Secure Diversity-Multiplexing Tradeoff of Zero-Forcing Transmit Scheme at Finite-SNR

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Abstract—In this paper, we address the finite Signal-to-Noise Ratio (SNR) Diversity-Multiplexing Tradeoff (DMT) of the Multiple Input Multiple Output (MIMO) wiretap channel, where a Zero-Forcing (ZF) transmit scheme, that intends to send the secret information in the orthogonal space of the eavesdropper channel, is used. First, we introduce the secrecy multiplexing gain at finite-SNR that generalizes the definition at high-SNR. Then, we provide upper and lower bounds on the outage probability under secrecy constraint, from which secrecy diversity gain estimates of ZF are derived. Through asymptotic analysis, we show that the upper bound underestimates the secrecy diversity gain, whereas the lower bound is tight at high-SNR, and thus its related diversity gain estimate is equal to the actual asymptotic secrecy diversity gain of the MIMO wiretap channel.

Index Terms—Diversity-Multiplexing tradeoff, finite-SNR, Secrecy outage probability, Secrecy capacity, MIMO wiretap, Zero-forcing transmit scheme.

I. INTRODUCTION

The wiretap channel in which a source communicates with a receiver through a discrete, memoryless channel (DMC) and a wire-tapper observes the output of this channel via another DMC, has been introduced by Wyner [1]. In this seminal work, it has been shown that if the physical channel to the eavesdropper is noisier than the channel to the legitimate receiver, then there exists an encoding-decoding scheme such that reliable communication with perfect-secrecy is possible (without the use of any encryption key). Motivated by this positive result, many other authors have recently addressed the impact of fading on secure communications. For instance, the effect of fading on secure communication for single-antenna wiretap and broadcast channels has been studied in [2]–[4] where the secrecy-capacity along with the optimal power allocation and/or rate-adaptation strategies at the source have been derived under different Channel State Information (CSI) assumptions. The secrecy-capacity of a multiple-antenna wiretap channel, with fixed channel matrices (that are hence known to the transmitter) has been addressed recently in [5]–[13].

While the ergodic capacity is meaningful in fast-fading regime, where the codeword length spans many coherence periods, it is unfortunately unachievable in slow-fading regime, where the channel remains constant during the transmission of the codeword [14]. In communication under secrecy constraint, the need for outage arises when 1) the transmitter does not have full CSI about the main channel or the eavesdroppers channel or 2) the transmitter has full CSI but is subject to delay constraints. Therefore, resorting to an outage analysis in order to derive performance limits of communication systems is the only option. To this end, it has been shown in, e.g., [15]–[17] that in a quasi-static fading channel and in contrast to the Gaussian channel, secure communication is possible even if the average Signal-to-Noise Ratio (SNR) of the main channel is less than that of the wire-tapper (or one of the wire-tappers in a multiple eavesdroppers case as discussed in [18]). Moreover, if a high level of outage is to be tolerated, then the outage secrecy rate of the fading channel can even be higher than the secrecy capacity of the Gaussian wiretap channel for similar average SNR levels.

A more global performance measure of a Multiple Input Multiple Output (MIMO) channel potential at high-SNR is the diversity-multiplexing tradeoff (DMT) of Zheng and Tse [19]. In a seminal work, they showed that in a MIMO channel, both the diversity gain and the multiplexing gain can be achieved with an optimal tradeoff. This tradeoff is a high-SNR characterization of the maximum diversity gain available at each multiplexing gain. Recently, the high-SNR secrecy DMT of a MIMO wiretap channel has been investigated in [20], where it has been shown that when all terminals perfectly know all the channels, a Zero-Forcing (ZF) transmit scheme that intends to send the secret information in the orthogonal space of the eavesdropper channel, is asymptotically (at infinite SNR) DMT-optimal.

At low to moderate SNR (typically 3 – 20 dB), where no secrecy constraint is considered, it has been shown that the asymptotic DMT is an optimistic upper bound on the finite-SNR DMT [21]. Indeed, the achievable diversity gains at realistic SNRs are significantly lower than the asymptotic values. From a communication theoretical point of view, the finite-SNR framework also emphasizes an important open problem: How to design communication schemes that are not only optimal at high-SNR, but more importantly at finite-SNR too. The aim of this paper is to analyze the finite-SNR DMT of ZF transmit scheme under secrecy constraint, where all terminals have perfect CSI. Section III presents the system model and introduces the related definitions. In section IV outage analysis is provided. Upper and lower bounds on the secrecy outage probability of ZF transmit scheme and their respective diversity gain estimates are presented in section V. Results related to a Gaussian approximation of ZF...
performances are reported in Section VII. Numerical results are reported in Section VIII. Section IX concludes the paper.

II. CHANNEL MODEL AND RELATED DEFINITIONS

We consider a Gaussian wiretap MIMO channel in which the source attempts to communicate confidential messages $w$ using $N_t$ transmit antennas to the destination with $N_m$ receive antennas. The outputs at both the legitimate destination and the eavesdropper, at symbol time index $i = 1, \ldots, L$, are expressed, respectively by:

$$
\begin{align*}
\{ y_m(i) &= H_m x(i) + n_m(i) \\
\bar{y}_e(i) &= H_e x(i) + n_e(i)
\end{align*}
$$

