

CONVERGENCE OF DISCONTINUOUS GALERKIN METHODS FOR INCOMPRESSIBLE TWO-PHASE FLOW IN HETEROGENEOUS MEDIA*

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Abstract. A class of discontinuous Galerkin methods with interior penalties is presented for incompressible two-phase flow in heterogeneous porous media with capillary pressures. The semidiscrete approximate schemes for fully coupled system of two-phase flow are formulated. In highly heterogeneous permeable media, the saturation is discontinuous due to different capillary pressures, and therefore, the proposed methods incorporate the capillary pressures in the pressure equation instead of saturation equation. By introducing a coupling approach for stability and error estimates instead of the conventional separate analysis for pressure and saturation, the stability of the schemes in space and time and a priori hp error estimates are presented in the $L^2(H^1)$ for pressure and in the $L^\infty(L^2)$ and $L^2(H^1)$ for saturation. Two time discretization schemes are introduced for effectively computing the discrete solutions.

Key words. two-phase flow, discontinuous Galerkin method, stability, error estimates

AMS subject classifications. 65M60, 65M15

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1. Introduction. The discontinuous Galerkin (DG) methods [2, 14, 28, 36, 41] have recently gained great popularity due to their local mass conservative virtue. The DG methods are conveniently applied on unstructured meshes and mesh adaptation, and they utilize the interior penalties to impose the interelement continuity weakly. For more attractive features of DG, we refer to [36]. The DG methods have been widely applied to solve the convection–diffusion equation and transport problems, for example, [5, 15, 34, 35, 36, 37]. The DG methods have been employed to solve the miscible displacement problem in porous media, for example, [12, 30, 38, 42].

Numerical simulation for incompressible two-phase flow in porous media is an important topic in hydrology and petroleum reservoir engineering. Mathematical models of two-phase flow in porous media can be established from the phase mass conservation and Darcy’s law. This model consists of a coupled system of nonlinear partial differential equations, in which the nonlinearity arises mainly from relative permeabilities and capillary pressure describing the interactions between the permeable media and the fluids. It has been shown that heterogeneity in capillary pressure has a significant influence on flow paths [20, 21, 27, 39]. In heterogeneous media with different distributions of permeability, different capillary pressure functions are employed within the rocks of different permeability type, and as a result, the capillary pressure functions of different permeable media may show the discontinuity on the interface

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of rocks. In this case, however, the capillary pressure is continuous, and therefore, the continuity of capillary pressure results in discontinuity of saturation [21]. The classical fractional flow formulation (for example, used in [7, 8, 9, 17]) incorporates the capillarity into the saturation equation and requires the continuity of saturation in spatial dimension. This formulation cannot be well used for the case of different capillary pressure functions for multiple rock types, because of the discontinuity of saturation across the rock interface [21].

Recently, a two-phase flow formulation has been introduced in [21], which is based on the conception that the wetting-phase pressure is always continuous as long as none of the phases are immobile, and can correctly describe discontinuity in the saturation due to continuity of the capillary pressure and discontinuity of different capillary pressure functions on the interface of regions. The main difference from the classical model is that this formulation incorporates the capillary pressure in the pressure equation instead of the saturation equation. Based on the fact that the saturation is discontinuous in the entire domain, the DG methods are preferable because they use discontinuous polynomial spaces. In this paper, based on the formulation used in [21], we will propose DG schemes for both the pressure and saturation equations.

A number of numerical methods for simulating two-phase flow in porous media have been developed in the literature, for example, [1, 13, 21, 22, 32, 33, 39]. The coupled mixed finite element and Galerkin methods or finite volume methods have been presented for two-phase flow in [8, 29]. In [19, 21, 24, 26], the combined mixed finite element and DG methods have recently been employed to compute the two-phase flow problems. Based on the traditional formulation of two-phase flow, a fully coupled DG scheme for both the pressure and saturation equations is proposed in [16, 18], and the convergence and error estimates of numerical solutions are analyzed in [17]. Differently from the work in [16, 17], we will propose and analyze the fully coupled DG schemes for both the pressure and saturation equations based on the two-phase flow formulation used in [21], which is often used in practical applications, but with great difficulty for mathematical analysis of numerical methods.

According to the two-phase flow formulation used in [21], the capillarity is incorporated into the pressure equation instead of the saturation equation. In the classical two-phase flow formulation, the capillarity is reformed into the saturation equation as a diffusion term of saturation, which can facilitate the theoretical analysis of the numerical schemes. Unlike the cases of the classical formulation, the analysis of the methods based on the up to date two-phase flow formulation is much more elusive. The main technical difficulty is the lack of control on the saturation in the pressure equation, which arises from capillarity. Another difficulty is the analysis of the nonlinear intrinsic relationship between pressure and saturation. The technique used in [17] treats the pressure and saturation equations separately, but it has great difficulty in dealing with the broken gradient of saturation in the pressure equation. We will introduce a coupling approach by virtue of the coupling relationship between the pressure equation and the saturation equation, and analyze the error estimates of the proposed schemes. The stability of numerical solutions and a priori hp error estimates are obtained in the $L^2(H^1)$ for the pressure and in the $L^\infty(L^2)$ and $L^2(H^1)$ for the saturation.

The rest of this paper is organized as follows. In section 2, we describe the two-phase flow model and the formulation proposed in [21]. In section 3, the fully coupled DG methods for two-phase flow are presented. The stability of numerical schemes is analyzed in section 4. A priori error analysis of numerical schemes is carried out in section 5. The global existence of discrete solutions is demonstrated in section 6,

and two time discretization schemes are introduced to effectively compute the discrete solutions in section 7. Concluding remarks are provided in section 8.

2. Mathematical model. The mathematical model of incompressible and immiscible two-phase fluid flow in porous media is established in the phase mass conservation law, Darcy's law, and the constraint relations [9]. The wetting phase is denoted by a subscript w , and the nonwetting phase is denoted by n . We denote by ϕ the porosity of the medium, the saturation of phase α by S_α , and Darcy's velocity of phase α by \mathbf{u}_α . Let q_α be the external mass flow rate of phase α , and the mass conservation equation of each phase is given by

$$(2.1) \quad \phi \frac{\partial S_\alpha}{\partial t} + \nabla \cdot \mathbf{u}_\alpha = q_\alpha, \quad \alpha = w, n.$$

Darcy's velocity of each phase is described as

$$(2.2) \quad \mathbf{u}_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha} \mathbf{K}(\nabla p_\alpha + \rho_\alpha g \nabla z), \quad \alpha = w, n,$$

where g is the gravity acceleration and z is the depth. Here, \mathbf{K} denotes the absolute permeability tensor in a porous medium, and the other coefficients $k_{r\alpha}$, μ_α , p_α , and ρ_α are the relative permeability, viscosity, pressure, and density of each phase, respectively.

The saturations of two phases satisfy $S_w + S_n = 1$. The capillary pressure is defined as the difference between the wetting phase and nonwetting phase pressures:

$$(2.3) \quad p_c := p_n - p_w.$$

In practical applications, the capillary pressures are often viewed as the functions of the wetting-phase saturation S_w . In this paper, we always assume that $p_c(S)$ is a decreasing function of S and uniformly bounded.

In this model, the absolute permeability tensor \mathbf{K} and viscosity are given by the properties of a porous medium and fluids, respectively, while the capillary pressure and relative permeabilities are the given functions of the wetting-phase saturation. The primary unknowns are the pressures and saturations of two phases, but the number of unknowns can be reduced to two by using the saturation constraint and capillarity relation. Here, we choose the pressure and saturation of the wetting phase as the principal unknown variables and drop their subscripts for simplicity; that is, let $p := p_w$ and $S := S_w$. For the sake of simplicity, we disregard the effect of gravity, but the results presented in this paper can be suitable to the case involving gravity. We define the phase mobility as $\lambda_\alpha := k_{r\alpha}/\mu_\alpha$ and the total mobility as $\lambda_t := \lambda_w + \lambda_n$.

Summing the mass conservation equations of two phases, and using Darcy's velocities of two phases and the capillary pressure definition, we can obtain

$$(2.4) \quad -\nabla \cdot (\lambda_t \mathbf{K} \nabla p) - \nabla \cdot (\lambda_n \mathbf{K} \nabla p_c) = q_t,$$

where $q_t := q_w + q_n$. The combination of mass conservation equation and Darcy's velocity of the wetting phase gives us

$$(2.5) \quad \phi \frac{\partial S}{\partial t} - \nabla \cdot (\lambda_w \mathbf{K} \nabla p) = q_w.$$

The coupled system consisting of (2.4) and (2.5) is the formulation used in [21]. In the above formulation for two-phase flow, we can use (2.4) to compute the pressure

of the wetting phase, so it is called the pressure equation; we use (2.5) to update the saturation and call it the saturation equation.

Assume that $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a convex, polygonal, and bounded domain. The boundary $\partial\Omega$ of the domain is divided into two parts Γ_D and Γ_N , and $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. The boundary conditions for the pressure and the saturation are given by

$$(2.6) \quad S = S_B, \quad p = p_B \quad \text{on } \Gamma_D,$$

$$(2.7) \quad -\lambda_w \mathbf{K} \nabla p \cdot \mathbf{n} = -\lambda_n \mathbf{K} \nabla p_n \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N.$$

It is clear that Γ_D denotes both inflow and outflow boundaries, whereas Γ_N indicates the no-flow boundary. It follows by the relation between the wetting-phase and no-wetting-phase pressures that the boundary condition (2.7) implies $-(\lambda_t \mathbf{K} \nabla p + \lambda_n \mathbf{K} \nabla p_c) \cdot \mathbf{n} = 0$ on Γ_N . The initial condition for the saturation is given by

$$(2.8) \quad S = S^0 \quad \text{on } \Omega, \quad t = 0.$$

We make some assumptions on the above two-phase flow model as below.

