Analysis of the Nonlinear Static and Dynamic Behavior of Offshore Structures

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ABSTRACT

Analysis of the Nonlinear Static and Dynamic Behavior of Offshore Structures

Feras Khaled Alfosail

Understanding static and dynamic nonlinear behavior of pipes and risers is crucial for the design aspects in offshore engineering fields. In this work, we examine two nonlinear problems in offshore engineering field: vortex induced vibration of straight horizontal pipes, and boundary layer static solution of inclined risers. In the first study, we analyze the effect of the internal velocity of straight horizontal pipe and obtain the vortex induced vibration forces via coupling the pipe equation of motion with the recently modified Van Der Pol oscillator governing the lift coefficient. Our numerical results are obtained for two different pipe configurations: hinged-hinged, and clamped-clamped. The results show that the internal velocity reduces the vibration and the oscillation amplitudes. Also, it is shown that the clamped-clamped pipe configuration offers a wider range of internal velocities before buckling instability occurs. The results also demonstrate the effect of the end condition on the amplitudes of vibration. In the second study, we develop a boundary layer perturbation static solution to govern and simulate the static behavior of inclined risers. In the boundary layer analysis, we take in consideration the effects of the axial stretch, applied tension, and internal velocity. Our numerical simulation results show good agreement with the exact solutions for special cases. In addition, our developed method overcomes the mathematical and numerical limitations of the previous methods used before.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXAMINATION COMMITTEE APPROVALS FORM</td>
<td>2</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>3</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>4</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>5</td>
</tr>
<tr>
<td>LIST OF ABBREVIATIONS</td>
<td>7</td>
</tr>
<tr>
<td>LIST OF SYMBOLS</td>
<td>8</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>10</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>11</td>
</tr>
<tr>
<td>1 INTRODUCTION</td>
<td>12</td>
</tr>
<tr>
<td>1.1 VORTEX INDUCED VIBRATION MODELS</td>
<td>14</td>
</tr>
<tr>
<td>1.2 RISERS STATIC CONFIGURATION MODELS</td>
<td>16</td>
</tr>
<tr>
<td>1.3 LITERATURE REVIEW</td>
<td>18</td>
</tr>
<tr>
<td>1.3.1 VIV ANALYSIS OF PIPES</td>
<td>18</td>
</tr>
<tr>
<td>1.3.2 PERTURBATION SOLUTIONS OF INCLINED RISERS</td>
<td>19</td>
</tr>
<tr>
<td>1.4 MOTIVATION AND RESEARCH CONTRIBUTION</td>
<td>20</td>
</tr>
<tr>
<td>2 ANALYTICAL VIV MODELING OF STRAIGHT HORIZONTAL PIPE</td>
<td>21</td>
</tr>
<tr>
<td>2.1 HORIZONTAL PIPE EQUATION OF MOTION</td>
<td>21</td>
</tr>
<tr>
<td>2.2 COUPLED VAN DER POL OSCILLATOR</td>
<td>23</td>
</tr>
<tr>
<td>2.3 NON-DIMENSIONLIZATION OF EQUATIONS</td>
<td>24</td>
</tr>
<tr>
<td>2.4 METHOD OF SOLUTION</td>
<td>25</td>
</tr>
<tr>
<td>2.5 CONVERGENCE ANALYSIS OF SOLUTION</td>
<td>29</td>
</tr>
<tr>
<td>3 STATIC EQUILIBRIUM SOLUTIONS OF INCLINED RISER</td>
<td>31</td>
</tr>
<tr>
<td>3.1 REDUCED ORDER MODEL VIA GALERKIN APPROXIMATION</td>
<td>32</td>
</tr>
<tr>
<td>3.2 EXACT SOLUTION OF THE STATIC EQUATION</td>
<td>34</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>3.3 BOUNDARY LAYER PERTURBATION SOLUTION</td>
<td>37</td>
</tr>
<tr>
<td>3.3.1 OUTER SOLUTION</td>
<td>38</td>
</tr>
<tr>
<td>3.3.2 INNER SOLUTION NEAR $x=0$</td>
<td>39</td>
</tr>
<tr>
<td>3.3.3 INNER SOLUTION NEAR $x=1$</td>
<td>41</td>
</tr>
<tr>
<td>3.3.4 COMPOSITE SOLUTION</td>
<td>42</td>
</tr>
<tr>
<td>3.4 SUMMARY</td>
<td>43</td>
</tr>
<tr>
<td>4 RESULTS AND DISCUSSION</td>
<td>44</td>
</tr>
<tr>
<td>4.1 VIV OF STRAIGHT HORIZONTAL PIPE</td>
<td>45</td>
</tr>
<tr>
<td>4.1.1 STATIC RESULTS AND FREQUENCY SHIFT</td>
<td>45</td>
</tr>
<tr>
<td>4.1.2 DYNAMIC ANALYSIS OF VIV</td>
<td>48</td>
</tr>
<tr>
<td>4.2 STATIC SOLUTION OF INCLINED RISERS</td>
<td>51</td>
</tr>
<tr>
<td>4.2.1 STATIC PROFILE OF RISERS WITHOUT AXIAL STRETCH</td>
<td>52</td>
</tr>
<tr>
<td>4.2.2 STATIC PROFILE OF RISER WITH AXIAL STRETCH</td>
<td>56</td>
</tr>
<tr>
<td>4.3 SUMMARY</td>
<td>58</td>
</tr>
<tr>
<td>5 CONCLUSIONS AND FUTURE WORK</td>
<td>59</td>
</tr>
<tr>
<td>5.1 STATIC SOLUTION OF RISER WITH MEAN FLOW DRAG EFFECTS</td>
<td>60</td>
</tr>
<tr>
<td>5.2 GENERAL STATIC SOLUTION OF RISERS WITH AXIAL STRETCH</td>
<td>60</td>
</tr>
<tr>
<td>5.3 VIV OF STATICALLY DEFORMED PIPE</td>
<td>61</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>62</td>
</tr>
<tr>
<td>APPENDICIES</td>
<td>66</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>67</td>
</tr>
<tr>
<td>APPENDIX B</td>
<td>72</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>-----------------------------------</td>
</tr>
<tr>
<td>VIV</td>
<td>Vortex Induced Vibration</td>
</tr>
<tr>
<td>CFD</td>
<td>Computational Fluid Dynamics</td>
</tr>
<tr>
<td>FD</td>
<td>Finite Difference</td>
</tr>
<tr>
<td>FEA</td>
<td>Finite Element Analysis</td>
</tr>
<tr>
<td>TDP</td>
<td>Touch Down Point</td>
</tr>
<tr>
<td>WKB</td>
<td>Wentzel-Kramers-Brillouin</td>
</tr>
<tr>
<td>IVP</td>
<td>Initial Value Problem</td>
</tr>
<tr>
<td>BVP</td>
<td>Boundary Value Problem</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$T_e$</td>
<td>Pipe Pretension</td>
</tr>
<tr>
<td>$We$</td>
<td>Effective Weight of Pipe</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angle of Inclination with respect to Horizontal</td>
</tr>
<tr>
<td>$L$</td>
<td>Length of Pipe</td>
</tr>
<tr>
<td>$C_D$</td>
<td>Drag Coefficient</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>Density of Sea Water</td>
</tr>
<tr>
<td>$D$</td>
<td>External Pipe Diameter</td>
</tr>
<tr>
<td>$U_e$</td>
<td>External Velocity</td>
</tr>
<tr>
<td>$C_{L0}$</td>
<td>Constant Lift Coefficient</td>
</tr>
<tr>
<td>$U_i$</td>
<td>Velocity of Internal Fluid</td>
</tr>
<tr>
<td>$m_f$</td>
<td>Mass of Internal Fluid</td>
</tr>
<tr>
<td>$PE$</td>
<td>Potential Energy</td>
</tr>
<tr>
<td>$\Omega_s$</td>
<td>Vortex Shedding Frequency</td>
</tr>
<tr>
<td>$P$</td>
<td>Wake Coefficient</td>
</tr>
<tr>
<td>$C_A$</td>
<td>Added Mass Coefficient</td>
</tr>
<tr>
<td>$\rho_e$</td>
<td>Density of Sea Water</td>
</tr>
<tr>
<td>$A_e$</td>
<td>Pipe External Area</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>Density of Internal Fluid</td>
</tr>
<tr>
<td>$A_i$</td>
<td>Pipe Internal Area</td>
</tr>
<tr>
<td>$\rho_p$</td>
<td>Density of Pipe</td>
</tr>
<tr>
<td>$A_p$</td>
<td>Cross Section Area of Pipe</td>
</tr>
<tr>
<td>$E$</td>
<td>Modulus of Elasticity</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS (CONTINUED)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>Area Moment of Inertia</td>
</tr>
<tr>
<td>$m_f$</td>
<td>Mass of Internal Fluid</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational Constant</td>
</tr>
<tr>
<td>$KE$</td>
<td>Kinetic Energy</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Wake Coefficient</td>
</tr>
<tr>
<td>$q$</td>
<td>Wake Variable</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Figure 1.1 Typical Design of Riser in Production Operations .................................. 13
Figure 1.2 Motion of the Pipe as a Result of Interaction with the Wake ................... 13
Figure 2.1 Horizontal Pipe Subjected to Hydrodynamic Loading .......................... 23
Figure 3.1 Boundary Layer Segments in the Riser ................................................. 37
Figure 4.1 Static Deflection of the Pipe due to the effect of the Internal Flow ....... 46
Figure 4.2 Natural Frequency Shift of the Pipe due to the Effect of Internal Flow ... 47
Figure 4.3 Convergence Analysis .......................................................................... 49
Figure 4.4 VIV Results ............................................................................................ 50
Figure 4.5 Agreement of Boundary Layer Solution with Exact and Numerical ...... 53
Figure 4.6 Divergence of Exact Solution ............................................................... 54
Figure 4.7 Numerical Solution Deficiency ............................................................. 55
Figure 4.8 Effect of Internal Velocity ..................................................................... 55
Figure 4.9 Static Profile Results with Axial Stretch .............................................. 56
Figure 4.10 Static Profile Results Using Stiff Numerical Scheme .......................... 57
Figure 4.11 Axial Stretch at Different Values of Applied Axial Tension .......... 57
Figure A.1 General Description of pipe under static and hydrodynamic loading .... 67
LIST OF TABLES

Table 2.1 The Natural Frequency Associated to the Mode Shape of Clamped Configuration .......................................................................................................................... 27

Table 3.1 Natural Frequency of Clamped-Hinged Configuration Mode Shape ........ 34

Table 4.1 Properties of the Pipe Used in the Analysis ................................................. 45

Table 4.2 Properties of the Riser Used in the Analysis .............................................. 52
CHAPTER 1

INTRODUCTION

The fluid-structure interaction is one of the research areas that has been explored extensively by many researchers. This research focuses on applications in the areas, such as offshore structures, bridges, aircraft design, and piping systems [1]. These structures are subjected to either hydrostatic or hydrodynamic excitation forces that cause the structure to deflect. The parameters involved in these applications such as internal velocity, external velocity and vibration amplitudes need to be studied carefully in order to avoid catastrophic failure of the structures. Recently, such problems have been proposed for energy harvesting application [2].

The extensive use of cables and pipes in ocean engineering design motivated researchers to focus more on the design aspects of pipe risers and cables and their interactions with the environment. Riser is a long slender pipe used in oil and gas exploration, production and shipping operations to transport the fluid from the well or the
seabad to the platform structure or to a ship as shown in figure 1.1.

**Figure 1.1:** Typical design of riser in production operations

The study of the riser configuration during installation i.e. laying operations, and production is essential to avoid static failure of the pipe by buckling [3]. Moreover, when the fluid flows around a stationary circular pipe it generate wakes as shown in figure 1.2.

**Figure 1.2:** Motion of the pipe as a result of interaction with the wake

If the frequencies of the generated wakes get close to the frequency of the circular pipe then the pipe undergoes vortex induced vibration (VIV). This mean that the pipe vibrates
violently as a result of the wake interaction with the structure and its vibration amplitudes can reach up to twice the diameter of the cylinder [4]. On that basis, there are two important aspects of cylindrical pipes in ocean engineering to be studied: the VIV of pipes and risers, and the static configuration of risers.

1.1 Vortex Induced Vibration Models

The VIV of a pipe occurs when the value of the vortex shedding frequency $f_v$ gets close to the value of the natural frequency of the structure $f_n$. This is known as pipe lock-in condition. Strouhal number $St$ is the parameter used to present the oscillatory flow condition relating to the external current velocity $U_e$, and the diameter of the pipe $D$, as shown in equation (1-1).

$$St = \frac{f_v D}{U_e} \quad (1-1)$$

A wide range of experimental and analytical researches have been devoted to model VIV due to the challenges involved in solving Navier-Stokes fluid equation with moving boundary conditions [5]. In addition, the time required to solve the equations numerically utilizing computational fluid dynamics (CFD) methods is considered quite expensive when modeling large structure or long spanning risers [6]. Therefore, semi-empirical formulations were introduced in the literature to estimate the lift forces. These formulations are coupled with the equation of motion of the structure to study VIV interaction.

During the lock-in condition, the structure with the wake undergoes a self-limiting motion, which can be described by a self-excited oscillator shown in equation (1-2). The
lock-in condition occurs in the subcritical range of Reynolds number between $300 < \text{Re} < 1.5 \times 10^5$.[4]

\[
\ddot{q} + \lambda \omega_s (q^2 - 1)\dot{q} + \omega_s^2 q = \text{forcing} \tag{1-2}
\]

The lift coefficient in equation (1-2) is described by the dimensionless wake variable $q$ and the dot indicates the derivative with respect to time while $\omega_s$ represent the generated wake frequency and $\lambda$ is lock-in scaling parameter obtained from experiments.

The use of semi-empirical formulation was first suggested by Griffin and Skop [7]. In their study, a Van Der Pol oscillator was utilized in conjunction with introducing a new parameter called Skop-Griffin parameter. The parameter relates damping of the structure to the mass ratio. Their formulation captured the response of the structure experimentally for air and water under various configurations. Their research was followed by the work of Currie and Turnbull [8] who studied the inline response of VIV of circular cylinders. They suggested that the force coupling term in the Van Der Pol oscillator consists of velocity and displacement terms. On that basis, they were able to fit the experimental data in the model and obtain the value of the parameters associated with the forcing terms. Later, Skop and Balasubramanian [9] suggested modifying the Van Der Pol oscillator model with the velocity forcing term by including a stall parameter. The stall parameter in the formulation accounts for the low structural damping at large vibration amplitudes. Their formulation was also extended [10] to study sheared current flow by introducing a diffusive term in the Van Der Pol oscillator. The model was further improved by Facchinetti et al. [11] who proposed to have an acceleration coupling in the Van Der Pol oscillator instead of velocity and displacement couplings. This was based on
experimental results showing that the velocity and displacement couplings failed to describe the VIV behavior at low values of mass ratios. Similarly, Mathelin and de Langre [12] extended the model further to study sheared current flow. In their model, they accounted for the sheared current by introducing a diffusive term in the Van Der Pol oscillator. Furthermore, Violette et al. [13] carried out direct numerical simulation to validate the oscillator model for long slender pipes while including flexural rigidity in the structure equation of motion. Recently Xu et al. [14], modified the parameters in the Van Der Pol oscillator wake oscillator to have piece-wise values accounting for before and after lock-in regimes, which showed better agreement.

