

On the Capacity Region of the Intensity-Modulation Direct-Detection Optical Broadcast Channel

Anas Chaaban, Zouheir Rezki, and Mohamed-Slim Alouini

Abstract

The capacity of the intensity-modulation direct-detection free-space optical broadcast channel (OBC) is investigated. The Gaussian model with input-independent Gaussian noise is used, with both average and peak intensity constraints. An outer bound on the capacity region is derived by adapting Bergmans' approach to the OBC. Inner bounds are derived by using superposition coding with either truncated-Gaussian distributions or discrete distributions. While the discrete input distribution achieves higher rates than the truncated-Gaussian distribution, the latter allows expressing the achievable rate region in a closed form. At high signal-to-noise ratio (SNR), it is shown that the truncated-Gaussian distribution is nearly optimal. It achieves the symmetric-capacity within a constant gap (independent of SNR), which approaches half a bit as the number of users grows large. It also achieves the capacity region within a constant gap, which depends on the number of users. At low SNR, it is shown that on-off keying with time-division multiple-access (TDMA) is optimal, as it achieves any point on the boundary of the developed outer bound. This is interesting in practice since both OOK and TDMA have low complexity. At moderate SNR (typically $[0, 8]$ dB), a discrete distribution with a small alphabet size achieves a fairly good performance in terms of symmetric rate.

Index Terms

Free-space optical; intensity-modulation; optical broadcast; capacity region; truncated-Gaussian.

The authors are with the Division of Computer, Electrical, and Mathematical Sciences and Engineering (CEMSE) at King Abdullah University of Science and Technology (KAUST), Thuwal, Saudi Arabia. Email: {anas.chaaban,zouheir.rezki,slim.alouini}@kaust.edu.sa.

This work is supported in part by King Abdulaziz City of Science and Technology (KACST) under grant AT-34-145. This is an extended version of work submitted to IEEE Globecom Workshop on Optical Wireless Communication, San Diego, CA, 2015 [1].

I. INTRODUCTION

Free-space optical (FSO) communication has gained lots of research attention recently. FSO refers to communication using a modulated light source to send information through free-space to a receiver, which employs an optical detector to capture the transmit signal. It can be used for long-range outdoors communications using lasers, and for indoors communication using LED lighting fixtures which is commonly known as visible-light communications (VLC) [2]–[4]. The advantage of FSO for outdoors communications is that it does not need a license, it can provide high data-rates, and it requires less infrastructure in comparison to wired optical communications (with optical fibers). The advantage of VLC for indoors communications besides providing high data-rates is that it is more secure/localized, and has reduced electromagnetic interference in comparison to radio-frequency communications. Thus, it can be a good solution for providing coverage in hospitals or airplanes, where electromagnetic compatibility is an essential requirement.

For these reasons, researchers have investigated several aspects of FSO recently [5]–[10]. Focus has been towards characterizing performance limits of FSO systems that employ intensity modulation and direct detection (IM-DD). In such systems, communication is established by modulating light intensity at the source and using an intensity detector at the destination. Although this implementation is practically simple, studying its performance in terms of achievable rates is rather difficult. The main reason is that the transmit signal of an IM-DD system has non-negativity, peak, and average constraints [9], and thus does not belong to the intensively-studied Gaussian point-to-point (P2P) channel with a second-moment constraint constraint [11]. In fact, the capacity of the IM-DD P2P channel is still unknown in closed form. Nevertheless, we know that the capacity achieving input distribution is discrete [12]. Furthermore, some fairly tight bounds exist on the capacity of this channel [8]–[10], and its capacity at high and low SNR is known [9], [10].

Consider a scenario where a light fixture in an office is used to sending information to multiple work-stations (users) using IM-DD for instance. The resulting system is an optical broadcast channel (OBC), and has been studied earlier in [13]–[15]. The common factor in these works is the use of orthogonal codes for code-division multiple-access. This allows serving multiple users in the OBC without interference between the different streams, and can also be accomplished by time-division multiple access (TDMA), where one user is served at a time. By orthogonalizing users this way, the channel from the transmitter to each receiver reduces to an IM-DD P2P, and capacity results on IM-DD P2P channels can be applied. However, as shown in [16], for a Gaussian broadcast channel which is physically degraded by nature, superposition coding is optimal and orthogonalizing users

is not efficient. Thus, one could apply superposition coding in the OBC. However, it is not clear which input distribution is optimal. Note that Gaussian distributions are not admissible due to the non-negativity and the peak intensity constraints.

The optimal input distribution for the IM-DD P2P is unknown, although it is known to be discrete [12]. Consequently, finding the optimal input distribution for the OBC is even more challenging, although one would expect the optimal distribution to be also discrete. At high SNR however, one could use an exponential or a truncated-Gaussian input distribution as well, based on [9], [10]. The advantage of these distributions is that their achievable rate can be expressed in closed-form contrary to a discrete input distribution. Since such distributions are optimal at high SNR for the IM-DD P2P, they might deliver good performance in the OBC.

The goal of this paper is to study the performance limits of the N -user OBC, which is interesting due to its practical importance for indoors applications. The main focus is finding simple closed-form statements on the channel capacity. This would require developing outer and inner bounds on the capacity region of the channel. To this end, we modify Bergmans' outer bound [17] to fit the OBC, and obtain outer bounds on the capacity region which are written as functions of outer bounds on the capacity of the IM-DD P2P in [9], [10]. Then, we develop inner bounds on the capacity region based on superposition coding, where the source sends the sum of several codewords each of which is desired by one user, and the symbols of each codeword follow a truncated-Gaussian distribution or a discrete input distribution. To obtain the desired signal, assuming users are ordered in decreasing order of their received SNR, user i decodes the signals intended to users $N, N-1, \dots, i$ successively in this order, each time treating the remaining signals as noise. For the truncated-Gaussian distribution, we express the achievable rate region in closed-form. For the discrete one, we use an input distributions with finitely many uniformly spaced points as in [18], and we use a successive optimization approach to optimize the distribution of the users. The achievable rate region can then be written as a function of the obtained distribution. Then, we focus on three SNR regimes: high, low, and moderate.

At high SNR, we provide an outer bound that can be written in a simple closed-form. Then, we study the symmetric-capacity, i.e., the highest rate R at which we can transmit simultaneously to all users. We show that a truncated-Gaussian input distribution achieves the high-SNR symmetric-capacity of the channel within a small constant gap. This gap is 0.34 nats per channel use at most for the 2-user case, and is proportional to $\frac{1}{N} \log(N)$ for the N -user case. Superposition coding with a truncated-Gaussian distribution is better than TDMA with the same distribution in general. We also

show that the same distribution achieves the high-SNR capacity region within a small constant gap. In particular, for an N -user OBC, if the rate tuple $(R_1, R_2, \dots, R_{N-1}, R_N)$ is inside our capacity outer bound, then the rate tuple $(R_1, R_2, \dots, R_{N-1}, R_N - \delta)$ is achievable, where $\delta = 0.68$ nats per channel use at most for $N = 2$, and δ scales as $\frac{1}{2} \log(N)$ in the N -user case. Those gaps are calculated analytically, although numerically the gap is much less.

At low SNR, we show that on-off keying (OOK) combined with TDMA is optimal as it achieves the channel's low-SNR capacity region, and hence also the low-SNR symmetric-capacity. The capacity region in this case is also given in a simple closed-form.

At moderate SNR ($[0, 8]$ dB), the discrete distribution achieves a larger rate region, and is fairly close to the capacity outer bound. Furthermore, using a discrete input distribution with a small number of mass points suffices to approach the capacity region. For instance, for a 2-user OBC with average and peak constraints \mathcal{E} and $\mathcal{A} = 2\mathcal{E}$, and noise variances σ_1^2 and $\sigma_2^2 = 4\sigma_1^2$ at users 1 and 2, respectively, with $\frac{\mathcal{E}}{\sigma_2^2} = 5$ dB, an input distribution with 4 symbols achieves a fair performance in terms of symmetric rate. If $\mathcal{A} = 5\mathcal{E}$ e.g., then this number of symbols becomes 8.

The main contributions of the paper can thus be summarized as follows:

- 1) Deriving outer and inner bounds on the capacity region of the OBC,
- 2) characterizing the high-SNR symmetric-capacity and capacity region within a small constant gap,
- 3) characterizing the low-SNR capacity region.

Organization: The rest of the paper is organized as follows. We define the OBC formally in Sec. II. Then, we focus on the 2-user case. We derive capacity outer bounds in Sec. III and inner bounds in Sec. IV. We study the high-SNR capacity in Sec. V and the low-SNR capacity in Sec. VI. We discuss the moderate-SNR regime in Sec. VII. The results are extended to the N -user case in Sec. VIII. Finally, we conclude in Sec. IX.

Notation: Throughout the paper, we use $g_{\mu,\nu}(x)$ to denote the Gaussian distribution with mean μ and variance ν^2 , and $G_{\mu,\nu}(x)$ its CDF. We use normal-face fonts to denote scalars and bold-face fonts to denote vectors. We use $\text{CH}(\cdot)$ to denote the convex hull of a set.

II. THE OPTICAL BROADCAST CHANNEL

Consider an N -user optical broadcast system, where information needs to be conveyed from a light source to users $i \in \mathcal{N} = \{1, \dots, N\}$ using intensity-modulation and direct-detection (IM-DD) (Fig. 1). Denote the light intensity by X . Clearly X has to be positive (light intensity). Additionally, it has

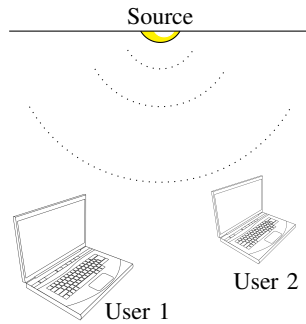


Fig. 1: The optical broadcast channel models a scenario, where light intensity is used to send information to N users (workstations in an office for instance). Here, $N = 2$.

to satisfy a peak constraint \mathcal{A} and an average constraint \mathcal{E} , due to safety and practical considerations. Thus, $X \in [0, \mathcal{A}]$ and $\mathbb{E}[X] \leq \mathcal{E}$. It might be further required that the average constraint is satisfied with equality $\mathbb{E}[X] = \mathcal{E}$ to guarantee a desired lighting condition in an office environment for instance. We denote the ratio $\frac{\mathcal{E}}{\mathcal{A}}$ by $\alpha \in [0, 1]$.

The received signal at user $i \in \mathcal{N}$ is

$$Y_i = X + Z_i, \quad (1)$$

where $Z_i \sim g_{0, \sigma_i}(z_i)$ represents the noise at user i , and has i.i.d. realizations. We assume that the noise is input-independent as in [9] although it might be dependent on the optical constraints \mathcal{A} and \mathcal{E} [10]. We further assume, without loss of generality, that $\sigma_i^2 \geq \sigma_{i-1}^2$ for all $i \in \mathcal{N}$, where we formally define $\sigma_0^2 = 0$. We call the resulting channel an optical broadcast channel (OBC). Generally speaking, we say that the channel has high SNR if $\mathcal{E} \gg \sigma_N$, and low SNR if $\mathcal{A} \ll \sigma_1$. Note that the first condition implies both \mathcal{A} and \mathcal{E} are much larger than σ_i for all $i \in \mathcal{N}$, while the second implies that they are much smaller.

The message that is desired at user i is represented by a random variable W_i uniformly distributed over $\mathcal{W}_i = \{1, \dots, 2^{nR_i}\}$ for some $n \in \mathbb{N}$ and $R_i \geq 0$. The source encodes the independent messages (W_1, \dots, W_N) to a codeword of length n denoted $\mathbf{X} \in [0, \mathcal{A}]^n$, and sends this codeword. User i receives $\mathbf{Y}_i \in \mathbb{R}^n$ and decodes \widehat{W}_i . It is required to find the set of tuples (R_1, \dots, R_N) for which there exists a coding scheme that satisfies $P_e = \text{Prob}\{W_i \neq \widehat{W}_i \text{ for some } i \in \mathcal{N}\} \rightarrow 0$ as $n \rightarrow \infty$. These tuples are called ‘achievable’ rate tuples, and the set of all achievable rate tuples is the capacity region of the channel, denoted \mathcal{C} .

Note that due to the symmetry of the Gaussian distribution of noise, using $\mathbb{E}[X] = \mathcal{E}$ or $\mathbb{E}[X] = \mathcal{A} - \mathcal{E}$ achieves the same rate [9]. Thus, for simplicity, we restrict \mathcal{E} to $(0, \frac{\mathcal{A}}{2}]$, i.e., $\alpha \in (0, \frac{1}{2}]$. The results for an OBC with an average constraint $\mathcal{E} > \frac{\mathcal{A}}{2}$ can be then obtained from the results for an OBC with an average constraint $\mathcal{A} - \mathcal{E}$. Furthermore, as shown in [10], for $\mathcal{E} \in (0, \frac{\mathcal{A}}{2}]$, the optimal distribution satisfies the average constraint with equality. Thus, from this point on, we fix $\mathbb{E}[X] = \mathcal{E}$. In the following sections, we stick to $N = 2$. We generalize the results to $N \geq 2$ in Section VIII.

III. OUTER BOUND

Before presenting our capacity outer bounds, it is worth to mention that the OBC considered in this work belongs to the class of physically-degraded broadcast channels, for which the capacity is known [19]. The capacity region of a degraded broadcast channel satisfying the Markov chain $X \rightarrow Y_1 \rightarrow Y_2$ is given by convex hull of the closure of all rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_2 \leq I(U; Y_2), \quad (2)$$

$$R_1 \leq I(X; Y_1|U), \quad (3)$$

for some joint distribution $p(u)p(x|u)p(y_1, y_2|x)$ over $\mathcal{U} \times \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$, where the cardinality of \mathcal{U} is bounded by $|\mathcal{U}| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\}$. Since in our case $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathbb{R}$, then the last condition becomes $|\mathcal{U}| \leq |\mathcal{X}|$. Note that the distribution on X must be chosen so that the non-negativity, peak, and average constraints are satisfied. Specifying the optimal distribution $p(u)p(x|u)$ is a challenging problem in general, which prevents us from computing this capacity region. Our goal in this paper is to obtain computable bounds which give a better insight on the capacity of the channel.

An outer bound on the capacity region of the OBC can be derived by adapting Bergmans' outer bound [17]. An important step in Bergmans' outer bound for the broadcast channel satisfying $\mathbb{E}[X^2] \leq P$ only, involves using the fact that a Gaussian distribution maximizes differential entropy under a covariance constraint [19]. In particular, Bergmans' outer bound relies on $h(X + Z) \leq \frac{1}{2} \log(2\pi e(\sigma^2 + P))$ for $\mathbb{E}[X^2] \leq P$ and $Z \sim g_{0,\sigma}(z)$, which is achievable by setting $X \sim g_{0,\sqrt{P}}(x)$. Unfortunately, while the input distribution maximizing $h(X + Z)$ is known if $\mathbb{E}[X^2] \leq P$, it is unknown when X satisfies average and peak constraints instead. To circumvent this problem, we use the bound

$$h(X + Z) = I(X; X + Z) + h(Z) \leq \bar{C}_\alpha(\mathcal{A}, \sigma) + \frac{1}{2} \log(2\pi e\sigma^2), \quad (4)$$

where $\bar{C}_\alpha(\mathcal{A}, \sigma)$ is an outer bound on the capacity of an IM-DD P2P channel with the same constraints \mathcal{A} and \mathcal{E} , and noise variance σ^2 , and can be obtained from bounds in [9], [10]. In what follows, we use the following bounds.