(H1) *The permeability tensor \mathbf{K} is symmetric positive definite. It is uniformly bounded above and below; that is, there exist two positive constants k^b, k^a such that $k^b \xi^T \xi \leq \xi^T \mathbf{K} \xi \leq k^a \xi^T \xi \quad \forall \xi \in \mathbb{R}^d$.*

(H2) *The porosity ϕ is time independent and uniformly bounded below and above: $0 < \phi^b \leq \phi \leq \phi^a$.*

(H3) *The mobilities are nonnegative and uniformly bounded functions of saturations: $0 < \lambda_t^b \leq \lambda_t(S) \leq \lambda_t^a, \quad 0 \leq \lambda_w(S) \leq \lambda_w^a, \quad 0 \leq \lambda_n(S) \leq \lambda_n^a$.*

(H4) *$\lambda_w(S), \lambda_n(S),$ and $\lambda_t(S)$ are Lipschitz continuous with respect to S .*

3. Discontinuous Galerkin methods for two-phase flow.

3.1. Notation. Here, the traditional notation is used for the Lebesgue spaces $L^p(R)$, the Sobolev spaces $W^{k,p}(R)$ with $1 \leq p \leq \infty$ over a domain R . In particular, $W^{k,2}(R)$ is denoted by $H^k(R)$. The corresponding norms are denoted by $\|\cdot\|_{L^p(R)}$, $\|\cdot\|_{W^{k,p}(R)}$, and $\|\cdot\|_{H^k(R)}$, respectively. For simplicity, we also denote $\|\cdot\|_{W^{k,p}(R)}$ by $\|\cdot\|_{k,p,R}$ and $\|\cdot\|_{H^k(R)}$ by $\|\cdot\|_{k,R}$. The inner product of $L^p(\Omega)$ or $(L^p(\Omega))^d$ is denoted by (\cdot, \cdot) , and the inner product of the functions in the boundary Γ is denoted by $(\cdot, \cdot)_\Gamma$. For a given normed space X and $1 \leq p \leq \infty$, we define

$$L^p(0, T; X) := \left\{ \varphi : \varphi(t) \in X, \quad \|\varphi\|_X \in L^p(0, T) \right\},$$

which is a normed linear space equipped with the norm given by

$$\|\varphi\|_{L^p(0,T;X)} := \|(\|\varphi\|_X)\|_{L^p(0,T)}.$$

We use a partition \mathcal{E}_h of the domain Ω , which is composed of triangles or quadrilaterals if $d = 2$, or tetrahedra, prisms, or hexahedra if $d = 3$. Denote by h_E the diameter of $E \in \mathcal{E}_h$ and let $h = \max_{E \in \mathcal{E}_h} h_E, h/h_E \leq C$. We assume that \mathcal{E}_h is non-degenerate; that is, each element is convex, and that there exists $\rho > 0$ such that each of the subtriangles (for $d = 2$) or subtetrahedra (for $d = 3$) of element E contains a ball of radius ρh_E in its interior. We also make an assumption that no element crosses the boundaries of Γ_D or Γ_N .

For the sake of regularizing the heterogeneity of a porous medium, we assume that there exists a nondegenerate partition \mathcal{E}_h^0 of Ω such that the absolute permeability tensor \mathbf{K} is a piecewise-constant function within each $E \in \mathcal{E}_h^0$. In this paper, we use $\mathcal{E}_h = \mathcal{E}_h^0$ or a local refined mesh of \mathcal{E}_h^0 .

The set of all interior edges or faces for \mathcal{E}_h is denoted by Γ_h , and the sets of all edges or faces on the boundaries Γ_D and on Γ_N for \mathcal{E}_h are indicated by $\Gamma_{h,D}$ and $\Gamma_{h,N}$, respectively. For any $\gamma \in \Gamma_h$, a unit normal vector \mathbf{n}_γ is chosen, and for any $\gamma \in \Gamma_{h,D} \cup \Gamma_{h,N}$, the normal vector \mathbf{n}_γ coincides with the outward unit normal vector.

For $s \geq 0$, the broken Sobolov space is defined by

$$H^s(\mathcal{E}_h) := \left\{ \varphi \in L^2(\Omega) : \varphi|_E \in H^s(E), E \in \mathcal{E}_h \right\}.$$

In this paper, we often write $H^s(\mathcal{E}_h)$ as H^s for simplicity. For $0 \leq m \leq s$, the broken norm is defined as

$$\|\varphi\|_m := \left(\sum_{E \in \mathcal{E}_h} \|\varphi\|_{H^m(E)}^2 \right)^{1/2}.$$

Let $\gamma \in \Gamma_h$ be shared by the elements E_γ^1 and E_γ^2 with \mathbf{n}_γ exterior to E_γ^1 ; that is, $\gamma = E_\gamma^1 \cap E_\gamma^2$. The average and jump values for $\varphi \in H^s(\mathcal{E}_h)$ ($s > 1/2$) on γ are defined by

$$\langle \varphi \rangle := \frac{1}{2} \left((\varphi|_{E_\gamma^1})|_\gamma + (\varphi|_{E_\gamma^2})|_\gamma \right), \quad [\varphi] := (\varphi|_{E_\gamma^1})|_\gamma - (\varphi|_{E_\gamma^2})|_\gamma.$$

For $\gamma \in \Gamma_{h,D}$, we define $\langle \varphi \rangle := \varphi|_\gamma$ and $[\varphi] := \varphi|_\gamma$.

The DG methods utilize discontinuous piecewise polynomials to approximate the solutions of differential equations. Let $\mathbb{P}_r(E)$ denote the space of polynomials of (total) degree less than or equal to r on the element E , then we take the discontinuous finite element space as

$$\mathcal{D}_r(\mathcal{E}_h) := \{ \varphi \in L^2(\Omega) : \varphi|_E \in \mathbb{P}_r(E), E \in \mathcal{E}_h \}.$$

We now recall the following well-known inverse estimates [10, 31, 40].

LEMMA 3.1. *Let $E \in \mathcal{E}_h$, and $v \in \mathbb{P}_r(E)$, then there exists a constant C independent of v, r , and h_E such that*

$$(3.1) \quad \|D^{q+1}v\|_{0,E} \leq C \frac{r^2}{h_E} \|D^q v\|_{0,E}, \quad \|D^q v\|_{0,\partial E} \leq C \frac{r}{h_E^{1/2}} \|D^q v\|_{0,E}, \quad q \geq 0.$$

LEMMA 3.2 (see [6]). *For any $v \in \mathcal{D}_r(\mathcal{E}_h)$, there exists a constant C independent of h and r such that*

$$(3.2) \quad \|v\|_{0,\Omega}^2 \leq C \left(\sum_{E \in \mathcal{E}_h} \|\nabla v\|_{0,E}^2 + \sum_{\gamma \in \Gamma_h} |\gamma|^{d/(1-d)} \left(\int_\gamma [v] \right)^2 + \left(\int_{\Gamma_{h,D}} v \right)^2 \right),$$

where $|\gamma|$ denotes the measure of γ .

3.2. Continuous-in-time schemes. Here, we aim to find the solutions of pressure and saturation in the broken Sobolov space $H^1(\mathcal{E}_h)$ instead of the usual space $H^1(\Omega)$. We consider the pressure equation defined within each element

$$(3.3) \quad -\nabla \cdot \lambda_t(S) \mathbf{K} \nabla p - \nabla \cdot \lambda_n(S) \mathbf{K} \nabla p_c = q_t \quad \text{in } E \in \mathcal{E}_h.$$

Multiplying the equation (3.3) by $\varphi \in H^1(\mathcal{E}_h)$ and integrating over each element

$E \in \mathcal{E}_h$, we then apply Green’s formula to obtain

$$(3.4) \quad \int_E \lambda_t(S) \mathbf{K} \nabla p \cdot \nabla \varphi - \int_{\partial E} \lambda_t(S) \mathbf{K} \nabla p \cdot \mathbf{n}_{\partial E} \varphi + \int_E \lambda_n(S) \mathbf{K} \nabla p_c \cdot \nabla \varphi - \int_{\partial E} \lambda_n(S) \mathbf{K} \nabla p_c \cdot \mathbf{n}_{\partial E} \varphi = \int_E q_t \varphi.$$

Define two velocity variables as

$$\mathbf{u}_a = -\lambda_t \mathbf{K} \nabla p, \quad \mathbf{u}_c = -\lambda_n \mathbf{K} \nabla p_c.$$

The normal components of these velocities are continuous across the element interfaces [21]. Thus, using the continuity of \mathbf{u}_a and \mathbf{u}_c and taking into account the boundary conditions, we sum (3.4) over all elements to obtain

$$(3.5) \quad \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S) \mathbf{K} \nabla p \cdot \nabla \varphi - \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_t(S) \mathbf{K} \nabla p \cdot \mathbf{n}_{\gamma} \rangle [\varphi] - \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_t(S_B) \mathbf{K} \nabla p \cdot \mathbf{n}_{\gamma}) \varphi = \int_{\Omega} q_t \varphi - \sum_{E \in \mathcal{E}_h} \int_E \lambda_n(S) \mathbf{K} \nabla p_c \cdot \nabla \varphi + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_n(S) \mathbf{K} \nabla p_c \cdot \mathbf{n}_{\gamma} \rangle [\varphi] + \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_n(S_B) \mathbf{K} \nabla p_c \cdot \mathbf{n}_{\gamma}) \varphi.$$