1.2 Risers Static Configuration Models

The static profile of risers is generally obtained by either analytical models or by numerical techniques such as finite difference (FD) method and finite element analysis (FEA). One of the earliest models describing the static configuration is the catenary model [15]. In this model, the pipe equation is developed by considering the effect of curvature and neglecting the environmental loads; therefore, the pipe static equation becomes

\[
EI \frac{d^4 \hat{y}}{d\hat{x}^4} - T \frac{\left(\frac{d^2 \hat{y}}{d\hat{x}^2}\right)}{\sqrt{1 + \left(\frac{d\hat{y}}{d\hat{x}}\right)^2}} = -W_E
\]

(1-3)

Where \(\hat{y}\) is lateral deflection along the length of the pipe \(\hat{x}\), \(E\) is modulus of elasticity, \(I\) is area moment of inertia, \(T\) is the applied tension and \(W_E\) is the effective weight. By
dropping the bending term in equation (1-3) and solving the differential equation based on the applied tension and the effective weight the solution becomes

$$\hat{y}(\hat{x}) = \frac{T_H}{W_E} \cosh \left( \frac{W_E}{T_H} \hat{x} + C_1 \right) + C_2 \quad (1-4)$$

Equation (1-4) is used to obtain the static profile of risers where: $T_H$ is the horizontal component of tension, $C_1$ and $C_2$ are constants to be determined based on the essential boundary conditions of the problem. The work of Palmer et al. [3] extended the formulation in (1-3) by transforming the coordinate of the equation to the curved distance coordinate in order to account for the static curvature of the pipe during laying operations. The equation is then solved numerically using finite difference scheme with appropriate boundary conditions. Moreover, the work of Bernitsas et al. [39] addressed the static bulking loads of vertical risers. Their model involves the use of Airy’s functions which accept only small argument values. At higher argument values the solution is going to have numerical instabilities. These instabilities have been explained by Sampaio and Hundhausen [38] to result from the truncations in the series solution. Also, the use of asymptotic expansions form of Airy’s functions is only valid near small argument values [40]. The review work of Chakrabarti and Frampton [16] suggested that the axial tension term in the static equation is variable and accounts for the profile curvature in vertical or inclined risers. The model was improved further by Wu and Lou [17] and Chen et al. [18] to account for the effect of internal flow. Recent models such as the one developed by Pesce et al. [19] account for the soil forces on the static equation of the pipe. In addition, Bernitsas et al. [20] developed nonlinear equations in order to account for large deformations mathematically. These equations were developed to account for structural
nonlinearities and were solved using FEA method. Then, these equations were extended further by Chucheepsakul et al. [21] and later by Athisakul et al. [22] to account for the axial stretch of the pipe. In both approaches FEA analysis was implemented with an appropriate optimization method to find the value of axial stretch of the pipe.

1.3 Literature Review

The previous two sections addressed the background required on the development of a suitable VIV model and the derivation of the differential equation governing the static profile of the pipe. In this section, a summary of more relevant works is presented.

1.3.1 VIV Analysis of Pipes

After the development of the coupled Van Der Pol oscillator of Facchinetti et al. [11], many researchers studied the interaction of that model with equation of motion of the pipe structure. Violette et al. [13] considered the coupling between slender structure and the wake model. Their results showed good agreement when compared to numerical simulation under uniform and non-uniform shear flows. A multi-modal response was obtained when the shear flow is acting on the structure with a diffusive Van Der Pol oscillator. In another work, Xu et al. [23] studied the effect between the axial stretch of the pipe coupled with a piece-wise Van Der Pol oscillator. Their results showed better agreement with the previous experimental work compared to the CFD simulation. Dai et al. [24] extended the work to include the effect of internal velocity with axial stretch for a straight pipe under hinged configurations. In addition, Srinil et al. [25] investigated the multi-modal interactions of pipes and catenary risers under VIV coupled with a Van Der
Pol oscillator and examined parametrically the effect of mass ratio, shear flows, and multi-frequency lock-in. They also developed a reduced order model. Unlike the previous models obtained by Galerkin method, Chatjigeorgiou [26] and Meng and Chen [27] examined three dimensional vibration behavior of riser under large deformation. They applied FEA methods to obtain the natural frequencies and the mode shapes of the riser. The numerical comparison reveals the presence of a multi-modal interaction in the structure due to the static perturbation of the structure.

1.3.2 Perturbation Solutions of Inclined Risers

The earliest model which examined the static solution of inclined riser by perturbation was the work of Wu and Lou [17]. Their work included the effect of internal velocity and Poisson’s effect using singular perturbation techniques under hinged-hinged configuration. Later, Triantafyllou et al. [28] focused on developing the boundary layer perturbation asymptotic solution of the linearized dynamic equation of a vertical rod. They examined free and forced vibration problems with the estimated natural frequency of the system. In another study, Aranha et al. [29] developed a formulation considering one boundary layer at the Touch Down Point (TDP) while estimating the effect of dynamic bending using boundary layer analysis on the linear frequency domain on catenary risers. In addition, Stump et al. [30] developed a boundary layer solution to the dynamics of a catenary solution assuming twisting and bending of the riser. Their truncated solution showed a good agreement with near-catenary solution. The study was extended by Lecni et al. [31] to apply boundary layer perturbation on different configuration of pipes during laying operations. On the other hand, Chatjigeorgiou [32]
using boundary layer perturbation analysis obtained a close form of the mode shapes of a vertical top-tension riser and its associated natural frequencies. Recently, Hsu [33] developed a Wentzel-Kramers-Brillouin (WKB) perturbation catenary solution, which includes flexural rigidity. The perturbation analysis included hinged configuration and showed good agreement compared to other mathematical models.

1.4 Motivation and Research Contribution

In relevance to the subject of VIV, our work aims to extend the analytical study conducted in [24] to include other pipe configuration such as clamped-clamped model. This is due to the fact that clamped configuration is more representative of the physical constraints imposed by the soil. Therefore, we consider the equation of motion of the pipe coupled with a Van Der Pol oscillator [11] and then conduct a static and dynamic analysis on the straight pipe under clamped-clamped configuration.

Previous solutions on the static profile of inclined risers were obtained using boundary layer perturbation methods in the absence of the axial stretch. Our aim is to use the perturbation method to develop a simple approximate mathematical tool utilized by engineers and designers in an efficient way. Therefore, the boundary layer perturbation solution used in this thesis is obtained with consideration of the axial stretch. In addition, we adopt a numerical scheme to obtain the value of the axial stretch for different applied axial tension and internal velocities.
CHAPTER 2

ANALYTICAL VIV MODELING OF STRAIGHT HORIZONTAL PIPE

In this chapter, we present our mathematical model for VIV of a straight horizontal pipe. The effect of internal flow and axial stretch of the pipe are considered in the analysis. From geometrical point of view, we account for two different boundary condition configurations: hinged-hinged and clamped-clamped configurations. Then, we present the equation of motion and convergence analysis of the method. In this section, we aim to extend the published results in [24] and to include clamped-clamped configuration.

The hydrodynamic forces are obtained by coupling the equation of motion of the pipe with a Van Der Pol oscillator governing the lift coefficient [11]. The drag force is described using Morison formulation while the effect of mean drag is not considered since the flow is orthogonal to the plane of the pipe motion.

2.1 Horizontal Pipe Equation of Motion

The general equation of motion of a pipe at any configuration subjected to hydrostatic and hydrodynamic forces is presented in equation (2-1) [1].
\[
(C_A \rho_e A_e + \rho_l A_l + \rho_p A_p) \frac{\partial^2 \dot{\gamma}}{\partial t^2} + EI \frac{\partial^4 \dot{\gamma}}{\partial \xi^4} + 2m_f U_l \frac{\partial^2 \dot{\gamma}}{\partial \xi \partial t} + \left( m_f U i^2 - (T_e - We \ast \sin(\theta)(L - \hat{x})) \right) - \frac{EA_p}{2L} \int_0^L \left( \left( \frac{\partial \dot{\gamma}}{\partial \xi} \right)^2 \right) \, d\hat{x} = \frac{1}{4} C_{L0} \rho_e DU_e \frac{\partial \gamma}{\partial t} - We \ast \cos(\theta)
\]

(2-1)

Where \( C_A \rho_e A_e \) is the added mass, \( \rho_l A_l \) is mass of internal fluid, \( \rho_p A_p \) is the mass of the pipe, \( A_l \) pipe internal cross section area, \( A_e \) pipe external cross section area, \( EI \) is flexural rigidity, \( U_l \) is internal fluid flow \( T_e \) is the applied tension, \( \theta \) is the inclination angle, \( L \) is the length of the pipe, \( A_p \) is the cross section area of the pipe, \( C_D \) is the drag coefficient, \( \rho_e \) is the external fluid density, \( D \) is the diameter, \( U_e \) is the external fluid flow, \( C_{L0} \) is the lift coefficient, and \( q \) is the wake variable and is defined as \( \frac{q}{c_{L0}(\xi, \mu)} \). Full derivation of the equation of motion using extending Hamilton principle is available in Appendix A.

Equation of motion (2-1) is reduced for horizontal pipe by considering an angle of zero theta and neglecting the effects of the pipe weight, pipe structural damping and the applied tension force since their effect is not considered in this analysis for simplicity [24]. Thus, equation (2-1) of motion reduces to (2-2).

\[
\left( C_A \rho_e A_e + \rho_l A_l + \rho_p A_p \right) \frac{\partial^2 \dot{\gamma}}{\partial t^2} + EI \frac{\partial^4 \dot{\gamma}}{\partial \xi^4} + 2m_f U_l \frac{\partial^2 \dot{\gamma}}{\partial \xi \partial t} + \left( m_f U i^2 - \frac{EA_p}{2L} \int_0^L \left( \left( \frac{\partial \dot{\gamma}}{\partial \xi} \right)^2 \right) \, d\hat{x} \right) \left( \frac{\partial^2 \dot{\gamma}}{\partial \xi^2} \right) + \frac{1}{2} C_D \rho_e DU_e \frac{\partial \gamma}{\partial t} = \frac{1}{4} C_{L0} \rho_e DU_e \frac{\partial \gamma}{\partial t} \frac{1}{4} C_{L0} \rho_e DU_e \frac{\partial \gamma}{\partial t}
\]

(2-2)
Equation of motion (2-2) describes a pipe structure under external and internal flow forces as shown in the pipe configuration of figure 2.1.

![Pipe Configuration Diagram](image)

**Figure 2.1:** Horizontal pipe subjected to hydrodynamic loading

It is evident from equation (2-2) that Morison formulation introduces two additional terms: a damping force on the structure resulting from the drag effects and an inertial term adding to the total inertia of the pipe structure. The effect of the mean drag flow is not considered since the external flow is orthogonal to the direction of the motion of the pipe. Also, we notice that the internal velocity of the pipe contributes to the motion of the pipe by an axial force on the pipe and a coriolis force. Although the coriolis term has a time derivative, this time derivative does not contribute to damping but it offers coupling between the mode shapes [17].

### 2.2 Coupled Van Der Pol Oscillator

In this thesis, we propose for simplicity to utilize a Van Der Pol oscillator governing the wake and the lift coefficient reported in. [11]. The wake equation is coupled with the equation of motion of the structure via an acceleration term in equation (2-3)
\[
\ddot{q} + \lambda \omega_s (q^2 - 1)\dot{q} + \omega_s^2 q = \frac{P}{D} \frac{\partial^2 \hat{y}(x, t)}{\partial t^2}
\]

(2-3)

The values of the parameters \( \lambda \) and \( P \) are determined based on experimental results. In the case for horizontal pipe [11], the values are 0.3 and 12, respectively. The coupling proposed in equation (2-3) provides better description of the synchronization under various structural mass ratios.

### 2.3 Non-Dimensionlization of Equations

In order to characterize the equation of motion of the pipe, we introduce the following non-dimensional variables

\[
\eta = \frac{\vartheta}{D}; \quad \xi = \frac{x}{L};
\]

(2-4)

After the introduction of the non-dimensional variables into equation (2-4), we obtain

\[
\frac{\partial^2 \eta}{\partial \tau^2} + c \frac{\partial \eta}{\partial \tau} + 2 \sqrt{\beta} \nu \frac{\partial^2 \eta}{\partial \xi \partial \tau} + \frac{\partial^4 \eta}{\partial \xi^4} + \nu^2 \frac{\partial^2 \eta}{\partial \xi^2} - \gamma \int_0^1 \left( \frac{\partial \eta}{\partial \xi} \right)^2 \frac{\partial^2 \eta}{\partial \xi^2} \, d\xi = \alpha \eta
\]

(2-5)

Also, the Van Der Pol oscillator equation (2-3) is reduced to

\[
\frac{\partial^2 q}{\partial \tau^2} + \lambda \Omega_s (q^2 - 1) \frac{\partial q}{\partial \tau} + \Omega_s^2 q = P \frac{\partial^2 \eta}{\partial \tau^2}
\]

(2-6)

Where

\[
\tau = \left( \frac{E_l}{m} \right)^{0.5} \frac{t}{L^2}; \quad \nu = \left( \frac{m_f}{E_l} \right)^{0.5} \frac{U}{L}; \quad \beta = \frac{m_f}{m}; \quad \gamma = \frac{A_P D^2}{2I}; \quad c = \frac{C_u \rho D U_e L^2}{4 E I};
\]

\[
\alpha = \frac{C_{L_0} \rho D U_e L^2}{4 E I}; \quad \Omega_s = \Omega_s \sqrt{\frac{m}{E I} L^2};
\]

(2-7)
The use of the non-dimensional form of the equation enables the comparison and validation of results regardless of the geometric characteristics such as: pipe length, pipe diameter, and elastic modulus. Also, all the terms in equation (2-5) are linear except for the axial stretch term.

2.4 Method of Solution

In our mathematical procedure, we use the Galerkin approximation [34] to solve equation (2-5). This Galerkin method is considered an effective weight residual method because it minimizes the weighted error to zero based on the choice of comparison function. Hence, we use the linear undamped mode shapes of the structure. Therefore, the linear undamped unforced form of equation (2-5) is reduced to

$$\frac{\partial^2 \eta}{\partial \tau^2} + \frac{\partial^4 \eta}{\partial \xi^4} + \nu^2 \frac{\partial^2 \eta}{\partial \xi^2} = 0$$

(2-8)

The solution of equation (2-8) is assumed to be of the form

$$\eta(\xi, \tau) = \sum_{j=1}^{n} \phi_j(\xi)e^{-i\omega \tau}$$

(2-9)

Substituting the solution (2-9) back in equation (2-8) yields the following

$$\phi_j'''''(\xi) + \nu^2 \phi_j''''(\xi) - \omega^2 \phi_j(\xi) = 0$$

(2-10)

Equation (2-10) is a homogenous differential equation and is solved for a set of boundary conditions. The general solution of the equation is assumed to be

$$\phi_j(\xi) = Ae^{s \xi}$$

(2-11)

Substituting equation (2-11) into equation (2-10) yields the following algebraic equation

$$s^4 + \nu^2 s^2 - \omega^2 = 0$$

(2-12)

The solutions to the algebraic equation (2-12) are
After substituting equation (2-13) in (2-11) the solution can be rewritten using trigonometric functions as

$$\phi_j(\xi) = A \cos(s_1 \xi) + B \sin(s_1 \xi) + C \cosh(s_2 \xi) + D \sinh(s_2 \xi)$$

(2-14)

Where the real part of $s_1$ is considered in function (2-14). Also, $A$, $B$, $C$, and $D$ are constants to be solved for based on the applied boundary conditions. Also, the roots of the characteristic equation of the eigenvalue matrix determinant give the frequency associated with each mode shape.