(1)

where $x(i) \in C^{N_t}$ is the transmitted signal, and $H_m \in C^{N_m \times N_t}$, $H_e \in C^{N_e \times N_t}$ represent the main channel and the wiretap channel gains, respectively; and $n_m(i) \in C^{N_m}$, $n_e(i) \in C^{N_e}$ are circularly symmetric white Gaussian noises with covariance matrices $E[n_m(i)n_m(i)^\dagger] = I_{N_m}$ and $E[n_e(i)n_e(i)^\dagger] = I_{N_e}$. We first assume that $H_m$ and $H_e$ are fixed and deterministic matrices and that they are both known at the transmitter and the eavesdropper. We also assume that $H_m$ is known at the legitimate receiver. The source is constrained according to an average power constraint:

$$
\frac{1}{L} \sum_{i=1}^{L} \mathbb{E}[|x(i)|^2] \leq \eta,
$$

(2)

where $\eta$ is the mean SNR per channel use at each receive antenna. We assume that CSI is available at both receivers and at the transmitter as well. The level of uncertainty about the message $w$ at the eavesdropper is measured by the equivocation rate defined by:

$$
R_e := \frac{1}{L} H(w | y_e^{(L)})
$$

(3)

where $H(w | y_e)$ is the conditional entropy of $w$ given $y$ and $y_e^{(L)} = (y_e(1), \ldots, y_e(L))$. The higher the equivocation rate is, the smaller is the knowledge of the eavesdropper about the source. We focus on a constant secrecy rate transmission for which $R = R_e$. A rate $R$ is an achievable secrecy rate if for all $\epsilon > 0$, there exists an encoder-decoder $(2L^R, L, P_e)$ for which $2L^R$ represents the number of messages to be sent to the destination, $R_e \geq R - \epsilon$ and $P_e \leq \epsilon$, where $P_e$ is the average error probability defined by:

$$
P_e = \frac{1}{2L^R} \sum_{w=1}^{2L^R} \Pr[\hat{w} \neq w],
$$

(4)

where $\hat{w}$ is the output of the decoder at the intended receiver as a result of observing $y_e^{(L)}$. Furthermore, the secrecy capacity is given by: $C_s := \sup_{R \in R_e} R$, where $R_e$ is the set of achievable secrecy rates. The secrecy capacity of the MIMO wiretap channel in nats per channel use (npcu) is given by [8–10]:

$$
C_s = \max_{\Psi(Q, H) \neq \emptyset} \left[ \Psi(Q, H_m) - \Psi(Q, H_e) \right],
$$

(5)

where $\Psi(Q, H) = \log \det (HQH^\dagger + I)$. An achievability scheme of (5) is described as follows: Let $R_s = C_s - \delta$ be the secrecy rate, for an arbitrary small $\delta > 0$. To transmit one of the $2^{L^R}$ equally likely messages $w \in \{1, 2, \ldots, 2^{L^R}\}$, the sender employs $2^{L^R}$ codes, $C_w, w = 1, 2, \ldots, 2^{L^R}$. Each code $C_w$ contains $2^{L^R} \log (Q, H_e) - \epsilon$ Gaussian codewords $x^{(L)}$ of length $L$ symbols drawn from $CN(0, Q)$. To encode a message $w$, a codeword $x^{(L)}$ is chosen uniformly at random from $C_w$ and is transmitted through the channel, using $N_t$ antennas. By a random coding argument, it can be shown that there exists at least one such a codebook with an arbitrary small $P_e$ and with the equivocation rate $R_e$ arbitrary close to $R_s$, as $L \rightarrow \infty$ [11].

Next, we consider a quasi-static fading model in which $H_m$ and $H_e$ in (1) have now independent and identically-distributed complex Gaussian entries with zero-mean and unit-variance. The notions of secrecy capacity is still meaningful if the matrices are known to all terminals. However, it loses significance when either the matrices are unknown or the transmitter has to fix a target rate due to delay constraints. In the later case, it is clear that no matter how small the secrecy rate at which we want to communicate, there is a non zero probability that the pair $(H_m, H_e)$ is not able to support it. The performance limit in this case may be captured through an outage analysis as argued in Section III.

For an SNR-dependent secrecy rate $R_s(SNR)$, the secrecy multiplexing and diversity gains are defined by:

$$
r_s = \frac{R_s}{\log(1 + g \cdot \eta)},
$$

(6)

$$
d_s(r_s, \eta) = -\eta \frac{\partial \log P_e(r_s, \eta)}{\partial \eta},
$$

(7)

where $g$ is an array gain that is chosen to guarantee a fair comparison of the outage performance with secrecy constraint, across different antenna configurations, at low to medium SNR values. Indeed, by considering the MIMO secrecy rate at low SNR, it has been shown in [22] that at low SNR, the secrecy capacity can be expressed by:

$$
C_s = \left[ \lambda_{\text{max}}(H_m^\dagger H_m - H_e^\dagger H_e) \right]^+ \cdot \eta + o(\eta),
$$

(8)

where $\lambda_{\text{max}}(H_m^\dagger H_m - H_e^\dagger H_e)$ is the maximum eigenvalue of $H_m^\dagger H_m - H_e^\dagger H_e$. Thus $g$ is chosen such that $g = \mathbb{E}\{[\lambda_{\text{max}}(H_m^\dagger H_m - H_e^\dagger H_e)]^+\}$. Note that if $\lambda_{\text{max}}(H_m^\dagger H_m - H_e^\dagger H_e) \leq 0$, then $H_m^\dagger H_m \leq H_e^\dagger H_e$ and thus the main channel is a degraded version of the eavesdropper one, which implies that the secrecy capacity is equal to zero and so is the secrecy multiplexing gain $r_s$ in (6). As defined in (6), the secrecy multiplexing gain $r_s$ provides an indication as to how the secrecy rate scales with the capacity of an Additive White Gaussian Noise (AWGN) channel with an array gain $g$, as the SNR changes. Note that for a constant secrecy multiplexing gain, the secrecy rate increases as the SNR increases. Hence, the secrecy multiplexing gain $r_s$ provides an indication of the sensitivity of a secure rate adaptation strategy as the SNR changes. On the other hand, the secrecy diversity gain $d_s(r_s, \eta)$ of a secure rate adaptive system, with a fixed multiplexing gain $r_s$ at SNR $\eta$, represents the negative slope of the log-log plot of the average error probability versus SNR. It can then be used to estimate the additional
SNR required to decrease the average error probability $P_e(r_s, \eta)$ by a specific amount for a given multiplexing gain $r_s$. This interpretation of the diversity gain is meaningful no matter how $P_e$ scales with $\eta$. However, as the SNR $\eta$ tends toward infinity, the diversity gain in (7) coincides with the asymptotic (at high-SNR) one as defined in (19). Recall that the finite-SNR multiplexing and diversity gains have been first introduced in [24] to estimate the performance limits of multiple antenna communication systems, but without secrecy constraint. By properly extending the notion of finite-SNR DMT to secure communication, we intend to provide an estimate of its performance limits, not asymptotically [20], but at realistic SNR values.