Based on (3.5), we introduce a functional for the pressure equation as

$$(3.6) \quad B_p(p, \varphi; S) := \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S) \mathbf{K} \nabla p \cdot \nabla \varphi - \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_t(S) \mathbf{K} \nabla p \cdot \mathbf{n}_{\gamma} \rangle [\varphi] - \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_t(S_B) \mathbf{K} \nabla p \cdot \mathbf{n}_{\gamma}) \varphi + s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_t(S) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_{\gamma} \rangle [p] + s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_t(S_B) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_{\gamma}) p + J_{0,D}^{\sigma}(p, \varphi).$$

In our schemes, the wetting-phase saturation is permitted to be discontinuous on the element interfaces, but it is continuous within each element and its broken gradient over each element is well defined. Since the absolute permeability is element wise constant and the capillary pressure is a function of saturation, we have $\nabla p_c = p'_c(S) \nabla S$ within each element, which is apparently valid for (3.5). We also define a functional for the pressure equation

$$(3.7) \quad L_p(\varphi; S) := \int_{\Omega} q_t \varphi - \sum_{E \in \mathcal{E}_h} \int_E \lambda_c(S) \mathbf{K} \nabla S \cdot \nabla \varphi + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_c(S) \mathbf{K} \nabla S \cdot \mathbf{n}_{\gamma} \rangle [\varphi] + \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_c(S_B) \mathbf{K} \nabla S \cdot \mathbf{n}_{\gamma}) \varphi - s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_n(S) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_{\gamma} \rangle [p_c(S)] - J_0^{\sigma}(p_c(S), \varphi) + s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_t(S_B) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_{\gamma} p_B + J_D^{\sigma}(p_B, \varphi),$$

where

$$(3.8) \quad \lambda_c(S) := \lambda_n(S)p'_c(S).$$

Let $\gamma \in \Gamma_h$ be shared by the two elements E_γ^1 and E_γ^2 with \mathbf{n}_γ exterior to E_γ^1 ; then the upwind value of saturation $S^*|_\gamma$ can be defined as

$$(3.9) \quad S^*|_\gamma = \begin{cases} S|_{E_\gamma^1} & \text{if } \mathbf{u}_a \cdot \mathbf{n}_\gamma \geq 0, \\ S|_{E_\gamma^2} & \text{if } \mathbf{u}_a \cdot \mathbf{n}_\gamma < 0. \end{cases}$$

If $\gamma \in \Gamma_{h,D}$, then we take $S^*|_\gamma = S_B$. We further define two functions of the saturation S as

$$f_w(S) := \lambda_w(S)/\lambda_t(S), \quad f_n(S) := \lambda_n(S)/\lambda_t(S).$$

Similarly to [18], we introduce a nonlinear functional argument with the upwind scheme for the discrete saturation equation, which is defined by

$$(3.10) \quad \begin{aligned} B_S(S, \varphi; p) := & \sum_{E \in \mathcal{E}_h} \int_E \lambda_w(S) \mathbf{K} \nabla p \cdot \nabla \varphi - \sum_{\gamma \in \Gamma_h} \int_\gamma f_w(S^*) \langle \lambda_t(S) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \rangle [\varphi] \\ & - \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma \lambda_w(S_B) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \varphi - \omega J_0^\sigma(p_c(S), \varphi) \\ & + s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_\gamma f_w(S^*) \langle \lambda_t(S) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_\gamma \rangle [p] \\ & + s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma \lambda_w(S_B) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_\gamma p + J_{0,D}^\sigma(p, \varphi) + J_D^\sigma(S, \varphi). \end{aligned}$$

We also define a functional for the saturation equation as

$$(3.11) \quad \begin{aligned} L_S(\varphi) := & s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma \lambda_w(S_B) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_\gamma p_B \\ & + J_D^\sigma(p_B, \varphi) + J_D^\sigma(S_B, \varphi) + \int_\Omega q_w \varphi. \end{aligned}$$

In (3.6), (3.7), (3.10), and (3.11), we take $s_{\text{form}} \in \{-1, 0, 1\}$. It is noted that $s_{\text{form}} = -1$ for the Oden–Babuška–Baumann DG (OBB-DG) formulation [28] and nonsymmetric interior penalty Galerkin [31], $s_{\text{form}} = 0$ for incomplete interior penalty Galerkin [14], and $s_{\text{form}} = 1$ for symmetric interior penalty Galerkin [41]. In (3.10), ω is a positive constant and one choice is given in section 5.

The interior penalty terms in our schemes are defined by

$$\begin{aligned} J_0^\sigma(\psi, \varphi) &:= \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_\gamma}{h_\gamma} \int_\gamma [\psi][\varphi], & J_D^\sigma(\psi, \varphi) &:= \sum_{\gamma \in \Gamma_{h,D}} \frac{r^2 \sigma_\gamma}{h_\gamma} \int_\gamma \psi \varphi, \\ J_{0,D}^\sigma(\psi, \varphi) &:= J_0^\sigma(\psi, \varphi) + J_D^\sigma(\psi, \varphi), \end{aligned}$$

where σ is a discrete positive function that takes the constant value σ_γ on the edge γ . Although there is no penalty term in OBB-DG ($\sigma = 0$), σ is always assumed to be bounded below and above, i.e., $0 < \sigma_0 \leq \sigma \leq \sigma_m$, for the analysis of DG schemes in

this paper. We note that the saturation may be discontinuous at the element interfaces of \mathcal{E}_h^0 , but the pressures are always continuous throughout the entire region, and as a result, the penalty terms are imposed on the wetting-phase pressure p and the capillary pressure p_c , but the saturation is free of the penalty except at the boundary of the domain.

Based on the bilinear forms and linear functionals defined above and letting $J = (0, T]$ denote the time interval, we now state the schemes of DG for the pressure equation and the saturation equation: find continuous-in-time DG approximations $p_h(\cdot, t) \in D_r(\mathcal{E}_h)$ and $S_h(\cdot, t) \in \mathcal{D}_r(\mathcal{E}_h)$ such that

$$(3.12) \quad B_p(p_h, \varphi; S_h) = L_p(\varphi; S_h), \quad \varphi \in \mathcal{D}_r(\mathcal{E}_h), \quad t \in J,$$

$$(3.13) \quad \left(\phi \frac{\partial S_h}{\partial t}, \varphi \right) + B_S(S_h, \varphi; p_h) = L_S(\varphi), \quad \varphi \in \mathcal{D}_r(\mathcal{E}_h), \quad t \in J,$$

$$(3.14) \quad (S_h, \varphi) = (S^0, \varphi), \quad \varphi \in \mathcal{D}_r(\mathcal{E}_h), \quad t = 0.$$

In (3.13), the upwind values of $S_h^*|_\gamma$ are determined by using the rules in (3.9), but \mathbf{u}_a is replaced by the DG velocity

$$\begin{aligned} \mathbf{u}_a^{DG} &= -\lambda_t \mathbf{K} \nabla p_h \quad \text{in } E, \\ \mathbf{u}_a^{DG} \cdot \mathbf{n} &= -\langle \lambda_t(S_h) \mathbf{K} \nabla p_h \cdot \mathbf{n} \rangle + \frac{r^2 \sigma_\gamma}{h_\gamma} [p_h] \quad \text{on } \gamma \in \Gamma_h. \end{aligned}$$

With the boundary conditions, we have $\mathbf{u}_a^{DG} \cdot \mathbf{n} = 0$ on $\Gamma_{h,N}$.

Since the proposed schemes employ the discontinuous finite element spaces to approximate the saturations, it allows that the approximate saturations may be discontinuous at the element interfaces although they are continuous within each element.

Let p and S be the solutions of the system of equations (2.4) and (2.5), which satisfy the regularity requirements: $p \in L^2(J; H^s(\Omega))$, $S \in L^\infty(J; L^2(\Omega)) \cap L^2(J; H^s(\mathcal{E}_h^0))$, $\frac{\partial S}{\partial t} \in L^2(J; H^{s-1}(\Omega))$, and $S^0 \in H^{s-1}(\Omega)$, where $s > \frac{3}{2}$. Moreover, we assume that $p_c(S)$, \mathbf{u}_a , and \mathbf{u}_c are continuous in the entire domain, and ∇p and ∇S are essentially bounded in \mathcal{E}_h^0 . For $t \in J$, the following equalities hold:

$$(3.15) \quad B_p(p, \varphi; S) = L_p(\varphi; S), \quad \varphi \in H^s(\mathcal{E}_h),$$

$$(3.16) \quad \left(\phi \frac{\partial S}{\partial t}, \varphi \right) + B_S(S, \varphi; p) = L_S(\varphi), \quad \varphi \in H^s(\mathcal{E}_h).$$

Furthermore, for λ_c defined by (3.8), we know that $\lambda_c < 0$ and make the following assumption.

(H5) λ_c is uniformly bounded both below and above, and λ_c is also assumed to be Lipschitz continuous with respect to the saturation S ; that is, $0 < \lambda_c^b \leq |\lambda_c| \leq \lambda_c^a$, $|\lambda_c(S_1) - \lambda_c(S_2)| \leq L_c |S_1 - S_2|$.

