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Adaptive Distributed Parameter and Input Estimation in Linear Parabolic PDEs

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SUMMARY

In this paper we discuss the on-line estimation of distributed source term, diffusion and reaction coefficients of a linear parabolic partial differential equation using both distributed and interior point measurements. First, new sufficient identifiability conditions of the input and the parameter simultaneous estimation are stated. Then, by means of Lyapunov-based design, an adaptive estimator is derived in the infinite dimensional framework. It consists of a state observer and gradient-based parameter and input adaptation laws. The parameter convergence depends on the plant signal richness assumption, whereas the state convergence is established using a Lyapunov approach. The results of the paper are illustrated by simulation on tokamak plasma heat transport model using simulated data. Copyright © 0000 John Wiley & Sons, Ltd.

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KEY WORDS: Distributed parameter systems, input estimation, state and parameter estimation, interior point measurements, identifiability conditions, adaptive estimation.

1. INTRODUCTION

Many industrial, thermal and chemical systems are governed by partial differential equations (PDEs). In many cases it is difficult to get the nominal PDE characterizing the process due to incomplete physical knowledge. Adaptive estimation is one of the methods, belonging to estimation/identification theory, which attempts to solve this question on-line.

Parameter identification of distributed parameter systems (DPSs) was first addressed in [1], [2], [3], [4], [5], [6] for the case of known inputs (source term) and available distributed sensing. In [2], it was pointed out that an identifiability issue in PDE setting should be discussed in relation to both input and boundary conditions. In [7], the problem of boundary parameter estimation in parabolic PDEs using boundary measurements and known actuation (inputs) was considered. Some recent results on unknown boundary parameter estimation using boundary measurements for hyperbolic PDEs can be found in [8], [9] and references therein. Estimation of point source terms has attracted many research works (e.g. [10] and references therein) where the source term is assumed to be separable and can be written as a time-varying amplitude times a delta function which describes the spatial location. The objective was to estimate the intensity and the amplitude assuming mainly that the source term becomes inactive after a pre-determined finite time. In [11] the identifiability conditions of unbounded point sources was studied.

In this paper, we are interested in the simultaneous estimation of space-varying parameters and inputs in parabolic PDEs. This kind of problems arise when the system is heated by electromagnetic

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induction or by microwaves. To achieve this purpose, we assume that distributed or interior-point measurements are available. Even if, from a practical perspective, a distributing sensing systems is not available, we will assume it as a basis to address the problem of estimating simultaneously the unknown input and parameters. In this way, the case of interior-point measurements will be easier to handle. It also allows us to highlight the relationship between the problem of sensor location and DPS estimation problems. To the best of the author's knowledge, there is no clear way in how to estimate uniquely the spatially varying (and even constant) PDE parameters using boundary measurements [12].

As stated by [6], the key problem of estimating spatially varying parameters is the development of constructive identifiability conditions. In [13], identifiability conditions were developed based on the linear independence condition between the solution and its derivatives. In [1], richness-like identifiability conditions of the plant were established for the first time for distributed parameter systems but relied on the solution of the plant itself (whose coefficients are unknown) and was from this point of view passive (see [6] for more details). Constructive enforceable identifiability conditions were formulated in [6] for the case of distributed sensing with known inputs in terms of persistently exciting (PE) input signals. In [14], it is stated that unlike finite dimensional systems, in infinite dimensional systems the concept of PE of the input should be investigated in relation to the time variable, the spatial variable and the boundary conditions. Moreover, in infinite dimensional systems, even a constant input can be PE [14]. The PE assumption is important since it guarantees the convergence of parameter errors to zero [6], [14]. In this work, using on-line estimation, we study simultaneous unknown PDE parameters and input estimation in 1-D linear parabolic PDEs. All the parameters are assumed to be space-varying and we consider both distributed and interior point measurements, provided that these measurements satisfy optimal sampling conditions (see [15], [16] and references therein). The main contributions of this paper are (i) the proposition of new sufficient parameter and input identifiability conditions using both distributed and interior point measurements while assuming that the PDE solution is smooth and (ii) the development of an adaptive estimator able to estimate unknown parameters and inputs simultaneously using interior point measurements. The proposed on-line estimator is developed within the Lyapunov framework where the L_2-convergence of the state and parameter errors is established under the PE assumption. The problem of unknown input estimation arises in many applications (fault detection, machine tools, manipulator applications, chaotic systems, general inverse problems, ...) including tokamak plasma heat transport. Thermonuclear fusion is one of the best new energy source candidate to replace fossil fuels and support the world increasing demand on energy. Unfortunately, due to the problem's complexity, some critical practical and theoretical issues are preventing the implementation of fusion reactors. For instance, for the electron heat transport profile, various empirical, theoretical, mixed, computational models exist in the literature: Each model depends on the reactor itself, its operating mode, the discharge parameters, the temperature profiles, the shear effect, the safety factor and many other conditions to name a few (see [17] and references therein). These issues motivate us to select the tokamak plasma heat transport as a challenging application to test the performance of the proposed estimation method.

