overline{\mathcal{C}}_\mathcal{A}$ is also easier to compute than $\overline{\mathcal{C}}_{L2}$, and has a simple sphere-packing interpretation. Fig. 3 represents all scenarios with $\mathcal{A} \leq 2\mathcal{E}$ (including the case with only a peak constraint) since they have the same capacity [4].

Fig. 4 shows the gap $\Delta = \overline{\mathcal{C}}_{\text{High SNR}} - \mathcal{R}'$ at high SNR as a function of $\frac{\mathcal{E}}{\mathcal{A}}$. The rate $\mathcal{R}'$ given in Corollary 1 is optimized numerically. It can be seen that this gap is smaller than 0.1 nats for any $\frac{\mathcal{E}}{\mathcal{A}} \in (0,1]$. On the same figure, the gap $\Delta' = \min\{\overline{\mathcal{C}}_{\text{High SNR}}, \overline{\mathcal{C}}_{L1}\} - \mathcal{R}'$ is plotted, where $\overline{\mathcal{C}}_{L1}$ is the upper bound given in [4, (12)]. This gap is close to zero. This indicates that the truncated-Gaussian distribution is nearly optimal at high SNR. Based on this numerical evaluation, we state the following.

Claim 1: The high SNR capacity can be approximated as $\mathcal{C}_{\text{High SNR}} \approx \overline{\mathcal{C}}_{\text{High SNR}}$ given in (8) where the approximation gap is < 0.1 nats.

This simple high-SNR capacity approximation can be useful for extending IM-DD capacity results from the P2P configuration to more general configurations.
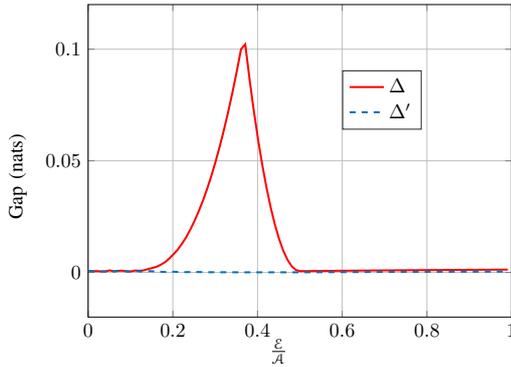
Fig. 4: The gap between $\overline{\mathcal{C}}_{\text{High SNR}}$ and $\mathcal{R}'$ for a channel with $\frac{\mathcal{E}}{\sigma} = 50$ dB.

VI. Conclusion

We have derived the achievable rate by a truncated-Gaussian input distribution in an IM-DD channel with both average and peak constraints. This achievable rate leads to a simple capacity lower bound. We prove that the gap between this lower bound and capacity upper bounds is upper bounded by a small constant at high SNR. This small gap is further shown to vanish when the parameters of the truncated-Gaussian distribution are optimized numerically. As a conclusion, the truncated-Gaussian input distribution is nearly optimal at high SNR. This leads to a simple high-SNR capacity approximation, with negligible gap to capacity. A new capacity upper bound is also derived by using sphere-packing arguments.

Appendix A
Derivation of I_2 in the Proof of Theorem 1

We start by substituting

$$f(y) = \eta g_{\mu,\sigma_y}(y)(G_{\mu',\nu'}(\mathcal{A}) - G_{\mu',\nu'}(0))$$

in the integral

$$I_2 = \int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x,y) \log (f(y)) \, dy dx \qquad (15)$$

to obtain

$$I_2 = \underbrace{\int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x,y) \log (\eta) \, dy dx}_{I_{21}}$$
$$+ \underbrace{\int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x,y) \log \left(g_{\mu,\sigma_y}(y)\right) dy dx}_{I_{22}}$$
$$+ \underbrace{\int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x,y) \log \left(G_{\mu',\nu'}(\mathcal{A}) - G_{\mu',\nu'}(0)\right) dy dx}_{I_{23}}.$$

The first term I_{21} is clearly equal to $\log(\eta)$. The last term I_{23} can be written as $I_{23} = \mathbb{E}_{X,Y}[\log (G_{\mu',\nu'}(\mathcal{A}) - G_{\mu',\nu'}(0))]$ where the expectation is with respect to the distribution $f(x,y)$. It remains to evaluate I_{22} in order to obtain the

achievable rate $\mathcal{R}$ in Theorem 1. We start by substituting $g_{\mu,\sigma_y}(y)$ in I_{22}, yielding

$$I_{22} = \int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x,y) \log \left(\sqrt{\frac{1}{2\pi\sigma_y^2}}\right) dy dx$$
$$+ \int_0^{\mathcal{A}} \int_{\mathbb{R}} f(x,y) \left(-\frac{(y-\mu)^2}{2\sigma_y^2}\right) dy dx \qquad (16)$$
$$= -\frac{1}{2} \log (2\pi\sigma_y^2)$$
$$- \frac{1}{2\sigma_y^2} \int_0^{\mathcal{A}} (\mathbb{E}_{Y|X}[Y^2] - 2\mu\mathbb{E}_{Y|X}[Y] + \mu^2)\tilde{g}_{\mu,\nu}(x)dx$$
$$= -\frac{1}{2} \log (2\pi\sigma_y^2) - \frac{1}{2\sigma_y^2} (\sigma^2 + \mathbb{E}_X[(X-\mu)^2]). \qquad (17)$$