Define a function of saturation as

$$(3.17) \quad \zeta(S) := \frac{\lambda_n(S)^{1/2} (M_c - p'_c(S))}{\lambda_t(S)^{1/2} |M_c p'_c(S)|^{1/2}},$$

where M_c is a positive constant. It is noted that $\zeta(S) \geq 0$ from the physical meanings of $\lambda_n(S)$, $\lambda_t(S)$, and $p'_c(S)$. Furthermore, we assume the following.

(H6) There exist positive constants M_c and δ such that $\max_{\mathcal{E}_h \times J} \zeta(S_h) \leq \delta < 2$. For the sake of simplification, we choose $M_c = 1$ in the theoretical analysis of this paper, but for the cases $M_c \neq 1$, the results can be obtained by making very small modifications only.

For the boundary and initial data, we assume that $p_B \in L^2(J; H^{1/2}(\Gamma_D)) \cap L^\infty(J \times \Gamma_D)$, $S_B \in L^\infty(J; H^{1/2}(\Gamma_D)) \cap L^\infty(J \times \Gamma_D)$, $q_w \in L^2(J; L^2(\Omega))$, $q_n \in L^2(J; L^2(\Omega))$. It is apparent that $q_t \in L^2(J; L^2(\Omega))$.

Throughout the paper, we use C to indicate a generic positive constant independent of h, r , and use ϵ to indicate a small positive constant. The values of C and ϵ are probably different in different occurrences.

4. Stability of discrete solutions. We now establish the stability of numerical solutions, which gives a priori estimates for the discrete pressure and saturation. For the purpose of theoretical analysis, we assume that there exist two constants $\theta_0 > 0, \theta_1 \geq 0$ such that

$$\theta_0 J_0^\sigma(S_h, S_h) - \theta_1 \leq -J_0^\sigma(p_c(S_h), S_h).$$

THEOREM 4.1. *If the penalty parameter σ_0 is chosen to be sufficiently large and $h \leq h_0$, then we have*

$$(4.1) \quad \begin{aligned} & \|S_h\|_{L^\infty(J; L^2)}^2 + \|\mathbf{K}^{1/2} \nabla p_h\|_{L^2(J; L^2)}^2 + \|\mathbf{K}^{1/2} \nabla S_h\|_{L^2(J; L^2)}^2 + \int_0^T J_{0,D}^\sigma(S_h, S_h) \\ & + \int_0^T J_{0,D}^\sigma(p_h, p_h) \leq C \|S^0\|_0^2 + C \int_0^T J_D^\sigma(p_B, p_B) + C \int_0^T J_D^\sigma(S_B, S_B) \\ & + C \|q_w\|_{L^2(J; L^2)}^2 + C \|q_n\|_{L^2(J; L^2)}^2 + C(p_{cmax}^2 + \theta_1), \end{aligned}$$

where p_{cmax} is the upper bound of p_c .

Proof. Choosing $\varphi = p_h - S_h$ in (3.12), we obtain

$$(4.2) \quad B_p(p_h, p_h - S_h; S_h) = L_p(p_h - S_h; S_h).$$

Furthermore, letting $\varphi = S_h$ in (3.13) gives us

$$(4.3) \quad \left(\phi \frac{\partial S_h}{\partial t}, S_h \right) + B_S(S_h, S_h; p_h) = L_S(S_h).$$

We now combine (4.2) and (4.3), and obtain a coupled equation

$$(4.4) \quad \begin{aligned} & \left(\phi \frac{\partial S_h}{\partial t}, S_h \right) + \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S_h) \mathbf{K} \nabla p_h \cdot \nabla p_h - \sum_{E \in \mathcal{E}_h} \int_E \lambda_c(S_h) \mathbf{K} \nabla S_h \cdot \nabla S_h \\ & + J_{0,D}^\sigma(p_h, p_h) - (\omega + 1) J_0^\sigma(p_c(S_h), S_h) + J_D^\sigma(S_h, S_h) \\ & = \sum_{E \in \mathcal{E}_h} \int_E (\lambda_n(S_h) - \lambda_c(S_h)) \mathbf{K} \nabla p_h \cdot \nabla S_h \\ & + (1 - s_{\text{form}}) \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla p_h \cdot \mathbf{n}_\gamma \rangle [p_h] \\ & + (1 - s_{\text{form}}) \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_t(S_B) \mathbf{K} \nabla p_h \cdot \mathbf{n}_\gamma) p_h + J_D^\sigma(p_B, p_h) \\ & + \int_\Omega q_t p_h + s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma \lambda_t(S_B) \mathbf{K} \nabla p_h \cdot \mathbf{n}_\gamma p_B \\ & + \sum_{\gamma \in \Gamma_h} \int_\gamma (\langle \lambda_c(S_h) \mathbf{K} \nabla S_h \cdot \mathbf{n}_\gamma \rangle + s_{\text{form}} f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla S_h \cdot \mathbf{n}_\gamma \rangle) [p_h] \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_c(S_B) + s_{\text{form}} \lambda_n(S_B)) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} p_h \\
 & - \sum_{\gamma \in \Gamma_h} \int_{\gamma} f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla p_h \cdot \mathbf{n}_{\gamma} \rangle [S_h] \\
 & - \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_n(S_B) \mathbf{K} \nabla p_h \cdot \mathbf{n}_{\gamma} S_h \\
 & - \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_c(S_h) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} \rangle [S_h] + J_D^{\sigma}(S_B, S_h) \\
 & - \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_c(S_B) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} S_h - s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_n(S_B) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} p_B \\
 & - s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_n(S_h) \mathbf{K} \nabla (p_h - S_h) \cdot \mathbf{n}_{\gamma} \rangle [p_c(S_h)] \\
 & - J_0^{\sigma}(p_c(S_h), p_h) - \int_{\Omega} q_n S_h.
 \end{aligned}$$

First of all, we estimate the terms on the right-hand side of (4.4). The first term is bounded by using assumption (H6) and the Cauchy–Schwarz inequality as

$$\begin{aligned}
 & \sum_{E \in \mathcal{E}_h} \int_E (\lambda_n(S_h) - \lambda_c(S_h)) \mathbf{K} \nabla p_h \cdot \nabla S_h \\
 & = \sum_{E \in \mathcal{E}_h} \int_E \zeta(S_h) \lambda_t(S_h)^{1/2} |\lambda_c(S_h)|^{1/2} \mathbf{K} \nabla p_h \cdot \nabla S_h \\
 & \leq \frac{\delta}{2} \left(\| \lambda_t(S_h) \mathbf{K}^{1/2} \nabla p_h \|_0^2 + \| |\lambda_c(S_h)| \mathbf{K}^{1/2} \nabla S_h \|_0^2 \right),
 \end{aligned}$$

where $\delta \in (0, 2)$. We now estimate the rest of the terms of the right-hand side in (4.4). The second and third terms on the right-hand side in (4.4) can be bounded as

$$\begin{aligned}
 & (1 - s_{\text{form}}) \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_t(S_h) \mathbf{K} \nabla p_h \cdot \mathbf{n}_{\gamma} \rangle [p_h] \\
 & + (1 - s_{\text{form}}) \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_t(S_B) \mathbf{K} \nabla p_h \cdot \mathbf{n}_{\gamma}) p_h \\
 & \leq \sum_{E \in \mathcal{E}_h} \frac{h_E}{Cr^2} \| \mathbf{K}^{1/2} \nabla p_h \cdot \mathbf{n}_{\partial E} \|_{0,\partial E}^2 + \sum_{\gamma \in \Gamma_h \cup \Gamma_{h,D}} \frac{Cr^2}{h_{\gamma}} \| [p_h] \|_{0,\gamma}^2 \\
 & \leq \epsilon_1 \| \mathbf{K}^{1/2} \nabla p_h \|_0^2 + \epsilon_2 J_{0,D}^{\sigma}(p_h, p_h),
 \end{aligned}$$

where the penalty parameter σ_0 is chosen to be sufficiently large. The fourth term is bounded trivially,

$$J_D^{\sigma}(p_B, p_h) \leq C J_D^{\sigma}(p_B, p_B) + \epsilon_3 J_D^{\sigma}(p_h, p_h) \leq C J_D^{\sigma}(p_B, p_B) + \epsilon_3 J_{0,D}^{\sigma}(p_h, p_h).$$

For the fifth term, we use Lemma 3.2 to obtain

$$\begin{aligned}
 \int_{\Omega} q_t p_h & \leq \| q_t \|_{0,\Omega} \| p_h \|_0 \leq C \| q_t \|_{0,\Omega} \left(\| \nabla p_h \|_0^2 + J_{0,D}^{\sigma}(p_h, p_h) \right)^{1/2} \\
 & \leq C \| q_t \|_{0,\Omega}^2 + \epsilon_4 \| \mathbf{K}^{1/2} \nabla p_h \|_0^2 + \epsilon_5 J_{0,D}^{\sigma}(p_h, p_h).
 \end{aligned}$$

Similarly to the boundedness of the second term, the sixth term is estimated by

$$s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_t(S_B) \mathbf{K} \nabla p_h \cdot \mathbf{n}_{\gamma} p_B \leq \epsilon_6 \|\mathbf{K}^{1/2} \nabla p_h\|_0^2 + C J_D^{\sigma}(p_B, p_B).$$