Lemma 1: The capacity of the IM-DD P2P channel $Y = X + Z$ where $X \in [0, \mathcal{A}]$, $\mathbb{E}[X] \leq \mathcal{E} = \alpha\mathcal{A}$, $\alpha \leq \frac{1}{2}$, and $Z \sim g_{0,\sigma}(z)$ is upper bounded by the following quantities

$$\begin{aligned}\bar{C}_\alpha^{[1]}(\mathcal{A}, \sigma) &= \inf_{\beta, \delta > 0} B(\beta, \delta), \\ \bar{C}_\alpha^{[2]}(\mathcal{A}, \sigma) &= \sup_{\beta \in [0, 1]} \left[\log \left(\frac{\mathcal{A}^2}{2\pi e \sigma^2} \right)^{\frac{\beta}{2}} - \log \left(\frac{\beta^\beta}{(1-\beta)^{\frac{3\beta-3}{2}}} \right) \right], \\ \bar{C}_\alpha^{[3]}(\mathcal{A}, \sigma) &= \frac{1}{2} \log \left(1 + \frac{\alpha(1-\alpha)\mathcal{A}^2}{\sigma^2} \right).\end{aligned}$$

where

$$\begin{aligned}B(\beta, \delta) &= \log \left(\beta e^{-\frac{\delta^2}{2\sigma^2}} + \sqrt{2\pi}\sigma \mathcal{Q} \left(\frac{\delta}{\sigma} \right) \right) + \frac{1}{2} \mathcal{Q} \left(\frac{\delta}{\sigma} \right) + \frac{\delta}{2\sqrt{2\pi}\sigma} e^{-\frac{\delta^2}{2\sigma^2}} \\ &\quad + \frac{\delta^2}{2\sigma^2} \left(1 - \mathcal{Q} \left(\frac{\delta + \mathcal{E}}{\sigma} \right) \right) + \frac{1}{\beta} \left(\delta + \mathcal{E} + \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\delta^2}{2\sigma^2}} \right) - \frac{1}{2} \log(2\pi e \sigma^2)\end{aligned}\quad (5)$$

The first and last bounds were given in [9], while the second was given in [10]. The upper bound on $h(X + Z)$ in (4) is generalized to the vector case $h(\mathbf{X} + \mathbf{Z})$ in the following lemma.

Lemma 2: For X satisfying and average constraint \mathcal{E} and a peak constraint \mathcal{A} , and $Z \sim g_{0,\sigma}(z)$, the entropy of the n -tuple $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ can be upper bounded by $h(\mathbf{Y}) \leq n\bar{C}_\alpha^{[j]}(\mathcal{A}, \sigma) + \frac{n}{2} \log(2\pi e \sigma^2)$ for any $j \in \{1, 2, 3\}$.

Proof: By the subadditivity of entropy, we have $h(\mathbf{Y}) \leq \sum_{k=1}^n h(Y(k)) \leq n \max_{p(x)} h(Y)$ where $Y(k)$ is the k -th component of \mathbf{Y} , and the maximization is over all distributions of X that satisfy the system constraints. By using (4) and Lemma 1, we obtain the statement of the lemma. ■

Using Lemma 2 and following Bergmans' approach, we can derive the following outer bound on the capacity region \mathcal{C} of the OBC.

Theorem 1: The capacity region \mathcal{C} of the 2-user OBC is outer bounded by $\bar{\mathcal{C}}^{[j]} = \bigcup_{\rho \in [0, 1]} \bar{\mathcal{R}}^{[j]}(\rho)$ where $\bar{\mathcal{R}}^{[j]}(\rho)$ is the set of $(R_1, R_2) \in \mathbb{R}_+^2$ that satisfy

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{\sigma_2^2}{\sigma_1^2} \left(e^{2\bar{C}_\alpha^{[j]}(\rho\mathcal{A}, \sigma_2)} - 1 \right) \right), \quad (6)$$

$$R_2 \leq \bar{C}_\alpha^{[j]}(\mathcal{A}, \sigma_2) - \bar{C}_\alpha^{[j]}(\rho\mathcal{A}, \sigma_2), \quad (7)$$

for any $j \in \{1, 2, 3\}$.

Proof: Using Lemma 2 and since conditioning reduces entropy, we have $h(\mathbf{Y}_2|W_2) \leq h(\mathbf{Y}_2) \leq n\bar{C}_\alpha^{[j]}(\mathcal{A}, \sigma_2) + \frac{n}{2} \log(2\pi e\sigma_2^2)$. On the other hand, $h(\mathbf{Y}_2|W_2) \geq h(\mathbf{Y}_2|W_2, \mathbf{X}) = h(\mathbf{Z}_2) = \frac{n}{2} \log(2\pi e\sigma_2^2)$. Thus, since $\bar{C}_\alpha^{[j]}(\mathcal{A}, \sigma_2)$ is increasing in \mathcal{A} and approaches zero as $\mathcal{A} \rightarrow 0$, then we can write

$$h(\mathbf{Y}_2|W_2) = n\bar{C}_\alpha^{[j]}(\rho\mathcal{A}, \sigma_2) + \frac{n}{2} \log(2\pi e\sigma_2^2) \quad (8)$$

for some $\rho \in [0, 1]$. To bound R_2 , we use the following steps

$$n(R_2 - \varepsilon_{2n}) \leq I(W_2; \mathbf{Y}_2) \quad (9)$$

$$= h(\mathbf{Y}_2) - h(\mathbf{Y}_2|W_2) \quad (10)$$

$$\leq n\bar{C}_\alpha^{[j]}(\mathcal{A}, \sigma_2) - n\bar{C}_\alpha^{[j]}(\rho\mathcal{A}, \sigma_2), \quad (11)$$

where the first step follows from Fano's inequality with $\varepsilon_{2n} \rightarrow 0$ as $n \rightarrow \infty$, and the last by applying Lemma 2 and (8). By dividing by n , and letting $n \rightarrow \infty$, we get the desired bound on R_2 . Now, we bound R_1 as follows

$$n(R_1 - \varepsilon_{1n}) \leq I(W_1; \mathbf{Y}_1) \quad (12)$$

$$\leq I(W_1; \mathbf{Y}_1|W_2) \quad (13)$$

$$\leq h(\mathbf{Y}_1|W_2) - h(\mathbf{Z}_1), \quad (14)$$

where the first step follows from Fano's inequality with $\varepsilon_{1n} \rightarrow 0$ as $n \rightarrow \infty$, the second follows from $I(W_1; \mathbf{Y}_1) \leq I(W_1; \mathbf{Y}_1, W_2) = I(W_1; \mathbf{Y}_1|W_2) + I(W_1; W_2) = I(W_1; \mathbf{Y}_1|W_2)$ by the independence of W_1 and W_2 , and the third step follows from the definition of mutual information and by writing $h(\mathbf{Y}_1|W_1, W_2) \geq h(\mathbf{Y}_1|W_1, W_2, \mathbf{X}) = h(\mathbf{Y}_1|\mathbf{X}) = h(\mathbf{Z}_1)$ due to the Markov chain $(W_1, W_2) \rightarrow \mathbf{X} \rightarrow \mathbf{Y}_1$ and the independence of \mathbf{X} and \mathbf{Z}_1 . Since $\sigma_2^2 \geq \sigma_1^2$, and since the capacity of the broadcast channel depends only on the marginal distributions of Y_i given X [20], we can write $Z_2 = Z_1 + \tilde{Z}_2$ where $\tilde{Z}_2 \sim g_{0, \sqrt{\sigma_2^2 - \sigma_1^2}}(\tilde{z}_2)$ independent of Z_1 . Thus, $\mathbf{Y}_2 = \mathbf{Y}_1 + \tilde{\mathbf{Z}}_2$, or in other words, \mathbf{Y}_2 is physically degraded with respect to \mathbf{Y}_1 . Using the conditional entropy-power inequality stating that $e^{\frac{2}{n}h(\mathbf{V}_1 + \mathbf{V}_2|W)} \geq e^{\frac{2}{n}h(\mathbf{V}_1|W)} + e^{\frac{2}{n}h(\mathbf{V}_2|W)}$ for conditionally independent \mathbf{V}_1 and \mathbf{V}_2 , we write

$$h(\mathbf{Y}_1|W_2) \leq \frac{n}{2} \log \left(e^{\frac{2}{n}h(\mathbf{Y}_2|W_2)} - e^{\frac{2}{n}h(\tilde{\mathbf{Z}}_2|W_2)} \right) \quad (15)$$

$$= \frac{n}{2} \log \left(e^{\frac{2}{n}h(\mathbf{Y}_2|W_2)} - 2\pi e(\sigma_2^2 - \sigma_1^2) \right) \quad (16)$$

$$= \frac{n}{2} \log \left(2\pi e\sigma_1^2 + 2\pi e\sigma_2^2 \left(e^{2\bar{C}_\alpha^{[j]}(\rho\mathcal{A}, \sigma_2)} - 1 \right) \right), \quad (17)$$

where the last step follows by applying (8). Plugging this inequality in (14), using $h(\mathbf{Z}_1) = \frac{n}{2} \log(2\pi e\sigma_1^2)$, dividing by n , and letting $n \rightarrow \infty$ yields the desired bound

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{\sigma_2^2}{\sigma_1^2} \left(e^{2\bar{C}_\alpha^{[b]}}(\rho^A, \sigma_2) - 1 \right) \right). \quad (18)$$

Finally, by taking the union over all $\rho \in [0, 1]$, we get the statement of the theorem. \blacksquare

To examine the tightness of this bound, we have to compare it with capacity inner bounds. Such inner bounds are derived in the next section.

IV. INNER BOUNDS

An inner bound on the capacity of the OBC can be obtained using Cover's superposition coding technique [20, Theorem 1]. For $N = 2$, this inner bound can be expressed by the region given in (2)-(3). This region is achievable by superposition coding where the $\mathbf{U} \in \mathcal{U}^n$ plays the role of the cloud center, and $\mathbf{X} \in \mathcal{X}^n$ the satellite codeword. User 2 decodes \mathbf{U} only, while user 1 decodes both \mathbf{U} and \mathbf{X} . To obtain the best inner bound, this region has to be maximized over input distributions $p(u, x)$ with $X \in [0, \mathcal{A}]$ and $\mathbb{E}[X] \leq \mathcal{E}$. Since the capacity of the IM-DD P2P channel is maximized by a discrete input distribution [12], we expect that the optimal $p(u, x)$ is also discrete. Unfortunately, the discreteness of $p(u, x)$ leads to a rate region which is not computable in closed form. We seek a simple computable rate region.

To this end, motivated by the high SNR optimality of the truncated-Gaussian (TG) input distribution for the IM-DD P2P channel [10], we derive an inner bound based on this distribution.

A. Truncated Gaussian Input

The TG distribution is described by the probability density function

$$\tilde{g}_{\mu, \nu}(x) = \begin{cases} \eta g_{\mu, \nu}(x), & x \in [0, \mathcal{A}] \\ 0 & \text{elsewhere,} \end{cases} \quad (19)$$

where

$$\eta = \frac{1}{G_{\mu, \nu}(\mathcal{A}) - G_{\mu, \nu}(0)}. \quad (20)$$

This distribution has the following mean and variance, respectively,

$$\tilde{\mu} = \nu^2(\tilde{g}_{\mu, \nu}(0) - \tilde{g}_{\mu, \nu}(\mathcal{A})) + \mu, \quad (21)$$

$$\tilde{\nu}^2 = \nu^2 \left(1 - (\mathcal{A} - \mu)\tilde{g}_{\mu, \nu}(\mathcal{A}) - \mu\tilde{g}_{\mu, \nu}(0) - \nu^2 (\tilde{g}_{\mu, \nu}(0) - \tilde{g}_{\mu, \nu}(\mathcal{A}))^2 \right). \quad (22)$$

In what follows, we use η_i , $\tilde{\mu}_i$, and $\tilde{\nu}_i$ to denote the parameters η , $\tilde{\mu}$, and $\tilde{\nu}$, respectively, corresponding to a TG distribution $\tilde{g}_{\mu_i, \nu_i}(x)$ defined over $[0, \mathcal{A}_i]$ for some i . The advantage of this distribution is that it leads to a simple rate region (similar to [20, Theorem 2]), and we expect it to be close to optimal at high SNR.

1) *Achievable Rate Region:* We first provide a general achievable rate region corresponding to this distribution, and then simplify it.

Definition 1: We denote by \mathcal{P} the set of all feasible $(\mathbf{p}_1, \mathbf{p}_2)$ given by

$$\mathcal{P} = \left\{ (\mathbf{p}_1, \mathbf{p}_2) \mid \mathbf{p}_i = (\mathcal{A}_i, \mu_i, \nu_i) \in \mathbb{R}^3, \mathcal{A}_i, \nu_i > 0, i \in \{1, 2\}, \mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}, \tilde{\mu}_1 + \tilde{\mu}_2 = \mathcal{E} \right\}.$$

Theorem 2: For some $(\mathbf{p}_1, \mathbf{p}_2) \in \mathcal{P}$, the region $\mathcal{R}_T(\mathbf{p}_1, \mathbf{p}_2)$ described by the set of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ bounded by

$$R_2 \leq \frac{1}{2} \log \left(\frac{\nu_2^2}{\tilde{\nu}_2^2} + \frac{\nu_2^2}{\tilde{\nu}_1^2 + \sigma_2^2} \right) - \phi_2, \quad (23)$$

$$R_1 \leq \frac{1}{2} \log \left(\frac{\nu_1^2}{\tilde{\nu}_1^2} + \frac{\nu_1^2}{\sigma_1^2} \right) - \phi_1, \quad (24)$$

is achievable, with $\phi_i = \log(\eta_i) + \frac{1}{2} ((\mathcal{A}_i - \mu_i) \tilde{g}_{\mu_i, \nu_i}(\mathcal{A}_i) + \mu_i \tilde{g}_{\mu_i, \nu_i}(0))$. The capacity region \mathcal{C} of the 2-user OBC is thus inner bounded by $\underline{\mathcal{C}}_T = \bigcup_{(\mathbf{p}_1, \mathbf{p}_2) \in \mathcal{P}} \mathcal{R}_T(\mathbf{p}_1, \mathbf{p}_2)$.