This paper is organized as follows. First the estimation problem is formulated in Section 2. In Section 3, parameter and input identifiability conditions are presented for the cases of distributed and interior point measurements. An adaptive estimator is considered in Section 4. The performances of the proposed identifier are depicted in Section 5 using simulated data describing electron heat transport in tokamaks.

Notation: The usual L_2-norm is denoted by $||.||_2$. The space of solution is given by $Q_T = C^1(0, t_f; C^2(\Omega))$, the space-time set is defined by $Q_{xt} = \Omega\times]0, t_f]$ and the space $H^1_{0,\{1\}}(\Omega)$ is the Hilbert space defined by

$$H^1_{0,\{1\}}(\Omega) = \{f \in L_2(\Omega) : \nabla f \in L_2(\Omega), f|_1 = 0\}.$$

The parameter spaces Q_{χ_e}, Q_b and Q_S are given by

$$Q_{\chi_e} = \Big\{f \in L_2(0,\, t_f; L_2(\Omega)) \cap C^0(0, t_f; C^1(\Omega)),\ \exists\, c_1, c_2 \in \mathbb{R}^*_+ :\ c_2 < f(x,t) < c_1\Big\},$$

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$$Q_b = \{g \in L_2(\Omega) \cap C^0(\Omega),\ \exists\, c_3 \in \mathbb{R}_+^* :\ g(x) > c_3\},$$

$$Q_S = L_2(0, t_f, L_2(\Omega)) \cap C^0(0, t_f\,;\, C^0(\Omega)).$$

2. PROBLEM FORMULATION

The general description of a diffusion-reaction equation with homogeneous mixed boundary conditions is given by

$$\begin{cases} \dfrac{\partial T}{\partial t} = div\,(\chi_e(x)\,\nabla T(x,t)) - b(x)\,T(x,t) + S(x,t), \;\; (x,t) \in Q_{xt} \\ \nabla T(0,t) = 0, \;\; T(1,t) = 0, \;\; t \in]0, t_f], \\ T(x,0) = 0, \quad x \in \Omega, \end{cases} \tag{1}$$

where $T := T(x,t)$ is a scalar state variable, χ_e is the diffusion coefficient, S is the source term, b is the reaction coefficient, $div(.)$ is the divergence operator, ∇ is the gradient operator, $\Omega :=]0,1[$ is the normalized space domain and t_f is the final time. To guarantee the existence, uniqueness and differentiability of PDE (1)'s classical solution i.e. $T \in Q_T$, the diffusion coefficient χ_e is assumed to belong to the space Q_{χ_e}, the reaction coefficient b in Q_b and the source term S in Q_S (see [18], chapter 07, page 375 for more details on these technical assumptions). Furthermore, these conditions ensure the a priori (structural) identifiability of PDE (1).
The objective is to estimate χ_e, b and S while the output measurement y is given by

$$\begin{cases} y(x,t) = T(x,t), & Case\ 1 \\ y_k(t) = T(x_k,t), \quad k = 1, ..., N < \infty, & Case\ 2 \end{cases} \tag{2}$$

where *Case* 1 refers to the distributed measurement assumption and *Case* 2 represents the interior point sensors. All measurements are assumed to be pre-filtered. For the case of interior point measurements, sensors are considered to be located such that optimal (in a predefined sense) data reconstruction is guaranteed. We refer to this assumption as the optimal sensing condition (more details are given in Subsection 3.2). In the literature and in practice, it is well stated that b-spline and wavelet reconstruction methods are powerful tools (less computational burden and less costly in terms of the number of sensors) for digital-to-analogue signal conversion (for more details see [19], [20],[15], [16] and references therein). In [21], it was proved that when cubic spline functions are used to reconstruct the point-wise measured signal (in order to get the distributed profile) the relative approximation error is of order equal to four (Δx^4, where Δx is the spatial sampling step). In this paper b-spline functions are chosen as basis functions in the approximation/reconstruction method. Note that optimal sensor location and scheduling are tightly related to DPS estimation problems [22] but such topics are beyond the scope of this work.

3. SUFFICIENT PARAMETER IDENTIFIABILITY CONDITIONS

In this paper, we first provide new sufficient parameter and input identifiability conditions for the case of distributed sensing and then extend them to the case of interior point measurements. Let $y := y(t)$ denote the output measurement given in (2)[†]. The state, output and parameter errors are

[†] $y(t) = y(x,t)$ for Case 1 and $y(t) = \{y_k(t)\}_{k=1}^N$ for Case 2

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given by

$$\begin{cases} \Delta T(x,t) = T(x,t) - \tilde{T}(x,t), \\ \Delta\chi_e(x) = \chi_e(x) - \tilde{\chi}_e(x), \;\; \Delta y(t) = y(t) - \tilde{y}(t), \\ \Delta b(x) = b(x) - \tilde{b}(x), \;\; \Delta S(x,t) = S(x) - \tilde{S}(x,t), \end{cases} \tag{3}$$

where $\tilde{T}$ is the solution of a model which has the same form as (1) where the unknown coefficients χ_e, b and S are replaced by the new unknowns $\tilde{\chi}_e$, $\tilde{b}$ and $\tilde{S}$ and where the boundary and initial conditions are equal. The output $\tilde{y}$ is obtained as in (2) where T is substituted by $\tilde{T}$.