Finally, we need to evaluate the expectation $\mathbb{E}_X[(X-\mu)^2]$. To this end, we use a change of variables $t = \frac{x-\mu}{\sqrt{2}\nu}$, followed by the identity $\int_a^b t^2 e^{-t^2} dt = \frac{1}{2} \left(\int_a^b e^{-t^2} dt - \left[te^{-t^2}\right]_a^b\right)$, to get $\mathbb{E}_X[(X-\mu)^2] = \nu^2 (1 - (\mathcal{A}-\mu)\tilde{g}_{\mu,\nu}(\mathcal{A}) - \mu\tilde{g}_{\mu,\nu}(0))$. Collecting the terms I_{21}, I_{22}, and I_{23} leads to I_2 as given in the proof of Theorem 1.

References

[1] M. A. Khalighi and M. Uysal, "Survey on free space optical communications: A communication theory perspective," *IEEE Communication Surveys and Tutorials*, vol. 16, no. 4, pp. 2231–2258, 4th quarter 2014.

[2] S. Arnon, J. Barry, G. Karagiannidis, R. Schober, and M. Uysal, *Advanced Optical Wireless Communication Systems*. Cambridge University Press, 2012.

[3] H. Elgala, R. Mesleh, and H. Haas, "Indoor optical wireless communication: Potential and state-of-the-art," *IEEE Comm. Magazine*, vol. 49, no. 9, pp. 56–62, Sep. 2011.

[4] A. Lapidoth, S. M. Moser, and M. Wigger, "On the capacity of free-space optical intensity channels," *IEEE Trans. on Info. Theory*, vol. 55, no. 10, pp. 4449–4461, Oct. 2009.

[5] A. A. Farid and S. Hranilovic, "Channel capacity and non-uniform signalling for free-space optical intensity channels," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 9, pp. 1–12, Dec. 2009.

[6] ——, "Capacity bounds for wireless optical intensity channels with Gaussian noise," *IEEE Trans. on Info. Theory*, vol. 56, no. 12, pp. 6066–6077, Dec. 2010.

[7] T. H. Chan, S. Hranilovic, and F. R. Kschischang, "Capacity-achieving probability measure for conditionally Gaussian channel with bounded input," *IEEE Trans. on Info. Theory*, vol. 51, no. 6, pp. 2073–2088, Jun. 2005.

[8] S. V. Kartalopoulos, *Free Space Optical Networks for Ultra-Broad Band Services*. John Wiley and Sons, Inc., 2011.

[9] M. Berger, *Geometry II*. Springer-Verlag, 1987.

[10] J. M. Morvan, *Generalized Curvatures*. Springer-Verlag, 2008.

[11] D. Cohen-Steiner and J. M. Morvan, "Restricted Delaunay triangulations and normal cycle," in *Proceedings of the Nineteenth Annual Symposium on Computational Geometry (SoCG), A.C.M. Press, New York, NY, USA*, 2003, pp. 312–321.

[12] S. Hranilovic and F. R. Kschischang, "Capacity bounds for power- and band-limited optical intensity channels corrupted by Gaussian noise," *IEEE Trans. on Info. Theory*, vol. 50, no. 5, pp. 784–795, May 2004.

[13] J. Karout, E. Agrell, K. Szczerba, , and M. Karlsson, "Optimizing constellations for single-subcarrier intensity-modulated optical Systems," *IEEE Trans. on Info. Theory*, vol. 58, no. 7, pp. 4645–4659, Apr. 2012.

[14] T. Cover and J. Thomas, *Elements of Information Theory (Second Edition)*. John Wiley and Sons, Inc., 2006.

[15] A. Chaaban, J.-M. Morvan, and M.-S. Alouini, "Free-space optical communications: Capacity bounds, approximations, and a new sphere-packing perspective," *KAUST Technical Report, http://hdl.handle.net/10754/552096*, Apr. 2015.

[16] C. E. Shannon, "Communication in the presence of noise," *Proceedings of the IRE*, vol. 37, no. 1, pp. 10–21, Jan. 1949.