In this analysis, two sets of boundary conditions are considered: pinned-pinned configuration and clamped-clamped configuration. The boundary conditions for the pinned-pinned configuration are

$$\phi_j(0) = 0; \quad \phi_j(1) = 0; \quad \phi_j''(0) = 0; \quad \phi_j''(1) = 0$$

(2-15)

On the other hand, the boundary conditions for clamped-clamped configuration are

$$\phi_j(0) = 0; \quad \phi_j(1) = 0; \quad \phi_j'(0) = 0; \quad \phi_j'(1) = 0$$

(2-16)

After applying these boundary conditions, the mode shape function obtained for pinned-pinned configuration is

$$\phi_j(\xi) = \sqrt{2} \sin(j\pi \xi)$$

(2-17)
Where \( j \) represents the \( j \)-th mode shape with each non dimensional frequency for pinned-pinned configuration. Similarly, the mode shape for the clamped-clamped configuration are

\[
\Phi_j(\xi) = \cosh(\sqrt{\omega_j} \xi) - \cos(\sqrt{\omega_j} \xi) - \sigma_j [ \sinh(\sqrt{\omega_j} \xi) - \sin(\sqrt{\omega_j} \xi) ]
\]

(2-18)

Where \( \omega_j \) corresponds to the natural frequency and the values of the first four natural frequencies of (2-18) as shown in table 2.1.

**Table 2.1**: The natural frequency associated with the mode shape of clamped configuration

<table>
<thead>
<tr>
<th>n</th>
<th>( \omega_j )</th>
<th>( \sigma_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22.3733</td>
<td>0.982502</td>
</tr>
<tr>
<td>2</td>
<td>61.6728</td>
<td>1.00078</td>
</tr>
<tr>
<td>3</td>
<td>120.903</td>
<td>0.999966</td>
</tr>
<tr>
<td>4</td>
<td>199.859</td>
<td>1.0</td>
</tr>
</tbody>
</table>

It is also assumed that the wake follows the motion of the structure and, therefore, we assume the same mode shape obtained for the structure [24] and the solution can be written as

\[
q(\xi, \tau) = \sum_{j=1}^{n} \Phi_j(\xi) q_j(\tau)
\]

(2-19)

In order to determine the buckling limits we obtain the static version of equation of 2-5, and eliminate the time dependent terms and the equation reduces to

\[
\frac{\partial^4 n}{\partial \xi^4} + \nu^2 \frac{\partial^2 n}{\partial \xi^2} - [\nu \int_0^1 \left( \frac{\partial \eta}{\partial \xi} \right)^2 d\xi] \frac{\partial^2 n}{\partial \xi^2} = 0
\]

(2-20)

Equation (2-20) provides that static equaiton of the pipe structure subjected to axial internal flow forces. The internal velocity of the pipe \( \nu \) induces a compressive force
on the pipe. If the internal velocity reaches a certain threshold value the pipe will experience buckling instability. The critical buckling instability associated with each configuration is obtained by solving the static equation [34]. This value of the buckling load is obtained when the frequency value is zero. Substituting the obtained frequencies of equation (2-18) and equation (2-17) into equation (2-13) and we solve the algebraic equation for the internal velocity value that will make zero frequency value. For the hinged-hinged and clamped-clamped configurations, the first critical buckling velocity is $\pi$ and $2\pi$, respectively.

The mode shapes obtained in (2-17) and (2-18) are substituted back into the original equation (2-5) with their solution in order to obtain the reduced order model temporal solution utilizing orthogonality of the independent modes shapes which results into

$$
\int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i(\xi) \ddot{\eta}_i(\tau) \, d\xi + c \int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i(\xi) \dot{\eta}_i(\tau) \, d\xi 
+ 2\sqrt{\beta} \nu \int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i'(\xi) \dot{\eta}_i(\tau) \, d\xi + \int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i'''(\xi) \eta_i(\tau) \, d\xi 
+ \nu^2 \int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i''(\xi) \eta_i(\tau) \, d\xi
- [\gamma \int_0^1 \left( \sum_{i=1}^n \phi_i'(\xi) \eta_i(\tau) \right)^2 \, d\xi] \int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i''(\xi) \eta_i(\tau) \, d\xi 
= \alpha \int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i(\xi) q_i(\tau) \, d\xi
$$
Similarly, the Van Der Pol equation \((2-6)\) is reduced using the same mode shapes of the structure as

\[
\int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i(\xi) \dot{q}_i(\tau) \, d\xi
+ \lambda \Omega s \int_0^1 \phi_j(\xi) \left( \left( \sum_{i=1}^n \phi_i(\xi) q_i(\tau) \right)^2 - 1 \right) \sum_{i=1}^n \phi_i(\xi) \dot{q}_i(\tau) \, d\xi
+ \Omega s^2 \int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i(\xi) q_i(\tau) \, d\xi = P \int_0^1 \phi_j(\xi) \sum_{i=1}^n \phi_i(\xi) \ddot{\eta}_i(\tau) \, d\xi
\]

\((2-22)\)

Equations \((2-21)\) and \((2-22)\), respectively, represent the reduced order models and initial value problems (IVP) which are integrated numerically using a fourth order Runge-Kutta scheme in Mathematica [35] to obtain the solution in time coordinates for different number of mode shapes with the following initial conditions

\[\eta_i(0) = 0.000025; \quad \dot{\eta}_i(0) = 0.000025; \quad \dot{q}_i(0) = 0.000025; \quad q_i(0) = 0.000025\]

\((2-23)\)

### 2.5 Convergence Analysis of Solution

To test convergence, we examine how many modes are sufficient to describe the motion of the structure. For that, the solution of different number of mode shapes are compared and a convergence is reached when addition of mode shape are no longer contributing significantly to the motion of the structure.

Then we examine the effects of internal and external velocities of the structure and we define two parameters: the reduced velocity and Strouhal frequency.
The Strouhal number $S$ has a value of 0.2 for a wide range of velocities [24]. To study the vibrations amplitudes, we vary the external reduced velocity controlling the lock-in frequency for the first mode shape with a particular value of internal velocity $\nu$. The overall VIV response of the pipe is obtained by recording the maximum deflection of the structure for each value of external and internal velocity.

\[ U_r = \frac{2\pi u_e}{\omega_1 D} \quad \Omega_s = U_r S \Omega_1 \]
CHAPTER 3

STATIC EQUILIBRIUM SOLUTIONS OF INCLINED RISER

In this chapter, we analyze the static solutions of the inclined riser static equation. Finding the static solution is the first crucial step into analyzing the dynamic of the system. Several mathematical methods are presented in this chapter with the purpose of validating them against boundary layer perturbation method. Therefore, this chapter is organized as follows:

- First, an approximated reduced order model is presented. The approximation is necessary to obtain a basis function that can be utilized as a comparison function in Galerkin method and obtain the static solution of the system.
- The mathematical derivation of the exact solution of the static equation is then presented for the purpose of validation.
- Finally, a boundary layer perturbation method is used to obtain a static solution of the equation. In the boundary layer method, the bending near the boundaries is
scaled such that a boundary layer solution exists near each boundary. Therefore, this enables obtaining the full solution of the system with the boundary layer perturbation method.

### 3.1 Reduced Order Model via Galerkin Approximation

The objective is to obtain a basis function for Galerkin method of the unloaded pipe using the approximation method [36] assuming that the pipe is initially straight, horizontal and weightless. The equation of motion of the inclined riser based on the mathematical derivation in Appendix A can be written as

\[
\frac{m}{\partial t^2} + \frac{E}{\partial \xi^4} + \frac{2m_f U_i}{\partial \xi \partial t}
+ \left( m_f U_i^2 - (T_e - We \cdot Sin(\theta) (L - \xi)) \right) \frac{EA_p}{2L} \int_0^L \left[ \left( \frac{\partial \hat{y}}{\partial \xi} \right)^2 \right] d\hat{\xi} \left( \frac{\partial^2 \hat{y}}{\partial \hat{\xi}^2} \right) \\
- We \cdot Sin(\theta) \left( \frac{\partial \hat{y}}{\partial \hat{\xi}} \right) + \frac{1}{2} C_D \rho_o D U_e \frac{\partial \hat{y}}{\partial t} = \frac{1}{4} C_{L_0} \rho_o D U_e^2 q - We \cdot Cos(\theta)
\]

(3-1)

Where \( m \) is the total mass, \( EI \) is flexural rigidity, \( m_f \) is mass of internal fluid, \( U_i \) is internal fluid flow \( T_e \) is the applied tension, \( \theta \) is the inclination angle, \( L \) is the length of the pipe, \( A_p \) is the cross section area of the pipe, \( C_D \) is the drag coefficient, \( \rho_o \) is the external fluid density, \( D \) is the diameter, \( U_e \) is the external fluid flow, \( C_{L_0} \) is the lift coefficient, and \( q \) is the wake variable. The tension of the pipe \( T_e \) is assumed to vary linearly with the weight of the pipe and, as a result, the eigenvalue problem of equation
(3-1) has no exact solution. Therefore, if the pipe becomes weightless and unforced then the equation is reduced to

\[ m \frac{\partial^2 \hat{y}}{\partial t^2} + EI \frac{\partial^4 \hat{y}}{\partial \hat{x}^4} + 2m_f U_i \frac{\partial^2 \hat{y}}{\partial \hat{x} \partial t} + \left( m_f U_i^2 - (T_e) - \frac{E A_p}{2L} \int_0^L \left[ \left( \frac{\partial \hat{y}}{\partial \hat{x}} \right)^2 \right] d\hat{x} \right) \left( \frac{\partial^2 \hat{y}}{\partial \hat{x}^2} \right) = 0 \]

(3-2)

The non-dimensional equation according to Section 2.3 can be written as

\[ \frac{\partial^2 \eta}{\partial \tau^2} + \frac{\partial^4 \eta}{\partial \xi^4} + 2v \sqrt{\beta} \frac{\partial^2 \eta}{\partial \xi \partial \tau} + \left( v^2 - T_e - \gamma \int_0^L \left[ \left( \frac{\partial \eta}{\partial \xi} \right)^2 \right] d\xi \right) \left( \frac{\partial^2 \eta}{\partial \xi^2} \right) = 0 \]

(3-3)

Equation 3-3 is the equation of motion describing a straight pipe. The mode shapes associated to the straight pipe are discussed in Section 2-4. However, the boundary conditions discussed are either clamped-clamped or hinged-clamped as follows

\[ \eta(0) = 0; \quad \eta'(0) = 0; \quad \eta(1) = 0; \quad \eta'(1) = 0 \]

Or

\[ \eta(0) = 0; \quad \eta'(0) = 0; \quad \eta(1) = 0; \quad \eta''(1) = 0 \]

(3-4)

The mode shape for clamped-clamped pipe is shown in Section 2-4 and following the same procedure the solution for a hinged-clamped pipe is

\[ \phi_h(\xi) = \text{Cosh}(\sqrt{\omega_j} \xi) - \text{Cos}(\sqrt{\omega_j} \xi) - \sigma_i \left[ \text{Sinh}(\sqrt{\omega_j} \xi) - \text{Sin}(\sqrt{\omega_j} \xi) \right] \]

(3-5)
Where the values of the parameters of equation (3-5) are shown in table 3.1.

**Table 3.1:** Parameters associated with the mode shape of a clamped-hinged beam

<table>
<thead>
<tr>
<th>n</th>
<th>( \omega_j )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15.4182</td>
<td>1.008</td>
</tr>
<tr>
<td>2</td>
<td>49.9649</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>104.248</td>
<td>1.0</td>
</tr>
<tr>
<td>4</td>
<td>178.27</td>
<td>1.0</td>
</tr>
</tbody>
</table>

The use of the method is limited to homogenous boundary conditions. Once, the boundary conditions are non-zero then we use another method to find the static equilibrium solution of inclined riser. In addition, other assumed mode shapes were used in literature [37] but the Galerkin method presented is considered an easier approach for obtaining the value of the axial stretch term [1].

### 3.2 Exact Solution of the Static Equation

The exact solution is obtained by solving the static differential equation as outlined in [38]. In addition, the derivation here includes the effect of the pipe axial stretch. The static equation is obtained by dropping all the time dependent terms in equation (3-1). Thus, the static equation is

\[
EI \frac{\partial^4 \gamma}{\partial \xi^4} + \left( m_f U i^2 - (T_e - We * Sin(\theta)(L - \bar{x})) - \frac{E A_p}{2 L} \int_0^L \left[ \left( \frac{\partial \gamma}{\partial \xi} \right)^2 \right] d\bar{x} \right) \left( \frac{\partial^2 \gamma}{\partial \xi^2} \right) = -We * Sin(\theta) \left( \frac{\partial \gamma}{\partial \xi} \right) = -We * Cos(\theta)
\]

(3-6)

Introducing the below non-dimensional variables in equation (3-6)

\[
y = \frac{\gamma}{D}; \quad x = \frac{\xi}{L};
\]

(3-7)
Therefore, equation 3-6 is rewritten in a non-dimensional form as

$$\frac{d^4 y}{dx^4} + \left( v^2 - (T - \sigma(1 - x)) - I' \right) \left( \frac{d^2 y}{dx^2} \right) - \sigma \left( \frac{dy}{dx} \right) = -Fs$$

(3-8)

Where

$$T = \frac{\tau_0 L^2}{EI}; \quad \sigma = \frac{We Sin(\theta) \mu^3}{EI}; \quad \eta = \frac{A_p D^2}{2l}; \quad Fs = We Cos(\theta) \frac{L^4}{EID};$$

$$v = \frac{\sqrt{I_0 m_f \eta}}{\sqrt{E}}$$

(3-9)

Where $I' = \eta \int_0^1 \left[ \left( \frac{dy}{dx} \right)^2 \right] dx$. In order to obtain an exact mathematical solution we assume that the axial stretch term $I'$ is constant and independent of $x$. Then we integrate equation (3-8), this leads to

$$\frac{d^3 y}{dx^3} + \left( v^2 - (T - \sigma(1 - x)) - I' \right) \left( \frac{dy}{dx} \right) = -Fs \ x + c1$$

(3-10)

$C1$ is a constant to be determined based on the boundary condition. Then, a linear change of variable is carried out such that the coefficient of $\frac{dy}{dx}$ is $-1$ [38]. Therefore, the transformation takes the form of

$$x = a \ X + b$$

(3-11)
Using the chain rule and substituting back in equation (3-10) the variables $a$ and $b$ become

\[ a = \left(\frac{1}{\sigma}\right)^3 \quad b = \frac{1}{\sigma}(v^2 - T + \sigma - \Gamma) \quad (3-12) \]

The transformation now can be written as

\[ x = \left(\frac{1}{\sigma}\right)^3 X + \frac{1}{\sigma}(v^2 - T + \sigma - \Gamma) \quad (3-13) \]

Equation (3-10) with the transformation of equation (3-13) is reduced to

\[ \frac{d^3y}{dx^3} - X \left(\frac{dy}{dx}\right) = \alpha X + 2b_3 \quad (3-14) \]

Where; \[ \alpha = -\left(\frac{1}{\sigma}\right)^4 Fs; \quad b_3 = \frac{1}{2}\left(-\frac{Fs}{\sigma^2} \left(v^2 - T_e + \sigma - \Gamma\right) + \frac{c_1}{\sigma}\right) \quad (3-15) \]

Substituting a change of variable \( \frac{dy}{dx} = z(X) \) into equation (3-14); then the equation becomes

\[ \frac{d^2z}{dx^2} - X z(X) = \alpha X + 2b_3 \quad (3-16) \]

The homogenous solution of equation (3-16) is given in terms of Airy’s differential equation. This solution is given in terms of Airy’s first and second kind functions \( Ai(X) \) \( Bi(X) \) as follows

\[ z_h(X) = c_2 Ai(X) + c_3 Bi(X) \quad (3-17) \]

In order to obtain the particular solution of equation (3-16), variations of parameter are carried out based on the homogenous solution and then integrate it. This leads to
\[ y(x) = \left( \frac{-1}{\sigma} \right)^{\frac{1}{3}} \left\{ \int_0^x \left( c_2 \ Ai(X) + c_3 \ Bi(X) + \left[ \pi \left( \int_0^x Bi(\zeta)(\alpha \zeta + 2b_3) \, d\zeta \right) \right] Ai(X) - \left[ \pi \left( \int_0^x Ai(\zeta)(\alpha \zeta + 2b_3) \, d\zeta \right) Bi(X) \right] \, dX + c_4 \right\} \]  

(3-18)

Equation (3-18) presents the exact analytical solution to equation (3-8) of an inclined riser with the axial stretch. The constants in the solution are obtained by applying the boundary conditions. The value of the axial stretch term \( \Gamma \) is obtained by applying a fixed point iteration (shooting) method by initial guess \( \Gamma_0 \) and the solution is updated with \( \Gamma \) at each iteration [43].