Definition: A set of parameters $\{\chi_e(x),\, b(x),\, S(x,t)\}$ of PDE (1) is said to be identifiable from the measured output y and with respect to the corresponding boundary conditions if and only if for all $x \in]0,1[,\; t \geq 0$:

$$\Delta y(t) = 0 \Rightarrow \Delta\chi_e = \Delta b = \Delta S = 0. \tag{4}$$

This condition is also known as the a posteriori identifiability condition.

3.1. Distributed sensing case

Sufficient conditions for simultaneous diffusion, reaction and source term estimation are given as follows.

Proposition 3.1
If the set G defined by

$$G(t) = \{x \in \Omega : \nabla T(x,t) = 0\}, \tag{5}$$

has zero Lebesgue measure a.e. in $[0, t_f]$, then the parameters χ_e, b and S are identifiable.

Proof 3.1
From definition 3, in distributed sensing[‡]

$$\begin{aligned} y(t) = \tilde{y}(t) \quad &\Leftrightarrow T(x,t) = \tilde{T}(x,t) \\ &\Rightarrow div\,(\Delta\chi_e\; \nabla T) - \Delta b\; T + \Delta S = 0, \end{aligned} \tag{6}$$

and thus, the following integral equality holds

$$\int_0^1 (div\,(\Delta\chi_e \nabla T) - \Delta b\, T + \Delta S)\, \psi dx = 0,\, \forall \psi \in H^1_{0,\{1\}}(\Omega). \tag{7}$$

Using Green's formula ([23], p. 12) and boundary conditions in (1), (7) becomes

$$\int_0^1 \Delta\chi_e \nabla T\; \nabla\psi\; dx + \int_0^1 (\Delta b\, T - \Delta S)\;\; \psi\; dx = 0,\;\; \forall \psi \in H^1_{0,\{1\}}(\Omega),$$

which implies that

$$\begin{cases} \Delta\chi_e\, \nabla T = 0, \\ \Delta b\, T - \Delta S = 0 \Leftrightarrow \Delta b\, T - \Delta S\; \mathbb{1}(x) = 0, \end{cases} \tag{8}$$

where $\mathbb{1}$ is the identity function for $x \in \Omega$. If the set G defined in (5) has zero Lebesgue measure a.e. in $[0, t_f]$ then from equality (8) $\Delta\chi_e \equiv 0$. In the other hand, if functions T and $\mathbb{1}$ are linearly

[‡]Occasionally, the spatial and time dependencies are dropped to alleviate the notations.

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independent then from equation (8) we have

$$\Delta b \equiv 0, \quad \Delta S \equiv 0. \tag{9}$$

To check the linear independence of pair $(T, \mathbb{1})$, we compute its Wronskian function W, which gives $W = \nabla T$. Hence, the linear independence of the pair $(T, \mathbb{1})$ turns out to be Proposition 3.1's condition on the set G.

Remark: The concept of persistency of the excitation input (PE) is equivalent to the a posteriori identifiability conditions [24]. In [5], the PE principle is defined in terms of the dynamics of the PDE operator. In [6], it was proved that if the plant is PE, the identifiability conditions are governed by the ability of expanding the PDE solution on an arbitrary basis in L_2. In this work, the plant is identifiable and thus persistently excited if the solution's profile does not include intervals on which it is constant. This condition is equivalent to the one presented in [6], since if the coefficients of the Fourier expansion of the PDE solution are linearly independent, the basis functions of this expansion cannot vanish on any interval in Ω and thus the set G defined in (5) has a zero Lebesgue measure.

3.2. Using interior point measurements

Let T_{rec} be the reconstruction of T obtained from interior point measurements $y(t) = \{T(x_k, t)\}_{k=1}^{N}$ such that

$$T(x,t) = T_{rec}(x,t) + e(x,t), \tag{10}$$

where e is the reconstruction error. The following assumptions are made.

Assumptions:

(i) $\lim_{t \to \infty} \|e(.,t)\|_2^2 = 0$,
(ii) The reconstruction error is bounded: $\|e(.,t)\|_2^2 \leqslant \epsilon_0, \;\; 0 < \epsilon_0 << 1$,
(iii) $T_{rec} \in B_n$ is the unique reconstruction function of order n such that $T_{rec}(x_k, t) = T(x_k, t), \;\; k = 1, ..., N, \;\; t \in [0, t_f]$, where B_n is the space of approximate solutions of order n.

Assumptions (i), (ii), and (iii) are referred as the optimal sensing condition.