The seventh and eighth terms are bounded similarly to the second term as

$$\begin{aligned} & \sum_{\gamma \in \Gamma_h} \int_{\gamma} (\langle \lambda_c(S_h) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} \rangle + s_{\text{form}} f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} \rangle) [p_h] \\ & \quad + \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_c(S_B) + s_{\text{form}} \lambda_n(S_B)) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} p_h \\ & \leq \frac{C}{\sigma_0} \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} \|\mathbf{K}^{1/2} \nabla S_h \cdot \mathbf{n}_{\gamma}\|_{0,\gamma}^2 + \epsilon \sum_{\gamma \in \Gamma_h \cup \Gamma_{h,D}} \frac{\sigma_{\gamma} r^2}{h_{\gamma}} \|[p_h]\|_{0,\gamma}^2 \\ & \leq \epsilon_7 \|\mathbf{K}^{1/2} \nabla S_h\|_0^2 + \epsilon_8 J_{0,D}^{\sigma}(p_h, p_h). \end{aligned}$$

The ninth and tenth terms on the right-hand side of (4.4) are bounded as

$$\begin{aligned} & - \sum_{\gamma \in \Gamma_h} \int_{\gamma} f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla p_h \cdot \mathbf{n}_{\gamma} \rangle [S_h] + \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_n(S_B) \mathbf{K} \nabla p_h \cdot \mathbf{n}_{\gamma}) S_h \\ & \leq \epsilon_9 \|\mathbf{K}^{1/2} \nabla p_h\|_0^2 + \epsilon_{10} J_{0,D}^{\sigma}(S_h, S_h), \end{aligned}$$

where we need to choose suitable penalty parameters to estimate this bound. The eleventh and thirteenth terms are bounded as

$$\begin{aligned} & - \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_c(S_h) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} \rangle [S_h] - \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_c(S_B) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} S_h \\ & \leq \epsilon_{12} \|\mathbf{K}^{1/2} \nabla S_h\|_0^2 + \epsilon_{11} J_{0,D}^{\sigma}(S_h, S_h). \end{aligned}$$

By the definition of J_D^{σ} , we bound the twelfth term

$$J_D^{\sigma}(S_B, S_h) \leq C J_D^{\sigma}(S_B, S_B) + \epsilon_{13} J_{0,D}^{\sigma}(S_h, S_h).$$

The fourteenth term is estimated like the seventh term,

$$-s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \langle \lambda_n(S_B) \mathbf{K} \nabla S_h \cdot \mathbf{n}_{\gamma} \rangle p_B \leq C J_D^{\sigma}(p_B, p_B) + \epsilon_{14} \|\mathbf{K}^{1/2} \nabla S_h\|_0^2.$$

The fifteenth term is estimated like the seventh term,

$$\begin{aligned} & -s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_n(S_h) \mathbf{K} \nabla (p_h - S_h) \cdot \mathbf{n}_{\gamma} \rangle [p_c(S_h)] \\ & \leq C p_{cmax}^2 + \epsilon_{15} \|\mathbf{K}^{1/2} \nabla p_h\|_0^2 + \epsilon_{16} \|\mathbf{K}^{1/2} \nabla S_h\|_0^2. \end{aligned}$$

The sixteenth term is bounded as $J_0^{\sigma}(p_c(S_h), p_h) \leq C p_{cmax}^2 + \frac{1}{2} J_0^{\sigma}(p_h, p_h)$. The last term is bounded by $\frac{1}{2} \|q_n\|_{0,\Omega}^2 + \frac{1}{2} \|S_h\|_0^2$.

Noticing that $(\phi \frac{\partial S_h}{\partial t}, S_h) = \frac{1}{2} \frac{d}{dt} \|\phi^{1/2} S_h\|_0^2$, we move (4.5) to the left-hand side of (4.4), combine the above estimates on the right-hand side of (4.4), and obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\phi^{1/2} S_h\|_0^2 + c_0 \|\mathbf{K}^{1/2} \nabla p_h\|_0^2 \\ & + c_1 \|\mathbf{K}^{1/2} \nabla S_h\|_0^2 + \min(\theta_0, 1) J_{0,D}^\sigma(S_h, S_h) + \frac{1}{2} J_{0,D}^\sigma(p_h, p_h) \\ & \leq \frac{1}{2} \|S_h\|_0^2 + (\varepsilon_1 + \varepsilon_4 + \varepsilon_6 + \varepsilon_9 + \varepsilon_{15}) \|\mathbf{K}^{1/2} \nabla p_h\|_0^2 \\ & + (\varepsilon_7 + \varepsilon_{12} + \varepsilon_{14} + \varepsilon_{16}) \|\mathbf{K}^{1/2} \nabla S_h\|_0^2 + (\varepsilon_2 + \varepsilon_3 + \varepsilon_5 + \varepsilon_8) J_{0,D}^\sigma(p_h, p_h) \\ & + (\varepsilon_{10} + \varepsilon_{11} + \varepsilon_{13}) J_{0,D}^\sigma(S_h, S_h) + C J_D^\sigma(p_B, p_B) \\ & + C J_D^\sigma(S_B, S_B) + C(\|q_w\|_{0,\Omega}^2 + \|q_n\|_{0,\Omega}^2) + C(p_{cmax}^2 + \theta_1), \end{aligned}$$

where c_0, c_1 are two positive constants depending on $\lambda_t, \lambda_c, \delta$. Furthermore, using the boundedness of ϕ, λ_t , and λ_c , and suitably adjusting the parameters $\varepsilon_i > 0$ such that

$$\begin{aligned} \varepsilon_1 + \varepsilon_4 + \varepsilon_6 + \varepsilon_9 + \varepsilon_{15} &= \frac{1}{2} c_0, & \varepsilon_7 + \varepsilon_{12} + \varepsilon_{14} + \varepsilon_{16} &= \frac{1}{2} c_1, \\ \varepsilon_2 + \varepsilon_3 + \varepsilon_5 + \varepsilon_8 &= \frac{1}{4}, & \varepsilon_{10} + \varepsilon_{11} + \varepsilon_{13} &= \frac{1}{2} \min(\theta_0, 1), \end{aligned}$$

we get

$$\begin{aligned} & \frac{d}{dt} \|S_h\|_0^2 + \|\mathbf{K}^{1/2} \nabla p_h\|_0^2 + \|\mathbf{K}^{1/2} \nabla S_h\|_0^2 + J_{0,D}^\sigma(S_h, S_h) + J_{0,D}^\sigma(p_h, p_h) \\ & \leq C \|S_h\|_0^2 + C J_D^\sigma(p_B, p_B) + C J_D^\sigma(S_B, S_B) \\ & + C(\|q_w\|_{0,\Omega}^2 + \|q_n\|_{0,\Omega}^2) + C(p_{cmax}^2 + \theta_1). \end{aligned}$$

Finally, taking into account the fact $\|S_h^0\|_0 \leq \|S^0\|_{0,\Omega}$, the inequality (4.1) can be obtained by Gronwall's inequality. \square

Remark. From Theorem 4.1 and Lemma 3.2, we know that $p_h \in L^2(J; H^1(\mathcal{E}_h))$ and $S_h \in L^\infty(J; L^2(\Omega)) \cap L^2(J; H^1(\mathcal{E}_h))$, where $J = (0, T]$.

5. Error estimates.

5.1. Approximation results. In order to derive the error estimates, we first recall the following well-known *hp*-approximation results, which can be proved using the techniques in [3, 4]. Let $E \in \mathcal{E}_h$ and $v \in H^s(E)$, then there exists a constant C independent of v, r , and h_E , and a sequence of $z_r^{h_E} \in \mathbb{P}_r(E), r \geq 1$, such that

$$(5.1) \quad \|v - z_r^{h_E}\|_{q,E} \leq C \frac{h_E^{\mu-q}}{r^{s-q}} \|v\|_{s,E}, \quad 0 \leq q < \mu,$$

$$(5.2) \quad \|v - z_r^{h_E}\|_{q,\partial E} \leq C \frac{h_E^{\mu-q-\frac{1}{2}}}{r^{s-q-\frac{1}{2}}} \|v\|_{s,E}, \quad 0 \leq q < \mu - \frac{1}{2},$$

where $\mu = \min(r + 1, s)$ and h_E is the diameter of E .

5.2. Error estimates. Let \hat{p} and \hat{S} be the interpolants of the pressure p and saturation S in $\mathcal{D}_r(\mathcal{E}_h)$ such that the *hp*-approximation results (5.1) and (5.2) hold.