### 3.3 Boundary Layer Perturbation Solution

In this section we solve the static equation (3-6) using boundary layer perturbation method. We follow the work of Nayfeh [40] and apply the method of composite expansion. The effect of the bending term in equation (3-6) is not dominant compared to the tension in the equation. This allows the stretching transformation for the bending term near each boundary of the pipe to obtain an approximate solution. Based on this method, we examine the static equation in three different segments as shown in figure 3.1.

![Figure 3.1: Boundary layer segments in the riser](image)
Two boundary layer solutions exist near each boundary and one solution exists outside
the boundary region. Based on that assumption, we scale the bending term in equation
(3-6) to obtain a boundary layer solution by introducing the following variables in
equation (3-6)

\[ y = \frac{\dot{y}}{D}; \quad x = \frac{\dot{x}}{L}; \]  

(3-19)

Therefore, equation 3-19 is

\[ \epsilon \frac{d^4 y}{dx^4} - \left( 1 - v^2 - \sigma + \sigma x + \eta \int_0^1 \left( \frac{dy}{dx} \right)^2 dx \right) \left( \frac{d^2 y}{dx^2} \right) - \sigma \frac{dy}{dx} = -F_s \]  

(3-20)

Where

\[ \sigma = \frac{We \cdot L \cdot \sin(\theta)}{T}; \quad \eta = \frac{EApD^2}{2TL^2}; \quad F_s = \frac{We \cdot L^2 \cdot \cos(\theta)}{DT}; \quad v = \frac{v_{le} \cdot m_f}{\sqrt{T}}; \quad \epsilon = \frac{EI}{TL^2}. \]

The complete boundary layer mathematical derivation is available in Appendix B.

The methodology and the solutions obtained are discussed in this section. It is assumed
that \( I = \eta \int_0^1 \left( \frac{dy}{dx} \right)^2 dx \) is constant and independent of \( x \). This is necessary in order to
obtain a boundary layer solution. The value of \( I \) is obtained using a numerical scheme
described later in section 3.3.3.

### 3.3.1 Outer Solution

Following [40] we seek an outer solution of equation (3-20) is obtained by
assuming a straight forward expansion of the outer solution in the form

\[ y^0(x; \epsilon) = y_0^0(x) + \epsilon \frac{1}{2} y_1^0(x) + \epsilon y_1^0(x) + O(\epsilon^{3/2}) \]  

(3-21)
Based on suggested straightforward expansion of equation (3-21) the solution of the bending term in the outer equation only appears at the third order. As a result, the outer solution is written as

\[
y^o = \frac{F_s}{\sigma} \frac{x}{x} + a1 \ln(\alpha + \sigma x) + a2 + \epsilon^2(a3 \ln(\alpha + \sigma x) + a4) \\
+ \epsilon \left( a5 \ln(\alpha + \sigma x) + a6 - \frac{2 \ a1 \ \sigma^2}{3(\alpha + \sigma x)^3} \right)
\]

(3-22)

Where \( \alpha = 1 - v^2 - \sigma + \Gamma \)

The boundary conditions considered for the boundary layer problem are

\[
y(0) = 0 \quad y'(0) = \beta_1 \quad y(1) = 0 \quad y'(1) = \beta_2
\]

(3-23)

Where \( \beta_1 \) and \( \beta_2 \) corresponds to the slope boundary condition at each end of the riser depending on the riser configuration. At this stage, the boundary conditions cannot be applied on the outer solution because two boundary layers exist near each boundary. Therefore, equation (3-20) is expanded near each boundary by introducing stretching transformation and applying the boundary conditions at each end.

### 3.3.2 Inner Solution Near \( x = 0 \)

In the neighborhood of \( x = 0 \) an interval of study i.e. a boundary layer is magnified to seek an outer solution assuming stretching transformation of the form

\[
\xi = x \frac{e^\lambda}{e^\lambda} \text{ where } x, \lambda \text{ are greater than zero.}
\]
Substituting the transformation in the main equation (3-20) yields that the dominant parts in the equation corresponds to $\lambda = \frac{1}{2}$ and the equation becomes

$$\frac{d^4 y^i}{d\xi^4} - \alpha \frac{d^2 y^i}{d\xi^2} - \sigma \frac{1}{\varepsilon^2} \frac{d^2 y^i}{d\xi^2} = -\varepsilon F_s \quad (3-24)$$

Seeking a solution of the form

$$y^i(\xi; \varepsilon) = y_0^i(\xi) + \varepsilon \frac{1}{\varepsilon^2} y_1^i(\xi) + \varepsilon y_1^i(\xi) + O(\varepsilon^{3/2}) \quad (3-25)$$

The boundary conditions are rewritten with the stretching transformation as

$$y^i(0) = 0 \quad \frac{dy^i}{d\xi}(0) = \frac{1}{\varepsilon^2} \beta_1 \quad (3-26)$$

Substituting equation (3-25) into equation (3-24), applying the boundary conditions and matching the inner solution with the outer solution yields the following

$$y^i(\xi; \varepsilon) = \varepsilon \left(-b_3 + \left(\beta_1 + b_3 \sqrt{\alpha}\right) \frac{x}{\varepsilon^2} + b_3 e^{-\sqrt{\alpha} \frac{x}{\varepsilon^2}}\right) + \varepsilon \left(\frac{7}{8} b_3 \alpha + f_3\right) +$$

$$\left(\sqrt{\alpha} \left(f_3 - \frac{7 b_3 \alpha}{8 \alpha \varepsilon}\right) + \frac{3 b_3 \alpha}{4 \alpha} - \frac{3 b_3 \alpha e^{-\sqrt{\alpha} \frac{x}{\varepsilon^2}}}{4 \alpha}\right) \frac{x}{\varepsilon^2} + \left(f_3 - \frac{7 b_3 \alpha}{8 \alpha^{3/2}}\right) e^{-\sqrt{\alpha} \frac{x}{\varepsilon^2}} +$$

$$\frac{\left(\frac{x}{\varepsilon^2}\right)^2}{2\alpha} \left(F_s - b_3 \sigma \sqrt{\alpha} - \sigma \beta_1 - b_3 \sigma \sqrt{\alpha} e^{-\sqrt{\alpha} \frac{x}{\varepsilon^2}}\right) \quad (3-27)$$

We obtain the values of the constants that appear in equation (3-27) by matching the inner solution equation (3-27) with the outer solution equation (3-22).
3.3.3 Inner Solution Near $x = 1$

Similar to Section 3.2.1, a stretching transformation on the boundary layer is expanded near $x = 1$ and it has the form of $\zeta = \frac{1-x}{e^x}$. Substituting the transformation with its derivatives in the main equation (3-20) yields that the dominant parts in the equation corresponds to $\lambda = \frac{1}{2}$ and the equation becomes

$$
\frac{d^4 y'}{d\zeta^4} - \mu \left( \frac{d^2 y'}{d\zeta^2} \right)^2 + \sigma \zeta \epsilon^{\frac{1}{2}} \left( \frac{d^2 y'}{d\zeta^2} \right) + \sigma \epsilon^{\frac{1}{2}} \left( \frac{dy'}{d\zeta} \right) = -\epsilon F_s
$$

(3-28)

Where $\mu = \alpha + \sigma = 1 - v^2 + \Gamma$. Seeking a solution of the form

$$
y'(\zeta; \epsilon) = y_0'(\zeta) + \epsilon^{\frac{1}{2}} y_1'(\zeta) + \epsilon y_1'(\zeta) + O(\epsilon^{3/2})
$$

(3-29)

The boundary conditions are rewritten with the stretching transformation as

$$
y(0) = 0 \quad y'(0) = -\epsilon^{\frac{1}{2}} \beta_2
$$

(3-30)

Substituting equation (3-30) into equation (3-29), applying the boundary conditions and matching the inner solution with the outer solution yields the following inner solution

$$
y'(\zeta; \epsilon) = \epsilon^{\frac{1}{2}} \left( -e3 + (e3\sqrt{\mu} - \beta_2) \frac{1-x}{e^2} + e3 \epsilon e^{-\sqrt{\mu} \frac{1-x}{e^2}} \right) + \epsilon \left( g3 + \frac{7\sigma e3}{8 \mu e^2} \right) + \\
\left( \sqrt{\mu} g3 + \frac{\sigma e3}{8 \mu} + \frac{3 \sigma e3}{4 \mu} \frac{1-x}{e^2} \right) \frac{1-x}{e^2} + \left( g3 + \frac{7\sigma e3}{8 \mu e^2} \right) e^{-\sqrt{\mu} \frac{1-x}{e^2}} + \\
F_s + \frac{1}{2} \sigma \sqrt{\mu} e3 e^{-\sqrt{\mu} \frac{1-x}{e^2}} + \sigma \sqrt{\mu} e3 - \beta_2 \sigma \frac{1-x}{2\mu e^2}
$$

(3-32)
We obtain the values of the constants that appear in equation (3-32) by matching the inner solution equation (3-32) with the outer solution equation (3-22).

3.3.4 Composite Solution

After substituting the values of the constants obtained by matching the outer solution with the inner solution at each end, the composite solution is written as

\[ y_c = y^o + y^i + y^l - (y^l)^o - (y^i)^o \]  
(3-33)

After simplifying the solution, it becomes

\[ y_c = \ln(\alpha + \sigma x) \ a1 + \frac{F_c}{\sigma} x + a2 + e^{\frac{1}{2}}(ln(\alpha + \sigma x) \ a3 + a4) + e \left( a5 \ln(\alpha + \sigma x) + \right. \]

\[ a6 - \frac{2a1\sigma^2}{3(\alpha + \sigma x)^3} \) + e^{\frac{1}{2}} \left( b3 e^{\frac{-\sqrt{\alpha}}{\epsilon^2}} \right) - \left( \frac{3b3\sigma e^{\frac{-\sqrt{\alpha}}{\epsilon^2}}}{4\alpha} \right) \right)^{\frac{1}{2}} \epsilon^2 x + e \left( f3 - \]

\[ \frac{7b3\sigma}{8\sigma^2} \ e^{\frac{-\sqrt{\alpha}}{\epsilon^2}} - \frac{x^2}{2\alpha} \ b3\sigma \sqrt{\alpha} e^{\frac{-\sqrt{\alpha}}{\epsilon^2}} + e^{\frac{1}{2}} \left( e3 e^{\frac{-\sqrt{\mu}}{\epsilon^2}} \right) \right) + \]

\[ \frac{1}{\epsilon^2} \left( \frac{3\sigma e3 e^{\frac{-\sqrt{\mu}}{\epsilon^2}}}{4\mu} \right) (1-x) + e \left( g3 + \frac{7\sigma e3}{8\mu^2} \right) e^{\frac{-\sqrt{\mu}}{\epsilon^2}} + \]

\[ (\frac{1}{2} \sigma \sqrt{\mu} e3 e^{\frac{-\sqrt{\mu}}{\epsilon^2}}) (1-x)^2 \]\n
(3-34)

The value of the axial stretch in the composite solution of equation (3-34) is unknown. Therefore, the value of the axial stretch i.e. gamma, is obtained numerically using the error function shown in equation (3-35).

\[ Error(\epsilon, \nu, \Gamma) = \Gamma - \eta \int_0^1 [y'_c(\epsilon, \nu, \Gamma)]^2 dx \]
(3-35)
The applied tension and the internal velocity are provided in equation (3-35) with an initial guess of gamma and the boundary conditions $\beta_1$ and $\beta_2$ need to be zero to obtain the root of the equation using any root finding software [42, 43]. On the other hand, any non-zero values of $\beta_1$ and $\beta_2$ cause the numerical procedure to diverge because the mathematical derivation in Appendix A exclude the contribution of the boundary conditions on the axial stretch equation [41].

### 3.4 Summary

In a summary, we presented several solution methods in this chapter governing the static behavior of inclined risers. We can overcome the numerical limitations and constrains imposed by the exact and the approximate eigenvalue solution methods by using the boundary layer solution method. Also we proposed a numerical scheme to evaluate the axial stretch in the solution for each particular value of applied tension and internal velocity.
CHAPTER 4

RESULTS AND DISCUSSION

In this chapter, we present the results of the numerical simulations of VIV of straight pipe equations developed in Chapter 2 and the boundary layer perturbation solution obtained in Chapter 3.

The numerical simulation allows us to study the behavior of the coupled pipe structure equation of motion and the Van Der Pol oscillator governing the wake variable. The static deflection solution and dynamic solution convergence is presented at the beginning. Then, we show the dynamic results of the simulation and examine the differences between the hinged-hinged configuration and the clamped-clamped configuration.

In addition, we validate the boundary layer solution of the inclined riser against other methods, such as numerical boundary value problem solver and Galerkin approximation. This is followed by showing the numerical and mathematical limitations that other methods have, such as the exact model and the numerical model. Finally, we
study the effect of the internal velocity and applied axial tension for different angle configurations.

4.1 VIV of Straight Horizontal Pipe

The numerical results involve the solution of the static equation (2-20), natural frequency shift equation (2-13) which are presented here for the purpose of validating the mathematical model. Then, the dynamic solution coupled with Van Der Pol oscillator equation (2-21) is presented. These equations are examined while varying the internal velocity $v$ and the external current recued velocity $U_r$. The data shown in Table 4.1 are used in all the numerical simulations.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outside Diameter, $D$</td>
<td>0.25 m</td>
</tr>
<tr>
<td>Inside Diameter, $D_i$</td>
<td>0.125 m</td>
</tr>
<tr>
<td>Modulus of Elasticity, $E$</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Density of Riser, $\rho_p$</td>
<td>7850 Kg/m$^3$</td>
</tr>
<tr>
<td>Density of Sea Water, $\rho_e$</td>
<td>1025 Kg/m$^3$</td>
</tr>
<tr>
<td>Density of Internal Fluid, $\rho_t$</td>
<td>870 Kg/m$^3$</td>
</tr>
<tr>
<td>Pipe Length</td>
<td>150 m</td>
</tr>
</tbody>
</table>

4.1.1 Static Results and Frequency Shift

The pipe under the application of an axial force i.e. internal velocity is at equilibrium. Ignoring the effect of gravity, the pipe in such a case deflects from the
equilibrium only if the internal velocity exceeds a critical value called the buckling limit. As a result, the pipe will buckle and deflect from its equilibrium. Figure 4.1 shows the different buckling limits for both clamped-clamped and the hinged-hinged configurations.