Finally, we consider perfect-secrecy communication in which we let the size of the sub-codebooks $C_w$ varies with the eavesdropper channel, i.e., $|C_w| = 2^L \Psi(Q_s, H_e)$. That is, to ensure a secrecy rate $R_s$, the transmission rate at the source is equal to:

$$ R = R_s + \Psi(Q_s, H_e). \quad (9) $$

### III. Outage Analysis and Related Background

#### A. Background

Recall that in a quasi-static channel model without secrecy constraint, a detection error happens as a result of three events: 1) the channel matrix is atypically ill-conditioned, 2) the additive noise is atypically large, or 3) some codewords are atypically close together. For large coherence block $L$ (which we assume in this paper) and using capacity-achieving codes, the randomness in the last two events is averaged out, and thus we can focus only on the bad channel event. That is, the error probability without secrecy constraint is mainly dominated by the outage events [19]. Now, introducing the secrecy constraint engenders a new error event at the legitimate receiver corresponding to the case where both the legitimate and the eavesdropper can decode the message correctly, and thus the secrecy is not achieved [25]. The later event error may be defined by:

$$ P\{ \text{secrecy is not achieved} \} = P \left\{ \frac{1}{L} H(w) \mid y_{eq}^L < R_s \right\}. $$

Hence, in order to compute the error probability of the scheme, we only need to analyze the probability that the main channel is in outage or the secrecy is not achieved. That is:

$$ P_e = P\{ \text{main channel is in outage or secrecy is not achieved} \}. \quad (10) $$

Note that (10) is rather general and does not depend on the coding scheme used at the transmitter. Therefore, in order to analyze the finite-SNR diversity-multiplexing tradeoff of the MIMO wiretap channel, a characterization of the main channel outage and “secrecy is not achieved” events is required.

When perfect CSI is available at all terminals, the outage probability has been characterized at high-SNR in [20]. The asymptotic secrecy DMT has been derived consequently. In particular, it has been shown that at high-SNR and under perfect CSI at all terminals, if $N_e < N_l$, then the secrecy DMT is a piecewise linear function joining the points $(l, d^o_s(l))$, where $l = 0, \ldots, \min(N_l - N_e, N_m)$, with $d^o_s(l)$ given by:

$$ d^o_s(l) = (N_l - N_e - l)(N_m - l). \quad (11) $$

If on the contrary $N_e \geq N_l$, then the secrecy DMT reduces to the single point $(0, 0)$. From a DMT perspective, the MIMO wiretap channel is equivalent to a reduced MIMO channel with $(N_l - N_e)$ and $N_m$ transmit and receive antennas, respectively. As the error probability is generally log-concave [26, 27], the diversity gain is an increasing function in SNR. Therefore, the finite-SNR secrecy diversity gain $d_s$ is equal to zero if $N_e < N_l$. Next, we assume that $N_e < N_l$ and we let $m = \min(N_l - N_e, N_m)$ and $k = \max(N_l - N_e, N_m)$, for notation convenience.

#### B. Outage Analysis

Since in our scheme, to transmit messages securely, we have used a family of capacity-achieving codes over the wiretap channel (please see our achievability scheme above), then by [23, Theorem 1], the secrecy is guaranteed and (10) becomes:

$$ P_e = P\{ \text{main channel is in outage} \}. \quad (12) $$

An outage occurs if the actual channel realization $(H_m, H_e)$ belongs to $O_1$, defined by: $O_1 = \left\{ H_m, H_e \mid \Psi(Q_s, H_m) < R \right\}$. Using (9), the main channel outage probability may be expressed by:

$$ P_{out} = \inf_{Q_s \geq 0, \Psi(Q_s) \geq \eta} P \left\{ \Psi(Q_s, H_m) - \Psi(Q_s, H_e) < R_s \right\}. \quad (13) $$

Since the structure of the optimal covariance matrix is generally not known for an arbitrary SNR, characterizing $P_{out}$ is still an insolvable problem. Nevertheless, any choice of $Q_s \geq 0$ that satisfies the trace constraint, provides an upper bound on the outage probability. Motivated by the results at high-SNR in [20] where it has been shown that a ZF transmit scheme is DMT-optimal, we investigate next the finite-SNR performance of the later scheme. Recall that a ZF transmit scheme consists of transmitting the secret information in the null space of the channel, which is equal to $H_e^\perp$. Hence, the input is Gaussian with covariance $\frac{1}{N_l - N_e} I_{N_l - N_e}$. This choice of $x$ provides the following upper bound on the outage probability:

$$ P_{out} \leq P_{ZF}^{out}, \quad (14) $$

where $P_{ZF}^{out} = P \left\{ \psi(H_{eq}) < R_s \right\}$, with $\psi(H_{eq}) = \log \det (I + \frac{\eta}{N_l - N_e} H_{eq} H_{eq}^H)$ and with $H_{eq} = (H_m A)$ an $N_m \times (N_l - N_e)$ channel matrix with i.i.d. complex Gaussian entries with zero-mean and unit-variance. $P_{ZF}^{out}$ represents the outage probability of an equivalent MIMO channel with $(N_l - N_e)$ and $N_m$ transmit and receive antennas, respectively, and where the input is Gaussian with covariance $\frac{1}{N_l - N_e} I_{N_l - N_e}$. From [19] we know that the $P_{ZF}^{out}$ achieves the high-SNR DMT of this MIMO channel, which is equal to $d^o_s$, given by (11), and thus ZF is also asymptotically optimal with respect to the secrecy DMT. At finite-SNR, we expect ZF to induce a power loss that may translate to a decrease of its secrecy diversity gain.
IV. ZF OUTAGE ANALYSIS AT FINITE-SNR