Proposition 3.2
If $T \in Q_T$, the optimal sensing condition is satisfied and the set G_k given by

$$G_k(t) = \{x \in]x_k - \epsilon, x_k + \epsilon[: \nabla T(x,t) = 0, \; \epsilon > 0\},$$

$k = 1, ..., N$, has zero Lebesgue measure a.e in $[0, t_f]$, then parameters $\chi_e, \; b$ and S are identifiable from measurement points $\{y(x_k)\}_{k=1}^{N}$.

Proof 3.2
The output error Δy is given by

$$\Delta y(t) = T(x_k, t) - \tilde{T}(x_k, t)$$

and

$$(\Delta y(t) = 0) \Rightarrow \Big(T(x_k, t) = \tilde{T}(x_k, t)\Big),$$

which leads to

$$div\,(\Delta \chi_e \nabla T)\Big|_{x_k} - \Delta b(x_k) T(x_k, t) + \Delta S(x_k, t) = 0. \tag{11}$$

Using delta function δ, one writes (11) as

$$\int_0^1 \delta(x - x_k)\left(div\,(\Delta \chi_e \nabla T) - \Delta b(x) T(x,t) + \Delta S(x,t)\right) dx = 0, \tag{12}$$

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which is equivqlent to the following inequality

$$\int_{x_k-\epsilon}^{x_k+\epsilon} \delta(x-x_k)\left(div\left(\Delta\chi_e\nabla T\right)-\Delta b(x)T(x,t)+\Delta S(x,t)\right)dx=0, \tag{13}$$

Delta function $\delta(x)$ can be written as

$$\delta(x)=\lim_{\sigma\to 0}\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-x^2/2\sigma^2}==\lim_{\sigma\to 0}g_\sigma(x), \tag{14}$$

where $g_\sigma(x)=\mathrm{e}^{-x^2/2\sigma^2}$.
Replacing (14) in (13), one obtains

$$\begin{aligned}
&\int_{x_k-\epsilon}^{x_k+\epsilon} \delta(x-x_k)\left(div\left(\Delta\chi_e\nabla T\right)-\Delta b(x)T(x,t)+\Delta S(x,t)\right)dx=0,\\
&\Leftrightarrow \int_{x_k-\epsilon}^{x_k+\epsilon} \lim_{\sigma\to 0} g_\sigma(x-x_k)\left(div\left(\Delta\chi_e\nabla T\right)-\Delta b(x)T(x,t)+\Delta S(x,t)\right)dx=0,\\
&\Leftrightarrow \lim\nolimits_{\sigma\to 0}\int_{x_k-\epsilon}^{x_k+\epsilon} g_\sigma(x-x_k)\left(div\left(\Delta\chi_e\nabla T\right)-\Delta b(x)T(x,t)+\Delta S(x,t)\right)dx=0,\\
&\Leftrightarrow g_\sigma(x-x_k)\left(div\left(\Delta\chi_e\nabla T\right)-\Delta b(x)T(x,t)+\Delta S(x,t)\right)=0,\ \ x\in]x_k-\epsilon,x_k+\epsilon[
\end{aligned} \tag{15}$$

which is satisfied if

$$div\left(\Delta\chi_e\nabla T\right)-\Delta bT+\Delta S=0,\ \ \forall x\in]x_k-\epsilon,x_k+\epsilon[. \tag{16}$$

Multiplying (16) by any function $\phi\in H^1_{0,\{1\}}$ and using the same steps as in proof 3.1, one obtains

$$\int_{x_k-\epsilon}^{x_k+\epsilon}\Delta\chi_e(x,t)\,\nabla T(x,t)\,\nabla\phi(x)\,dx-\int_{x_k-\epsilon}^{x_k+\epsilon}\left(\Delta b(x,t)\,T(x,t)-\Delta S(x,t)\right)\phi(x)\,dx=0,\ \ k=1,...,N. \tag{17}$$

Which yields to

$$\begin{cases}\Delta\chi_e(x,t)\,\nabla T(x,t)=0,\ \ \forall x\in]x_k-\epsilon,x_k+\epsilon[\\[2ex] \Delta b(x,t)\,T(x,t)-\Delta S(x,t)=0,\end{cases} \tag{18}$$

and if the set $G_k(t)$ defined in Proposition 3.2 has zero Lebesgue measure a.e in $[0,t_f]$ and $\forall k=1,...,N$ then $\Delta\chi_e=\Delta b=\Delta S=0$. This asserts conditions under which the parameters are identifiable at point measurements $\{x_k\}_{k=1}^N$ and their neighborhoods ($]x_k-\epsilon,x_k+\epsilon[$), i.e. the sets $\{\chi_e(x_k)\}_{k=1}^N$, $\{b(x_k)\}_{k=1}^N$ and $\{S(x_k,t)\}_{k=1}^N$ are uniquely identifiable. However, this does not prove parameter identifiability all over Ω. Nevertheless, if interior point measurements satisfy the optimal sensing condition then functions $\chi_e(x)$, $b(x)$ and $S(x)$ can be obtained uniquely by interpolation from their unique discrete formulations $\{\chi_e(x_k)\}_{k=1}^N$, $\{b(x_k)\}_{k=1}^N$ and $\{S(x_k,t)\}_{k=1}^N$. The accuracy of the identification process relies only on the approximation/ reconstruction method.