![Graph showing buckling limits](image)

**Figure 4.1**: Static deflection of the pipe due to the effect of the internal flow

a) Hinged-hinged configuration b) Clamped-clamped configuration

We notice from the static solution on figure 4.1 that the buckling limit of the hinged pipe is $\pi$ while the clamped pipe is $2\pi$ giving in figure 4.1 (a) and (b). This means that clamped-clamped has more range for internal velocity to increase before the pipe
deflects. In addition, the post-buckled shape of the clamped-clamped pipe deflects more than that of the hinged-hinged configuration. These results matches the earlier the work of [34] for buckling solutions of beams under axial forces.

Moreover, the natural frequency of the pipe shifts due to the effect of the internal velocity based on pipe equation (2-13). As the internal velocity of the pipe increases the natural frequencies generally decreases. Also, the frequency range in the clamped configuration is more than that of the hinged pipe as shown in figure 4.2

![Figure 4.2](image_url)

**Figure 4.2:** Natural frequencies shift of the pipe due to the effect of internal flow.

a) Hinged-hinged configuration b) Clamped-clamped configuration
Also, we notice that the value of the natural frequency of the first mode shape decreases more than higher mode shapes. The reported numerical results in the figure matches the results reported by [1].

4.1.2 Dynamic Analysis of VIV

We obtain the dynamic solution of the system numerically by integrating the reduced order equation (2-21). However, the first step in the analysis is to determine the number of mode shapes that are required to achieve convergence of solution. Therefore, the contributions of the symmetric mode shapes are considered and the non-symmetric mode shapes are neglected because these do not contribute to the dynamics of the system [44]. If the system mathematically is subjected to a symmetric force with symmetric boundary conditions, the orthogonality condition of the mode shapes results into zero anti-symmetric forcing terms.

On that basis, the number of mode shapes required for convergence is four based on the results shown in figure 4.3 (a) and figure 4.3 (b) under hinged-hinged configuration. In the first case (a), the number of mode shapes is studied by plotting vibration amplitude of the center of the pipe versus the external reduced velocity while keeping the internal velocity fixed at zero. We observe from these results that the four modes expanded solution overlay the results obtained using three modes. Therefore, four mode shapes are sufficient to describe the dynamics of the system.
The second case in figure 4.3 (b) is to examine the number of mode shapes during lock-in keeping the reduced velocity fixed at a value of $Ur = 5$ while the internal velocity of the pipe is varied. This value of reduced velocity implies mathematically that the vortex shedding frequency is equal to the natural frequency of the pipe structure.

Figure 4.3 a: Convergence Analysis: Varying reduced velocity

Figure 4.3 b: Convergence Analysis: Varying internal velocity
From the results obtained in figure 4.3 (b) we conclude that four modes are sufficient to describe the dynamics of the system.

As a result, the vibration amplitudes of the pipe are obtained and shown in figure 4.4.

![VIV results: a) Hinged Configuration b) Clamped Configuration](image)

**Figure 4.4:** VIV results: a) Hinged Configuration b) Clamped Configuration

The plots show the vibration amplitude of the center of the pipe versus reduced velocity
for various values of internal pipe velocities. Based on equation (2-24), Strouhal number of 0.2 corresponds to a reduced velocity equal to five for lock-in to occur. This is the condition where the vortex shedding frequency is equal to the natural frequency of the structure. Also, as it is evident from figure 4.4 the cubic nonlinearity in the axial stretching term causes a hardening behavior which matches the experimental results reported in literature [24]. Also, the vibrations amplitude decreases with increasing internal velocity. This is due to the effect of the velocity contribution as an axial force. The internal velocity causes stiffening of the structure and as a result the natural frequency shifts. In addition, the lock-in region is characterized by a jump phenomenon, which can be observed at both results. However, the clamped-clamped configuration at zero internal velocity exhibits higher vibration amplitudes due to the effect of the geometric nonlinearity [45]. The induced axial stretch as a result from vibration is higher in the clamped-clamped configuration than that of the hinged-hinged configuration. In addition, exceeding the critical velocity of the hinged-hinged configuration yields a chaotic unstable behavior [24].

4.2 Static Solution of Inclined Risers

In this section, we study the static solutions of inclined risers in two different cases. The first case is when the axial stretch is zero. This allows us mathematically to use non-homogenous boundary conditions to obtain the solution. The second case is when including the axial stretch term while the boundary conditions are homogenous. In all cases, the methods of exact, Galerkin and boundary layer pertubation are validated against each other with a numerical boundary value problem solver solution using Rung
Kutta method in Mathematica. The values shown in table 4.2 have been used to obtain the static profile of different solution methods.

For simplicity we shall refer to the solution of Section 3.1 as reduced order model, solution of Section 3.2 as exact solution, solution of Section 3.3 as boundary layer solution and the boundary value problem solver solution as numerical solution.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outside Diameter, $D$</td>
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</tr>
<tr>
<td>Inside Diameter, $D_i$</td>
<td>0.2 m</td>
</tr>
<tr>
<td>Modulus of Elasticity, $E$</td>
<td>207 GPa</td>
</tr>
<tr>
<td>Offset at the Top</td>
<td>75 m</td>
</tr>
<tr>
<td>Density of Riser, $\rho_p$</td>
<td>7850 Kg/m$^3$</td>
</tr>
<tr>
<td>Density of Sea Water, $\rho_e$</td>
<td>1025 Kg/m$^3$</td>
</tr>
<tr>
<td>Density of Internal Fluid, $\rho_l$</td>
<td>998 Kg/m$^3$</td>
</tr>
<tr>
<td>Depth of Sea</td>
<td>150 m</td>
</tr>
</tbody>
</table>

4.2.1 Static Profile of Risers without Axial Stretch

In the absence of the axial stretch term we can utilize non-zero boundary conditions based on the mathematical derivation. Thus, the techniques that could be utilized are the boundary layer perturbation, analytical and the numerical solution. The boundary conditions applied here are
The use of the boundary layer above is in order to account for the physical constrains such as those introduced by the seabed configuration. The boundary layer solution is similar to the mathematical work introduced by [17]. The static profile obtained in figure 4.5 shows the good agreement between the exact, numerical and boundary layer perturbation solution at \( \epsilon = 0.0035 \) while the internal velocity is zero.

\[
\begin{align*}
y(0) &= 0 & y'(0) &= \frac{L}{D} \tan(\theta) = \beta_1 \\
y(1) &= 0 & y'(1) &= \frac{L}{D} \cot(\theta) = \beta_2
\end{align*}
\]

**Figure 4.5:** Boundary layer solution with exact and numerical at \( \epsilon = 0.0035, \nu=0 \)

From the figure we can observe close to \( x=0.2 \) that the boundary layer solution slightly deviate because the boundary layer did not fully form under the application of low tension. It is also noteworthy that the computation time of the exact solution is quite expensive. This is due to the time required to perform the integration of the hypergeometric series which was about 60 hours using on a regular personal computer PC. Moreover, if we increase the applied tension i.e. decreasing epsilon, then the exact
solution does not converge at values of epsilon where the boundary layer solution method exist due to the limitation caused by the Airy functions as shown in figure 4.6.

![Graph showing divergence of exact solution at \( \epsilon = 0.0028, \nu = 0 \).](image)

**Figure 4.6**: Divergence of exact solution at \( \epsilon = 0.0028, \nu = 0 \)

(---) Boundary Layer Solution, (---) Numerical BVP Solution, (---) Exact Solution

The exact solution reaches a turning point where the Airy function changes the behavior from monotonic to oscillatory in figure 4.6. This is caused because the argument value in equation (3-13) changes from a positive value to a negative value. Decreasing epsilon further limits the mathematical and numerical methods except the boundary layer method. This is mainly due the fact that the finite difference matrix in the numerical boundary value problem solver becomes stiff containing very small numerical values that causes numerical errors due to truncations [43]. Figure 4.7 shows the deficiency of the numerical solution at value of epsilon at \( \epsilon = 0.0015, \nu = 0 \).
The internal velocity of the pipe contributes to the pipe by an applied compressive force. Thus, increasing internal velocity causes the overall tension of the pipe to decrease and, therefore, the maximum deflection of the pipe increases as shown in figure 4.8.

**Figure 4.7:** Numerical solution deficiency at $\epsilon = 0.0015$, $\nu=0$

**Figure 4.8:** Effect of internal velocity at $\epsilon = 0.0028$
4.2.2 Static Profile of Risers with Axial Stretch

In contrast to the previous section, the boundary conditions in this case are homogenous. The homogenous boundary conditions represents a structure configuration different than that of Section 4.2.1 since the slope values at the ends of the pipe are considered zero, which allows the computation of the axial stretch term. However, the validation here is carried out using the method described in Section 3.1 where the basis function of the mode shapes are that of a straight pipe. Moreover, we obtain the axial stretch term in the boundary layer perturbation method and the numerical method using a root finding algorithm. The exact solution described in Section 3.2 is neglected here because the time required computing the axial stretch is expensive. In addition, the shooting algorithm to find the axial stretch did not converge.

The results in figure 4.9 show that the boundary layer solution agrees very well with the reduced order model solution using seven mode shapes and the numerical (BVP) solution. The value of the axial stretch is computed using the error function on the boundary layer solution. In addition, it is computed numerically in the boundary value problem solver by generating a function relating the axial stretch with the deflection.

![Graph showing Static profile results with axial stretch at $\epsilon = 0.0025, \nu = 0$](image)

**Figure 4.9:** Static profile results with axial stretch at $\epsilon = 0.0025, \nu = 0$
Then the axial stretch is computed based on the numerical BVP solution using the same error function. However, the numerical solution showed stiffness at $\epsilon = 0.0025$ as noted close to $x = 0.2$. This result is due to the effect of the axial stretch term that made the overall numerical scheme to become stiff. On the other hand, we can overcome the stiffness issue in the numerical scheme by using more stiff numerical methods with controlled precision as in figure 4.10. However, utilizing a stiff numerical scheme increases the cost of numerical computation.

Figure 4.10: Static profile at $\epsilon = 0.0025$, $\nu = 0$ using stiff numerical scheme

Moreover, the effect of the applied tension with the offset on the axial stretch is plotted in figure 4.11. The results show that increasing the applied axial tension which implied decreasing value of epsilon lowers the axial stretch.

Figure 4.11: Axial stretch at different values of applied axial tensions, $\nu = 0$
In addition, increasing the pipe offset at a fixed applied axial tension subjects the pipe to a large static force, which results into increasing the axial stretch of the pipe due to decreasing inclination angle.

4.3 Summary

In this chapter, the results of the nonlinear analysis on the dynamics of VIV of a straight horizontal pipe and boundary layer perturbation solution were shown. The results of the VIV problem showed that the use of Van Der Pol oscillator is sufficient to study the vibrations of the pipe in comparison to previously published work. Moreover, the boundary layer perturbation method showed good agreement to numerical, exact and even simplified eigenvalue solution in two different configurations. The boundary layer perturbation method offers a great advantage to obtain the static solution where other methods fail or have some limitations.
CONCLUSIONS AND FUTURE WORK

In conclusion, in this thesis we analyzed two case studies in offshore engineering fields. In the first study, the effect of the internal velocity on the VIV response of horizontal pipes was examined utilizing the modified Van Der Pol wake oscillator. The results showed that the increase of the internal velocity of the pipe causes the amplitude of vibration to decrease for different pipe end conditions such as hinged-hinged and clamped-clamped. Also, the numerical analysis of the clamped-clamped pipe configuration revealed higher range of internal velocities. The results are in good agreement with previously published experimental results. The axial stretch in the equation showed interesting dynamic features when hinged-hinged configuration is compared to clamped-clamped one. The hardening behavior as a result of the cubic nonlinearity showed higher vibration amplitude in the clamped-clamped configuration than that of the hinged-hinged configuration.

On the other part of the thesis, we developed a boundary layer static solution governing the static profile of inclined risers under applied tension loads. The effects of the internal velocity, applied tension, and axial stretch were accounted for in the
An error function was proposed to calculate the value of the axial stretch using a root finding method. The boundary layer solution showed good agreement compared to other methods such as analytical, Galerkin, and numerical boundary value problem techniques. In addition, the boundary layer solution offers a great advantage by overcoming the mathematical and numerical constraints that other methods have.

Moreover, the results of this work inspire research to be continued in a variety of ways. This is due to the wide research that is related to the topic presented in each area. These topics are presented here in this chapter.

### 5.1 Static Solution of Inclined Riser with Mean Flow Drag Effects

The boundary layer perturbation solution presented in Chapter 3 can be extended to include the effect of mean drag flow in Morison formulation. This occurs when the external current flow is in the curvature plane of the pipe. Therefore, the solution obtained is not only a function of the applied tension and internal velocity but also it includes the effects of external flow current. The normal mean drag force causes a static deflection of the structure. Therefore, it becomes very useful and handy tool if the effect of the mean drag and its interaction with the structure is captured in the analysis. The equation of motion of pipe in that case needs to be modified to account for the effects of the mean drag flow. Then the boundary layer method is applied to the modified equation in order to obtain a solution.

### 5.2 General Static Solution of Riser with Axial Stretch

The boundary layer perturbation static solution of inclined riser was obtained with the limitation that the axial stretch can only be obtained under homogenous boundary
conditions. The work carried out by [31] during laying operation emphasized on obtaining the solution for different riser configurations but did not account for the axial stretch. A great advantage is the ability to obtain an analytical or perturbation model that specifies the static profile of inclined pipe at any boundary condition with the axial stretch. This is based on the physical constrain and the stretch of the pipe that occurs in practice. Therefore, a new mathematical model shall be derived governing the axial stretch of the riser and under any end configuration.

5.3 VIV of Statically Deformed Pipe

Upon subjecting the pipe to static deformation the dynamic behavior of the pipe will change. It is, therefore, imperative to examine the VIV of the pipe under the circumstances mentioned either as a future work or presented in chapter 4. This will offer an insight on the interaction of the static profile with the overall vibration behavior of the pipe. Nayfeh and Emmam [34] showed in their work that axial loaded beams showed interesting nonlinear dynamic behaviour when the beams vibrate under post-buckled condition. Some features such as: multi-modal interaction and internal resonances might be excited. The natural frequency of the post-buckled clamped-clamped pipe configuration shows 2:1 internal resonance. Therefore, the dynamics of the pipe is examined by substituting the static solution in the original equation of motion.
REFERENCES


APPENDICES
Appendix A

Deriving the Equation of Motion of an Inclined Riser Conveying Fluid

This section provides the mathematical formulation to derive the equation of motion of inclined risers conveying fluid using extended Hamilton principle. The pipe is assumed to be in an inclined configuration with an initial angle $\theta$. The equilibrium static configuration of the riser results from applying the hydrostatic loads and the internal fluid force. It is also assumed that the direction of the flow is orthogonal to the plane of pipe curvature; thus, it is safe to neglect the effect of the mean drag forces on the riser.

The following are assumed while deriving the equation of motion of the riser:

1. The internal fluid flow is one-dimensional plug laminar flow steady and uniform flow.

2. The fluid internal is inviscid, incompressible, and irrotational.

3. The Strouhal number is assumed as $St = 0.2$ for the subcritical range of Reynolds number between $300 < Re < 1.5 \times 10^5$.