Proof: The transmitter encodes the codewords W_1 and W_2 to two independent signals \mathbf{X}_1 and \mathbf{X}_2 so that the instances of \mathbf{X}_i are independent and identically distributed according to the TG distribution $\tilde{g}_{\mu_i, \nu_i}(x)$ over $[0, \mathcal{A}_i]$. The transmitter sends $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$. Due to the average and peak constraints, we require $\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}$ and $\tilde{\mu}_1 + \tilde{\mu}_2 = \mathcal{E}$. Both users decode \mathbf{X}_2 , and user 1 decodes \mathbf{X}_1 after subtracting \mathbf{X}_2 from \mathbf{Y}_1 . The achievable rates can be expressed as

$$R_2 \leq I(X_2; Y_2), \quad (25)$$

$$R_1 \leq I(X_1; Y_1 | X_2), \quad (26)$$

which corresponds to choosing $U = X_2$ in (2)-(3). Consider $I(X_2; Y_2)$ first. This can be written as $h(X_2) - h(X_2 | Y_2)$. The entropy of X_2 is given by

$$h(X_2) = \frac{1}{2} \log(2\pi e \nu_2^2) - \phi_2. \quad (27)$$

On the other hand, $h(X_2 | Y_2)$ can be upper bounded by $h(\overline{X}_2 | \overline{Y}_2)$ where $(\overline{X}_2, \overline{Y}_2)$ are jointly Gaussian with covariance matrix $\begin{pmatrix} \tilde{\nu}_2^2 & \tilde{\nu}_2^2 \\ \tilde{\nu}_2^2 & \tilde{\nu}_1^2 + \tilde{\nu}_2^2 + \sigma_2^2 \end{pmatrix}$. This follows since the Gaussian distribution maximizes conditional differential entropy under a covariance constraint [21]. Thus,

$$h(X_2 | Y_2) \leq \frac{1}{2} \log \left(2\pi e \frac{\tilde{\nu}_2^2 (\tilde{\nu}_1^2 + \sigma_2^2)}{\tilde{\nu}_1^2 + \tilde{\nu}_2^2 + \sigma_2^2} \right). \quad (28)$$

Therefore, $I(X_2; Y_2) \geq \frac{1}{2} \log \left(\frac{\nu_2^2}{\tilde{\nu}_2^2} + \frac{\nu_2^2}{\tilde{\nu}_1^2 + \sigma_2^2} \right) - \phi_2$ and the rate R_2 stated in the theorem is achievable. Similarly, we can show that $I(X_1; Y_1 | X_2) \geq \frac{1}{2} \log \left(\frac{\nu_1^2}{\tilde{\nu}_1^2} + \frac{\nu_1^2}{\sigma_1^2} \right) - \phi_1$ leading to the achievable rate R_1 in theorem. This proves the achievability of $\mathcal{R}_T(\mathbf{p}_1, \mathbf{p}_2)$. By taking the union over \mathcal{P} , we obtain the capacity region inner bound $\underline{\mathcal{C}}_T$. \blacksquare

Note that the region defined by (25)-(25) can be computed numerically for the chosen input distribution. The region $\mathcal{R}_T(\mathbf{p}_1, \mathbf{p}_2)$ on the other hand serves as an inner bound which is relatively simple to compute. However, an optimal choice of the parameters \mathbf{p}_1 and \mathbf{p}_2 is difficult to find. Next, we propose \mathbf{p}_1 and \mathbf{p}_2 which lead to a simpler inner bound, within a constant gap of the outer bound $\bar{\mathcal{C}}$ at high SNR.

2) *Achievable Region Simplification:* Note that if we choose $\mu \in [0, \mathcal{A}]$ and $\nu = \frac{1}{3} \min\{\mu, \mathcal{A} - \mu\}$, the distribution $\tilde{g}_{\mu, \nu}(x)$ becomes almost identical to a Gaussian distribution $g_{\mu, \nu}(x)$, since in this case, $\eta \approx 1$. We use this to simplify $\mathcal{R}(p_1, p_2)$ as follows.

Proposition 1: The rate region $\underline{\mathcal{C}}'_T = \bigcup_{\beta \in [0, 1]} \mathcal{R}'_T(\beta)$ is achievable, where $\mathcal{R}'_T(\beta)$ is the set of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{(\alpha(1-\beta)\mathcal{A} - \epsilon_\mu)^2}{\alpha^2 \beta^2 \mathcal{A}^2 + 9\sigma_2^2} \right) - \epsilon_\phi, \quad (29)$$

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{(\alpha\beta\mathcal{A} - \epsilon_\mu)^2}{9\sigma_1^2} \right) - \epsilon_\phi, \quad (30)$$

with $\epsilon_\phi = 0.016$ and $\epsilon_\mu = 0.0044$.

Proof: First, we choose $\mathcal{A}_1 = \beta\mathcal{A}$ and $\mathcal{A}_2 = (1-\beta)\mathcal{A}$, and we fix $\tilde{\mu}_i = \alpha\mathcal{A}_i$. This guarantees $\tilde{\mu}_1 + \tilde{\mu}_2 = \mathcal{E}$. Since $\alpha \leq \frac{1}{2}$, then $\tilde{\mu}_i \leq \frac{\mathcal{A}_i}{2}$, $i = 1, 2$. Furthermore, in this case, $\mu_i \leq \tilde{\mu}_i$ and hence $\mu_i \leq \frac{\mathcal{A}_i}{2}$. Then we choose $\nu_i = \frac{\mu_i}{3}$ with $\mu_i \geq 0$. This choice leads to $\phi_i \leq \epsilon_\phi$, $\mu_i \leq \tilde{\mu}_i < \mu_i + \epsilon_\mu$, and $(1 - \epsilon_\nu)\nu_i^2 \leq \tilde{\nu}_i^2 \leq \nu_i^2$, where $\epsilon_\phi = 0.016$, $\epsilon_\mu = 0.0044$, and $\epsilon_\nu = 0.0267$ (see Appendix A). With this in mind, we can write

$$\frac{1}{2} \log \left(\frac{\nu_2^2}{\tilde{\nu}_2^2} + \frac{\nu_2^2}{\tilde{\nu}_1^2 + \sigma_2^2} \right) - \phi_2 \geq \frac{1}{2} \log \left(1 + \frac{\nu_2^2}{\nu_1^2 + \sigma_2^2} \right) - \epsilon_\phi \quad (31)$$

$$\frac{1}{2} \log \left(\frac{\nu_1^2}{\tilde{\nu}_1^2} + \frac{\nu_1^2}{\sigma_1^2} \right) - \phi_1 \geq \frac{1}{2} \log \left(1 + \frac{\nu_1^2}{\sigma_1^2} \right) - \epsilon_\phi. \quad (32)$$

Furthermore, since $\tilde{\mu}_1 = \alpha\beta\mathcal{A}$, $\nu_1 = \frac{\mu_1}{3}$, and $\mu_1 \leq \tilde{\mu}_1 \leq \mu_1 + \epsilon_\mu$, we have $\frac{\alpha\beta\mathcal{A} - \epsilon_\mu}{3} \leq \nu_1 \leq \frac{\alpha\beta\mathcal{A}}{3}$, and thus,

$$\frac{1}{2} \log \left(1 + \frac{\nu_1^2}{\sigma_1^2} \right) \geq \frac{1}{2} \log \left(1 + \frac{(\alpha\beta\mathcal{A} - \epsilon_\mu)^2}{9\sigma_1^2} \right). \quad (33)$$

Similarly, we have

$$\frac{1}{2} \log \left(1 + \frac{\nu_2^2}{\nu_1^2 + \sigma_2^2} \right) \geq \frac{1}{2} \log \left(1 + \frac{(\alpha(1-\beta)\mathcal{A} - \epsilon_\mu)^2}{\alpha^2\beta^2\mathcal{A}^2 + 9\sigma_2^2} \right), \quad (34)$$

which concludes the proof. \blacksquare

While this inner bound is simple and fairly tight at high SNR, it is not at moderate/low SNR. Next, we provide another inner bound achievable by using a discrete input distribution in the spirit of [18].

B. Discrete Input

Similar to [18], we consider discrete distributions with a uniform spacing between the mass points. That is, X_i , $i = 1, 2$, follows the distribution

$$p_{X_i}(x_i) = \sum_{k=0}^{K_i} a_{ik} \delta(x - k\ell_i), \quad (35)$$

for some K_i and ℓ_i such that $K_i\ell_i = \mathcal{A}_i$ and $\mathbb{E}[X_i] = \mathcal{E}_i$ ($\delta(\cdot)$ is the Dirac delta). The transmitter sends $X = X_1 + X_2$, and the average and peak constraints are satisfied if $\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}$ and $\mathcal{E}_1 + \mathcal{E}_2 = \mathcal{E}$.

Note that if $K+1$ is the number of mass points of X , then $K_1 + K_2 + 1 \leq K + 1 \leq (K_1 + 1)(K_2 + 1)$, where the lower bound corresponds to the case $\ell_1 = \ell_2$, and the upper bound to the case where $k_1\ell_1 \neq k_2\ell_2$ for all $k_i \in \{1, \dots, K_i\}$, $i = 1, 2$. For simplicity, we focus on the second case.

1) Sequential Distribution Optimization: To derive distributions p_{X_1} and p_{X_2} which achieve good performance, we use a sequential approach. In this approach, we first find the distribution p_{X_1} which maximizes $H(X_1)$ for a given K_1 . This problem has been solved in [18]. Then, given $H(X_1)$, we find the distribution p_{X_2} which maximizes $H(X_1 + X_2)$ for a given K_2 . This optimization determines the probabilities a_{ik} , $i \in \{1, 2\}$, $k \in \{0, \dots, K_i\}$, and allows us to evaluate the achievable rates $R_1 = I(X_1; Y_1 | X_2)$ and $R_2 = I(X_2; Y_2)$.

The reason for using this approach is the following. The achievable rate of user 1 is given by

$$R_1 = I(X_1; Y_1 | X_2) \quad (36)$$

$$= h(X_1 + Z_1) - h(Z_1) \quad (37)$$

$$\geq \frac{1}{2} \log(e^{2H(X_1)} + 2\pi e\sigma_1^2) - \frac{1}{2} \log(2\pi e\sigma_1^2) \quad (38)$$

$$= \frac{1}{2} \log \left(\frac{e^{2H(X_1)}}{2\pi e\sigma_1^2} + 1 \right), \quad (39)$$

where the inequality follows from the entropy power inequality (EPI). To maximize this rate, one should maximize $I(X_1; Y_1 | X_2)$. Due to the difficulty of this optimization problem, one can resort to

maximizing $H(X_1)$ instead. This does not necessarily maximize $I(X_1; Y_1|X_2)$, but leads to a fairly good performance [18]. Given $H(X_1)$, a similar argument can be applied to $I(X_2; Y_2)$. In particular, we have

$$R_2 = I(X_2; Y_2) \tag{40}$$

$$= h(X_1 + X_2 + Z_2) - h(X_1 + Z_2) \tag{41}$$

$$\geq \frac{1}{2} \log(e^{2H(X_1+X_2)} + 2\pi e\sigma_2^2) - h(X_1 + Z_2), \tag{42}$$

by the EPI. Now for a given p_{X_1} , the last term is fixed, and the first term can be maximized by finding p_{X_2} which maximizes the entropy of $X = X_1 + X_2$ given p_{X_1} . Note that by optimizing p_{X_1} first followed by p_{X_2} , each step of the sequential optimization procedure involves optimizing with respect to only one distribution. Due to the imposed condition $k_1\ell_1 \neq k_2\ell_2 \forall k_i \in \{1, \dots, K_i\}$, the mapping from (X_1, X_2) to $X = X_1 + X_2$ is one-to-one, and thus $H(X_1|X) = 0$. Hence,

$$H(X) = H(X, X_1) - H(X_1|X) = H(X, X_1) = H(X_1) + H(X_2). \tag{43}$$

Consequently, to maximize $H(X)$ for a given $H(X_1)$, it suffices to maximize $H(X_2)$.

The optimization problem for $H(X_i)$ can be written as

$$\max_{a_{ik}} H(X_i) \tag{44}$$

$$\text{s.t.} \quad \sum_{k=0}^{K_i} a_{ik} = 1, \quad \sum_{k=0}^{K_i} a_{ik} k \ell_1 = \mathcal{E}_i, \tag{45}$$

with $K_i \ell_i = \mathcal{A}_i$. The solution of this optimization problem has been given in [18] as

$$a_{ik} = \frac{t_i^k}{\sum_{j=0}^{K_i} t_i^j}, \quad k = 0, \dots, K_i, \tag{46}$$

where $t_i \in [0, 1]$ is the solution of $\sum_{k=0}^{K_i} \left(1 - \frac{k\mathcal{A}_i}{K_i\mathcal{E}_i}\right) t^k = 0$, which exists if $\mathcal{E}_i \leq \frac{\mathcal{A}_i}{2}$ [18]. To guarantee the existence of $t_1, t_2 \in [0, 1]$, we choose $\mathcal{E}_i = \alpha\mathcal{A}_i$ which suffices since $\alpha \leq \frac{1}{2}$.

Note that the obtained distribution maximizes the entropy of X_1 and $X_1 + X_2$ as long as $k_1\ell_1 \neq k_2\ell_2 \forall k_i \in \{1, \dots, K_i\}$. However, if this condition is not satisfied, then we still can use these distribution to obtain an achievable rate.

2) *Achievable Rate Region*: To obtain the achievable rate region corresponding to these distributions, we perform the following steps. First \mathcal{A} is split to \mathcal{A}_1 and \mathcal{A}_2 such that $\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}$. Then, we set $\mathcal{E}_i = \alpha\mathcal{A}_i$, $i = 1, 2$, and we choose $K_1, K_2 \geq 1$. Next, given the peak and average constraints \mathcal{A}_i and \mathcal{E}_i , we use (35) and (46) to obtain the distributions on X_1 and X_2 which are then used to evaluate $R_2 = I(X_2; Y_2)$ and $R_1 = I(X_1, Y_1|X_2)$. This is stated formally in the following.

Definition 2: Denote the set of feasible parameters $(\mathbf{q}_1, \mathbf{q}_2)$ by \mathcal{Q} , where

$$\mathcal{Q} = \{(\mathbf{q}_1, \mathbf{q}_2) | \mathbf{q}_i = (\mathcal{A}_i, K_i) \in \mathbb{R}_+ \times \mathbb{N}_+, i \in \{1, 2\}, \mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}\}. \quad (47)$$

Further, denote by $p_{X_1}^*$ and $p_{X_2}^*$ the distributions on $X_1 \in [0, \mathcal{A}_1]$ and $X_2 \in [0, \mathcal{A}_2]$ satisfying $\mathbb{E}[X_1] = \mathcal{E}_1 = \alpha\mathcal{A}_1$ and $\mathbb{E}[X_2] = \mathcal{E}_2 = \alpha\mathcal{A}_2$ obtained from (35) and (46).

Theorem 3: For some $(\mathbf{q}_1, \mathbf{q}_2) \in \mathcal{Q}$, the region $\mathcal{R}_D(\mathbf{q}_1, \mathbf{q}_2)$ described by the set of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ bounded by

$$R_2 \leq I(X_2; Y_2), \quad (48)$$

$$R_1 \leq I(X_1; Y_1 | X_2) \quad (49)$$

where $X_1 \sim p_{X_1}^*$ and $X_2 \sim p_{X_2}^*$ is achievable. Thus, the capacity region \mathcal{C} of the 2-user OBC is inner bounded by $\underline{\mathcal{C}}_D = \bigcup_{(\mathbf{q}_1, \mathbf{q}_2) \in \mathcal{Q}} \mathcal{R}_D(\mathbf{q}_1, \mathbf{q}_2)$.

Proof: The region $\mathcal{R}_D(\mathbf{q}_1, \mathbf{q}_2)$ is achievable by superposition coding as in [20], [22] using the distributions $p_{X_1}^*$ and $p_{X_2}^*$, and the inner bound follows by taking the union over the set of feasible parameters \mathcal{Q} . ■

In the following sections, we focus on the capacity region at high and low SNR. We start with the high SNR regime.

V. HIGH SNR ANALYSIS

The channel has high SNR if $\frac{\mathcal{E}}{\sigma^2} \gg 1$. In this case, it can be shown that the bounds $\overline{C}_\alpha^{[1]}(\mathcal{A}, \sigma)$ and $\overline{C}_\alpha^{[2]}(\mathcal{A}, \sigma)$ in Lemma 1 reduce to the simple expression $\frac{1}{2} \log(c\mathcal{A}^2/\sigma^2)$ [9], [10] where

$$c = \min \left\{ \frac{1}{2\pi e}, \frac{e\alpha^2}{2\pi} \right\}. \quad (50)$$

Thus, the following high SNR capacity upper bound holds for the IM-DD P2P channel

$$C_\alpha(\mathcal{A}, \sigma) \leq \frac{1}{2} \log \left(1 + \frac{c\mathcal{A}^2}{\sigma^2} \right) \quad \text{at high SNR.} \quad (51)$$

Using this bound, the outer bound of Theorem 1 becomes as given in the following corollary.