4. ADAPTIVE ESTIMATION DESIGN

In this section, we assume that the input S is only space varying. Our objective is to demonstrate that a stable adaptive estimator exists and guarantees the L_2-convergence of both the state and the parameters using distributed sensing and interior point measurements. Let $\hat{T}$, $\hat{\chi}_e$, $\hat{b}$ and $\hat{S}$ be the estimated state, diffusion, reaction and source term respectively. Let the adaptive estimator be described by the following equations (19)-(20)

$$\begin{cases} \dfrac{\partial \hat{T}}{\partial t}(x,t) = div\left(\hat{\chi}_e(x,t)\,\nabla \hat{T}\right) - \hat{b}(x)\,\hat{T}(x,t) + \hat{S}(x,t) - \vartheta_0\,(\hat{T}(x,t) - y(t)), \quad (x,t) \in Q_t, \\ \nabla \hat{T}(0,t) = \hat{T}(1,t) = 0, \quad \hat{T}(x,0) = \hat{T}_0(x) \geq 0, \end{cases} \tag{19}$$

and

$$\begin{cases} \dfrac{\partial \hat{\chi}_e}{\partial t}(x,t) = \vartheta_1 \nabla\left(\hat{T}(x,t) - y(t)\right)\,\nabla \hat{T}(x,t), \;\; \hat{\chi}_e(x,0) = \hat{\chi}_{e_0}(x), \\ \dfrac{\partial \hat{b}}{\partial t}(x,t) = \vartheta_2\,(\hat{T}(x,t) - y(t))\,\hat{T}(x,t), \;\; \hat{b}(x,0) = \hat{b}_0(x), \\ \dfrac{\partial \hat{S}}{\partial t}(x,t) = -\vartheta_3\,(\hat{T}(x,t) - y(t)), \;\; \hat{S}(x,0) = \hat{S}_0(x), \end{cases} \tag{20}$$

where ΔT is the state estimation error, $y(t)$ is the output measurement defined as in (2), $\vartheta_i \geq 0$, $i = 0,1,2,3$ are the adaptation gains, $\hat{\chi}_{e_0}(x) \in Q_{\chi_e}$ is a smooth function, $\hat{b}_0(x) \in Q_b$ and $\hat{S}_0(x) \in Q_S$ are continuous functions.

4.1. *Case of distributed sensing $y(t) = T(x,t)$*

Theorem 4.1
If the plant (1) is identifiable, the adaptive identification law given by (19) combined with the parameters' identifiers in (20) ensure the L_2 convergence of the state and parameter deviations

$$\lim_{t \to +\infty} \int_0^1 \left\{(\Delta T(x,t))^2 + (\Delta S)^2 + (\Delta b(x,t))^2 + (\Delta \chi_e(x,t))^2\right\} dx = 0.$$

Proof 4.1
The proof is an extension of the one presented in [6] to the case of input estimation.
Since $\hat{\chi}_{e_0}(x)0$ is a smooth function, $\hat{b}_0(x)0$ and $\hat{S}_0(x) \in L_2(\Omega)$ are bounded continuous functions, the PDE (19) is well-posed [18]. We start by introducing the variable δT given by

$$\delta T(x,t) = T(x,t) - T^*(x),$$

where $T^*(x)$ is the steady state solution of (1). Define the state, parameter and input deviations as

$$\begin{cases} \Delta T(x,t) = \hat{T}(x,t) - T(x,t), \\ \nabla(\Delta T)(0,t) = \Delta T(1,t) = 0, \end{cases} \tag{21}$$

and

$$\Delta \chi_e(x,t) = \hat{\chi}_e(x,t) - \chi_e(x), \quad \Delta b(x,t) = \hat{b}(x,t) - b(x), \quad \Delta S(x,t) = \hat{S}(x,t) - S(x), \tag{22}$$

The first derivatives of $\delta T(x,t)$, $\Delta T(x,t)$, $\Delta \chi_e(x,t)$, $\Delta b(x,t)$ and $\Delta S(x,t)$ are given by

$$\begin{cases} \dfrac{\partial \delta T}{\partial t} = div\,(\chi_e(x)\nabla(\delta T)) - b(x)\delta T(x,t), \\ \nabla(\delta T)(0,t) = \delta T(1,t) = 0, \end{cases}$$