In this section, upper and lower bounds on the secrecy outage probability of ZF transmit scheme are derived. In order to characterize the secrecy diversity gain of this transmission scheme, these bounds are then used to provide insightful diversity estimates.

A. Upper Bound on ZF Secrecy Outage Probability

Our results is summarized in the following theorem.

**Theorem 1 (Upper bound on \( P_{out}^{ZF} \))**: An upper bound on the outage probability of ZF transmit scheme is given by:

\[
P_{out}^{ZF} \leq \prod_{l=1}^{m} \Gamma_{inc}(\xi(r), k-l+1) \left[ 1 - \prod_{l=1}^{m} \left[ 1 - \frac{\Gamma_{inc}(\xi(b_l), k-l+1)}{\Gamma_{inc}(\xi(r), k-l+1)} \right] \right],
\]

where \( \Gamma_{inc} \) is the incomplete Gamma function defined by \( \Gamma_{inc}(x, a) = \frac{1}{\Gamma(a)} \int_{0}^{x} t^{a-1} e^{-t} dt \), where \( \xi(x) = \frac{N-N_e}{N_e} (1 + g\eta)^{x-1} \) and where \( b_l, l = 1, \ldots, m \), are arbitrary positive coefficients that satisfy \( 0 \leq b_m \leq \ldots \leq b_1 \) along with \( r_s = \sum_{l=1}^{m} b_l \).

**Proof**: Let \( H = QR \) be the orthogonal triangular (QR) decomposition of the matrix \( H_{eq}A \), where \( Q \) is an \( N_m \times N_m \) unitary matrix and where \( R \) is an \( N_m \times (N_t - N_e) \) upper triangular matrix with independent entries. The square magnitudes of the diagonal entries of \( R \), \( |R_{il}|^2 \), are chi-square distributed with \( 2(k-l+1) \) degrees of freedom, \( l = 1, \ldots, m \). The off-diagonal elements of \( R \) are i.i.d. Gaussian variables, with zero mean and unit variance. Using the fact that \( \det(I + RR^t) \geq \Pi_{i=1}^{m} (1 + |R_{il}|^2) \), the following lower bound is obtained:

\[
\Psi(H_{eq}) \geq \prod_{i=1}^{m} \log \left( 1 + \frac{\eta}{N_t - N_e} |R_{il}|^2 \right) \geq \prod_{l=1}^{m} \left[ 1 - \prod_{l=1}^{m} \left[ 1 - \frac{\Gamma_{inc}(\xi(b_l), k-l+1)}{\Gamma_{inc}(\xi(r), k-l+1)} \right] \right].
\]

Then \( P_{out}^{ZF} \) can be upper bounded as follows:

\[
P_{out}^{ZF} \leq P \left[ \prod_{l=1}^{m} \log \left( 1 + \frac{\eta}{N_t - N_e} |R_{il}|^2 \right) < R_s \right] = P \left[ \log \left( 1 + \frac{\eta}{N_t - N_e} |R_{il}|^2 \right) < R_s, l = 1, \ldots, m \right] \leq P \left[ \log \left( 1 + \frac{\eta}{N_t - N_e} |R_{il}|^2 \right) \geq R_s, l = 1, \ldots, m \right].
\]

B. Lower Bound on ZF Secrecy Outage Probability

Our result is stated in the following theorem.

**Theorem 2 (Lower Bound on \( P_{out}^{ZF} \))**: A lower bound on the outage probability of ZF transmit scheme is given by:

\[
P_{out}^{ZF} \geq \prod_{i=1}^{m} \Gamma_{inc}(\xi(a_l), k - m + 2l - 1),
\]

where \( a_l, l = 1, \ldots, m \), are arbitrary positive coefficients that satisfy \( 0 \leq a_m \leq \ldots \leq a_1 \) along with \( r_s = \sum_{l=1}^{m} a_l \).

**Proof**: Similarly to the proof of Theorem 1 and using the fact that \( \det(T) \leq \Pi_{i=1}^{m} |T_{il}| \), for any nonnegative-definite matrix \( T \), the following upper bound is obtained:

\[
\Psi(H_{eq}) \leq \prod_{i=1}^{m} \log \left( 1 + \frac{\eta}{N_t - N_e} A_i \right).
\]
where $\Delta_l = \sum_{k=1}^{m} |R_{lk}|^2$, $l = 1, \ldots, m$, is the $l$th diagonal entry of $RR^T$. Then $P_{\text{out}}^Z$ can be lower bounded as follows:

$$P_{\text{out}}^Z \geq P \left[ \prod_{l=1}^{m} \left( 1 + \frac{\eta}{N_l - N_e} \Delta_l \right) < (1 + g \eta)^m \right] \geq P \left[ \prod_{l=1}^{m} \text{Prob}(\Delta_l < \xi(a_l)) \right].$$

from which the lower bound in (20) follows because $\Delta_l$ is chi-square distributed with $2(k + m - 2l + 1)$ degrees of freedom.