Corollary 1: The high-SNR capacity region of the 2-user OBC is outer bounded by $\overline{\mathcal{C}}_{\text{High SNR}} = \bigcup_{\rho \in [0, 1]} \overline{\mathcal{R}}_{\text{High SNR}}(\rho)$ where $\overline{\mathcal{R}}_{\text{High SNR}}(\rho)$ is the set of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ that satisfy

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{c\rho^2\mathcal{A}^2}{\sigma_1^2} \right) \quad (52)$$

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{c(1-\rho^2)\mathcal{A}^2}{\sigma_2^2 + c\rho^2\mathcal{A}^2} \right), \quad (53)$$

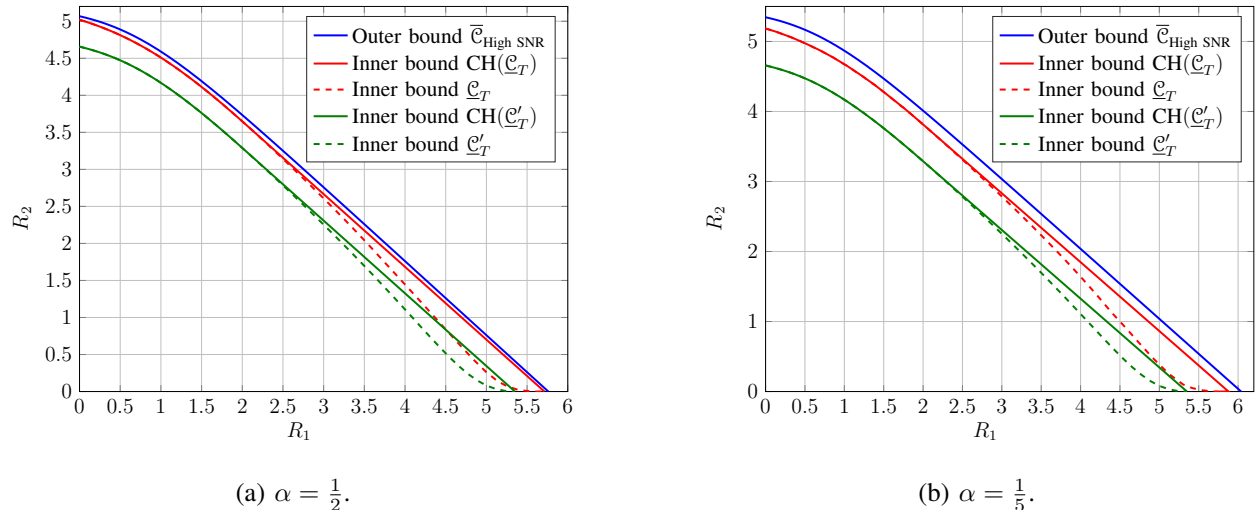


Fig. 2: Capacity region outer and inner bounds (nats per channel use) for an OBC with $\frac{\epsilon}{\sigma_2} = 25$ dB and $\sigma_2 = 2\sigma_1$. Here $\text{CH}(\cdot)$ denotes the convex-hull of a set obtained by time-sharing.

and c is given in (50).

Proof: Follows from Theorem 1 and the upper bound (51). ■

This outer bound is easily computable. Figure 2 shows this outer bound for two cases, $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{5}$, along with the truncated-Gaussian inner bounds $\underline{\mathcal{C}}_T$ and $\underline{\mathcal{C}}'_T$ (Theorem 2 and Proposition 1). The convexified inner bounds $\text{CH}(\underline{\mathcal{C}}_T)$ and $\text{CH}(\underline{\mathcal{C}}'_T)$ are also shown, where convexification is achieved by time-sharing. The achievable rate due Theorem 3 is not shown since its performance at high SNR is comparable to $\text{CH}(\underline{\mathcal{C}}_T)$. It can be seen from this figure that $\text{CH}(\underline{\mathcal{C}}_T)$ is fairly tight. The simpler inner bound $\text{CH}(\underline{\mathcal{C}}'_T)$ is not as tight, but it is useful for bounding the capacity region within a constant gap as we show in the next paragraphs. We start by characterizing the symmetric-capacity of the channel at high SNR within a constant gap.

A. High SNR Symmetric Capacity within a Constant

The symmetric-capacity is defined as the largest R so that $(R, R) \in \mathcal{C}$. We denote the high SNR symmetric-capacity by C_h . We can find an upper bound on C_h by equating the rate constraints in Corollary 1, and solving for ρ . First, we relax this upper bound slightly by dropping σ_2^2 from the

second constraint to obtain

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{c\rho^2 \mathcal{A}^2}{\sigma_1^2} \right) \quad (54)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{1}{\rho^2} \right). \quad (55)$$

Next, we equate the two bounds, and solve for ρ . By plugging the obtained ρ in one of the constraints, we obtain the symmetric-capacity upper bound

$$\bar{C}_h = \frac{1}{2} \log \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4c\mathcal{A}^2}{\sigma_1^2}} \right). \quad (56)$$

On the other hand, a symmetric-capacity lower bound can be obtained from the achievable rate region in Proposition 1. Since we are interested in the high SNR performance, we will ignore ϵ_μ and ϵ_ϕ from this rate region. Furthermore, we will neglect the impact of σ_2^2 in the constraint on R_2 at high SNR to obtain the following constraint

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{(1-\beta)^2}{\beta^2} \right), \quad (57)$$

Now, to obtain an achievable symmetric rate, we need to equate the rate constraint on R_1 from Proposition 1 and this constraint on R_2 , and solve for β . The result, denoted β^* , is given by

$$\beta^* = \sqrt{\frac{2\sigma_1}{\alpha\mathcal{A}}}, \quad (58)$$

which when substituted in one of the constraints leads to an achievable symmetric rate, and consequently the lower bound on the symmetric-capacity given by

$$\underline{C}_h = \frac{1}{2} \log \left(1 + \frac{\alpha\mathcal{A}}{3\sigma_1} \right). \quad (59)$$

This leads to the following corollary.

Corollary 2: The high SNR symmetric-capacity of the 2-user OBC C_h satisfies $\underline{C}_h \leq C_h \leq \bar{C}_h$ where \underline{C}_h and \bar{C}_h are given in (56) and (59). Furthermore, the gap between the two bounds is $\frac{1}{2} \log \left(\frac{3\sqrt{c}}{\alpha} \right) \leq 0.34$ nats per channel use.

Proof: The bounds follow from the above discussion. The gap is bounded by approximating the bounds at high SNR by $\frac{1}{2} \log \left(\frac{\alpha\mathcal{A}}{3\sigma_1} \right)$ and $\frac{1}{2} \log \left(\frac{\sqrt{c}\mathcal{A}}{\sigma_1} \right)$, and noting that their difference is $\frac{1}{2} \log \left(\frac{3\sqrt{c}}{\alpha} \right)$. For $\alpha < \frac{1}{e}$, $c = \frac{e\alpha^2}{2\pi}$ and this gap evaluates to 0.34 nats. Otherwise, if $\alpha \geq \frac{1}{e}$, then $c = \frac{1}{2\pi e}$ and this gap is maximum if $\alpha = \frac{1}{e}$, with a maximum value of 0.34 nats. ■

We recall that the achievability of this symmetric-capacity lower bound is simple. The transmitter simply splits \mathcal{A} into $\mathcal{A}_1 = \beta^*\mathcal{A}$ and $\mathcal{A}_2 = (1-\beta^*)\mathcal{A}$ where β^* is given in (58), and uses superposition coding $X = X_1 + X_2$ where X_i follows a TG distribution over $[0, \mathcal{A}_i]$ with the parameters given in Section IV-A2.

B. Symmetric Capacity versus TDMA

It is interesting to compare the achievable symmetric rate using superposition coding with that of simple time-division multiple access (TDMA), both under a truncated-Gaussian input distribution. In TDMA, the transmitter sends to user 1 a fraction $\tau \in [0, 1]$ of the time, and to user 2 in the remaining time. The OBC decomposes into two P2P channels. Over each such a channel, the transmitter uses a TG distribution over $[0, \mathcal{A}]$ with average \mathcal{E} . Denote the highest achievable rate of users i using this scheme by \bar{R}_i . According to Theorem 2, this rate can be written as

$$\bar{R}_i = \max_{\mu, \nu} \frac{1}{2} \log \left(\frac{\nu^2}{\bar{\nu}^2} + \frac{\nu^2}{\sigma_i^2} \right) - \phi, \quad (60)$$

where ϕ is as defined in Theorem 2, and the maximization is over all μ and ν such that $\tilde{\mu} = \mathcal{E}$ ($\tilde{\mu}$ is defined in (21)). Thus, we can write the achievable rate of TDMA for a given $\tau \in [0, 1]$ as

$$R_1 \leq \tau \bar{R}_1, \quad (61)$$

$$R_2 \leq (1 - \tau) \bar{R}_2. \quad (62)$$

The maximum achievable symmetric rate can be shown to be given by half the harmonic mean of \bar{R}_1 and \bar{R}_2 , i.e.,

$$R_h^{\text{TDMA}} = \frac{\bar{R}_1 \bar{R}_2}{\bar{R}_1 + \bar{R}_2}. \quad (63)$$

For $\sigma_1 = \sigma_2$, we have $\bar{R}_1 = \bar{R}_2$, and $R_h^{\text{TDMA}} = \frac{\bar{R}_1}{2}$. Note that the truncated-Gaussian distribution achieves the high SNR capacity of the P2P IM-DD channel, which can be approximated by the right-hand-side of (51) [10]. Thus, for $\sigma_1 = \sigma_2$, we have $R_h^{\text{TDMA}} \approx \frac{1}{4} \log \left(\frac{cA^2}{\sigma_1^2} \right)$ which coincides with the symmetric-capacity upper bound given in Corollary 2 at high SNR. Thus, TDMA is optimal at high SNR if $\sigma_1 = \sigma_2$.

Since R_h^{TDMA} is achieved by time-sharing between achievable rate pairs in $\underline{\mathcal{C}}_T$, then the achievable symmetric rate in $\text{CH}(\underline{\mathcal{C}}_T)$ is in general higher than R_h^{TDMA} . In fact, the achievable symmetric rate in $\underline{\mathcal{C}}_T$, $\text{CH}(\underline{\mathcal{C}}'_T)$, and $\underline{\mathcal{C}}'_T$ can also be larger than R_h^{TDMA} in some cases. Figure 3 shows the achievable symmetric rate at different values of α at high SNR. From this figure, we can see that when $\sigma_1 = \sigma_2$, then TDMA becomes nearly optimal confirming the above statement. When $\sigma_2 > \sigma_1$, then the performance of TDMA becomes worse in comparison with superposition coding. We can also note in this figure that the achievable symmetric rates due to $\underline{\mathcal{C}}_T$ and $\underline{\mathcal{C}}'_T$ are almost equal to their time-sharing variants $\text{CH}(\underline{\mathcal{C}}_T)$ and $\text{CH}(\underline{\mathcal{C}}'_T)$, respectively. Consequently, to achieve these symmetric rates, time-sharing is not necessary. The gap between $\text{CH}(\underline{\mathcal{C}}'_T)$ and the upper bound can be seen to be less than 0.34 nats, confirming the statement of Corollary 2.

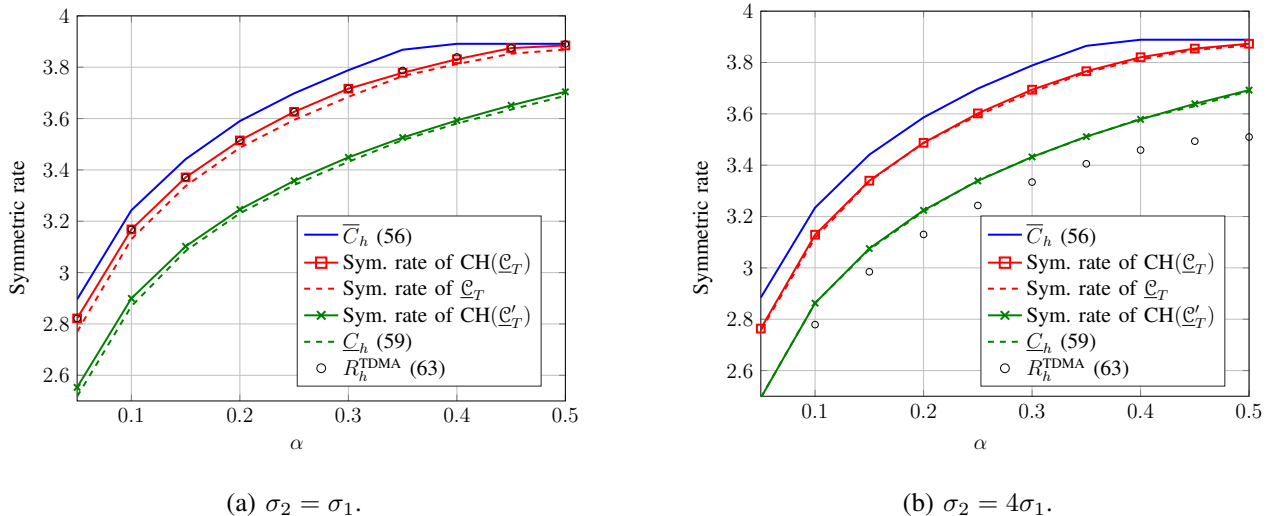


Fig. 3: Achievable symmetric rates (in nats per channel use) as a function of α for an OBC with $\frac{A}{\sigma_1} = 40$ dB. The upper bound (56) is also shown for comparison.

In conclusion, TDMA is a good strategy (close to optimal) if the receivers have similar channel qualities to the transmitter. This can be the case for instance when the receivers are computers in the same relative position to the light source in an office environment. If this is not the case, then superposition coding is better, but time-sharing is not necessary at high SNR.

Next, we extend this result to prove that \underline{C}'_T and $\bar{C}_{\text{High SNR}}$ are within a constant gap at high SNR.

C. High SNR Capacity Region within a Constant

First, we describe the boundaries of $\bar{C}_{\text{High SNR}}$ and \underline{C}'_T by a function $R_2 = f(R_1)$. We start with the outer bound whose boundary¹ is described by

$$R_2 = \frac{1}{2} \log \left(\frac{\sigma_2^2 + c\mathcal{A}^2}{\sigma_2^2 + \sigma_1^2(e^{2R_1} - 1)} \right). \quad (64)$$

On the other hand, the boundary of the inner bound \underline{C}'_T is describe by

$$R_2 = \frac{1}{2} \log \left(\frac{9\sigma_2^2 + \alpha^2\beta^2\mathcal{A}^2 + \alpha^2(1-\beta)^2\mathcal{A}^2}{9\sigma_2^2 + 9\sigma_1^2(e^{2R_1} - 1)} \right). \quad (65)$$

Hence, the gap between the two functions is

$$\Delta = \frac{1}{2} \log \left(\frac{\sigma_2^2 + c\mathcal{A}^2}{\sigma_2^2 + \frac{1}{9}\alpha^2\mathcal{A}^2(\beta^2 + (1-\beta)^2)} \right). \quad (66)$$

¹We refer here and henceforth to the Pareto-boundary, which is in this case the boundary of the region excluding the axes

Note that this gap achieves its maximum at $\beta = \frac{1}{2}$. Since the region $\underline{\mathcal{C}}'_T$ is not convex, the maximum gap corresponding to $\beta = \frac{1}{2}$ lies in the region of non-convexity of the boundary of $\underline{\mathcal{C}}'_T$. To reduce this gap, we should convexify this region by time-sharing (see Fig. 2).