The effects of shear, rotary inertia, structural damping and Poisson’s ratio are not considered in the analysis.
The riser in the unreformed state is assumed to be straight and we examine the forces on the riser in the local x,y coordinates:

Due to the pipe inclination angle the distributed weight load is decomposed to an axial component and a lateral component in which the axial component contributes to the applied tension force as follows

\[ T_a = T_e - W_{\text{eff}} \sin(\theta)(L - \hat{x}) \]  

(A-1)

Where

\[ W_{\text{eff}} = (\rho_i A_i + \rho_p A_p - \rho_e A_e)g \]  

(A-2)

The effective weight \( W_{\text{eff}} \) takes into account the static buoyancy force which opposes the weight of the riser structure and internal fluid. The total potential energy \( PE \) of the

**Figure A.1:** General description of the pipe under static and hydrodynamic loading
riser resulting from the effects of bending moment and the axial stretch of the riser is expressed as

\[ PE_{\text{riser}} = \frac{EA}{2} \int_0^L (u' + \frac{\hat{\gamma}'^2}{2})^2 \, d\hat{x} + \frac{EI}{2} \int_0^L (\hat{\gamma}'')^2 \, d\hat{x} \quad (A-3) \]

Neglecting the effect of the axial inertia, the kinetic energy \( KE \) is a result from contribution of the riser structure, internal fluid and added mass:

\[ KE_{\text{riser}} = \frac{A_p}{2} \int_0^L \rho_p \, \hat{y}^2 \, d\hat{x} \quad (A-4) \]

\[ KE_{\text{fluid}} = \frac{dA_l}{2} \int_0^L \left[ (\hat{\psi} + U_l \hat{\gamma}')^2 + U_l^2 \right] \, d\hat{x} \quad (A-5) \]

\[ KE_{\text{Added Mass}} = \frac{1}{2} \int_0^L C_A \rho e \frac{\pi}{4} D^2 \, \dot{\hat{\gamma}} \, d\hat{x} \quad (A-6) \]

The work done by the non-conservative forces includes the applied tension and the vortex induced vibration forces

\[ W_{\text{NC}} = \int_0^L F(\hat{x}, t) \, \dot{\hat{x}} \, d\hat{x} - T_a \, u(l) \quad (A-7) \]

Where the forces are

\[ F(x, t) = \frac{1}{2} \rho \cdot D U_e^2 C_L(\hat{x}, t) - \frac{1}{2} C_D \rho \cdot D U_e \hat{y} - W_{\text{eff}} \cos(\theta) \quad (A-8) \]

It is noteworthy that the fluid drag forces where obtained mathematically using Morison formulation which will contribute to an inertial and a damping components.

Applying the extended Hamilton principle

\[ \int_{t_1}^{t_2} (\delta KE - \delta PE + \delta W_{\text{NC}}) \, dt = 0 \quad (A-9) \]
Substituting the previous equations into equation A-9 gives the following outcome

\[ \delta PE = \int_{t_1}^{t_2} \left\{ EA \left( u' + \frac{\ddot{y}^2}{2} \right) \delta u \bigg|_0^L - EA \int_0^L \left( u' + \frac{\ddot{y}^2}{2} \right)' \delta u \, d\hat{x} + EA \left( u' + \frac{\ddot{y}^2}{2} \right) y' \delta y \bigg|_0^L - EA \int_0^L \left( u' + \frac{\ddot{y}^2}{2} \right) \dot{v}' \delta \hat{y} \, d\hat{x} + EI \dddot{y} \delta \ddot{y} \bigg|_0^L - EI \dddot{y}'' \delta \dot{y} \bigg|_0^L + EI \int_0^L \dddot{y}''' \, d\hat{x} \right\} \]

\[ W_{eff} \sin(\theta) (L - x) \dddot{y} \delta \ddot{y} \bigg|_0^L + \int_0^L \left( W_{eff} \sin(\theta) (L - x) \dddot{y}' \delta \dot{y} \bigg|_0^L \right) \, dt \quad (A-10) \]

\[ \delta KE = A_p p_p \int_0^L \left\{ - \int_{t_1}^{t_2} \dddot{y} \delta \dot{y} \, dt \right\} \, d\hat{x} + \rho_i A_i \int_0^L \left\{ - \int_{t_1}^{t_2} (\dddot{y} + 2 U_i \dot{y} + U_i^2 \dot{y}'') \delta \ddot{y} \, dt \right\} \, d\hat{x} + \rho_i A_i (\dddot{U}_i + U_i^2 \dot{y}') \delta \ddot{y} \bigg|_0^L + \int_0^L \left\{ - \int_{t_1}^{t_2} \dddot{U}_i \delta \dot{y} \, dt \right\} \, d\hat{x} \quad (A-11) \]

\[ \delta W_{NC} = \int_0^L F(\hat{x}, t) \delta \dot{y} \, d\hat{x} + T_a \delta u(l) \quad (A-12) \]

For the integrals to be satisfied each group of terms in A-10, A-11 and A-12 shall vanish.

This will result into

\[ EI y''' + \left( A_p p_p + \rho_i A_i + \frac{1}{2} C_A \rho_e \frac{\pi}{4} D^2 \right) \dddot{y} + \rho_i A_i 2 U_i \dddot{y} + \rho_i A_i U_i^2 \dot{y}'' \]

\[ - EA \left( u' + \frac{\ddot{y}^2}{2} \right) \dot{y}' = F(\hat{x}, t) \quad (A-13) \]

\[ EA \left( u' + \frac{\ddot{y}^2}{2} \right)' = 0 \quad (A-14) \]

The above two equations A-13 and A-14 respectively govern the motion of the riser structure subjected to vortex induces vibration forces. The system of equations after reducing and further manipulation can be written as
\[
m \frac{\partial^2 \ddot{y}}{\partial t^2} + EI \frac{\partial^4 \ddot{y}}{\partial \dddot{x}^4} + 2m_f U_i \frac{\partial^2 \ddot{y}}{\partial \dddot{x} \partial t} \\
+ \left( m_f U_i^2 - (T_e - We * Sin(\theta)(L - \dddot{x})) - \frac{EA_p}{2L} \int_0^L \left( \left( \frac{\partial \ddot{y}}{\partial \dddot{x}} \right)^2 \right) d\dddot{x} \right) \left( \frac{\partial^2 \ddot{y}}{\partial \dddot{x}^2} \right) \\
- We * Sin(\theta) \left( \frac{\partial \ddot{y}}{\partial \dddot{x}} \right) + \frac{1}{2} C_d \rho_e D U_e \frac{\partial \ddot{y}}{\partial t} = \frac{1}{4} C_{lo} \rho_e D U_e^2 q - We * Cos(\theta)
\]

(A-15)

Where \( m = (C_A \rho_e A_e + \rho_i A_i + \rho_p A_p) \)

Since the riser in the assumed configuration is resting on the seabed or touch down point (TDP) the boundary condition is to be fixed-fixed as below

\[
y(0) = 0 \quad y'(0) = \left( \frac{\text{Depth}}{\text{Offset}} \right) = \text{Tan}(\theta) \\
y(L) = 0 \quad y'(L) = \left( \frac{\text{Offset}}{\text{Depth}} \right) = \text{Tan}(90 - \theta)
\]

(A-16)

Equation A-15 is coupled with the lift equation that is governed by a van der pol oscillator as shown

\[
\frac{\partial^2 q}{\partial t^2} + \lambda \omega_s (q^2 - 1) \frac{\partial q}{\partial t} + \omega_s^2 q = P \frac{\partial^2 y}{\partial t^2}
\]

(A-17)

Where

\[
q = \frac{2C_L(\ddot{x},t)}{C_{lo}}
\]

(A-18)
Appendix B

Boundary Layer Perturbation Solution of the Static Equation of Inclined Risers

The objective of the boundary layer perturbation is to establish an approximate static closed form solution to the inclined riser configuration by resolving the bending near the boundaries of the riser throughout perturbation method.

\[ EI \frac{d^4 \phi}{d \xi^4} + \left( m_f U_i^2 - (T - We \sin(\theta)(L - \xi)) - \frac{E_A p}{2L} \int_0^L \left[ \frac{d \phi}{d \xi} \right] d \xi \right) \left( \frac{d^2 \phi}{d \xi^2} \right) = 0 \]

\[ We \sin(\theta) \frac{d \phi}{d \xi} = -We \cos(\theta) \quad (B-1) \]

It is assumed herein that \( T \) i.e. the axial tension is dominant over the bending moment and other contributions; thereby we can divide the equation by \( T \) and introducing Non-dimensional variables

\[ y = \frac{\phi}{D} \quad x = \frac{\xi}{L} \quad \sigma = \frac{We L \sin(\theta)}{T} \quad \eta = \frac{E_A p D^2}{2TL^2} \quad F_s = \frac{We L^2 \cos(\theta)}{D T} \]

\[ \nu = \frac{U_i \sqrt{m_f}}{\sqrt{T}} \quad \varepsilon = \frac{EI}{TL^2} \]

Where;

\( D \): Pipe Diameter  \( L \): Pipe Length  \( T \): Pipe Pre-Tension  \( U_i \): Internal Fluid Velocity

\( We \): Effective Weight = \( (\rho_i A_i + \rho_p A_p - \rho_e A_e) g \)  \( EI \): Flexural Rigidity
The equation then becomes

\[ \epsilon \frac{d^4y}{dx^4} - (1 - \nu^2 - \sigma + \sigma x + \eta \int_0^1 \left( \frac{dy}{dx} \right)^2 dx) \left( \frac{d^2y}{dx^2} \right) - \sigma \frac{dy}{dx} = -F_s \]  

(B-2)

The boundary condition are written as

\[ y(0) = 0 \quad y'(0) = \left( \frac{\text{Depth}}{\text{Offset}} \right) = \frac{L}{D} \tan(\theta) = \beta_1 \]

\[ y(1) = 0 \quad y'(1) = \frac{L}{D} \cot(\theta) = \beta_2 \]  

(B-3)

Now we let \( \Gamma = \eta \int_0^1 \left( \frac{dy}{dx} \right)^2 dx \) and treat it as a constant throughout the derivation process. At a later stage an iterative numerical procedure will be applied to obtain the solution based on root finding technique.

For simplicity we denote \( \alpha = 1 - \nu^2 - \sigma + \Gamma \) and substitute it in the equation. The scale was chosen such that the force is of the same order as the bending. The equation is scaled such that \( \epsilon \) represent small scale:

\[ \epsilon \frac{d^4y}{dx^4} - (\alpha + \hat{x}) \left( \frac{d^2y}{dx^2} \right) - \sigma \frac{dy}{dx} = -F_s \]  

(B-4)

Assuming a straight forward expansion of the solution of the form

\[ y^0(x; \epsilon) = y^0_0(x) + \epsilon^2 y^1_1(x) + \epsilon y^1_0(x) + O(\epsilon^{3/2}) \]

\[ \epsilon \frac{d^4y_0^0}{dx^4} + \frac{3}{2} \frac{d^4y_1^0}{dx^4} + \epsilon^2 \frac{d^4y_2^0}{dx^4} - (\alpha) \left( \frac{d^2y_0^0}{dx^2} + \epsilon^2 \frac{d^2y_1^0}{dx^2} + \epsilon \frac{d^2y_2^0}{dx^2} \right) - \sigma \hat{x} \left( \frac{d^2y_0^0}{dx^2} + \epsilon^2 \frac{d^2y_1^0}{dx^2} + \epsilon^2 \frac{d^2y_2^0}{dx^2} \right) \]

\[ \epsilon \frac{d^2y_2^0}{dx^2} - \sigma \left( \frac{dy_0^0}{dx} + \frac{1}{2} \frac{dy_1^0}{dx} + \epsilon \frac{dy_2^0}{dx} \right) = -F_s \]  

(B-5)
As $\epsilon \to 0$ we seek an outer straightforward expansion of equation B-5 written as

\[
O(\epsilon^0): \quad \alpha \frac{d^2y_0}{dx^2} + \sigma \hat{x} \frac{d^2y_0}{dx^2} + \sigma \frac{dy_0}{dx} = F_s \quad \text{(B-6a)}
\]

\[
O\left(\frac{1}{\epsilon^2}\right): \quad \alpha \frac{d^2y_1}{dx^2} + \sigma \hat{x} \frac{d^2y_1}{dx^2} + \sigma \frac{dy_1}{dx} = 0 \quad \text{(B-6b)}
\]

\[
O(\epsilon): \quad \alpha \frac{d^2y_2}{dx^2} + \sigma \hat{x} \frac{d^2y_2}{dx^2} + \sigma \frac{dy_2}{dx} = -\frac{d^4y_0}{dx^4} \quad \text{(B-6c)}
\]

The general solution of the outer expansion of equation B-6a, B-6b and B-6c is:

\[
y_0^0(\tilde{x}) = \frac{F_s}{\sigma} x + a1 \ln(\alpha + \sigma x) + a2 \quad \text{(B-7a)}
\]

\[
y_1^0(\tilde{x}) = a3 \ln(\alpha + \sigma x) + a4 \quad \text{(B-7b)}
\]

\[
y_2^0(\tilde{x}) = a5 \ln(\alpha + \sigma x) + a6 - \frac{2a1}{3(\alpha + \sigma x)^3} \sigma^2 \quad \text{(B-7c)}
\]

And therefore the total outer solution becomes

\[
y^0 = \frac{F_s}{\sigma} x + a1 \ln(\alpha + \sigma x) + a2 + \frac{1}{\epsilon^2}(a3 \ln(\alpha + \sigma x) + a4)
\]

\[+ \epsilon \left(a5 \ln(\alpha + \sigma x) + a6 - \frac{2a1}{3(\alpha + \sigma x)^3} \sigma^2\right) \quad \text{(B-8)}
\]

The outer solution will not satisfy any boundary condition; instead, the outer solution will be matched with the inner solutions that exist near $x = 0$ and $x = 1$

**Inner Expansion near $x = 0$**

In the neighborhood of $x = 0$ an interval of study i.e. a boundary layer is magnified to seek an outer solution assuming a variable transformation of the form:
\[ \xi = \frac{x}{e^\lambda} \] where \( x, \lambda \) are greater than zero.

\[
x = \xi \ e^\lambda
\]

\[
\frac{dy}{dx} = \frac{dy}{d\xi} \frac{d\xi}{dx} = \frac{dy}{d\xi} \ e^{-\lambda} ; \quad \frac{d^2y}{dx^2} = \frac{1}{\epsilon^{2\lambda}} \frac{d^2y}{d\xi^2} ; \quad \frac{d^4y}{dx^4} = \frac{1}{\epsilon^{4\lambda}} \frac{d^4y}{d\xi^4}
\]

Substituting the above in the main equation B1.5 becomes

\[
e^{1-4\lambda} \frac{d^4y^i}{d\xi^4} - \alpha \left( e^{-2\lambda} \frac{d^2y^i}{d\xi^2} \right) - \sigma \xi \ e^\lambda \left( e^{-2\lambda} \frac{d^2y^i}{d\xi^2} \right) - \sigma \left( e^{-\lambda} \frac{dy^i}{d\xi} \right) = -F_y \quad \text{(B-9)}
\]

As \( \epsilon \to 0 \) the dominant parts corresponds to \( \lambda = \frac{1}{2} \) and the equation becomes

\[
\frac{d^4y^i}{d\xi^4} - \alpha \frac{d^2y^i}{d\xi^2} - \sigma \xi \left( \frac{1}{\epsilon^2} \frac{d^2y^i}{d\xi^2} \right) - \sigma \left( \frac{1}{\epsilon} \frac{dy^i}{d\xi} \right) = -\epsilon F_y \quad \text{(B-10)}
\]

Seeking a solution \( y^i(\xi; \epsilon) = y_0^i(\xi) + \epsilon^{\frac{1}{2}} y_1^i(\xi) + \epsilon y_2^i(\xi) + O(\epsilon^{3/2}) \); then we have

\[
O(\epsilon^0): \quad \frac{d^4y_0^i}{d\xi^4} - \alpha \frac{d^2y_0^i}{d\xi^2} = 0 \quad \text{(B-11a)}
\]

\[
O\left( \epsilon^{\frac{1}{2}} \right): \quad \frac{d^4y_2^i}{d\xi^4} - \alpha \frac{d^2y_2^i}{d\xi^2} = \sigma \xi \left( \frac{d^2y_0^i}{d\xi^2} \right) + \sigma \left( \frac{dy_0^i}{d\xi} \right) \quad \text{(B-11b)}
\]

\[
O(\epsilon): \quad \frac{d^4y_1^i}{d\xi^4} - \alpha \frac{d^2y_1^i}{d\xi^2} = \sigma \xi \left( \frac{d^2y_1^i}{d\xi^2} \right) + \sigma \left( \frac{dy_1^i}{d\xi} \right) - F_y \quad \text{(B-11c)}
\]

And the boundary conditions are

\[
y^i(0) = 0 \quad \frac{dy^i}{dx}(0) = \left( \frac{\text{Depth}}{\text{Offset}} \right) = \beta_1
\]

Which translate in terms of \( \xi \) to
The general solution of equation B-11a is

\[ y_i^i(0) = 0 \quad \frac{dy_i}{d\xi}(0) = \epsilon^{\frac{1}{2}}\beta_1 \]

Note that the other exponential growth has been omitted for matching purposes.