The lower bound in (20) is then maximized over all $a_l$, $l = 1, \ldots, m$, that satisfy $0 \leq a_m \leq \ldots \leq a_1$ along with $r_s = \sum_{l=1}^{m} a_l$ to obtain a better result.

**Corollary 2 (Secrecy Diversity Gain Estimate $\hat{d_s}$):** A secrecy diversity gain estimate $\hat{d_s}$ of ZF transmission scheme is given by:

$$\hat{d_s}(r_s, \eta) = \frac{N_t - N_e}{\eta} \sum_{l=1}^{m} f_{2l-m}(a_l)$$

**Proof:** The proof follows directly from (20) and (7).

Note that the secrecy diversity gain estimate $\hat{d_s}(r_s, \eta)$ in (22) is again a finite sum that can be easily evaluated. To conclude this section, few remarks are worthwhile:

1. When $N_t - N_e = 1$ or $N_m = 1$, the upper bound (15) and the lower bound (20) coincide, the secrecy outage probability of ZF transmit scheme is completely characterized at all SNR values, and the secrecy diversity gain is equal to:

$$d_s(r_s, \eta) = \frac{N_t - N_e}{\eta} f_1(r_s).$$

2. From the proof of Theorem 2 it is easy to see that $P_{\text{out}}^Z \leq \prod_{l=1}^{m} \Gamma_{\text{inc}}(\xi(r_s), k - l + 1)$. This “naive” upper bound is however loose as it fails to provide any positive diversity gain estimate for $r_s \geq 1$ (cf. Appendix B for details). The upper bound in (15) provides better results.

3. By (14), the upper bound in (15) is also an upper bound on the outage probability of the MIMO wiretap channel.

**V. Asymptotic Analysis**

In this section, we analyze the derived results in section [V] at low secrecy multiplexing gain regime ($r_s \to 0$), which captures the maximum secrecy diversity gain estimates for a given SNR value, and at high-SNR regime ($\eta \to \infty$), which characterizes the asymptotic DMT achieved by the upper and the lower bounds reported in Theorem 1 and Theorem 2, respectively.

**Corollary 3 (Maximum Diversity Estimates):** The maximum secrecy diversity estimates provided by $\hat{d_s}$ and $\hat{d_s}$ are respectively given by:

$$\hat{d_s}^{\text{U,max}} = \lim_{r_s \to 0} \hat{d_s}^U(r_s)$$

and

$$\hat{d_s}^{\text{L,max}} = \lim_{r_s \to 0} \hat{d_s}^L(r_s)$$

**Proof:** For convenience, the proof is presented in Appendix A. By letting $\eta$ tends toward zero in (24) and (25), it can be seen that $\hat{d_s}^{\text{L,max}}$ underestimates the maximum diversity gain at high-SNR, which is equal to $m k$; whereas, $\hat{d_s}^{\text{L,max}}$ matches the later asymptotic maximum diversity gain. In fact, at high-SNR regime, $\hat{d_s}^U$ underestimates the diversity gain at any multiplexing gain, whereas, $\hat{d_s}$ matches the asymptotic secrecy diversity gain at all multiplexing gains, as stated below.

**Theorem 3 (High-SNR Estimates of Secrecy Diversity Gain):** At high-SNR, we have:

$$\lim_{\eta \to \infty} \hat{d_s}^U(r_s) = m k \left( 1 - \frac{m - 1}{2k} \right) \left( 1 + \frac{g \eta}{(1 + g \eta) \ln(1 + g \eta)} \right)$$

$$\lim_{\eta \to \infty} \hat{d_s}^L(r_s) = m k \left( 1 - \frac{g \eta}{(1 + g \eta) \ln(1 + g \eta)} \right)$$

**Proof:** For convenience, the proof is presented in Appendix B.

**VI. ZF Outage Analysis Using A Gaussian Approximation**

Instead of the bounding techniques used in section [V], we present in this section performance of ZF transmit scheme using a Gaussian approximation. Gaussian approximation of the channel mutual information $\Psi[H_{eq}]$, initially proposed by Smith and Shafi in [28], has been widely used since then, albeit in a different context, to provide a more tractable analytical tool. While this approximation does not provide much insight as to how secure DMT of ZF transmit scheme varies with SNR and with the multiplexing gain $r_s$, the accuracy of the results as shown by simulation (cf. section [VII]), especially at finite-SNR, and the analytical tractability, both justify its presentation in this section.

**A. Outage Probability Approximation**

Recall that the mean $\mu_{eq}$ and the variance $\sigma^2_{eq}$ of $\Psi[H_{eq}]$ have been derived in integral forms by Telatar in [29] and by Smith and Shafi in [28], respectively. Kang and Alouini succeeded to obtain these quantities in closed forms [30], eqs. (34) and (35). Introducing the generalized hypergeometric function $pF_q(a; b; z)$, the Meijer’s G function $G^{a,n}_{b,n}(z; a_1, \ldots, a_p; b_1, \ldots, b_q)$, and the exponential integral function $E_n(z)$ [31], it can be shown that their expressions may be further simplified as shown at the top of the next page, where the Gamma function in (28) is
defined by: \( \Gamma(\alpha, z) = \int_0^\infty t^{\alpha-1}e^{-zt} \, dt \). Using the above Gaussian approximation, the upper bound in (14) may be approximated by:

\[
\rho_{ZFou}^{\text{est}} \approx \mathcal{Q}
\left( \frac{\mu_G - R_s}{\sqrt{\sigma_{eq}^2}} \right),
\]

(30)

where \( \mathcal{Q}(\cdot) \) represents the Gaussian Q-function.