Define

$$\begin{aligned} E_p &= p - p_h, & E_p^I &= \widehat{p} - p, & E_p^A &= E_p^I + E_p = \widehat{p} - p_h, \\ E_S &= S - S_h, & E_S^I &= \widehat{S} - S, & E_S^A &= E_S^I + E_S = \widehat{S} - S_h. \end{aligned}$$

Moreover, it is assumed that for $h \leq h_0$, there exist positive constants a_0, a_1 such that

$$a_0 J_0^\sigma(E_S^A, E_S^A) \leq -J_0^\sigma(p_c(\widehat{S}) - p_c(S_h), E_S^A) + C \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}},$$

$$J_0^\sigma(p_c(\widehat{S}) - p_c(S_h), p_c(\widehat{S}) - p_c(S_h)) \leq a_1 J_0^\sigma(E_S^A, E_S^A) + C \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.$$

Subtracting the pressure DG scheme (3.12) from the weak formulation (3.15), we can obtain

$$\begin{aligned} (5.3) \quad & \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S_h) \mathbf{K} \nabla E_p^A \cdot \nabla \varphi + J_{0,D}^\sigma(E_p^A, \varphi) - J_{0,D}^\sigma(E_p^I, \varphi) \\ & - \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S_h) \mathbf{K} \nabla E_p^I \cdot \nabla \varphi + \sum_{E \in \mathcal{E}_h} \int_E (\lambda_t(S) - \lambda_t(S_h)) \mathbf{K} \nabla p \cdot \nabla \varphi \\ & - \sum_{\gamma \in \Gamma_h} \int_\gamma \langle (\lambda_t(S) - \lambda_t(S_h)) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \rangle [\varphi] - \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma \rangle [\varphi] \\ & - \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_t(S_B) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma) \varphi + s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_\gamma \rangle [E_p] \\ & + s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_t(S_B) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_\gamma) E_p \\ & = L_p(\varphi; S) - L_p(\varphi; S_h), \quad \varphi \in \mathcal{D}_r(\mathcal{E}_h). \end{aligned}$$

We note that the jumps $[p]$ are zero almost everywhere on the interior edges (or faces), and thus some terms vanish. Subtracting the saturation DG scheme (3.13) from the weak formulation (3.16), and letting $\varphi = E_S^A$, we can obtain

$$\begin{aligned} (5.4) \quad & \left(\phi \frac{\partial E_S^A}{\partial t}, E_S^A \right) - \omega J_0^\sigma(p_c(\widehat{S}) - p_c(S_h), E_S^A) + J_D^\sigma(E_S^A, E_S^A) \\ & = \left(\phi \frac{\partial E_S^I}{\partial t}, E_S^A \right) - \sum_{E \in \mathcal{E}_h} \int_E (\lambda_w(S) - \lambda_w(S_h)) \mathbf{K} \nabla p \cdot \nabla E_S^A \\ & - \sum_{E \in \mathcal{E}_h} \int_E \lambda_w(S_h) \mathbf{K} \nabla E_p \cdot \nabla E_S^A + \sum_{\gamma \in \Gamma_h} \int_\gamma (f_w(S^*) \langle \lambda_t(S) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \rangle \\ & \quad - f_w(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \rangle) [E_S^A] \\ & + \sum_{\gamma \in \Gamma_h} \int_\gamma f_w(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma \rangle [E_S^A] \\ & + \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_w(S_B) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma) E_S^A \\ & - J_{0,D}^\sigma(p - p_h, E_S^A) + J_D^\sigma(E_S^I, E_S^A) \\ & - s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_\gamma f_w(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla E_S^A \cdot \mathbf{n}_\gamma \rangle [E_p] \\ & - s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_w(S_B) \mathbf{K} \nabla E_S^A \cdot \mathbf{n}_\gamma) E_p + \omega J_0^\sigma(p_c(S) - p_c(\widehat{S}), E_S^A). \end{aligned}$$

In the conventional technique for the error analysis of two-phase flow, for example [16, 17], one first analyzes the pressure equation and the saturation equation separately, and then combines them together. This technique is conveniently carried out for the classical two-phase flow formulation; however, for our formulation, there is a great difficulty handling the broken gradients of saturation in the pressure equation. In order to overcome this difficulty, we introduce a coupling approach for the pressure and saturation equations, and then obtain a coupled error equation, which will conveniently yield the required error estimates. Choosing $\varphi = E_p^A - E_S^A$ in (5.3), and then adding the obtained equation to (5.4), after some simplifications, we obtain a coupled error equation

$$\begin{aligned}
 (5.5) \quad & \left(\phi \frac{\partial E_S^A}{\partial t}, E_S^A \right) - (\omega + 1) J_0^\sigma \left(p_c(\widehat{S}) - p_c(S_h), E_S^A \right) + J_D^\sigma(E_S^A, E_S^A) \\
 & + \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S_h) \mathbf{K} \nabla E_p^A \cdot \nabla E_p^A + J_{0,D}^\sigma(E_p^A, E_p^A) \\
 & - \sum_{E \in \mathcal{E}_h} \int_E \lambda_c(S_h) \mathbf{K} \nabla E_S^A \cdot \nabla E_S^A \\
 & = \sum_{E \in \mathcal{E}_h} \int_E (\lambda_n(S_h) - \lambda_c(S_h)) \mathbf{K} \nabla E_p^A \cdot \nabla E_S^A + G_p + G_S,
 \end{aligned}$$

where

$$\begin{aligned}
 G_p := & \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S_h) \mathbf{K} \nabla E_p^I \cdot \nabla E_p^A - \sum_{E \in \mathcal{E}_h} \int_E (\lambda_t(S) - \lambda_t(S_h)) \mathbf{K} \nabla p \cdot \nabla E_p^A \\
 & + \sum_{\gamma \in \Gamma_h} \int_\gamma \langle (\lambda_t(S) - \lambda_t(S_h)) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \rangle [E_p^A] + \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma \rangle [E_p^A] \\
 & + \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_t(S_B) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma) E_p^A - s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla E_p^A \cdot \mathbf{n}_\gamma \rangle [E_p] \\
 & - s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_t(S_B) \mathbf{K} \nabla E_p^A \cdot \mathbf{n}_\gamma) E_p + J_{0,D}^\sigma(E_p^I, E_p^A) \\
 & + \sum_{E \in \mathcal{E}_h} \int_E \lambda_c(S_h) \mathbf{K} \nabla E_S^I \cdot \nabla E_p^A - \sum_{E \in \mathcal{E}_h} \int_E (\lambda_c(S) - \lambda_c(S_h)) \mathbf{K} \nabla S \cdot \nabla E_p^A \\
 & + \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_c(S_h) \mathbf{K} \nabla E_S \cdot \mathbf{n}_\gamma \rangle [E_p^A] + \sum_{\gamma \in \Gamma_h} \int_\gamma \langle (\lambda_c(S) - \lambda_c(S_h)) \mathbf{K} \nabla S \cdot \mathbf{n}_\gamma \rangle [E_p^A] \\
 & + \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma \lambda_c(S_B) \mathbf{K} \nabla E_S \cdot \mathbf{n}_\gamma E_p^A - J_0^\sigma(p_c(S) - p_c(S_h), E_p^A), \\
 G_S := & - \sum_{E \in \mathcal{E}_h} \int_E \lambda_n(S_h) \mathbf{K} \nabla E_p^I \cdot \nabla E_S^A + \sum_{E \in \mathcal{E}_h} \int_E (\lambda_n(S) - \lambda_n(S_h)) \mathbf{K} \nabla p \cdot \nabla E_S^A \\
 & - \sum_{\gamma \in \Gamma_h} \int_\gamma (f_n(S^*) \langle \lambda_t(S) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \rangle - f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \rangle) [E_S^A] \\
 & - \sum_{\gamma \in \Gamma_h} \int_\gamma f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma \rangle [E_S^A] - \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_n(S_B) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma) E_S^A
 \end{aligned}$$

$$\begin{aligned}
 &+ s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla E_S^A \cdot \mathbf{n}_{\gamma} \rangle [E_p] \\
 &+ s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_n(S_B) \mathbf{K} \nabla E_S^A \cdot \mathbf{n}_{\gamma}) E_p - \sum_{E \in \mathcal{E}_h} \int_E \lambda_c(S_h) \mathbf{K} \nabla E_S^I \cdot \nabla E_S^A \\
 &+ \sum_{E \in \mathcal{E}_h} \int_E (\lambda_c(S) - \lambda_c(S_h)) \mathbf{K} \nabla S \cdot \nabla E_S^A - \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_c(S_h) \mathbf{K} \nabla E_S \cdot \mathbf{n}_{\gamma} \rangle [E_S^A] \\
 &- \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle (\lambda_c(S) - \lambda_c(S_h)) \mathbf{K} \nabla S \cdot \mathbf{n}_{\gamma} \rangle [E_S^A] - \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_c(S_B) \mathbf{K} \nabla E_S \cdot \mathbf{n}_{\gamma} E_S^A \\
 &- s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_n(S_h) \mathbf{K} \nabla (E_p^A - E_S^A) \cdot \mathbf{n}_{\gamma} \rangle [p_c(S) - p_c(S_h)] \\
 &+ (\omega + 1) J_0^{\sigma}(p_c(S) - p_c(\widehat{S}), E_S^A) + J_D^{\sigma}(E_S^I, E_S^A) + \left(\phi \frac{\partial E_S^I}{\partial t}, E_S^A \right).
 \end{aligned}$$

For the first term on the right-hand side of (5.5), using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
 (5.6) \quad &\sum_{E \in \mathcal{E}_h} \int_E (\lambda_n(S_h) - \lambda_c(S_h)) \mathbf{K} \nabla E_p^A \cdot \nabla E_S^A \\
 &= \sum_{E \in \mathcal{E}_h} \int_E \zeta(S_h) \lambda_t(S_h)^{\frac{1}{2}} |\lambda_c(S_h)|^{\frac{1}{2}} \mathbf{K} \nabla E_p^A \cdot \nabla E_S^A \\
 &\leq \frac{\delta}{2} \|(\lambda_t(S_h) \mathbf{K})^{\frac{1}{2}} \nabla E_p^A\|_0^2 + \frac{\delta}{2} \|(|\lambda_c(S_h)| \mathbf{K})^{1/2} \nabla E_S^A\|_0^2,
 \end{aligned}$$

where we have used the assumption on $\zeta(S)$ and $\delta \in (0, 2)$. We substitute (5.6) into (5.5) and obtain

$$\begin{aligned}
 (5.7) \quad &\left(\phi \frac{\partial E_S^A}{\partial t}, E_S^A \right) + (\omega + 1) a_0 J_0^{\sigma}(E_S^A, E_S^A) + J_D^{\sigma}(E_S^A, E_S^A) + J_{0,D}^{\sigma}(E_p^A, E_p^A) \\
 &+ \left(1 - \frac{\delta}{2} \right) \left(\|(\lambda_t(S_h) \mathbf{K})^{1/2} \nabla E_p^A\|_0^2 + \|(|\lambda_c(S_h)| \mathbf{K})^{1/2} \nabla E_S^A\|_0^2 \right) \leq G_p + G_S + C \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.
 \end{aligned}$$

In addition, we can see that $\left(\phi \frac{\partial E_S^A}{\partial t}, E_S^A \right) = \frac{1}{2} \frac{d}{dt} \|\phi^{1/2} E_S^A\|_0^2 \geq \frac{\phi^b}{2} \frac{d}{dt} \|E_S^A\|_0^2$ because of the boundedness of porosity. Furthermore, since λ_t and $|\lambda_c|$ are bounded below, the inequality (5.7) can be changed into

$$\begin{aligned}
 (5.8) \quad &\frac{\phi^b}{2} \frac{d}{dt} \|E_S^A\|_0^2 + (\omega + 1) a_0 J_0^{\sigma}(E_S^A, E_S^A) + J_D^{\sigma}(E_S^A, E_S^A) + \kappa_1 \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 \\
 &+ \kappa_2 \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + J_{0,D}^{\sigma}(E_p^A, E_p^A) \leq G_p + G_S + C \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}},
 \end{aligned}$$

where κ_1 and κ_2 depend on $\delta, \lambda_t, \lambda_c$.