To obtain the boundary of the convexified region, one has to find the value of β for which the tangent to the boundary of $\underline{\mathcal{C}}'_T$ passes through the point $(\bar{R}'_1, 0)$, where $\bar{R}'_1 = \max_{(R_1, R_2) \in \underline{\mathcal{C}}'_T} R_1$. Instead of finding this β , we use β^* given in (58). This is motivated by the above observation that the achievable symmetric rates in $\underline{\mathcal{C}}'_T$ and $\text{CH}(\underline{\mathcal{C}}'_T)$ are close at high SNR. This does not necessarily convexify the region, but suffices for our aim.

The boundary of the inner bound obtained by combining the region $\underline{\mathcal{C}}'_T$ with time sharing between $(\bar{R}'_1, 0)$ and the symmetric rate point $(\underline{C}_h, \underline{C}_h)$ is described by

$$R_2 = \frac{1}{2} \log \left(\frac{9\sigma_2^2 + \alpha^2 \beta^2 \mathcal{A}^2 + \alpha^2 (1 - \beta)^2 \mathcal{A}^2}{9\sigma_2^2 + 9\sigma_1^2 (e^{2R_1} - 1)} \right) \quad R_1 \leq \underline{C}_h \quad (67)$$

$$R_2 = \frac{\underline{C}_h (R_1 - \bar{R}'_1)}{\underline{C}_h - \bar{R}'_1}, \quad R_1 > \underline{C}_h, \quad (68)$$

where $\bar{R}'_1 = \frac{1}{2} \log \left(1 + \frac{\alpha^2 \mathcal{A}^2}{9\sigma_1^2} \right)$ (30). The first equation corresponding to condition $R_1 \leq \underline{C}_h$ describes the boundary for $\beta \in [0, \beta^*]$, and the second equations describes the remaining portion. It is worth to note that the slope of the second portion of this boundary is -1 at high SNR.

In the first portion, the gap Δ is maximum for $\beta = \beta^*$ since Δ is increasing in $\beta \in [0, \frac{1}{2}]$ and $\beta^* < \frac{1}{2}$ for \mathcal{A} large. By noting that β^* approaches 0 as SNR grows, we conclude that Δ converges to $\frac{1}{2} \log \left(\frac{9c}{\alpha^2} \right)$. For the remaining portion of the boundary, we bound the gap by noting the following. The slope of the boundary of the outer bound $\bar{\mathcal{C}}_{\text{High SNR}}$ described by (64) is in $[0, -1]$. In particular, it is equal to -1 where the outer bound meets the R_1 axis. We denote this point of intersection by $(\bar{C}_1, 0)$, where \bar{C}_1 is given by $\frac{1}{2} \log \left(1 + \frac{c\mathcal{A}^2}{\sigma_1^2} \right)$ according to (51). Therefore, the region defined by the two axes and the tangent to $\bar{\mathcal{C}}_{\text{High SNR}}$ at $(\bar{C}_1, 0)$ is an outer bound on the capacity region. This outer bound is described by the boundary

$$R_2 = -R_1 + \bar{C}_1. \quad (69)$$

Since both the inner bound (68) and this outer bound are linear with slopes -1 (at high SNR), the gap can be calculated at one of the extremes $R_1 = \underline{C}_h$ or $R_1 = \bar{R}'_1$. This gap is given by $\frac{1}{2} \log \left(\frac{9c}{\alpha^2} \right)$ as for the first portion. Consequently, we have the following corollary.

Corollary 3: The high SNR capacity region \mathcal{C} of the 2-user OBC is bounded as $\bar{\mathcal{C}}'_{\text{High SNR}} \subset \mathcal{C} \subseteq \bar{\mathcal{C}}_{\text{High SNR}}$ where $\bar{\mathcal{C}}_{\text{High SNR}}$ is defined in Corollary 1 and $\bar{\mathcal{C}}'_{\text{High SNR}}$ is defined as the set $\{(R_1, R_2) \in \mathbb{R}_+^2 | (R_1, R_2 + \delta) \in \bar{\mathcal{C}}_{\text{High SNR}}\}$ with $\delta = \frac{1}{2} \log \left(\frac{9c}{\alpha^2} \right)$. Furthermore, $\delta \leq 0.68$ nats per channel use.

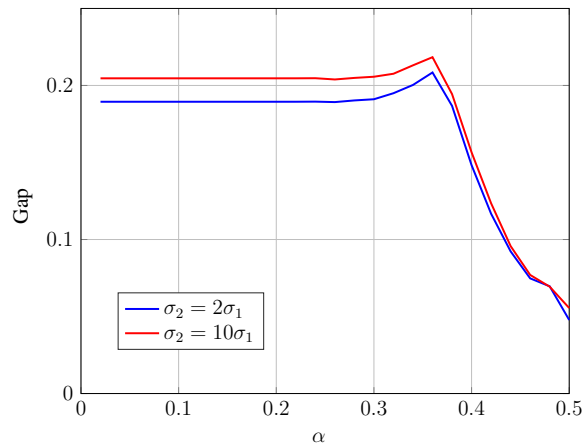


Fig. 4: The maximum gap (in nats per channel use) between the outer bound $\bar{\mathcal{C}}_{\text{High SNR}}$ and the inner bound $\text{CH}(\underline{\mathcal{C}}_T)$ in the R_2 direction for an OBC with $\frac{\epsilon}{\sigma_2} = 25$ dB.

Proof: The outer bound follows from Corollary 1. The inner bound follows from the above arguments showing that time-sharing between points in the inner bound $\underline{\mathcal{C}}'$ in Proposition 1 achieves the outer bound within a gap of $\frac{1}{2} \log\left(\frac{9c}{\alpha^2}\right)$ in the R_2 direction. This gap is equal to 0.68 nats for $\alpha < \frac{1}{e}$ and less than 0.68 nats otherwise. ■

Using this corollary, the boundary of the capacity region at high SNR lies between $f(R_1) - 0.68$ nats and $f(R_1)$ where $f(R_1)$ is the function defined by the right-hand side of (64). It is important to note at this point, that the inner bound $\underline{\mathcal{C}}_T$ (Theorem 2) combined with time-sharing, i.e., $\text{CH}(\underline{\mathcal{C}}_T)$, is closer to the outer bound than $\text{CH}(\underline{\mathcal{C}}'_T)$, and hence the gap is smaller than 0.68 nats. Fig. 4 shows the maximum over R_1 of the gap in the R_2 direction between $\bar{\mathcal{C}}_{\text{High SNR}}$ and $\underline{\mathcal{C}}_T$ at high SNR.

VI. LOW SNR ANALYSIS

The tightest upper bound on the capacity of the IM-DD P2P channel at low SNR is given by the bound $\bar{\mathcal{C}}_\alpha^{[3]}(\mathcal{A}, \sigma)$ in Lemma 1, since it coincides with the low-SNR capacity of the P2P [9]. The corresponding outer bound $\bar{\mathcal{C}}^{[3]}$ given in Theorem 1 can be described as $\bar{\mathcal{C}}^{[3]} = \bigcup_{\rho \in [0,1]} \bar{\mathcal{R}}^{[3]}(\rho)$ where $\bar{\mathcal{R}}^{[3]}(\rho)$ is the set of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ that satisfy

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{\rho^2 \alpha (1 - \alpha) \mathcal{A}^2}{\sigma_1^2} \right) \quad (70)$$

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{(1 - \rho^2) \alpha (1 - \alpha) \mathcal{A}^2}{\sigma_2^2 + \rho^2 \alpha (1 - \alpha) \mathcal{A}^2} \right). \quad (71)$$

This outer bound is tight at low SNR as we shall see next.

At low SNR, the inner bound $\underline{\mathcal{C}}_D$ becomes better than $\underline{\mathcal{C}}_T$. Unfortunately, $\underline{\mathcal{C}}_D$ does not have a closed form expression. However, we know that with the discrete distribution given in Sec. IV-B, we can achieve the capacity of the IM-DD P2P channel [9]. In particular, at low SNR, on-off keying (OOK) is optimal. This leads to the following statement.

Theorem 4: The capacity region \mathcal{C} of the 2-user OBC coincides with $\bar{\mathcal{C}}^{[3]}$ at low SNR, and is achieved by OOK and TDMA.

Proof: At low SNR ($\frac{A}{\sigma_1} \rightarrow 0$), the outer bound $\bar{\mathcal{C}}^{[3]}$ converges to the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq \frac{\rho^2 \alpha (1 - \alpha) \mathcal{A}^2}{2\sigma_1^2} \quad (72)$$

$$R_2 \leq \frac{(1 - \rho^2) \alpha (1 - \alpha) \mathcal{A}^2}{2\sigma_2^2}, \quad (73)$$

which follows by neglecting $\rho^2 \alpha (1 - \alpha) \mathcal{A}^2$ with respect to σ_2^2 , and using $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$. The boundary of this outer bound can be described by

$$R_2 = \frac{\alpha(1 - \alpha) \mathcal{A}^2}{2\sigma_2^2} - \frac{\sigma_1^2}{\sigma_2^2} R_1, \quad (74)$$

which is linear in R_1 . Now the low SNR capacity of the P2P channel formed by the transmitter and user 1 is $C_\alpha(\mathcal{A}, \sigma_1) \approx \frac{\alpha(1-\alpha)\mathcal{A}^2}{2\sigma_1^2}$ [9] achievable using OOK². Similarly, for the channel between the transmitter and user 2 we have the low SNR capacity $C_\alpha(\mathcal{A}, \sigma_2) \approx \frac{\alpha(1-\alpha)\mathcal{A}^2}{2\sigma_2^2}$. By time sharing between $C_\alpha(\mathcal{A}, \sigma_1)$ and $C_\alpha(\mathcal{A}, \sigma_2)$ with a time sharing parameter $\rho \in [0, 1]$, we achieve any point on the boundary in (74), and hence we achieve the outer bound which proves the statement of the theorem. ■

This implies that at low SNR, the simple combination of OOK and TDMA suffices for achieving the capacity region. The low SNR symmetric-capacity is given in the following corollary.

Corollary 4: The symmetric-capacity of the OBC at low SNR is equal to $C_l \approx \frac{\alpha(1-\alpha)\mathcal{A}^2}{2(\sigma_1^2 + \sigma_2^2)}$.

Proof: By setting $\rho^2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$ in Theorem 4. ■

Figure 5 shows the capacity region of the OBC at low SNR, for a setting with $\sigma_2 = 2\sigma_1$. In this figure, the achievable rate region $\underline{\mathcal{C}}_D$ is plotted with $K_1 = K_2 = 1$. Keep in mind that while this region corresponds to binary X_1 and X_2 , X is not binary but quaternary. However, the points of maximum R_1 and R_2 (intersection with the axes) correspond to $\mathcal{A}_2 = 0$ and $\mathcal{A}_1 = 0$, respectively, and the channel input X is binary at these points. Thus, the extremes of $\underline{\mathcal{C}}_D$ are achievable by OOK. The figure shows that time-sharing between these extremes is optimal.

²Here, we say that $f(x) \approx g(x)$ at $x = x_0$ if $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1$

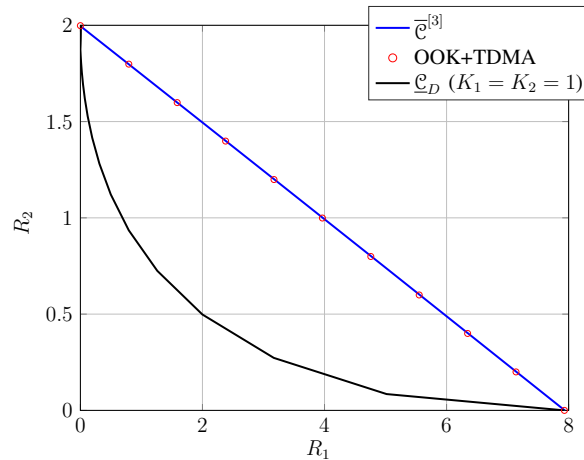


Fig. 5: Capacity at low SNR ($\times 10^{-3}$ nats per channel use) of an OBC with $\alpha = 0.2$, $\sigma_2 = 2\sigma_1$, and $\frac{A}{\sigma_1} = -5$ dB.

VII. MODERATE SNR DISCUSSION

We have seen in the last two sections that truncated-Gaussian inputs perform good at high SNR, where they achieve the capacity region within a small gap. We have also seen that at low SNR, OOK combined with TDMA is optimal. Next, we investigate the moderate SNR regime, where SNR can neither be described as high ($\frac{\mathcal{E}}{\sigma_2} \gg 1$) nor low ($\frac{A}{\sigma_1} \ll 1$). At moderate SNR, truncated-Gaussian does not perform that good. But the discrete input distribution discussed above performs relatively good.

Figure 6 shows the achievable rate regions and outer bounds for a channel with $\frac{\mathcal{E}}{\sigma_2} = 5$ dB. It can be seen that superposition coding with a discrete input distribution achieves rates fairly close to the outer bound. We have restricted each of K_1 and K_2 to $\{1, \dots, 4\}$ in Fig. 6a and to $\{1, \dots, 10\}$ in Fig. 6b. We have also plotted the achievable rate region for some values of K_1 and K_2 . This shows that even with low K_1 and K_2 , a discrete input outperforms truncated-Gaussian in the OBC at moderate SNR. The only drawback is that the achievable rate is not computable in closed form when the input distribution is discrete.

Nevertheless, the discrete input distribution is a good choice for the moderate SNR regime due to its practical simplicity (relative to a continuous distribution). Another advantage is that it achieves an acceptable symmetric rate at moderate SNR, even with a low number of mass points ($K_1 = K_2 = 1$ in Fig. 6a and $K_1 = K_2 \leq 3$ in Fig. 6b).

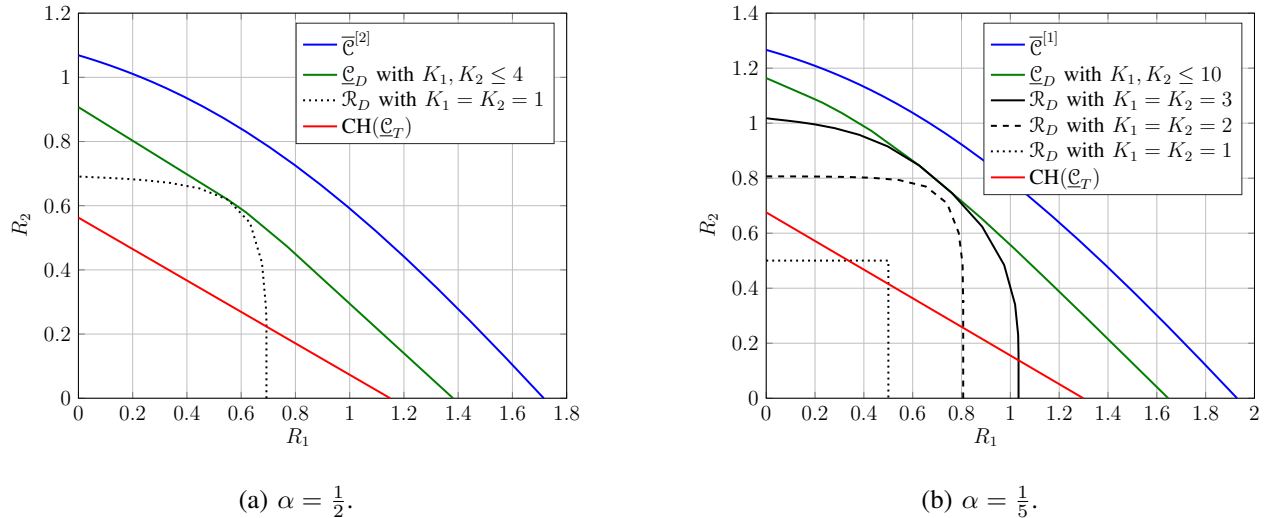


Fig. 6: Capacity region outer and inner bounds (nats per channel use) for an OBC with $\frac{\varepsilon}{\sigma_2} = 5$ dB and $\sigma_2 = 2\sigma_1$.