### B. Secure DMT Approximation

Using (30) along with (7), an estimate of ZF secure diversity gain \( \hat{d}_s^G(r_s, \eta) \) (G for Gaussian) can be expressed by:

\[
\hat{d}_s^G(r_s, \eta) = \frac{\eta}{\mathcal{Q}
\left( \frac{\mu_G - R_s}{\sqrt{\sigma_{eq}^2}} \right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu_G - R_s)^2}{2\sigma_{eq}^2}} h(r_s, \eta),
\]

(31)

where \( h(r_s, \eta) \) is defined by:

\[
h(r_s, \eta) = \frac{(R_s - \mu_G)(\sigma_{eq}^2)^{-\frac{1}{2}}}{2(\sigma_{eq}^2)^{3/2}} - \frac{(R_s - \mu_G')}{(\sigma_{eq}^2)^{1/2}} - \frac{R_s - \mu_{G'}'}{(\sigma_{eq})^{1/2}},
\]

where \( f' \) represents the derivative of \( f \) with respect to \( \eta \). Expressions of \( R_s, \mu_G' \), and \( (\sigma_{eq}^2)^{1/2} \) are given by (51), (52), and (53), respectively, in Appendix C.

Using the asymptotic representation of the Q-function:

\[
\mathcal{Q}(x) \approx e^{-x^2/2}\left(1 + \frac{1}{2\sqrt{\pi}}x + o\left(\frac{1}{\sqrt{x}}\right)\right),
\]

letting \( t = \frac{\mu_G - R_s}{\sqrt{\sigma_{eq}^2}} \) for convenience, and applying the log \((\cdot)\) function to the RHS of (30), we have at high-SNR:

\[
\log \left( \mathcal{Q}(t) \right) \approx -\frac{t^2}{2} + o(t).
\]

But at high-SNR, \( t \approx (m - r_s) \log(\eta) \) [54] which combined with (52) give:

\[
\log \left( \mathcal{Q}(t) \right) \approx -\frac{(r_s - m)^2}{2} \log(\eta^2).
\]

That is, the asymptotic secrecy diversity gain estimate provided by the Gaussian approximation diverges for all multiplexing gains.

---

**VII. Numerical results**

In this section, numerical results of ZF transmit scheme for a MIMO wiretap channel with \( (N_t, N_m, N_e) = (3, 2, 1) \) and \( (N_t, N_m, N_e) = (4, 2, 1) \), are presented. In Fig. 1 and Fig. 2 ZF secrecy outage probability obtained using computer simulation, along with the upper, the lower bounds and the Gaussian approximation given respectively by (15), (20) and (30), are plotted for secrecy multiplexing gains \( r_s = 0.5 \) and \( r_s = 1 \) and for \( (N_t, N_m, N_e) = (3, 2, 1) \) and \( (N_t, N_m, N_e) = (4, 2, 1) \), respectively. As can be seen in Fig. 1 and Fig. 2 the curves corresponding to the bounds follow the same shape as the exact curves for all SNR values, supporting the results reported in Theorem 1 and Theorem 2. The lower bound curves are however closer to the exact curves than the upper bound ones, as shown in Fig. 1 and Fig. 2. The Gaussian approximation curves are enough accurate to approximate the ZF secrecy outage probability at low SNR (below 12 dB in Fig. 1 and up to 20 dB in Fig. 2). However, as SNR increases, a discrepancy between the Gaussian approximation curves and the curves obtained by simulations shows up at high-SNR, suggesting that the Gaussian approximation is somehow too pessimistic to be useful in this case. The secrecy diversity gain and its estimates given by (19), (22) and (31) are reported in Fig. 3 and Fig. 4 for different antenna configurations and different SNR values. In these figures, it can be seen that the asymptotic secrecy diversity gain provides an optimistic upper bound on the secrecy diversity gain at finite-SNR. Furthermore, the secrecy diversity estimates provided by the upper bound is closer to the exact secrecy diversity gain at high multiplexing gains \( r_s \geq 1 \). As the multiplexing gain decreases, the secrecy diversity gain estimate provided by the lower bound is more accurate and converges to the exact curves as \( r_s \to 0 \). In both Fig. 3 and Fig. 4 the secrecy diversity gain estimate provided by the Gaussian approximation is the closest to the exact curves for almost all multiplexing gains which again justify the presentation of this approximation in this work. Also shown in Fig. 4 is the asymptotic (at high-SNR) diversity estimate \( d_s^G \). Finally, the asymptotic behavior of the diversity estimates is illustrated in Fig. 5 where the diversity estimate curves are plotted for an SNR \( \eta = 60 \) dB. Figure 5 confirms the results presented in Theorem 3.
VIII. Conclusion

In this paper, the finite-SNR DMT of ZF transmit scheme under secrecy constraint has been analyzed. Performance limits of such a transmit scheme have been characterized in terms of secrecy outage probability, where analytically tractable upper and lower bounds have been derived. Related secrecy diversity gain estimates have been provided. These diversity estimates give insight as to how the performance of ZF varies with SNR and with the secrecy multiplexing gain. Through asymptotic analysis, we characterize ZF transmit scheme in some limit conditions. Motivated by its accuracy, we have also presented performance of the Gaussian approximation and its related diversity estimates. Simulation results have been provided to confirm the accuracy of the finite-SNR DMT characterizations. Furthermore, beside the fact that the diversity estimates can be used to characterize the potential limit of a particular transmit scheme, it can also be used, for instance, as a metric to improve error performance of certain adaptive communications under secrecy constraint.