**Equation B-12**

\[ y_0^i(\xi) = c_1 + c_2\xi + c_3 e^{-\sqrt{\alpha} \xi} \]

\[ c_1 + c_3 = 0 \quad \text{(B-13a)} \]

\[ c_2 - c_3 \sqrt{\alpha} = 0 \quad \text{(B-13b)} \]

The solution equation B-12 is written as

**Equation B-14**

\[ y_0^i(\xi) = c_3(\sqrt{\alpha}\xi + e^{-\sqrt{\alpha} \xi} - 1) \]

To obtain the value of the constant we match a one-term inner expansion with one term outer expansion

**Outer Expansion in terms of \( \xi \)**

\[ y^o = a_1 \ln(\alpha + \sigma \epsilon^{\frac{1}{2}}\xi) + \frac{F_s}{\sigma} \epsilon^{\frac{1}{2}}\xi + a_2 \quad \text{(B-15)} \]

Expanded for small value of \( \epsilon \)

**Equation B-16**

\[ y^o = a_1 \left( \ln(\alpha) + \frac{\sigma \epsilon^{\frac{1}{2}}\xi}{\alpha} \right) + \frac{F_s}{\sigma} \epsilon^{\frac{1}{2}}\xi + a_2 \]

Which becomes for in terms of \( x \)

**Equation B-17**

\[ y^o = a_1 \ln(\alpha) + a_2 \]

Inner expansion in terms of \( x \)

**Equation B-18**

\[ c_3 \left( \sqrt{\alpha} \frac{x}{\epsilon^{\frac{1}{2}}} - 1 \right) + E.S.T. \]

Expanding for small value of \( \epsilon \) the \( \hat{x} \) term has to be zero for matching purposes:
\[ c3 = 0 \]  \hspace{1cm} (B-19a)

\[ a2 = -a1 \ln(\alpha) \]  \hspace{1cm} (B-19b)

Thus equation B1.12 reduces to

\[ y_0^i(\xi) = 0 \]  \hspace{1cm} (B-19b)

Substituting back into B-11b becomes

\[ \frac{d^4 y_1^i}{d\xi^4} - \alpha \frac{d^2 y_1^i}{d\xi^2} = 0 \]  \hspace{1cm} (B-20)

\[ \dot{y}^i(0) = 0, \quad \frac{dy^i}{d\xi}(0) = \frac{1}{\sqrt{\alpha}} \beta_1 \]

The solution of equation B-20 is

\[ y_1^i(\xi) = b1 + b2 \xi + b3 e^{-\sqrt{\alpha}} \xi \]  \hspace{1cm} (B-21)

Applying the boundary conditions on equation B-21 we obtain the following

\[ b1 + b3 = 0 \]  \hspace{1cm} (B-22a)

\[ b2 - b3\sqrt{\alpha} = \beta_1 \]  \hspace{1cm} (B-22b)

Therefore equation B1.21 is rewritten as

\[ y_1^i(\xi) = -b3 + (\beta_1 + b3\sqrt{\alpha})\xi + b3 e^{-\sqrt{\alpha}} \xi \]  \hspace{1cm} (B-23)

**Two Terms Matching**

Outer Expansion in terms of \( \xi \)

\[ y^o = a1 \ln \left( \alpha + \sigma \xi \frac{1}{\varepsilon} \right) + \frac{F}{\sigma} \xi \frac{1}{\varepsilon} + a2 + \varepsilon^{\frac{1}{2}} \left( a3 \ln \left( \alpha + \sigma \xi \frac{1}{\varepsilon} \right) + a4 \right) \]  \hspace{1cm} (B-24)
Expanded for small value of $\epsilon$

$$y^o = a1 \left( \ln(\alpha) + \frac{\frac{1}{\alpha} e^\xi}{\sigma} \right) + \frac{F_s}{\sigma} e^{\frac{1}{\epsilon^2} \xi} + a2 + e^{\frac{1}{\epsilon^2}} (a3(\ln(\alpha)) + a4)$$  \hspace{1cm} (B-25)

Which becomes in terms of $x$

$$y^o = a1 \left( \ln(\alpha) + \frac{\sigma x}{\alpha} \right) + \frac{F_s}{\sigma} x + a2 + e^{\frac{1}{\epsilon^2}} (a3(\ln(\alpha)) + a4)$$  \hspace{1cm} (B-26)

Inner Expansion in terms of $x$

$$\frac{1}{\epsilon^2} \left( -b3 + (\beta_1 + b3\sqrt{\alpha}) \frac{x}{\epsilon} + b3 e^{-\sqrt{\alpha} \frac{x}{\epsilon^2}} \right)$$  \hspace{1cm} (B-27)

Which reduces for small value of $\epsilon$ to

$$-e^{\frac{1}{\epsilon^2}} b3 + (\beta_1 + b3\sqrt{\alpha})x + E.S.T.$$  \hspace{1cm} (B-28)

Comparing the two equation for order results into

$$-b3 = a3 \ln(\alpha) + a4$$  \hspace{1cm} (B-29a)

$$\left( \frac{\sigma}{\alpha} \right) a1 + \frac{F_s}{\sigma} = (\beta_1 + b3\sqrt{\alpha})$$  \hspace{1cm} (B-29b)

Substituting back solution $B-23$ into $B-11c$ the solution becomes

$$y_{2}^{(1)}(\xi) = f1 + \left( f2 - \frac{3b3\sigma e^{-\sqrt{\alpha} \xi}}{4 \alpha} \right) \xi + \left( f3 - \frac{7b3\sigma}{8 \alpha^{3/2}} \right) e^{-\sqrt{\alpha} \xi} \xi$$

$$+ \frac{\xi^2}{2\alpha} \left( F_s - b3 \sigma \sqrt{\alpha} - \sigma \beta_1 - b3 \sigma \sqrt{\alpha} e^{-\sqrt{\alpha} \xi} \right)$$  \hspace{1cm} (B-30)
Applying the boundary conditions on equation B-30 we obtain the following

\[ f\mathbf{1} + f\mathbf{3} - \frac{7}{8} \frac{b3\sigma}{a^2} = 0 \]  
\[ \text{(B-31a)} \]

\[ f\mathbf{2} - \frac{3}{4} \frac{b3\sigma}{a} - \sqrt{\alpha} \left( f\mathbf{3} - \frac{7}{8} \frac{b3\sigma}{a^2} \right) = 0 \]  
\[ \text{(B-31b)} \]

Thus solution B-30 can be rewritten as

\[ y_2^i(\xi) = \left( \frac{7}{8} \frac{b3\sigma}{a^2} - f\mathbf{3} \right) + \left( \sqrt{\alpha} \left( f\mathbf{3} - \frac{7}{8} \frac{b3\sigma}{a^2} \right) + \frac{3}{4} \frac{b3\sigma}{a} - \frac{3}{4} \frac{b3\sigma}{a} e^{-\sqrt{\alpha} \xi} \right) \xi + \left( f\mathbf{3} - \frac{7}{8} \frac{b3\sigma}{a^2} \right) e^{-\sqrt{\alpha} \xi} + \frac{\xi^2}{2\alpha} \left( f_s - b3\sigma\sqrt{\xi} - \sigma \beta_1 - b3\sigma\sqrt{\alpha} e^{-\sqrt{\alpha} \xi} \right) \]  
\[ \text{(B-32)} \]

Three Terms Matching

Outer Expansion in terms of \( \xi \)

\[ y^o = a1 \ln \left( \alpha + \sigma \xi e^{-\frac{1}{\xi}} \right) + \frac{F_s}{\sigma} \xi e^{-\frac{1}{\xi}} + a2 + e^{\frac{1}{\xi}} \left( a3 \ln \left( \alpha + \sigma \xi e^{-\frac{1}{\xi}} \right) + a4 \right) \]

\[ + \epsilon \left( a5 \ln \left( \alpha + \sigma \xi e^{-\frac{1}{\xi}} \right) + a6 - \frac{2 a1 \sigma^2}{3 \left( \alpha + \sigma \xi e^{-\frac{1}{\xi}} \right)^3} \right) \]  
\[ \text{(B-33a)} \]

Expanded for small value of \( \epsilon \)

\[ y^o = a1 \left( \ln(\alpha) + \frac{1}{\sigma} \alpha e^{-\frac{1}{\xi}} / 2a^2 \right) + \frac{F_s}{\sigma} \frac{1}{\xi} e^{-\frac{1}{\xi}} + a2 + e^{\frac{1}{\xi}} \left( a3 \left( \ln(\alpha) + \frac{1}{\sigma} \alpha e^{-\frac{1}{\xi}} / 2a^2 \right) + a4 \right) + \epsilon \left( a5 \left( \ln(\alpha) + \frac{1}{\sigma} \alpha e^{-\frac{1}{\xi}} / 2a^2 \right) + a6 - \frac{a1 \sigma^2}{3 \alpha^3} + \frac{2 a1 \sigma^3}{a^4} e^{-\frac{1}{\xi}} \right) \]  
\[ \text{(B-33b)} \]
Which becomes in terms of $x$

$$y^o = a1 \left( \ln(\alpha) + \frac{\sigma x}{\alpha} - \frac{\sigma^2 x^2}{2\alpha^2} \right) + \frac{F_s}{\sigma} x + a2 + \varepsilon^2 \left( a3 \left( \ln(\alpha) + \frac{\sigma x}{\alpha} \right) + a4 \right) +$$

$$\varepsilon \left( a5 \left( \ln(\alpha) + \frac{\sigma x}{\alpha} \right) + a6 - \frac{a1}{3} \frac{\sigma^2}{\alpha^3} + \frac{2 a1}{\alpha^4} \frac{\sigma^3}{\alpha^3} x \right)$$

(B-33c)

Inner Expansion in terms of $x$

$$\varepsilon^2 \left( b3 + (\beta_1 + b3\sqrt{\alpha}) \frac{x}{\varepsilon^2} + b3 e^{-\sqrt{\alpha} \frac{x}{\varepsilon^2}} \right) + e \left( \left( \frac{7 b3\sigma}{8 \alpha^3} - f3 \right) + \left( \sqrt{\alpha} \left( f3 - \frac{7 b3\sigma}{8 \alpha^3} \right) \right) \right)$$

$$\left( \frac{3 b3\sigma}{4 \alpha} - \frac{3 b3\sigma}{4 \alpha} e^{-\sqrt{\alpha} \frac{x}{\varepsilon^2}} \right) \frac{x}{\varepsilon^2} + \left( f3 - \frac{7 b3\sigma}{8 \alpha^{3/2}} \right) e^{-\sqrt{\alpha} \frac{x}{\varepsilon^2}} + \frac{x^2}{2\alpha} \left( F_s - b3 \sigma\sqrt{\alpha} - \sigma \beta_1 - b3\sigma\sqrt{\alpha} e^{-\sqrt{\alpha} \frac{x}{\varepsilon^2}} \right)$$

(B-33d)

Which reduces for small value of $\varepsilon$ to

$$-\varepsilon^2 b3 + (\beta_1 + b3\sqrt{\alpha}) x + e \left( \frac{7 b3\sigma}{8 \alpha^3} - f3 \right) + \varepsilon^2 x \left( \sqrt{\alpha} \left( f3 - \frac{7 b3\sigma}{8 \alpha^3} \right) + \frac{3 b3\sigma}{4 \alpha} \right) +$$

$$\frac{x^2}{2\alpha} \left( F_s - b3 \sigma\sqrt{\alpha} - \sigma \beta_1 \right) + E.S.T.$$ (B-33e)

Comparing the two equation for order results into

$$\left( \frac{7 b3\sigma}{8 \alpha^3} - f3 \right) = a5 \ln(\alpha) + a6 - \frac{a1}{3} \frac{\sigma^2}{\alpha^3}$$

(B-34a)
\[
\left( \sqrt{\alpha} \left( f 3 - \frac{7 b 3 \sigma}{8 \alpha^2} \right) + \frac{3 b 3 \sigma}{4 \alpha} \right) = a 3^{\frac{\sigma}{\alpha}} \tag{B-34b}
\]

\[
\left( \frac{F_s}{\sigma} - b 3 \sqrt{\alpha} - \beta_1 \right) = -a \frac{\sigma}{\alpha} \tag{B-34c}
\]

**Inner Expansion near \( x = 1 \)**

In the neighborhood of \( x = 1 \) an interval of study i.e. a boundary layer is magnified to seek an outer solution assuming a variable transformation of the form:

\[
\zeta = \frac{1 - x}{\epsilon^\lambda}
\]

\[
x = 1 - \zeta \epsilon^\lambda
\]

\[
\frac{dy^l}{dx} = \frac{dy}{d\zeta} \frac{d\zeta}{dx} = -\frac{dy^l}{d\zeta} \epsilon^{-\lambda} \quad ; \quad \frac{d^2 y^l}{dx^2} = \frac{1}{\epsilon^{2\lambda}} \frac{d^2 y^l}{d\zeta^2} \quad ; \quad \frac{d^4 y}{dx^4} = \frac{1}{\epsilon^{4\lambda}} \frac{d^4 y^l}{d\zeta^4}
\]

Substituting the above into the main equation B1.5 becomes

\[
\epsilon^{1-4\lambda} \frac{d^4 y^l}{d\zeta^4} - \alpha \left( \epsilon^{-2\lambda} \frac{d^2 y^l}{d\zeta^2} \right) - \sigma \left( 1 - \zeta \epsilon^\lambda \right) \left( \epsilon^{-2\lambda} \frac{d^2 y^l}{d\zeta^2} \right) - \sigma \left( \epsilon^{-\lambda} \frac{dy^l}{d\zeta} \right) = -F_s \tag{B-35}
\]

Equation B-35 can be written as

\[
\epsilon^{1-4\lambda} \frac{d^4 y^l}{d\zeta^4} - (\alpha + \sigma) \left( \epsilon^{-2\lambda} \frac{d^2 y^l}{d\zeta^2} \right) + \sigma \zeta \epsilon^\lambda \left( \epsilon^{-2\lambda} \frac{d^2 y^l}{d\zeta^2} \right) + \sigma \left( \epsilon^{-\lambda} \frac{dy^l}{d\zeta} \right) = -F_s \tag{B-36}
\]

We denote \( \mu = \alpha + \sigma = 1 - \nu^2 + \Gamma \)