VIII. N -USER OBC

In this section, we extend the above results to the N -user case.

A. Outer Bound

We start with the outer bound which is expressed for the N -user case in the following theorem.

Theorem 5: The capacity region \mathcal{C} of the N -user OBC is outer bounded by $\bar{\mathcal{C}}^{[j]} = \bigcup_{\rho \in \mathcal{S}_\rho} \bar{\mathcal{R}}^{[j]}(\rho)$ where $j \in \{1, 2, 3\}$ and $\bar{\mathcal{R}}^{[j]}(\rho)$ is the set of $(R_1, \dots, R_N) \in \mathbb{R}_+^N$ that satisfy

$$R_i \leq \frac{1}{2} \log \left(\frac{\sigma_i^2 + \sigma_N^2 \left(e^{2\bar{C}_\alpha^{[j]}(\rho_{i+1} \mathcal{A}, \sigma_N)} - 1 \right)}{\sigma_i^2 + \sigma_N^2 \left(e^{2\bar{C}_\alpha^{[j]}(\rho_i \mathcal{A}, \sigma_N)} - 1 \right)} \right), \quad i \in \mathcal{N}, \quad (75)$$

$\rho = (\rho_1, \dots, \rho_N)$, and $\mathcal{S}_\rho = \{\rho \in [0, 1]^N \mid \rho_i \leq \rho_{i+1} \forall i \in \mathcal{N}, \rho_1 = 0, \rho_{N+1} = 1\}$.

Proof: The proof is similar to that of Theorem 1. Details are given in Appendix B. \blacksquare

B. Inner Bound

The achievable rate region by the truncated-Gaussian distribution is stated next for the N -user case.

Theorem 6: Let

$$\mathcal{P} = \left\{ (\mathbf{p}_1, \dots, \mathbf{p}_N) \mid \mathbf{p}_i = (\mathcal{A}_i, \mu_i, \nu_i) \in \mathbb{R}^3, \mathcal{A}_i, \nu_i > 0, i \in \mathcal{N}, \sum_{i \in \mathcal{N}} \mathcal{A}_i = \mathcal{A}, \sum_{i \in \mathcal{N}} \tilde{\mu}_i = \mathcal{E} \right\},$$

where $\tilde{\mu}_i$ is defined in (21). Then, for some $(\mathbf{p}_1, \dots, \mathbf{p}_N) \in \mathcal{P}$, the region $\mathcal{R}_T(\mathbf{p}_1, \dots, \mathbf{p}_N)$ described by the set of rate tuples $(R_1, \dots, R_N) \in \mathbb{R}_+^N$ bounded by

$$R_i \leq \frac{1}{2} \log \left(\frac{\nu_i^2}{\tilde{\nu}_i^2} + \frac{\nu_i^2}{\sum_{j=1}^{i-1} \tilde{\nu}_j^2 + \sigma_i^2} \right) - \phi_i, \quad i \in \mathcal{N}, \quad (76)$$

is achievable, where $\phi_i = \log(\eta_i) + \frac{1}{2} ((\mathcal{A}_i - \mu_i) \tilde{g}_{\mu_i, \nu_i}(\mathcal{A}_i) + \mu_i \tilde{g}_{\mu_i, \nu_i}(0))$ and $\tilde{\nu}_i$ as defined in (22). The capacity region \mathcal{C} of the N -user OBC is thus inner bounded by $\underline{\mathcal{C}}_T = \bigcup_{(\mathbf{p}_1, \dots, \mathbf{p}_N) \in \mathcal{P}} \mathcal{R}_T(\mathbf{p}_1, \dots, \mathbf{p}_N)$.

Proof: The transmitter sends $X = X_1 + X_2 + \dots + X_N$ where X_i follows a truncated-Gaussian distribution $\tilde{g}_{\mu_i, \nu_i}(x_i)$ as defined in (19). User i decodes X_N, X_{N-1}, \dots, X_i in this order. The derivation of the achievable rates is similar to the proof of Theorem 2. \blacksquare

This inner bound can be simplified similar to Proposition 1 leading to the following simplified inner bound.

Proposition 2: The rate region $\underline{\mathcal{C}}'_T = \bigcup_{\boldsymbol{\beta} \in \mathcal{S}_\beta} \mathcal{R}'_T(\boldsymbol{\beta})$ is achievable, where $\mathcal{R}'_T(\boldsymbol{\beta})$ is the set of rate tuples $(R_1, \dots, R_N) \in \mathbb{R}_+^N$ satisfying

$$R_i \leq \frac{1}{2} \log \left(1 + \frac{(\alpha \beta_i \mathcal{A} - \epsilon_\mu)^2}{\sum_{j=1}^{i-1} \alpha^2 \beta_j^2 \mathcal{A}^2 + 9\sigma_i^2} \right) - \epsilon_\phi, \quad i \in \mathcal{N}, \quad (77)$$

with $\epsilon_\phi = 0.016$, $\epsilon_\mu = 0.0044$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_N)$, and $\mathcal{S}_\beta = \{\boldsymbol{\beta} \in [0, 1]^N \mid \sum_{i \in \mathcal{N}} \beta_i = 1\}$.

Proof: Follows from Theorem 6 by using the parameters $\mathcal{A}_i = \beta_i \mathcal{A}$ where $\sum_{i \in \mathcal{N}} \beta_i = 1$, and choosing μ_i and $\nu_i = \frac{\mu_i}{3}$ such that $\tilde{\mu}_i = \alpha \mathcal{A}_i$. The derivation of the achievable rates is similar to the proof of Proposition 1. \blacksquare

Next, we extend the achievable inner bound using the discrete input distribution to the N -user case. Let $X = X_1 + X_2 + \dots + X_N$, where $X_i \in [0, \mathcal{A}_i]$ has mean $\mathcal{E}_i = \alpha \mathcal{A}_i$ and is distributed according to a discrete input distribution of the form

$$p_{X_i}^*(x_i) = \sum_{k=0}^{K_i} a_{ik} \delta(x - k\ell_i), \quad (78)$$

with

$$a_{ik} = \frac{t_i^k}{\sum_{j=0}^{K_i} t_i^j}, \quad k = 0, \dots, K_i, \quad (79)$$

$K_i \in \mathbb{N}_+$, $\ell_i = \frac{\mathcal{A}_i}{K_i}$, and $t_i \in [0, 1]$ is the solution of $\sum_{k=0}^{K_i} \left(1 - \frac{k\mathcal{A}_i}{K_i \mathcal{E}_i}\right) t^k = 0$. Furthermore, $\sum_{i \in \mathcal{N}} \mathcal{A}_i = \mathcal{A}$. User i decodes the signals X_N, \dots, X_i successively in this order. The achievable rate region is given as follows.

Theorem 7: Let

$$\mathcal{Q} = \left\{ (\mathbf{q}_1, \dots, \mathbf{q}_N) \mid \mathbf{q}_i = (\mathcal{A}_i, K_i) \in \mathbb{R}_+ \times \mathbb{N}_+, i \in \mathcal{N}, \sum_{i \in \mathcal{N}} \mathcal{A}_i = \mathcal{A} \right\}. \quad (80)$$

Then, for some $(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathcal{Q}$, the region $\mathcal{R}_D(\mathbf{q}_1, \dots, \mathbf{q}_N)$ described by the set of rate tuples $(R_1, \dots, R_N) \in \mathbb{R}_+^N$ bounded by

$$R_N \leq I(X_N; Y_N) \quad (81)$$

$$R_i \leq I(X_i; Y_i \mid X_{i+1}, \dots, X_N), \quad i \in \mathcal{N} \setminus \{N\}, \quad (82)$$

where $X_i \sim p_{X_i}^*$ defined in (78), is achievable. The capacity region \mathcal{C} of the N -user OBC is inner bounded by $\underline{\mathcal{C}}_D = \bigcup_{(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathcal{Q}} \mathcal{R}_D(\mathbf{q}_1, \dots, \mathbf{q}_N)$.

Proof: Using superposition coding as in [20], [22] with the distribution $p_{X_i}^*$ (78). \blacksquare

C. High SNR Analysis

By replacing the upper bound $\bar{C}_\alpha^{[j]}(\mathcal{A}, \sigma)$ in Theorem 5 by the high-SNR bound $\frac{1}{2} \log \left(1 + \frac{c\mathcal{A}^2}{\sigma^2} \right)$ as in (51), where c is defined in (50), we can outer bound the capacity region of the N -user OBC at high SNR as follows.

Corollary 5: The high-SNR capacity region of the N -user OBC is outer bounded by $\bar{\mathcal{C}}_{\text{High SNR}} = \bigcup_{\boldsymbol{\rho} \in \mathcal{S}_\rho} \bar{\mathcal{R}}_{\text{High SNR}}(\boldsymbol{\rho})$ where $\bar{\mathcal{R}}_{\text{High SNR}}(\boldsymbol{\rho})$ is the set of rate tuples $(R_1, \dots, R_N) \in \mathbb{R}_+^N$ that satisfy

$$R_i \leq \frac{1}{2} \log \left(\frac{\sigma_i^2 + c\rho_{i+1}^2 \mathcal{A}^2}{\sigma_i^2 + c\rho_i^2 \mathcal{A}^2} \right), \quad i \in \mathcal{N}, \quad (83)$$

with $\boldsymbol{\rho} = (\rho_1, \dots, \rho_N)$, $\mathcal{S}_\rho = \{\boldsymbol{\rho} \in [0, 1]^N \mid \rho_i \leq \rho_{i+1} \forall i \in \mathcal{N}, \rho_1 = 0, \rho_{N+1} = 1\}$, and where c is given in (50).

Proof: Follows from Theorem 5 and the upper bound (51). \blacksquare

Using this corollary, we can bound the symmetric-capacity of the N -user OBC as follows.

Corollary 6: The high-SNR symmetric-capacity of the N -user OBC C_h satisfies $\underline{C}_h \leq C_h \leq \bar{C}_h$ where

$$\underline{C}_h = \frac{1}{2} \log \left(1 + \frac{1}{2} \left(\frac{2\alpha^2 \mathcal{A}^2}{9N^2 \sigma_1^2} \right)^{\frac{1}{N}} \right), \quad (84)$$

$$\bar{C}_h = \frac{1}{2} \log \left(1 + \left(\frac{c\mathcal{A}^2}{\sigma_1^2} \right)^{\frac{1}{N}} \right). \quad (85)$$

Furthermore, the gap between the two bounds is $\frac{1}{2} \log \left(2 \left(\frac{9cN^2}{2\alpha^2} \right)^{\frac{1}{N}} \right)$ nats per channel use, which is less than or equal to $\frac{1}{2} \log \left(2 \left(\frac{9eN^2}{4\pi} \right)^{\frac{1}{N}} \right)$.

Proof: See Appendix C. ■

Note that the maximum gap $\frac{1}{2} \log \left(2 \left(\frac{9eN^2}{4\pi} \right)^{\frac{1}{N}} \right)$ decreases in N . The maximum gap at $N = 2$ is 0.86 nats per channel use, and the gap approaches $\frac{1}{2} \log(2)$ as N grows. Corollary 2 provides a tighter characterization for the case $N = 2$. The capacity region of the N -user OBC can be bounded as given next.

Corollary 7: The high SNR capacity region \mathcal{C} of the N -user OBC is bounded as $\bar{\mathcal{C}}'_{\text{High SNR}} \subset \mathcal{C} \subseteq \bar{\mathcal{C}}_{\text{High SNR}}$ where $\bar{\mathcal{C}}_{\text{High SNR}}$ is defined in Corollary 5 and $\bar{\mathcal{C}}'_{\text{High SNR}}$ is defined as the set $\{(R_1, \dots, R_N) \in \mathbb{R}_+^N | (R_1, \dots, R_{N-1}, R_N + \delta) \in \bar{\mathcal{C}}_{\text{High SNR}}\}$ with $\delta = \frac{1}{2} \log \left(\frac{9eN}{\alpha^2} \right)$. Furthermore, $\delta \leq 0.68 + \frac{1}{2} \log(N)$ nats per channel use.

Proof: See Appendix D. ■

Note that the gap here is in one direction only, namely, in the R_N direction. If we fix a tuple $\mathbf{R} = (R_1, \dots, R_{N-1}, R_N + \delta)$ on the boundary of $\bar{\mathcal{C}}_{\text{High SNR}}$, and choose an achievable rate tuple with back-off from \mathbf{R} in the i -th direction, $i \in \mathcal{N} \setminus \{N\}$, then the achievable rate of the N -th user becomes higher than R_N . The achievable rate in this case can be expressed as $(R_1 - \xi_1, \dots, R_{N-1} - \xi_{N-1}, R_N + \xi_N)$. Roughly speaking, this distributes the gap δ between the N directions, so that the gap per user becomes proportional to $\frac{1}{N} \log(N)$. The gap here is bounded without using time-sharing in combination with the inner bound in Proposition 2. Hence, this gap can be reduced by incorporating time-sharing in a manner similar to the proof of Corollary 3. Next, we consider the low SNR regime.

D. Low SNR Analysis

Using the upper bound $\bar{\mathcal{C}}_\alpha^{[3]}(\mathcal{A}, \sigma)$ given in Lemma 1 in Theorem 5, we can obtain a simple outer bound on the capacity region, which is tight at low SNR. This is stated in the following Theorem.

Theorem 8: The capacity region of the N -user OBC is outer bounded by the region $\bar{\mathcal{C}}_{\text{Low SNR}} = \bigcup_{\boldsymbol{\rho} \in \mathcal{S}_\rho} \bar{\mathcal{R}}_{\text{Low SNR}}(\boldsymbol{\rho})$ where $\bar{\mathcal{R}}_{\text{Low SNR}}(\boldsymbol{\rho})$ is the set of rate tuples $(R_1, \dots, R_N) \in \mathbb{R}_+^N$ that satisfy

$$R_i \leq \frac{1}{2} \log \left(\frac{\sigma_i^2 + \rho_{i+1}^2 \alpha (1 - \alpha) \mathcal{A}^2}{\sigma_i^2 + \rho_i^2 \alpha (1 - \alpha) \mathcal{A}^2} \right), \quad i \in \mathcal{N}, \quad (86)$$

where $\boldsymbol{\rho} = (\rho_1, \dots, \rho_N)$ and $\mathcal{S}_\rho = \{\boldsymbol{\rho} \in [0, 1]^N | \rho_i \leq \rho_{i+1} \forall i \in \mathcal{N}, \rho_1 = 0, \rho_{N+1} = 1\}$. Furthermore, at low SNR, this outer bound is tight, since it is achievable by OOK combined with TDMA.

Proof: The outer bound follows from Theorem 5 and the upper bound $\bar{\mathcal{C}}_\alpha^{[3]}(\mathcal{A}, \sigma)$ in Lemma 1. The achievability at low SNR is shown in Appendix E. ■

This concludes the high and low SNR analysis of the N -user OBC. It remains to say that at moderate SNR, the discrete input distribution performs better than the truncated-Gaussian, although its achievable rate region is not computable in closed form.