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$$\left\{\begin{array}{rcl}\dfrac{\partial \Delta T}{\partial t} & = & div\left(\chi_e(x)\nabla(\Delta T)\right) - b(x)\,\Delta T(x,t) + \Delta S(x,t) + div\left(\Delta\chi_e(x,t)\nabla(\hat{T})\right) \\ & & -\vartheta_0 \Delta T(x,t) - \Delta b(x,t)\hat{T}(x,t), \\ \dfrac{\partial \Delta \chi_e}{\partial t} & = & \vartheta_1 \nabla(\hat{T})\,\nabla(\Delta T), \quad \dfrac{\partial \Delta b}{\partial t} = \vartheta_2\,\Delta T(x,t)\,\hat{T}(x,t), \quad \dfrac{\partial \Delta S}{\partial t} = -\vartheta_3\,\Delta T(x,t). \end{array}\right. \tag{23}$$

Let us introduce the following Lyapunov candidate functional

$$V(t) = \frac{1}{2} \int_0^1 \Big([\delta T(x,t)]^2 + [\Delta T(x,t)]^2 + \frac{1}{\vartheta_1}[\Delta\chi_e(x,t)]^2 + \frac{1}{\vartheta_2}[\Delta b(x,t)]^2 + \frac{1}{\vartheta_3}[\Delta S(x,t)]^2\Big)\, dx. \tag{24}$$

Using (20), (19) and Gauss divergence formula [23], the derivative of V is as follows

$$\begin{array}{rcl}\dot{V}(t) & = & -\displaystyle\int_0^1 \chi_e(x)\left(\nabla(\delta T)\right)^2 dx - \int_0^1 b(x)[\delta T(x,t)]^2 dx - \int_0^1 \Delta\chi_e(x,t)\nabla(\Delta T)\,\nabla(\hat{T}) dx \\ & & -\displaystyle\int_0^1 \left(b(x)+\vartheta_0\right)\,[\Delta T(x,t)]^2 dx + \int_0^1 \Delta S(x,t)\Delta T(x,t) dx \\ & & +\displaystyle\int_0^1 \Delta\chi_e(x,t)\nabla(\hat{T})\,\nabla(\Delta T) dx - \int_0^1 \Delta b(x,t)[\Delta T(x,t)]^2 dx \\ & & -\displaystyle\int_0^1 \Delta S(x,t)\Delta T(x,t) dx - \int_0^1 \chi_e(x)\left(\nabla(\Delta T)\right)^2 dx, \\ & = & -\displaystyle\int_0^1 \chi_e(x)\left(\nabla(\delta T)\right)^2 dx - \int_0^1 b(x)[\delta T(x,t)]^2 dx - \int_0^1 \chi_e(x,t)\left(\nabla(\Delta T)\right)^2 dx \\ & & -\displaystyle\int_0^1 \left(b(x)+\vartheta_0\right)[\Delta T(x,t)]^2 dx. \end{array} \tag{25}$$

Since χ_e and b are strictly positive functions, one can write

$$\dot{V}(t) \le -\vartheta_0\,\|\Delta T(.,t)\|_2^2 \le 0, \tag{26}$$

Inequality (26) implies the boundedness of the Lyapunov function $V(t) \le V(0) < \infty$ and the L_2-boundedness of the state observation error ΔT. Since the adaptation laws (20) are just integrations, system (23) turns out to be time-invariant and using [theorm 4.3.4 of [25]], all the trajectories of the system deviations converge to the invariant set for which $\dot{V} = 0$, which leads to

$$\Delta T(x,t) = 0 \Longrightarrow div\Big(\Delta\chi_e(x,t)\frac{\partial T}{\partial x}\Big) - \Delta b(x,t)\,T(x,t) + \Delta S(x,t) = 0.$$

By assuming that the plant is persistently excited (and thus identifiable), it follows that

$$\lim_{t\to+\infty} \int_0^1 \left\{(\Delta T)^2 + (\Delta b)^2 + (\Delta S)^2 + (\Delta\chi_e)^2\right\} dx = 0.$$

As demonstrated for the case of adaptive parameter estimation in [6], because $\chi_e \in Q_{\chi_e}$, $b \in Q_b$, $S \in Q_S$ and the domain Ω is compact, the point-wise parameter and input convergence is easily obtained using Fourier expansions . It is well established that the proposed identifier (19)-(20) can also handle slowly time varying parameters and input by averaging analysis [3], [5], [26].

4.2. Case of interior point measurements

In this case, the output y is defined by $y(t) = T_{rec}(x,t)$. The correction term $\big(\hat{T}(x,t) - y(t)\big)$ in (19)-(20) can be written as

$$\hat{T}(x,t) - y(t) = \hat{T}(x,t) - T_{rec}(x,t) = \Delta T(x,t) + e(x,t), \tag{27}$$

where $e(x,t)$ was defined in (10).