Appendix A

Proof of Corollary 1

First, note that since \( r_s \) tends toward zero, then so are all \( b_l, l = 1, \ldots, m \). Then, it is easy to verify that:

\[
\lim_{r_s \to 0} \frac{\beta l}{\alpha l} \left( f(b_l) - f(r_s) \right) \prod_{k=1}^{m} \left( 1 - \frac{\beta l}{\alpha l} \right) = 0. \tag{34}
\]

This follows from the facts that:

\[
0 \leq \frac{\beta l}{\alpha l} \left( 1 - \frac{\beta l}{\alpha l} \right) \leq 1, \tag{35}
\]

along with:

\[
\lim_{b_i \to 0} f(b_i) = \left( 1 - \frac{g \eta}{(1 + g \eta)} \log(1 + g \eta) \right) (k - l + 1), \tag{36}
\]

\[
\lim_{r_s \to 0} f(r_s). \tag{37}
\]

Equation (34) implies that this term does not provide any diversity contribution at low secrecy multiplexing gain. Using
This problem is convex and can be easily solved by the Lagrangian method. We skip the details and give below the optimal solution at high-SNR:

\[(b_l - r_s)(k - l + 1) = -\delta, \quad (43)\]

where \(\delta = \frac{m - r_s}{\sum_{l=1}^{m} r_l}\), for all \(b_l, l = 1, \ldots, m\). Thus, the asymptotic diversity gain provided by the second term on the RHS of (15) is equal to \(\delta\). Summing the two diversity gains, we have:

\[\lim_{\eta \to \infty} d_s^U(r_s) = (1 - r_s) \sum_{l=1}^{m} (k - l + 1) + \frac{m - 1}{\sum_{l=1}^{m} r_l} r_s, \quad (44)\]

for \(r_s \in [0, 1]\), from which the first relation in (26) is immediate.

- \(r_s \geq 1\)

On the contrary to the first case, when \(\eta \to \infty\), \(\alpha_l\) tends toward 1, \(l = 1, \ldots, m\), which implies that the first term \((\prod_{l=1}^{m} a_l)\) on the right hand side (RHS) of (15) does not provide any diversity gain in this case. Hence, minimizing the upper bound (13) is equivalent to solving the following optimization problem at high-SNR:

\[
\begin{align*}
\min_{b_l} & \sum_{l=1}^{m} \eta^{(b_l-1)(k-l+1)} \\
\text{s.t.} & \sum_{l=1}^{m} b_l = r_s, \\
& 0 \geq b_1 \geq \ldots \geq b_m.
\end{align*}
\]

(45)

which can be solved similarly to the first case to obtain the following optimal solution:

\[(b_l - 1)(k - l + 1) = -\gamma, \quad (46)\]

where \(\gamma = \frac{m - r_s}{\sum_{l=1}^{m} r_l}\) for all \(b_l, l = 1, \ldots, m\). Hence, for \(r_s \in [1, m]\), the asymptotic secrecy diversity gain is equal to \(\gamma\), which proves the second relation in (26) too.

- \(r_s < 1\)

First, it can be easily verified that:

\[
\begin{align*}
\forall i \in N, f_i(r_s) &= \frac{\eta}{N_i - N_r} (k - l + 1) (1 - r_s) + o(\eta) \\
\forall i \in N, f_i(b_l) &= \frac{\eta}{N_i - N_r} (k - l + 1) (1 - b_l) + o(\eta).
\end{align*}
\]

(47)

(48)

Thus the diversity contribution of the first term \(\frac{N_r}{\eta} \sum_{i=1}^{m} f_i(r_s)\) on the RHS of (19) is equal to \((1 - r_s) \sum_{l=1}^{m} (k - l + 1)\). We are then left to analyze the second term on the RHS of (19). Combining (40), (41), (47) and (48), along with the fact that \(\prod_{l=1}^{m} (1 - b_l) \to 1\), as \(\eta\) tends toward infinity, the limit as \(\eta \to \infty\), of the diversity estimate in (19) can be expressed by:

\[
\lim_{\eta \to \infty} d_s^U(r_s) = (1 - r_s) \sum_{l=1}^{m} (k - l + 1) + \frac{m - 1}{\sum_{l=1}^{m} r_l} r_s, \quad (49)
\]

This problem is convex and can be easily solved by the Lagrangian method. We skip the details and give below the optimal solution at high-SNR:

\[(b_l - r_s)(k - l + 1) = -\delta, \quad (43)\]

where \(\delta = \frac{m - r_s}{\sum_{l=1}^{m} r_l}\), for all \(b_l, l = 1, \ldots, m\). Thus, the asymptotic diversity gain provided by the second term on the RHS of (15) is equal to \(\delta\). Summing the two diversity gains, we have:

\[\lim_{\eta \to \infty} d_s^U(r_s) = (1 - r_s) \sum_{l=1}^{m} (k - l + 1) + \frac{m - 1}{\sum_{l=1}^{m} r_l} r_s, \quad (44)\]

for \(r_s \in [0, 1]\), from which the first relation in (26) is immediate.

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On the contrary to the first case, when \(\eta \to \infty\), \(\alpha_l\) tends toward 1, \(l = 1, \ldots, m\), which implies that the first term \((\prod_{l=1}^{m} a_l)\) on the right hand side (RHS) of (15) does not provide any diversity gain in this case. Hence, minimizing the upper bound (13) is equivalent to solving the following optimization problem at high-SNR:

\[
\begin{align*}
\min_{b_l} & \sum_{l=1}^{m} \eta^{(b_l-1)(k-l+1)} \\
\text{s.t.} & \sum_{l=1}^{m} b_l = r_s, \\
& 0 \geq b_1 \geq \ldots \geq b_m.
\end{align*}
\]

(45)

which can be solved similarly to the first case to obtain the following optimal solution:

\[(b_l - 1)(k - l + 1) = -\gamma, \quad (46)\]

where \(\gamma = \frac{m - r_s}{\sum_{l=1}^{m} r_l}\) for all \(b_l, l = 1, \ldots, m\). Hence, for \(r_s \in [1, m]\), the asymptotic secrecy diversity gain is equal to \(\gamma\), which proves the second relation in (26) too.