IX. CONCLUSION

We have studied the capacity region of the optical broadcast channel employing IM-DD. This channel models a situation where visible-light communication is used to send information from a source to multiple destinations. For this channel, we have derived capacity region outer and inner bounds. The outer bounds are derived by combining existing upper bounds on the capacity of the IM-DD P2P channel and Bergmans' approach. The inner bounds are achieved using superposition coding and either a truncated-Gaussian input distribution or a discrete input distribution.

We have shown that a truncated-Gaussian input achieves the capacity region within a constant gap at high SNR. The best achievable region using a truncated-Gaussian input requires time-sharing. However, as far as the symmetric-capacity is concerned, time-sharing is not necessary at high SNR. On the other hand, we have shown that on-off keying (OOK) combined with TDMA is optimal at low SNR. This is particularly interesting since both OOK and TDMA are practically simple to implement. Thus, in an office environment with strong noise for instance (strong background radiation or noisy equipment), the optimal scheme is simply OOK with TDMA. For moderate SNR, superposition coding with a discrete input distribution is better than truncated-Gaussian.

Interesting problems related to the optical broadcast channel include studying the impact of fading induced by motion of receivers (vibrations of a computer e.g.) and by reflections of light on moving objects. This can be studied using frameworks similar to [23]–[25]. Another important aspect is secrecy since broadcasting is vulnerable to eavesdropping. As such, an interesting direction for future work include investigating secure broadcast and optical wiretap channels.

APPENDIX A

BOUNDING $\tilde{\mu}$, $\tilde{\nu}$, AND ϕ FOR $\nu = \frac{\mu}{3}$

Consider a truncated-Gaussian distribution $\tilde{g}_{\mu,\nu}(x)$ defined over $[0, \mathcal{A}]$ with $\nu = \frac{\mu}{3}$ and $0 \leq \mu \leq \frac{\mathcal{A}}{2}$. We show that the mean and variance of this distribution are approximately μ and ν^2 , and that ϕ is close to zero in this case.

We start by considering $\tilde{\mu}$ which is given by

$$\tilde{\mu} = \nu^2(\tilde{g}_{\mu,\nu}(0) - \tilde{g}_{\mu,\nu}(\mathcal{A})) + \mu \quad (87)$$

$$= \frac{\eta}{\sqrt{2\pi}} \left(e^{-\frac{\mu^2}{2\nu^2}} - e^{-\frac{(\mathcal{A}-\mu)^2}{2\nu^2}} \right) + \mu, \quad (88)$$

where $\eta = \frac{1}{G_{\mu,\nu}(\mathcal{A}) - G_{\mu,\nu}(0)}$. Since $e^x > 0$, we get

$$\tilde{\mu} < \frac{\eta}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\nu^2}} + \mu = \frac{\eta}{\sqrt{2\pi}e^9} + \mu. \quad (89)$$

On the other hand, since $\mu \leq \frac{\mathcal{A}}{2}$, then $G_{\mu,\nu}(\mathcal{A}) - G_{\mu,\nu}(0) > G_{\mu,\nu}(2\mu) - G_{\mu,\nu}(0) > 0.997$, which implies that $\eta < 1.0027$. Therefore, $\tilde{\mu} < 0.0044 + \mu$. On the other hand, clearly for $\mu \leq \frac{\mathcal{A}}{2}$, $\mu \leq \tilde{\mu}$, and hence $\mu \leq \tilde{\mu} < \mu + 0.0044$. Similarly we can bound $\tilde{\nu}^2$ given by (22) as

$$\tilde{\nu}^2 = \nu^2 \left(1 - (\mathcal{A} - \mu)\tilde{g}_{\mu,\nu}(\mathcal{A}) - \mu\tilde{g}_{\mu,\nu}(0) - \nu^2 (\tilde{g}_{\mu,\nu}(0) - \tilde{g}_{\mu,\nu}(\mathcal{A}))^2 \right). \quad (90)$$

First note that $\tilde{g}_{\mu,\nu}(0) \geq \tilde{g}_{\mu,\nu}(\mathcal{A})$ for $\mu \leq \frac{\mathcal{A}}{2}$. Thus, $\nu^2 (\tilde{g}_{\mu,\nu}(0) - \tilde{g}_{\mu,\nu}(\mathcal{A}))^2 \leq (\nu\tilde{g}_{\mu,\nu}(0))^2 = \frac{\eta^2}{2\pi e^9}$. Furthermore, since the function xe^{-x^2} is decreasing for $x > \frac{1}{\sqrt{2}}$, and since $\mathcal{A} - \mu \geq \mu$ for $\mu \leq \frac{\mathcal{A}}{2}$, we have $(\mathcal{A} - \mu)\tilde{g}_{\mu,\nu}(\mathcal{A}) \leq \mu\tilde{g}_{\mu,\nu}(0) = \frac{3\eta}{\sqrt{2\pi}e^9}$. Thus,

$$\tilde{\nu}^2 \geq \nu^2 \left(1 - \frac{6\eta}{\sqrt{2\pi}e^9} - \frac{\eta^2}{2\pi e^9} \right) = 0.9733\nu^2. \quad (91)$$

On the other hand, since $\mu \geq 0$, then $\tilde{\nu}^2 \leq \nu^2$, and hence $0.9733\nu^2 \leq \tilde{\nu}^2 \leq \nu^2$. Finally, we bound ϕ given by

$$\phi = \log(\eta) + \frac{1}{2} \left((\mathcal{A} - \mu)\tilde{g}_{\mu,\nu}(\mathcal{A}) + \mu\tilde{g}_{\mu,\nu}(0) \right), \quad (92)$$

using $(\mathcal{A} - \mu)\tilde{g}_{\mu,\nu}(\mathcal{A}) \leq \mu\tilde{g}_{\mu,\nu}(0) = \frac{3\eta}{\sqrt{2\pi}e^9}$ and $\eta < 1.0027$ to obtain $\phi \leq 0.016$.

APPENDIX B

OUTER BOUND FOR THE N -USER OBC

Here, we prove Theorem 5. Similar to the proof of Theorem 1, we can write

$$h(\mathbf{Y}_N | W_N) = n\bar{\mathcal{C}}_\alpha^{[j]}(\rho_N \mathcal{A}, \sigma_N) + \frac{n}{2} \log(2\pi e \sigma_N^2) \quad (93)$$

$$= \frac{n}{2} \log \left(2\pi e \sigma_N^2 + 2\pi e \sigma_N^2 \left(e^{2\bar{\mathcal{C}}_\alpha^{[j]}(\rho_N \mathcal{A}, \sigma_N)} - 1 \right) \right), \quad (94)$$

for some $\rho_N \in [0, 1]$. On the other hand,

$$h(\mathbf{Y}_N) \leq n\bar{\mathcal{C}}_\alpha^{[j]}(\mathcal{A}, \sigma_N) + \frac{n}{2} \log(2\pi e \sigma_N^2) \quad (95)$$

$$= \frac{n}{2} \log \left(2\pi e \sigma_N^2 + 2\pi e \sigma_N^2 \left(e^{2\bar{\mathcal{C}}_\alpha^{[j]}(\mathcal{A}, \sigma_N)} - 1 \right) \right). \quad (96)$$

Therefore, we can upper bound R_N as given in (83). Now assume that $h(\mathbf{Y}_{i+1}|W_{i+1}, \dots, W_N)$ can be written similar to (94) as

$$h(\mathbf{Y}_{i+1}|W_{i+1}, \dots, W_N) = \frac{n}{2} \log \left(2\pi e \sigma_{i+1}^2 + 2\pi e \sigma_N^2 \left(e^{2\bar{c}_\alpha^{[j]}(\rho_{i+1}, \sigma_N)} - 1 \right) \right), \quad (97)$$

for some $i \in \mathcal{N} \setminus \{N\}$, and some $\rho_{i+1} \in [0, 1]$. Then, we have

$$h(\mathbf{Y}_i|W_i, \dots, W_N) \geq h(\mathbf{Y}_i|W_i, \dots, W_N, \mathbf{X}) \quad (98)$$

$$= h(\mathbf{Z}_i) \quad (99)$$

$$= \frac{n}{2} \log(2\pi e \sigma_i^2) \quad (100)$$

and

$$h(\mathbf{Y}_i|W_i, \dots, W_N) \leq h(\mathbf{Y}_i|W_{i+1}, \dots, W_N) \quad (101)$$

$$\leq \frac{n}{2} \log \left(e^{\frac{2}{n} h(\mathbf{Y}_{i+1}|W_{i+1}, \dots, W_N)} - 2\pi e (\sigma_{i+1}^2 - \sigma_i^2) \right) \quad (102)$$

$$= \frac{n}{2} \log \left(2\pi e \sigma_i^2 + 2\pi e \sigma_N^2 \left(e^{2\bar{c}_\alpha^{[j]}(\rho_{i+1}, \sigma_N)} - 1 \right) \right), \quad (103)$$

where the first step follows since conditioning reduces entropy and the second by using the entropy-power inequality, using $\sigma_{i+1}^2 \geq \sigma_i^2$. Thus, we can write

$$h(\mathbf{Y}_i|W_i, \dots, W_N) = \frac{n}{2} \log \left(2\pi e \sigma_i^2 + 2\pi e \sigma_N^2 \left(e^{2\bar{c}_\alpha^{[j]}(\rho_i, \sigma_N)} - 1 \right) \right), \quad (104)$$

for some $\rho_i \in [0, \rho_{i+1}]$. Using this equality, we can bound $h(\mathbf{Y}_i|W_{i+1}, \dots, W_N)$ as follows

$$h(\mathbf{Y}_i|W_{i+1}, \dots, W_N) \leq \frac{n}{2} \log \left(e^{\frac{2}{n} h(\mathbf{Y}_{i+1}|W_{i+1}, \dots, W_N)} - e^{\frac{2}{n} h(\tilde{\mathbf{Z}}_{i+1}|W_{i+1}, \dots, W_N)} \right) \quad (105)$$

$$= \frac{n}{2} \log \left(e^{\frac{2}{n} h(\mathbf{Y}_{i+1}|W_{i+1}, \dots, W_N)} - 2\pi e (\sigma_{i+1}^2 - \sigma_i^2) \right) \quad (106)$$

$$= \frac{n}{2} \log \left(2\pi e \sigma_i^2 + 2\pi e \sigma_N^2 \left(e^{2\bar{c}_\alpha^{[j]}(\rho_{i+1}, \sigma_2)} - 1 \right) \right), \quad (107)$$

where the first step follows by writing $\mathbf{Y}_{i+1} = \mathbf{Y}_i + \tilde{\mathbf{Z}}_{i+1}$ with $Z_{i+1} = Z_i + \tilde{Z}_{i+1}$ and $\tilde{Z}_{i+1} \sim g_{0, \sqrt{\sigma_{i+1}^2 - \sigma_i^2}}(\tilde{z}_2)$ independent of Z_i , and by using the conditional entropy-power inequality, and the last step follows by applying (97). Now we can write the following bound on R_i

$$n(R_i - \varepsilon_{in}) \leq I(W_i; \mathbf{Y}_i) \quad (108)$$

$$\leq I(W_i; \mathbf{Y}_i|W_{i+1}, \dots, W_N) \quad (109)$$

$$\leq h(\mathbf{Y}_i|W_{i+1}, \dots, W_N) - h(\mathbf{Y}_i|W_i, \dots, W_N), \quad (110)$$

which follows similar to the proof of Theorem 1. Substituting (104) and (107) in (110), and letting $n \rightarrow \infty$ yields the desired bound

$$R_i \leq \frac{1}{2} \log \left(\frac{\sigma_i^2 + \sigma_N^2 \left(e^{2\bar{C}_\alpha^{[j]}(\rho_{i+1}A, \sigma_N)} - 1 \right)}{\sigma_i^2 + \sigma_N^2 \left(e^{2\bar{C}_\alpha^{[j]}(\rho_i A, \sigma_N)} - 1 \right)} \right). \quad (111)$$

Finally, by taking the union over all $\rho_i \in [0, 1]$ with $\rho_i \leq \rho_{i+1}$, we get the statement of the Theorem 5. Note that we can set $\rho_1 = 0$ since $h(\mathbf{Y}_1|W_1, \dots, W_N) \geq h(\mathbf{Y}_1|W_1, \dots, W_N, \mathbf{X}) = h(\mathbf{Z}_1) = \frac{n}{2} \log(2\pi e\sigma_1^2)$.

APPENDIX C

SYMMETRIC-CAPACITY BOUNDS FOR THE N -USER OBC

Here, we prove Corollary 6. We start by deriving a symmetric-capacity upper bound. From Corollary 5, we have

$$R_i \leq \frac{1}{2} \log \left(\frac{\sigma_i^2 + c\rho_{i+1}^2 \mathcal{A}^2}{\sigma_i^2 + c\rho_i^2 \mathcal{A}^2} \right), \quad i \in \mathcal{N}, \quad (112)$$

where $0 \leq \rho_i \leq \rho_{i+1} \leq 1$, $\rho_1 = 0$ and $\rho_{N+1} = 1$. For convenience, we write this bounds as

$$R_i \leq \frac{1}{2} \log \left(1 + \frac{c\gamma_i^2 \mathcal{A}^2}{\sigma_i^2 + c \sum_{j=0}^{i-1} \gamma_j^2 \mathcal{A}^2} \right), \quad i \in \mathcal{N}, \quad (113)$$

where $\gamma_i^2 = \rho_{i+1}^2 - \rho_i^2$, $\gamma_0 = 0$, and $\sum_{i \in \mathcal{N}} \gamma_i^2 = 1$. Thus, the rate R_1 satisfies

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{c\gamma_1^2 \mathcal{A}^2}{\sigma_1^2} \right). \quad (114)$$

In the remaining rate constraints, we will neglect σ_i^2 in comparison to $c \sum_{j=0}^{i-1} \gamma_j^2 \mathcal{A}^2$ since we are focusing on the high SNR regime. Thus for those rates, we have

$$R_i \leq \frac{1}{2} \log \left(1 + \frac{\gamma_i^2}{\sum_{j=0}^{i-1} \gamma_j^2} \right), \quad i \in \mathcal{N} \setminus \{1\}. \quad (115)$$

To find an upper bound on the symmetric-rate, we have to equate the rate constraints for all $i \in \mathcal{N}$. This yields

$$\frac{\gamma_1^2}{\gamma^2} = \frac{\gamma_i^2}{\sum_{j=0}^{i-1} \gamma_j^2}, \quad \forall i \in \mathcal{N} \setminus \{1\}, \quad (116)$$

where $\gamma^2 = \frac{\sigma_1^2}{c\mathcal{A}^2}$. The solution of this system gives

$$\gamma_i^2 = \frac{\gamma_1^2}{\gamma^2} \left(1 + \frac{\gamma_1^2}{\gamma^2} \right)^{i-2} \gamma_1^2. \quad (117)$$

Substituting in $\sum_{i \in \mathcal{N}} \gamma_i^2 = 1$ leads to $f(\gamma_1^2) = 1$ where

$$f(\gamma_1^2) = \gamma_1^2 \left(1 + \frac{\gamma_1^2}{\gamma^2} \sum_{i=2}^N \left(1 + \frac{\gamma_1^2}{\gamma^2} \right)^{i-2} \right). \quad (118)$$

By solving this expression for $\frac{\gamma_1^2}{\gamma^2}$ and substituting in $R_1 \leq \frac{1}{2} \log \left(1 + \frac{\gamma_1^2}{\gamma^2} \right)$, we get a symmetric-capacity upper bound. Instead of solving this equation for $\frac{\gamma_1^2}{\gamma^2}$, we upper bound the solution. First, we note that $f(\gamma_1^2)$ is increasing in $\frac{\gamma_1^2}{\gamma^2}$. Thus, by lower bounding $f(\gamma_1^2)$ by some function $f_0(\gamma_1^2)$ and solving $f_0(\gamma_1^2) = 1$, we get an upper bound on the solution of $f(\gamma_1^2) = 1$. To this end, we lower bound $f(\gamma_1^2)$ as follows

$$f(\gamma_1^2) > \gamma_1^2 \left(1 + \frac{\gamma_1^2}{\gamma^2} \sum_{i=2}^N \left(\frac{\gamma_1^2}{\gamma^2} \right)^{i-2} \right) \quad (119)$$

$$= \gamma^2 \sum_{i=1}^N \left(\frac{\gamma_1^2}{\gamma^2} \right)^i \quad (120)$$

$$> \gamma^2 \left(\frac{\gamma_1^2}{\gamma^2} \right)^N \quad (121)$$

$$= f_0(\gamma_1^2). \quad (122)$$

Setting $f_0(\gamma_1^2) = 1$ yields $\frac{\gamma_1^2}{\gamma^2} = \left(\frac{1}{\gamma^2} \right)^{\frac{1}{N}}$. Therefore, the symmetric-capacity at high SNR is upper bounded by

$$\bar{C}_h = \frac{1}{2} \log \left(1 + \left(\frac{c\mathcal{A}^2}{\sigma_1^2} \right)^{\frac{1}{N}} \right), \quad (123)$$

which proves the upper bound in Corollary 6.