We estimate the two parts G_p and G_S in Appendixes A and B, respectively. Taking $\omega \geq \frac{a_1}{a_0}$ and substituting the estimates (A.1) and (B.1) of G_p and G_S into

(5.8), we obtain

$$\begin{aligned}
 (5.9) \quad & \frac{d}{dt} \|E_S^A\|_0^2 + \kappa_1 \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + \kappa_2 \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + \frac{1}{2} J_{0,D}^\sigma(E_p^A, E_p^A) \\
 & + a_0 J_0^\sigma(E_S^A, E_S^A) + J_D^\sigma(E_S^A, E_S^A) \\
 & \leq (\epsilon_1^p + \epsilon_2^p + \epsilon_5^p + \epsilon_6^p + \epsilon_8^p + \epsilon_9^p + \epsilon_{11}^p + \epsilon_{14}^p + \epsilon_{15}^p + \epsilon_5^s + \epsilon_6^s + \epsilon_{19}^s) \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 \\
 & + (\epsilon_3^p + \epsilon_4^p + \epsilon_7^p + \epsilon_{10}^p + \epsilon_{12}^p + \epsilon_{13}^p + \epsilon_{16}^p + \epsilon_{18}^p + \epsilon_{20}^p + \epsilon_8^s + \epsilon_{11}^s) J_{0,D}^\sigma(E_p^A, E_p^A) \\
 & + (\epsilon_{17}^p + \epsilon_{19}^p + \epsilon_1^s + \epsilon_2^s + \epsilon_9^s + \epsilon_{10}^s + \epsilon_{12}^s + \epsilon_{13}^s + \epsilon_{14}^s + \epsilon_{17}^s + \epsilon_{20}^s) \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 \\
 & + (\epsilon_3^s + \epsilon_4^s + \epsilon_7^s + \epsilon_{15}^s + \epsilon_{16}^s + \epsilon_{18}^s + \epsilon_{21}^s + \epsilon_{22}^s) J_{0,D}^\sigma(E_S^A, E_S^A) \\
 & + C \|E_S^A\|_0^2 + C (\|p\|_s^2 + \|S\|_s^2 + \|S_t\|_{s-1}^2) \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.
 \end{aligned}$$

Choosing sufficiently small ϵ_i^p and ϵ_i^s such that

$$\begin{aligned}
 \epsilon_1^p + \epsilon_2^p + \epsilon_5^p + \epsilon_6^p + \epsilon_8^p + \epsilon_9^p + \epsilon_{11}^p + \epsilon_{14}^p + \epsilon_{15}^p + \epsilon_5^s + \epsilon_6^s + \epsilon_{19}^s &= \frac{\kappa_1}{2}, \\
 \epsilon_3^p + \epsilon_4^p + \epsilon_7^p + \epsilon_{10}^p + \epsilon_{12}^p + \epsilon_{13}^p + \epsilon_{16}^p + \epsilon_{18}^p + \epsilon_{20}^p + \epsilon_8^s + \epsilon_{11}^s &= \frac{1}{4}, \\
 \epsilon_{17}^p + \epsilon_{19}^p + \epsilon_1^s + \epsilon_2^s + \epsilon_9^s + \epsilon_{10}^s + \epsilon_{12}^s + \epsilon_{13}^s + \epsilon_{14}^s + \epsilon_{17}^s + \epsilon_{20}^s &= \frac{\kappa_2}{2}, \\
 \epsilon_3^s + \epsilon_4^s + \epsilon_7^s + \epsilon_{15}^s + \epsilon_{16}^s + \epsilon_{18}^s + \epsilon_{21}^s + \epsilon_{22}^s &= \frac{1}{2} \min(a_0, 1),
 \end{aligned}$$

and taking into account $\|E_S^A\|_0(0) = 0$, by Gronwall's inequality, we obtain

$$\begin{aligned}
 (5.10) \quad & \|E_S^A\|_0^2(\tau) + \int_0^T \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + \int_0^T \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 \\
 & + \int_0^T J_{0,D}^\sigma(E_p^A, E_p^A) + \int_0^T J_{0,D}^\sigma(E_S^A, E_S^A) \\
 & \leq C \left(\|p\|_{L^2(J; H^s)}^2 + \|S\|_{L^2(J; H^s)}^2 + \|S_t\|_{L^2(J; H^{s-1})}^2 \right) \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}},
 \end{aligned}$$

where $\tau \in J = (0, T]$. Finally, by the triangle inequality and Lemma 3.2, and taking into account $\|E_S(0)\|_0 \leq C \|S^0\|_{H^{s-1}} \frac{h^{\min(r, s-1)}}{r^{s-1}}$, we obtain the error estimates for the pressure and saturation:

$$\begin{aligned}
 \|E_p\|_{L^2(J; L^2)} + \|\mathbf{K}^{1/2} \nabla E_p\|_{L^2(J; L^2)} &\leq C (\|p\|_{L^2(J; H^s)} + \|S\|_{L^2(J; H^s)} \\
 & + \|S_t\|_{L^2(J; H^{s-1})}) \frac{h^{\min(r, s-1)}}{r^{s-\frac{3}{2}}}, \\
 \|E_S\|_{L^\infty(J; L^2)} + \|\mathbf{K}^{1/2} \nabla E_S\|_{L^2(J; L^2)} &\leq C (\|p\|_{L^2(J; H^s)} + \|S\|_{L^2(J; H^s)} \\
 & + \|S_t\|_{L^2(J; H^{s-1})} + \|S^0\|_{H^{s-1}}) \frac{h^{\min(r, s-1)}}{r^{s-\frac{3}{2}}},
 \end{aligned}$$

where $r \geq 1$ and $s \geq 2$.

6. Existence of discrete solutions. For proving the global existence of approximate solutions, the key point is to construct a continuous mapping defined on a

bounded closed set. We first define two sets,

$$V_h^p := \left\{ v(\cdot, t) \in \mathcal{D}_r(\mathcal{E}_h) : \int_0^T \|\mathbf{K}^{1/2} \nabla(v - \widehat{p})\|_0^2 + \int_0^T J_{0,D}^\sigma(v - \widehat{p}, v - \widehat{p}) \leq C \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}} \right\},$$

$$V_h^S := \left\{ v(\cdot, t) \in \mathcal{D}_r(\mathcal{E}_h) : \|v - \widehat{S}\|_{L^\infty(J; L^2)}^2 + \int_0^T \|\mathbf{K}^{1/2} \nabla(v - \widehat{S})\|_0^2 + \int_0^T J_{0,D}^\sigma(v - \widehat{S}, v - \widehat{S}) \leq C \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}} \right\},$$

where C depends on $\|p\|_{L^2(J; H^s)}$, $\|S\|_{L^2(J; H^s)}$, $\|S_t\|_{L^2(J; H^{s-1})}$, and $\|S^0\|_{H^{s-1}}$. It is clear that V_h^p and V_h^S are bounded and closed. Let $\overline{S}_h(\cdot, t) \in V_h^S$ be given and $(\overline{S}_h, \varphi) = (S^0, \varphi)$ at $t = 0$ for any $\varphi \in \mathcal{D}_r(\mathcal{E}_h)$, then the pressure p_h and saturation S_h are computed by

$$(6.1) \quad (S_h, \varphi) = (S^0, \varphi), \quad t = 0,$$

$$(6.2) \quad B_p(p_h, \varphi; \overline{S}_h) = L_p(\varphi; \overline{S}_h), \quad t \in J,$$

$$(6.3) \quad \left(\phi \frac{\partial S_h}{\partial t}, \varphi \right) - \sum_{E \in \mathcal{E}_h} \int_E \lambda_c(\overline{S}_h) \mathbf{K} \nabla S_h \cdot \nabla \varphi + \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_c(\overline{S}_h) \mathbf{K} \nabla S_h \cdot \mathbf{n}_\gamma \rangle [\varphi] + \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_c(S_B) \mathbf{K} \nabla S_h \cdot \mathbf{n}_\gamma) \varphi + J_{0,D}^\sigma(S_h, \varphi) + B_S(\overline{S}_h, \varphi; p_h) = L_S(\varphi) + B_p(p_h, \varphi; \overline{S}_h) - s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma \lambda_t(S_B) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_\gamma p_B + J_0^\sigma(p_c(\overline{S}_h), \varphi) - J_D^\sigma(p_B, \varphi) + s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_n(\overline{S}_h) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_\gamma \rangle [p_c(\overline{S}_h)] - \int_\Omega q_t \varphi + J_{0,D}^\sigma(\overline{S}_h, \varphi), \quad t \in J,$$

for any $\varphi \in \mathcal{D}_r(\mathcal{E}_h)$.