As \( \epsilon \to 0 \) the dominant parts corresponds to \( \lambda = \frac{1}{2} \) and the equation becomes

\[
\frac{d^4 y^l}{d\zeta^4} - \mu \left( \frac{d^2 y^l}{d\zeta^2} \right) + \sigma \zeta \epsilon^{\frac{1}{2}} \left( \frac{d^2 y^l}{d\zeta^2} \right) + \sigma \epsilon^{\frac{1}{2}} \left( \frac{dy^l}{d\zeta} \right) = -\epsilon F_s \tag{B-37}
\]
Seeking a solution $y^l(\zeta; \varepsilon) = y_0^l(\zeta) + \varepsilon^{1/2} y_1^l(\zeta) + \varepsilon y_2^l(\zeta) + O(\varepsilon^{3/2})$ then we have

$$O(\varepsilon^0): \frac{d^4 y_0^l}{d\zeta^4} - \mu \left( \frac{d^2 y_0^l}{d\zeta^2} \right) = 0 \quad (B-38a)$$

$$O\left(\varepsilon^{1/2}\right): \frac{d^4 y_2^l}{d\zeta^4} - \mu \left( \frac{d^2 y_2^l}{d\zeta^2} \right) = -\sigma \zeta \left( \frac{d^2 y_0^l}{d\zeta^2} \right) - \sigma \left( \frac{d y_0^l}{d\zeta} \right) \quad (B-38b)$$

$$O(\varepsilon): \frac{d^4 y_2^l}{d\zeta^4} - \mu \left( \frac{d^2 y_2^l}{d\zeta^2} \right) = -\sigma \zeta \left( \frac{d^2 y_2^l}{d\zeta^2} \right) - \sigma \left( \frac{d y_2^l}{d\zeta} \right) - F_s \quad (B-38c)$$

As $x \to 1 \Rightarrow \zeta = 0$ and the boundary conditions transforms as follows

$$y(1) = 0 \quad y'(1) = \beta_2$$

$$y(0) = 0 \quad y'(0) = -\varepsilon^{1/2} \beta_2$$

The solution of equation B1.33 is

$$y_0^l(\zeta) = d1 + d2 \zeta + d3 \ e^{-\sqrt{\mu} \zeta} \quad (B-39)$$

Applying the boundary conditions on equation B-39 we obtain

$$d1 + d3 = 0 \quad (B-40a)$$

$$d2 - d3 \sqrt{\mu} = 0 \quad (B-40b)$$

The solution B-39 can be written as

$$y_0^l(\zeta) = d3(\zeta \sqrt{\mu} + e^{-\sqrt{\mu} \zeta} - 1) \quad (B-41)$$

Using one Term matching demands that $d3 = 0$ and $y_0^l(\zeta) = 0$ and yields that

$$\ln(\mu) a1 + \frac{F_s}{\sigma} + a2 = 0 \quad (B-42)$$
The general solution of equation B-38b

\[ y_1(\zeta) = e_1 + e_2 \zeta + e_3 e^{-\sqrt{\mu}} \zeta \]  

(B-43)

Applying the boundary conditions

\[ y(0) = 0 \quad \quad \quad y'(0) = -\frac{1}{\varepsilon^2} \beta_2 \]

(B-44a)

\[ e_1 + e_3 = 0 \]

(B-44b)

The equation becomes

\[ y_1(\zeta) = -e_3 + (e_3 \sqrt{\mu} - \beta_2) \zeta + e_3 e^{-\sqrt{\mu}} \zeta \]  

(B-45)

Matching the inner solution with the outer

The inner at \( x \to 1 \)

\[ -\varepsilon^2 e_3 + (e_3 \sqrt{\mu} - \beta_2)(1 - x) + E.S.T \]  

(B-46)

Two Terms Matching

Outer Expansion in terms of \( \zeta \)

\[ y^o = \frac{F_2}{\delta} - \frac{F_2}{\delta} \zeta^{\frac{1}{2}} + a_1 \ln \left( \mu - \sigma \zeta^{\frac{1}{2}} \right) + a_2 + \varepsilon^{\frac{1}{2}} \left( a_3 \ln \left( \mu - \sigma \zeta^{\frac{1}{2}} \right) + a_4 \right) \]  

(B-47)

Expanded for small value of \( \varepsilon \)

\[ y^o = a_1 \left( \ln(\mu) - \frac{\sigma \zeta^{\frac{1}{2}}}{\mu} \right) + \frac{F_2}{\delta} - \frac{F_2}{\delta} \zeta^{\frac{1}{2}} + a_2 + \varepsilon^{\frac{1}{2}} \left( a_3 \ln(\mu) + a_4 \right) \]  

(B-48)

Which becomes in terms of \( x \)
\[ y^\circ = a_1 (\ln(\mu) - \frac{\sigma (1-x)}{\mu}) + \frac{F_S}{\sigma} - \frac{F_S}{\sigma} (1 - x) + a_2 + e^{\frac{1}{2}}(a_3(\ln(\mu)) + a_4) \quad (B-49) \]

Matching with the inner solution of equal power terms yields

\[ -e_3 = a_3 \ln(\mu) + a_4 \quad (B-50a) \]

\[ (e_3 \sqrt{\mu} - \beta_2) = -\frac{\sigma}{\mu} a_1 - \frac{F_s}{\sigma} \quad (B-50b) \]

Substituting back solution \( B-45 \) back into equation \( B-38c \) the solution becomes

\[ y_2'(\zeta) = g_1 + \left( g_2 + \frac{3 \sigma e_3 e^{-\sqrt{\mu} \zeta}}{4 \mu} \right) \zeta + \left( g_3 + \frac{7\sigma e_3}{8 \mu^{3/2}} \right) e^{-\sqrt{\mu} \zeta} + \left( F_s + \frac{1}{2} \sigma \sqrt{\mu} e_3 e^{-\sqrt{\mu} \zeta} + \sigma \sqrt{\mu} e_3 - \beta_2 \sigma \right) \frac{\zeta^2}{2\mu} \quad (B-51) \]

Applying the boundary conditions on the solution \( B-51 \) we obtain the following

\[ g_1 + \left( g_3 + \frac{7\sigma e_3}{8 \mu^{3/2}} \right) = 0 \quad (B-52a) \]

\[ \left( g_2 + \frac{3 \sigma e_3}{4 \mu} \right) - \sqrt{\mu} \left( g_3 + \frac{7\sigma e_3}{8 \mu^{3/2}} \right) = 0 \quad (B-52b) \]

Substituting \( B-52a \) and \( B-52b \) into \( B-51 \) the solution can be rewritten as

\[ y_2'(\zeta) = -\left( g_3 + \frac{7\sigma e_3}{8 \mu^{3/2}} \right) + \left( \sqrt{\mu} g_3 + \frac{\sigma e_3}{8 \mu} + \frac{3\sigma e_3 e^{-\sqrt{\mu} \zeta}}{4 \mu} \right) \zeta + \left( g_3 + \frac{7\sigma e_3}{8 \mu^{3/2}} \right) e^{-\sqrt{\mu} \zeta} + \left( F_s + \frac{1}{2} \sigma \sqrt{\mu} e_3 e^{-\sqrt{\mu} \zeta} + \sigma \sqrt{\mu} e_3 - \beta_2 \sigma \right) \frac{\zeta^2}{2\mu} \quad (B-53) \]
Three Terms Matching

Outer Expansion in terms of $\zeta$

$$y^0 = \frac{F_z}{\sigma} - \frac{F_z}{\sigma} \zeta \epsilon^2 + a1 \ln \left( \mu - \sigma \zeta \epsilon^2 \right) + a2 + \epsilon^2 \left( a3 \ln \left( \mu - \sigma \zeta \epsilon^2 \right) + a4 \right) +$$

$$\epsilon \left( a5 \ln \left( \mu - \sigma \zeta \epsilon^2 \right) + a6 - \frac{2 a1 \sigma^2}{3 \left( \mu - \sigma \zeta \epsilon^2 \right)^3} \right)$$  \hspace{1cm} (B-54)

Expanded for small value of $\epsilon$

$$y^0 = a1 \left( \ln(\mu) - \frac{\sigma \zeta \epsilon^2}{\mu} - \frac{\sigma^2 \epsilon \zeta^2}{2 \mu^2} \right) + \frac{F_z}{\sigma} - \frac{F_z}{\sigma} \zeta \epsilon^2 + a2$$

$$+ \epsilon^2 \left( a3 \left( \ln(\mu) - \frac{\sigma \zeta \epsilon^2}{\mu} \right) + a4 \right)$$

$$+ \epsilon \left( a5 \left( \ln(\mu) - \frac{\sigma \zeta \epsilon^2}{\mu} \right) + a6 - \frac{2 a1 \sigma^2}{3 \mu^3} - \frac{2 a1 \sigma^3 \zeta \epsilon^2}{\mu^4} \right)$$  \hspace{1cm} (B-55)

Which becomes in terms of $x$

$$y^0 = a1 \left( \ln(\mu) - \frac{\sigma(1-x)}{\mu} - \frac{\sigma^2 (1-x)^2}{2 \mu^2} \right) + \frac{F_z}{\sigma} - \frac{F_z}{\sigma} (1-x) + a2$$

$$+ \epsilon^2 \left( a3 \left( \ln(\mu) - \frac{\sigma(1-x)}{\mu} \right) + a4 \right)$$

$$+ \epsilon \left( a5 \left( \ln(\mu) - \frac{\sigma(1-x)}{\mu} \right) + a6 - \frac{2 a1 \sigma^2}{3 \mu^3} - \frac{2 a1 \sigma^3 (1-x)}{\mu^4} \right)$$  \hspace{1cm} (B-56)
The inner expansion at \( x \to 1 \)

\[
-\varepsilon^2 e^3 + (e^3 \sqrt{\mu} - \beta_2)(1 - x) - \varepsilon \left( g^3 + \frac{7\sigma e^3}{8 \mu^2} \right) + \varepsilon^3 \left( \sqrt{\mu} g^3 + \frac{\sigma e^3}{8 \mu} \right)(1 - x) +
\]

\[
(F_s + \sigma \sqrt{\mu} e^3 - \beta_2 \sigma) \frac{(1 - x)^2}{2\mu} + E.S.T
\]  

(B-57)

Comparing equations B-56 and B-57 we obtain the following

\[-\left( g^3 + \frac{7\sigma e^3}{8 \mu^2} \right) = (a_5 \ln(\mu) + a_6 - \frac{2a_1 \sigma^2}{3 \mu^3}) \]  

(B-58a)

\[
\left( \sqrt{\mu} g^3 + \frac{\sigma e^3}{8 \mu} \right) = -\left( a_3 \frac{\sigma}{\mu} \right)
\]  

(B-58b)

\[
\left( \frac{F_s}{\sigma} + \sqrt{\mu} e^3 - \beta_2 \right) = -a_1 \frac{\sigma}{\mu}
\]  

(B-58c)

Equations B-19, B-29a, B-29b, B-34a, B-34b, B-42, B-50a, B-50b, B-58a, B-58b forms a set of equations that can be solved algebraically to obtain the values of the constants in the outer solution and obtain the composite solution

\[ a_2 = -\ln(\alpha) a_1 \]

\[ -b_3 = a_3 \ln(\alpha) + a_4 \]

\[
\left( \frac{\sigma}{\alpha} \right) a_1 + \frac{F_s}{\sigma} = (\beta_1 + b_3 \sqrt{\alpha})
\]

\[
\left( \frac{7b_3\sigma}{8 \alpha^2} - f^3 \right) = a_5 \ln(\alpha) + a_6 - \frac{a_1 \sigma^2}{3 \alpha^3}
\]

\[
\left( \sqrt{\alpha} \left( f^3 - \frac{7b_3\sigma}{8 \alpha^2} \right) + \frac{3b_3\sigma}{4 \alpha} \right) = a_3 \frac{\sigma}{\alpha}
\]

\[ a_1 \ln(\mu) + \frac{F_s}{\sigma} + a_2 = 0 \]
\[-e_3 = a_3 \ln(\mu) + a_4\]

\[(e_3 \sqrt{\mu} - \beta_2) = -\frac{\sigma}{\mu} a_1 - \frac{F_s}{\sigma}\]

\[\left(-\left(g_3 + \frac{7\sigma e_3}{8 \mu^2}\right) = (a_5 \ln(\mu) + a_6 - \frac{2a_1 \sigma^2}{3 \mu^3}\right)\]

\[\left(\sqrt{\mu} g_3 + \frac{\sigma e_3}{8 \mu}\right) = -\left(a_3 \frac{\sigma}{\mu}\right)\]  \hspace{1cm} \text{(B-59a)}

The following results are obtained

\[a_1 = -\frac{F_s}{\sigma \ln(\frac{\mu}{\alpha})}\]

\[a_2 = \frac{F_s \ln(\alpha)}{\sigma \ln(\frac{\mu}{\alpha})}\]

\[b_3 = \frac{1}{\sqrt{\alpha}} \left(-\frac{1}{\alpha \ln(\frac{\mu}{\alpha})} + \frac{F_s}{\sigma} - \beta_1\right)\]

\[e_3 = \frac{1}{\sqrt{\mu}} \left(\frac{1}{\mu \ln(\frac{\mu}{\alpha})} - \frac{F_s}{\sigma} + \beta_2\right)\]

\[a_3 = \frac{1}{\ln(\frac{\mu}{\alpha})} (b_3 - e_3)\]

\[a_4 = -e_3 - a_3 \ln(\mu)\]

\[f_3 = \frac{1}{\sqrt{\alpha}} \left(a_3 \frac{\sigma}{\alpha} - \frac{3}{4} b_3 \sigma\right) + \frac{7}{8} b_3 \sigma\]

\[g_3 = \frac{1}{\sqrt{\mu}} \left(-a_3 \frac{\sigma}{\mu} - \frac{\sigma e_3}{8 \mu}\right)\]

\[a_5 = \frac{1}{\ln(\frac{\mu}{\alpha})} \left(\frac{a_1 \sigma^2}{3} - \frac{2}{\mu^3} - \frac{1}{\alpha^3}\right) - \frac{7\sigma}{8} \left(\frac{b_3}{3 \alpha^2} + \frac{e_3}{\mu^2}\right) - (g_3 - f_3)\]
The general solution is expressed with values of constants as shown in B-59a, B-59b and the composite solution is written as

\[ y^c = y^o + y^l + y^l - (y^l)^o - (y^l)^o \]

\[ y^c = \frac{F_s}{\sigma} x + \ln(\alpha + \sigma x) a1 + a2 + e^{\frac{1}{2}}(\ln(\alpha + \sigma x) a3 + a4) + \epsilon \left( a5 \ln(\alpha + \sigma x) + \frac{2a1}{3} \sigma^2 \right) \]

\[ a6 = \left( \frac{7b3\sigma}{8a} - f3 \right) - a5 \ln(\alpha) + \frac{a1}{3} \sigma^2 \]

\[ + e^{\frac{1}{2}} \left( \beta_1 + b3\sqrt{\alpha} \right) \xi + b3 e^{-\sqrt{\alpha}} \xi \]
The boundary layer solution presented by equation B-60b takes into account the effect of bending in the outer and inner solution respectively. The solution B-60b at large magnitude of applied tension can be reduced to

\[ y^e = \ln(\alpha + \sigma x) \mathbf{a}_1 + \frac{F_S}{\sigma} x + \mathbf{a}_2 + \epsilon^2 (\ln(\alpha + \sigma x) \mathbf{a}_3 + \mathbf{a}_4) + \epsilon \left( \mathbf{b}_3 e^{-\sqrt{\alpha} \frac{x}{\epsilon^2}} \right) + \epsilon^2 \left( \mathbf{e}_3 e^{-\sqrt{\mu} \frac{(1-x)}{1/\epsilon^2}} \right) \]

(B-60c)