- \(r_s < 1\)

First, it can be easily verified that:

\[
\begin{align*}
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\forall i \in N, f_i(b_l) &= \frac{\eta}{N_i - N_r} (k - l + 1) (1 - b_l) + o(\eta).
\end{align*}
\]

(47)

(48)

Thus the diversity contribution of the first term \(\frac{N_r}{\eta} \sum_{i=1}^{m} f_i(r_s)\) on the RHS of (19) is equal to \((1 - r_s) \sum_{l=1}^{m} (k - l + 1)\). We are then left to analyze the second term on the RHS of (19). Combining (40), (41), (47) and (48), along with the fact that \(\prod_{l=1}^{m} (1 - b_l) \to 1\), as \(\eta\) tends toward infinity, the limit as \(\eta \to \infty\), of the diversity estimate in (19) can be expressed by:

\[
\lim_{\eta \to \infty} d_s^U(r_s) = (1 - r_s) \sum_{l=1}^{m} (k - l + 1) + \frac{m - 1}{\sum_{l=1}^{m} r_l} r_s, \quad (49)
\]
\[ R' = \frac{tN_r}{1 + N_r \eta} + \frac{r_s N_m}{1 + N_m \eta} \]  
(51)

\[ \mu'_{eq} = \sum_{k=0}^{m-1} \frac{\mu'_{eq}}{k!(n-m+k)!} \sum_{l=0}^{2k} \frac{\Gamma(l-k)(n-m)!}{\Gamma(-k)!} \times \sum_{q=1}^{n-m+q+1} (-l + m - n + q - 1) \eta^{-l+m-n+q-2} \Gamma(-l + m - n + q - 1, \frac{1}{\eta}) \]
(52)

\[ (\sigma_{eq}^2)' = \sum_{k=0}^{m-1} \frac{\sigma_{eq}^2}{(n-m+k)!} \sum_{l=0}^{2k} \frac{2 \Gamma(l-k)(n-m)!}{\Gamma(-k)!} \times 3F_2(-k,-l,-l+m-n;k-l+1,-m+n+1;-1) \]

\[ - \frac{\sigma_{eq}^2}{\eta^2 k(n-m+k)!} \sum_{l=0}^{2k} \frac{2 \Gamma(l-k)(n-m)!}{\Gamma(-k)!} \times 3F_2(-k,-l,-l+m-n;k-l+1,-m+n+1;-1) \]

\[ \times \sum_{q=0}^{l-1} \frac{(-1)^q \eta^q}{n+q} \left( G_{3,4}^{1,0} \left( \frac{1}{\eta} \mid l + m - n + 1, q, q, q \right) - G_{2,3}^{3,0} \left( \frac{1}{\eta} \mid l + m - n + 1, q, q \right) \right) \]

\[ - \frac{\sigma_{eq}^2}{\eta^2 (n-m+k)!} \sum_{l=0}^{2k} \frac{2 \Gamma(l-k)(n-m)!}{\Gamma(-k)!} \times 3F_2(-k,-l,-l+m-n;k-l+1,-m+n+1;-1) \]

\[ \times \sum_{q=0}^{l-1} \frac{(-1)^q \eta^q}{n+q} \left( G_{3,4}^{1,0} \left( \frac{1}{\eta} \mid l + m - n + 1, q, q, q \right) \right) \]

\[ - \frac{m+1}{2} \sum_{l=0}^{m+1} \sum_{j=0}^{l} \frac{\eta^2 (n-m+k)!}{j!} \left( \Gamma(k-i) \Gamma(l-j)(n-m)! \Gamma(l+m+n+1) \Gamma(l-m+n+1) \sum_{q=1}^{k+l+m+n+1} E_{k+l-m-n+q+2} \left( \frac{1}{\eta} \right) \right) \]

\[ \times \frac{\eta^2}{k+l+m+n+1} \sum_{l=0}^{k+l+m+n+1} \Gamma(k-i) \Gamma(l-j)(n-m+k)! \Gamma(-l) \Gamma(l-j)(n-m+k)! \Gamma(l+m+n+1) \Gamma(l-m+n+1) \sum_{q=1}^{k+l+m+n+q+2} E_{k+l-m-n+q+2} \left( \frac{1}{\eta} \right) \]

\[ \times \frac{\eta^2}{k+l+m+n+1} \sum_{l=0}^{k+l+m+n+1} \Gamma(k-i) \Gamma(l-j)(n-m+k)! \Gamma(-l) \Gamma(l-j)(n-m+k)! \Gamma(l+m+n+1) \Gamma(l-m+n+1) \sum_{q=1}^{k+l+m+n+q+2} E_{k+l-m-n+q+2} \left( \frac{1}{\eta} \right) \]

(53)

Recall that \( b_i \)'s in (40) are those that solve (42), and thus by (43), we have:

\[ \lim_{\eta \to \infty} d_{r_s} = (1 - r_s) \sum_{l=1}^{m} (k - l + 1) + \delta, \]  
(50)

from which the first relation in (26) follows.

\( r_s \geq 1 \)

To prove (27), we use the fact that when \( \eta \to \infty, a_i \) tends toward \( 1, l = 1, \ldots, m \), along with the fact that \( f_i(r_s) = o(\eta^{-N}) \), for all \( N \geq 0 \). The proof is then completed using similar machinery than the one utilized in the first case.

**Appendix C**

**Expressions of \( R'_r, \mu'_{eq} \) and \( (\sigma_{eq}^2)' \)**

Expressions of \( R'_r, \mu'_{eq} \) and \( (\sigma_{eq}^2)' \) are given by (51), (52) and (53), respectively, at the top of the page.

**References**


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