Assumptions:

(iv) All the parameter spaces are compact

$$\exists\, c_1, c_2, c_3 \in \mathbb{R}_+^* :\ \|\Delta\chi_e(.,t)\|_2^2\ \le c_1,\ \ \|\Delta b(.,t)\|_2^2\ \le c_2,\ \ \|\Delta S(.,t)\|_2^2\ \le c_3. \tag{28}$$

(v) The state estimate and its derivative are bounded

$$\exists\, c_4, c_5 \in \mathbb{R}_+^* :\ \|\hat{T}(.,t)\|_2^2\ \le c_4,\ \ \|\nabla\hat{T}(.,t)\|_2^2\ \le c_5. \tag{29}$$

These assumptions are guaranteed since the domain Ω is bounded.

Theorem 4.2
If the plant (1) is identifiable and assumptions (i), (ii), (iii) and (iv) are satisfied, the adaptive estimator (19)-(20) guarantees

$$\lim_{t\to\infty} \|\Delta T(.,t)\|_2^2 = 0,$$

$$\lim_{t\to\infty} \Delta T(.,t) = \lim_{t\to\infty} \Delta\chi_e(.,t) = \lim_{t\to\infty} \Delta b(.,t) = \lim_{t\to\infty} \Delta S(.,t) = 0.$$

Proof 4.2
By replacing the correction term $(\hat{T}(x,t) - y(t))$ in (19)-(20) by (27), the derivative of the Lyapunov candidate function (24) becomes

$$\begin{aligned}
\dot{V}(t) \ = \ & -\int_0^1 \chi_e(x)\left(\nabla\delta T(x,t)\right)^2 dx - \int_0^1 b(x)[\delta T(x,t)]^2 dx - \int_0^1 \chi_e(x)\left(\nabla\Delta T(x,t)\right)^2 dx \\
& - \int_0^1 (b(x)+\vartheta_0)[\Delta T(x,t)]^2 dx - \int_0^1 e(x,t)\,\Delta T(x,t)\,dx \\
& + \int_0^1 \Delta\chi_e(x,t)\,\nabla e(x,t)\,\nabla\hat{T}(x,t) dx - \int_0^1 \Delta b(x,t)\,\hat{T}(x,t)\,e(x,t)\,dx \\
& + \int_0^1 \Delta S(x,t) e(x,t) dx.
\end{aligned} \tag{30}$$

Using Cauchy-Schwartz inequality and assumptions (i)-(iv)-(v), we obtain

$$\dot{V}(t) \leqslant -\vartheta_0(1-e_0)\,\|\Delta T(.,t)\|_2^2 + K\,\|e(.,t)\|_2^2, \tag{31}$$

where $K = c_1\,c_5 + c_2\,c_3 + c_4$. Integrating (31) over time gives

$$\vartheta_0(1-e_0)\int_0^\infty \|\Delta T(.,t)\|_2^2 dt \quad \leqslant V(0) - V(\infty) + K\int_0^\infty \|e(.,t)\|_2^2 dt < \infty \tag{32}$$

and thus

$$\lim_{t\to\infty} \|e(.,t)\|_2^2 = 0 \Rightarrow \lim_{t\to\infty} \|\Delta T(.,t)\|_2^2 = 0,$$

and from Theorem 1 of [27], the identifiability condition and assumption (v), we may conclude that

$$\lim_{t\to\infty} \Delta T(.,t) = 0 \Rightarrow \lim_{t\to\infty} \Delta\chi_e(.,t) = \lim_{t\to\infty} \Delta b(.,t) = \lim_{t\to\infty} \Delta S(.,t) = 0.$$

5. NUMERICAL RESULTS

In this section, using simulated data, the performance of the proposed adaptive estimator are evaluated for the estimation of the coefficients which govern the electron temperature profile in tokamak plasma, mainly the diffusion coefficient and the source term. The electron temperature inside a tokamak is described by a linear non-homogeneous diffusion-reaction equation [17] as in (1) with the same boundary and initial conditions.
Throughout this section, the continuous Galerkin cubic b-splines semi-discrete scheme is considered. The temperature, the diffusion coefficient and source term profiles presented in Figure 1 are proposed for estimation. The reaction coefficient b is considered to be constant. Generally, in plasma heat transport models the reaction coefficient is a confinement time which models particle plastic collisions and therefore is assumed to be time and space independent [17]. The choice of

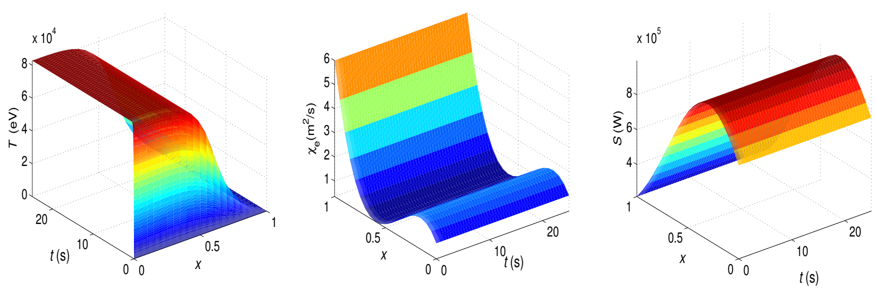

Figure 1. T, χ_e and S for the in silico example

χ_e, b and S is motivated by plasma physics $a\ priori$ assumptions [17] and references therein. It is assumed that the diffusion coefficient has a positive increasing function from the tokamak center to the edge and that the heating source undergoes a spatial Gaussian form. They are generated by the following system