Now we prove the lower bound. Based in Proposition 2, the rates given by

$$R_i \leq \frac{1}{2} \log \left(1 + \frac{(\alpha\beta_i\mathcal{A} - \epsilon_\mu)^2}{\sum_{j=1}^{i-1} \alpha^2\beta_j^2\mathcal{A}^2 + 9\sigma_i^2} \right) - \epsilon_\phi, \quad i \in \mathcal{N}, \quad (124)$$

are achievable, where with $\epsilon_\phi = 0.016$, $\epsilon_\mu = 0.0044$, $\beta_i \in [0, 1]$, and $\sum_{i \in \mathcal{N}} \beta_i = 1$. Let us first neglect ϵ_ϕ and ϵ_μ . Then, we neglect $9\sigma_i^2$ in comparison to $\sum_{j=1}^{i-1} \alpha^2\beta_j^2\mathcal{A}^2$ at high SNR. This yields,

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{\alpha^2\beta_1^2\mathcal{A}^2}{9\sigma_1^2} \right) \quad (125)$$

$$R_i \leq \frac{1}{2} \log \left(1 + \frac{\beta_i^2}{\sum_{j=1}^{i-1} \beta_j^2} \right), \quad i \in \mathcal{N} \setminus \{1\}. \quad (126)$$

To find the achievable symmetric-rate, we have to equate the rate constraints as we have done above. This yields the same solution as (117), with γ_i replaced by β_i , and γ by $\beta = \frac{9\sigma_1^2}{\alpha^2\mathcal{A}^2}$. However, contrary to γ_i , the constraint on β_i is linear, given by $\sum_{i \in \mathcal{N}} \beta_i = 1$. Thus, we have $f'(\beta_1) = 1$ where

$$f'(\beta_1) = \beta_1 \left(1 + \frac{\beta_1}{\beta} \sum_{i=2}^N \left(1 + \frac{\beta_1^2}{\beta^2} \right)^{\frac{i-2}{2}} \right). \quad (127)$$

Similar to above, this function is increasing in $\beta_1 > 0$, and thus, a lower bound on the solution of $f'(\beta_1) = 1$ can be obtained by upper bounding $f'(\beta_1)$. To this end, we write

$$f'(\beta_1) < \beta_1 \left(1 + \frac{\sqrt{2}\beta_1}{\beta} \sum_{i=2}^N \left(2 \frac{\beta_1^2}{\beta^2} \right)^{\frac{i-2}{2}} \right) \quad (128)$$

$$= \frac{\beta}{\sqrt{2}} \sum_{i=1}^N \left(\frac{\beta_1 \sqrt{2}}{\beta} \right)^i \quad (129)$$

$$< N \frac{\beta}{\sqrt{2}} \left(\frac{\beta_1 \sqrt{2}}{\beta} \right)^N \quad (130)$$

$$= f'_0(\beta_1), \quad (131)$$

where we have chosen $\beta_1 > \beta$ since choosing $\beta_1 < \beta$ leads to the trivial zero symmetric-rate. Setting $f'_0(\beta_1) = 1$ yields $\frac{\beta_1}{\beta} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{N\beta} \right)^{\frac{1}{N}}$. Recall that this is a lower bound on the solution of $f'(\beta_1) = 1$. Substituting this lower bound in (125) leads to the symmetric-capacity lower bound

$$\underline{C}_h = \frac{1}{2} \log \left(1 + \frac{1}{2} \left(\frac{2\alpha^2 \mathcal{A}^2}{9N^2 \sigma_1^2} \right)^{\frac{1}{N}} \right), \quad (132)$$

which proves the lower bound in Corollary 6.

With these upper and lower bounds, bounding the gap becomes simple. We calculate the difference $\overline{C}_h - \underline{C}_h$ after using $\lim_{x \rightarrow \infty} \log(1+x) = \log(x)$, to obtain

$$\overline{C}_h - \underline{C}_h \leq \frac{1}{2} \log \left(2 \left(\frac{9cN^2}{2\alpha^2} \right)^{\frac{1}{N}} \right). \quad (133)$$

By substituting the value of c , we get the maximum gap $\frac{1}{2} \log \left(2 \left(\frac{9eN^2}{4\pi} \right)^{\frac{1}{N}} \right)$ as in Corollary 6.

APPENDIX D

CAPACITY REGION GAP FOR THE N -USER OBC

Here, we prove Corollary 7. Using (113), we can upper bound R_N for a given (R_1, \dots, R_{N-1}) by

$$R_N \leq \frac{1}{2} \log \left(\frac{\sigma_N^2 + c\mathcal{A}^2}{\sigma_N^2 + \sum_{i=1}^{N-1} \sigma_i^2 (e^{2R_i} - 1) \prod_{k=i+1}^{N-1} e^{2R_k}} \right). \quad (134)$$

Similarly, using (124) (after neglecting ϵ_ϕ and ϵ_ν), the achievable rate R_N for a given achievable (R_1, \dots, R_{N-1}) is constrained by

$$R_N \leq \frac{1}{2} \log \left(\frac{9\sigma_N^2 + \alpha^2 \sum_{i \in \mathcal{N}} \beta_i^2 \mathcal{A}^2}{9\sigma_N^2 + 9 \sum_{i=1}^{N-1} \sigma_i^2 (e^{2R_i} - 1) \prod_{k=i+1}^{N-1} e^{2R_k}} \right), \quad (135)$$

given $\sum_{i \in \mathcal{N}} \beta_i = 1$. The gap between the upper bound on R_N (134) and the achievable R_N (135) is thus

$$\Delta = \frac{1}{2} \log \left(\frac{\sigma_N^2 + c\mathcal{A}^2}{\sigma_N^2 + \frac{1}{9}\alpha^2 \sum_{i \in \mathcal{N}} \beta_i^2 \mathcal{A}^2} \right). \quad (136)$$

This gap is maximum when $\sum_{i=1}^N \beta_i^2$ is minimum, subject to $\sum_{i \in \mathcal{N}} \beta_i = 1$. This minimum is achieved if $\beta_i = \frac{1}{N}$ for all $i \in \mathcal{N}$. The corresponding maximum gap is then

$$\Delta = \frac{1}{2} \log \left(\frac{\sigma_N^2 + c\mathcal{A}^2}{\sigma_N^2 + \frac{1}{9N}\alpha^2 \mathcal{A}^2} \right). \quad (137)$$

At high SNR, this gap can be approximated by $\frac{1}{2} \log \left(\frac{9cN}{\alpha^2} \right)$. By substituting c by its value, and maximizing the gap with respect to α , we conclude that the gap is upper bounded by $\frac{1}{2} \log \left(\frac{9eN}{2\pi} \right)$. This proves Corollary 7.

APPENDIX E

CAPACITY REGION OF THE N -USER OBC AT LOW SNR

First, we write the outer bound in Theorem 8 in terms of $\gamma_i^2 = \rho_{i+1}^2 - \rho_i^2$ as follows

$$R_i \leq \frac{1}{2} \log \left(1 + \frac{\gamma_i^2 \alpha (1 - \alpha) \mathcal{A}^2}{\sigma_i^2 + \sum_{j=0}^{i-1} \gamma_j^2 \alpha (1 - \alpha) \mathcal{A}^2} \right), \quad i \in \mathcal{N}, \quad (138)$$

where $\gamma_0 = 0$ and $\sum_{i \in \mathcal{N}} \gamma_i^2 = 1$. At low SNR, ($\frac{\mathcal{A}}{\sigma_1} \rightarrow 0$), this upper bound becomes

$$R_i \leq \frac{\gamma_i^2 \alpha (1 - \alpha) \mathcal{A}^2}{2\sigma_i^2}, \quad i \in \mathcal{N}, \quad (139)$$

which follows by neglecting $\sum_{j=0}^{i-1} \gamma_j^2 \alpha (1 - \alpha) \mathcal{A}^2$ with respect to σ_i^2 and using $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$.

The boundary of this outer bound is the plane defined by

$$\sum_{i \in \mathcal{N}} \frac{2\sigma_i^2 R_i}{\alpha(1 - \alpha)\mathcal{A}^2} = 1. \quad (140)$$

This plane intersects with the R_i -axis at $R_i = \frac{\alpha(1-\alpha)\mathcal{A}^2}{2\sigma_i^2}$. This corner point of the outer bound is achievable at low SNR by using OOK [9] to transmit to user i only, i.e., $X_j = 0, \forall j \neq i$. By time sharing between the N corner points with time sharing parameters γ_i^2 , any point in the plane boundary of the outer bound is achievable. Therefore, the outer bound is tight, which proves Theorem 8.

REFERENCES

- [1] A. Chaaban, Z. Rezki, and M.-S. Alouini, "On the capacity of the 2-user IM-DD optical broadcast channel," in *submitted to IEEE Globecom Workshop on Optical Wireless Communication*, San Diego, CA, Dec. 2015.
- [2] S. Arnon, J. Barry, G. Karagiannidis, R. Schober, and M. Uysal, *Advanced Optical Wireless Communication Systems*. Cambridge University Press, 2012.
- [3] M. A. Khalighi and M. Uysal, "Survey on free space optical communications: A communication theory perspective," *IEEE Communication Surveys and Tutorials*, vol. 16, no. 4, pp. 2231–2258, 4th quarter 2014.
- [4] H. Elgala, R. Mesleh, and H. Haas, "Indoor optical wireless communication: Potential and state-of-the-art," *IEEE Comm. Magazine*, vol. 49, no. 9, pp. 56–62, Sep. 2011.
- [5] S. M. Moser, "Capacity results of an optical intensity channel with input-dependent Gaussian noise," *IEEE Trans. on Info. Theory*, vol. 58, no. 1, pp. 207–223, Jan. 2012.
- [6] M. A. Kashani and M. Uysal, "Outage performance and diversity gain analysis of free-space optical multi-hop parallel relaying," *IEEE/OSA Journal of Optical Communications and Networking*, vol. 5, no. 8, pp. 901–909, Aug. 2013.
- [7] A. A. Farid and S. Hranilovic, "Diversity gain and outage probability for MIMO free-space optical links with misalignment," *IEEE Trans. on Communications*, vol. 60, no. 2, pp. 479–487, Feb. 2012.
- [8] —, "Capacity bounds for wireless optical intensity channels with Gaussian noise," *IEEE Trans. on Info. Theory*, vol. 56, no. 12, pp. 6066–6077, Dec. 2010.
- [9] A. Lapidoth, S. M. Moser, and M. Wigger, "On the capacity of free-space optical intensity channels," *IEEE Trans. on Info. Theory*, vol. 55, no. 10, pp. 4449–4461, Oct. 2009.
- [10] A. Chaaban, J.-M. Morvan, and M.-S. Alouini, "Free-space optical communications: Capacity bounds, approximations, and a new sphere-packing perspective," *KAUST Technical Report*, <http://hdl.handle.net/10754/552096>, Apr. 2015.
- [11] C. E. Shannon, "A mathematical theory of communication," *The Bell System Technical Journal*, vol. 27, pp. 379–423, 623–656, 1948.
- [12] T. H. Chan, S. Hranilovic, and F. R. Kschischang, "Capacity-achieving probability measure for conditionally Gaussian channel with bounded input," *IEEE Trans. on Info. Theory*, vol. 51, no. 6, pp. 2073–2088, Jun. 2005.
- [13] J. M. H. Elmirghani and R. Cryan, "Hybrid PPM-CDMA systems utilising OOC for indoor wireless infrared communication," *Microwave and Opt. Tech. Lett.*, vol. 8, no. 1, pp. 44–47, Jan 1995.
- [14] B. M. Ghaffari, M. Matinfar, and J. A. Salehi, "Wireless optical CDMA LAN: Digital design concepts," *IEEE Trans. on Communications*, vol. 56, no. 12, pp. 2145–2155, Dec. 2008.
- [15] U. N. Griner and S. Arnon, "Multiuser diffuse indoor wireless infrared communication using equalized synchronous CDMA," *IEEE Trans. on Communications*, vol. 54, pp. 1654–1662, Sep. 2006.
- [16] T. M. Cover, "Broadcast channels," *IEEE Trans. on Info. Theory*, vol. IT-18, no. 1, pp. 2–14, Jan. 1972.
- [17] P. Bergmans, "A simple converse for broadcast channels with additive white Gaussian noise," *IEEE Trans. on Info. Theory*, vol. 20, no. 2, pp. 279–280, March 1974.
- [18] A. A. Farid and S. Hranilovic, "Channel capacity and non-uniform signalling for free-space optical intensity channels," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 9, pp. 1–12, Dec. 2009.
- [19] T. Cover and J. Thomas, *Elements of Information Theory (Second Edition)*. John Wiley and Sons, Inc., 2006.
- [20] T. M. Cover, "Comments on broadcast channels," *IEEE Trans. on Info. Theory*, vol. 44, no. 6, pp. 2524–2530, Oct. 1998.
- [21] J. A. Thomas, "Feedback can at most double Gaussian multiple access channel capacity," *IEEE Trans. on Info. Theory*, vol. 33, no. 5, pp. 711–716, Sep. 1987.
- [22] P. Bergmans, "Random coding theorem for broadcast channels with degraded components," *IEEE Trans. on Info. Theory*, vol. IT-19, no. 2, pp. 197–207, March 1973.

- [23] H. AlQuwaiee, I. S. Ansari, and M.-S. Alouini, "On the performance of free-space optical communication systems over double generalized Gamma channel," *IEEE Journal on Selected Areas in Communications*, May 2015.
- [24] H. Kazemi, Z. Mostaani, M. Uysal, and Z. Ghassemlooy, "Outage performance of MIMO FSO systems in Gamma-Gamma fading channels," in *Proc. of IEEE 18th European Conference on Network and Optical Communications*, Graz, Austria, Jul. 2013.
- [25] A. García-Zambrana, C. Castillo-Vázquez, and B. Castillo-Vázquez, "On the capacity of FSO links over Gamma-Gamma atmospheric turbulence channels using OOK signaling," *EURASIP J. Wirel. Commun. Netw.*, Jan. 2010.