$$\begin{cases} \chi_e(x,t) = (1 + 9\,x - 36\,x^2 + 32\,x^3)\mathbf{1}(t), \\ S(x,t) = \dfrac{10^6}{\sqrt{2\,\pi\,\sigma}}\,\exp\left(\dfrac{-(x-\mu)^2}{2\,\sigma^2}\right)\,\mathbf{1}(t), \\ b = 0.1(s),\ x \in [0,1],\ t \in [0,25],\ dx = 0.05,\ dt = 0.01. \end{cases} \tag{33}$$

χ_e and S are considered constant in time ($\mathbf{1}(t) = 1 \forall t \in [0,25]$). The time and the spatial discretization steps (dt and dx) are equal to those generally provided by the experimental data. Since the cubic b-spline functions are chosen to reconstruct the spatial profile from point-wise measurement, the relative reconstruction error is of order $dx^4 \simeq 10^-6$ [21]. In the adaptive estimator (19)-(20), the tuning gains were chosen as follows $\vartheta_0 = 10^3,\ \vartheta_1 = 1850,\ \vartheta_2 = 750$ and $\vartheta_2 = 850.$ The L_2-norms of the relative estimation errors of T, χ_e, b and S in Figures 2 and 3 show that using the adaptive estimator (19)-(20) and the adopted tuning gains, the parameters (χ_e, b) and the input S profiles are reconstructed efficiently. Note that the efficiency of the distributed sensing version of this estimator was tested using experimental data in [28] and gave good performance.

6. CONCLUSIONS

In this paper we have studied and tested state, input and parameter adaptive estimation for a linear parabolic PDE representing heat transport in tokamak plasma. First a new sufficient identifiability condition based on the gradient of the PDE solution was formulated for the cases of distributed and interior point measurements. If the profile of the available measurements is not flat on an interval then distributed parameters and input are identifiable. In the second step, an adaptive identifier was proposed. This estimator can handle both distributed and interior point measurements. The

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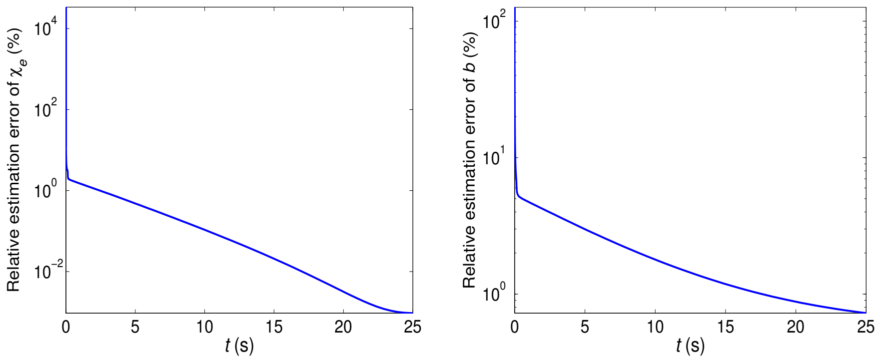
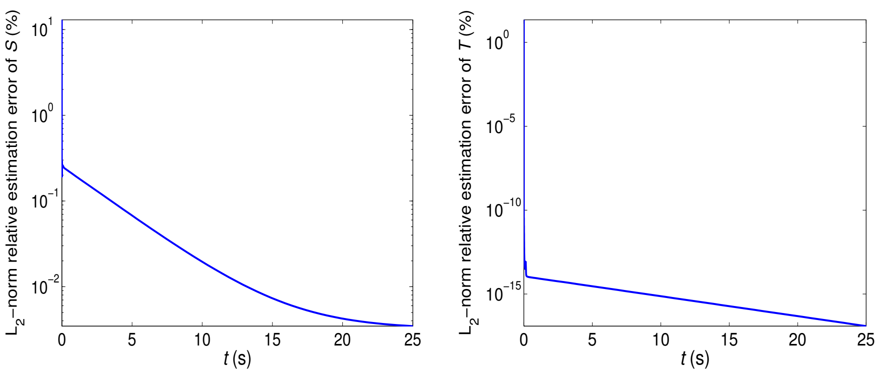


(a) L_2-norm of the estimation error of χ_e (b) L_2-norm of the estimation error of b

Figure 2. Estimation errors of χ_e, b for the insilico example



(a) L_2-norm of the estimation error of S (b) L_2-norm of the estimation error of T

Figure 3. Estimation errors of S and T for the insilico example

convergence of the state, parameter and input errors was also demonstrated. Simulations using simulated data prove the performances of the proposed technique. However, the effect of noisy measurements was not investigated. This will be addressed in our future works where the case of space and time varying parameters presented in [28], [29] but using interior point measurements will also be considered.

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