The system of equations (6.1)–(6.3) is linear, and thus, from the theory of the linear DG system, it has a unique solution (p_h, S_h) . The approximate solution of pressure is unique for $\Gamma_D \neq \emptyset$ or for $\Gamma_D = \emptyset$ with the compatibility conditions. In order to establish the existence of numerical solutions, we introduce a map \mathbb{T}_h such that $S_h = \mathbb{T}_h(\overline{S}_h)$, where S_h solves the equations (6.1)–(6.3).

Using approaches similar to those used to analyze the error estimates in section 5, we can prove that $p_h \in V_h^p$ and $S_h \in V_h^S$. For two given $\overline{S}_h^i \in V_h^S$, $\overline{S}_h^j \in V_h^S$, let (p_h^i, S_h^i) and (p_h^j, S_h^j) be the solutions of (6.1)–(6.3) when taking $\overline{S}_h = \overline{S}_h^i$ and $\overline{S}_h = \overline{S}_h^j$, respectively. Denote $S_{ij} = S_h^i - S_h^j$, $\overline{S}_{ij} = \overline{S}_h^i - \overline{S}_h^j$, and $p_{ij} = p_h^i - p_h^j$.

LEMMA 6.1. *Let $s > 1 + \frac{d}{2}$, $0 < h \leq h_0$, $r \geq 2$. If the penalty parameter σ_0 is chosen to be sufficiently large, then we have*

$$\|S_{ij}\|_{L^\infty(J; L^2)}^2 + \int_0^T \left(\|\mathbf{K}^{1/2} \nabla p_{ij}\|_0^2 + \|\mathbf{K}^{1/2} \nabla S_{ij}\|_0^2 + J_{0,D}^\sigma(S_{ij}, S_{ij}) + J_{0,D}^\sigma(p_{ij}, p_{ij}) \right) \leq C \left(\|\overline{S}_{ij}\|_{L^\infty(J; L^2)}^2 + \|\mathbf{K}^{1/2} \nabla \overline{S}_{ij}\|_0^2 + J_{0,D}^\sigma(\overline{S}_{ij}, \overline{S}_{ij}) \right),$$

where C depends on r, p, S .

Lemma 6.1 can also be proved by approaches similar to those used in section 5, and it demonstrates the continuity of the map \mathbb{T}_h . Therefore, we can conclude the global existence of discrete solutions in V_h^p and V_h^S by the Schauder fixed-point theorem.

7. The time discretization schemes. In this section, we will discuss the time discretization schemes applied to the nonlinear system obtained by the proposed DG schemes. To do this, we first divide the total time interval $[0, T]$ into N uniform time steps as $0 = t_0 < t_1 < \dots < t_N = T$ and denote by $\tau = T/N$ the time step length, but our methods can be used for the cases of nonuniform time steps as well. Let v^i denote the value of a function $v(t)$ at the time point t_i . For $t = 0$, we can obtain S_h^0 by the initial conditions of the saturation. In what follows, we will propose two time discretization schemes for solving the system of nonlinear ordinary differential equations obtained by the proposed DG schemes.

7.1. IMPES-DG scheme. The IMplicit Pressure EXplicit Saturation (IMPES) method [11] is a popular time-discretization scheme for simulating two-phase flow in porous media. Here, we use IMPES for (3.12) and (3.13) and obtain that

$$(7.1) \quad B_p(p_h^{i+1}, \varphi; S_h^i) = L_p(\varphi; S_h^i),$$

$$(7.2) \quad \left(\phi \frac{S_h^{i+1} - S_h^i}{\tau}, \varphi \right) + B_S(S_h^i, \varphi; p_h^{i+1}) = L_S(\varphi)$$

for $i \geq 0$ and any $\varphi \in \mathcal{D}_r(\mathcal{E}_h)$. In the saturation equation (7.2), the upwind values of saturations are determined by

$$(7.3) \quad S_h^{i*}|_\gamma = \begin{cases} S_h^i|_{E_\gamma^1} & \text{if } \mathbf{u}_a^{DG,i+1} \cdot \mathbf{n}_\gamma \geq 0, \\ S_h^i|_{E_\gamma^2} & \text{if } \mathbf{u}_a^{DG,i+1} \cdot \mathbf{n}_\gamma < 0, \end{cases}$$

where the two elements E_γ^1 and E_γ^2 share γ with \mathbf{n}_γ exterior to E_γ^1 , and $\mathbf{u}_a^{DG,i+1} \cdot \mathbf{n}_\gamma$ is calculated by

$$(7.4) \quad \mathbf{u}_a^{DG,i+1} \cdot \mathbf{n}_\gamma = - \langle \lambda_t(S_h^i) \mathbf{K} \nabla p_h^{i+1} \cdot \mathbf{n} \rangle + \frac{r^2 \sigma_\gamma}{h_\gamma} [p_h^{i+1}] \quad \text{on } \gamma \in \Gamma_h.$$

For given S_h^i , we can calculate p_h^{i+1} by solving the pressure equation (7.1), which is a linear equation of p_h^{i+1} and possesses a unique solution by the theory of linear DG methods for elliptic equations, and then we calculate $\mathbf{u}_a^{DG,i+1} \cdot \mathbf{n}_\gamma$ for each $\gamma \in \Gamma_h \cup \Gamma_{h,D}$. Finally, we update the saturation S_h^{i+1} explicitly by using the saturation equation (7.2). This process can start with S_h^0 given by (3.14) and work well if the suitable small time step size is chosen to guarantee the stability. It is also noted that at each time step, we always use the implicit scheme for the boundary conditions.

7.2. Improved IMPES-DG scheme. The IMPES-DG scheme treats the saturation explicitly in the pressure equation (7.1), and this treatment may result in instability of the IMPES method [22] and the decoupling between the pressure equation (3.12) and the saturation equation (3.13). In order to improve the IMPES-DG scheme, based on the ideas in [22], we treat the gradient of the saturation implicitly in (3.12), but keep $\lambda_t(S_h)$ and $\lambda_c(S_h)$ explicitly. As a result, we obtain the following

pressure equation:

$$\begin{aligned}
(7.5) \quad & \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S_h^i) \mathbf{K} \nabla p_h^{i+1} \cdot \nabla \varphi - \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_t(S_h^i) \mathbf{K} \nabla p_h^{i+1} \cdot \mathbf{n}_{\gamma} \rangle [\varphi] \\
& - \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_t(S_B^{i+1}) \mathbf{K} \nabla p_h^{i+1} \cdot \mathbf{n}_{\gamma}) \varphi \\
& + s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_t(S_h^i) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_{\gamma} \rangle [p_h^{i+1}] \\
& + s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_t(S_B^{i+1}) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_{\gamma}) p_h^{i+1} + J_{0,D}^{\sigma}(p_h^{i+1}, \varphi) \\
= & s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_t(S_B^{i+1}) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_{\gamma} p_B^{i+1} + J_D^{\sigma}(p_B^{i+1}, \varphi) + \int_{\Omega} q_t^{i+1} \varphi \\
& - \sum_{E \in \mathcal{E}_h} \int_E \lambda_c(S_h^i) \mathbf{K} \nabla S_h^{i+1} \cdot \nabla \varphi + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_c(S_h^i) \mathbf{K} \nabla S_h^{i+1} \cdot \mathbf{n}_{\gamma} \rangle [\varphi] \\
& + \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_c(S_B^{i+1}) \mathbf{K} \nabla S_h^{i+1} \cdot \mathbf{n}_{\gamma}) \varphi \\
& - s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_n(S_h^i) \mathbf{K} \nabla \varphi \cdot \mathbf{n}_{\gamma} \rangle [D_c(S_h^{i+1})] \\
& - J_0^{\sigma}(D_c(S_h^{i+1}), \varphi), \quad \varphi \in \mathcal{D}_r(\mathcal{E}_h),
\end{aligned}$$

where $D_c(S_h^{i+1}) = p_c(S_h^i) + p'_c(S_h^i)(S_h^{i+1} - S_h^i)$.

In the improved IMPES-DG scheme, for given S_h^i , the pressure p_h^{i+1} and saturation S_h^{i+1} solve the two linear equations (7.5) and (7.2). In order to efficiently solve this coupled system, we compute the upwind saturations in (7.2) as

$$(7.6) \quad S_h^{i*}|_{\gamma} = \begin{cases} S_h^i|_{E_{\gamma}^1} & \text{if } \mathbf{u}_a^{DG,i} \cdot \mathbf{n}_{\gamma} \geq 0, \\ S_h^i|_{E_{\gamma}^2} & \text{if } \mathbf{u}_a^{DG,i} \cdot \mathbf{n}_{\gamma} < 0. \end{cases}$$

For $i = 0$, $S_h^{0*}|_{E_{\gamma}^1} = S_h^0|_{E_{\gamma}^1} = S_h^0|_{E_{\gamma}^2}$ since $[S^0] = 0$ almost everywhere. In this case, we can substitute (7.2) into (7.5) to reduce the saturation S_h^{i+1} of (7.5), and thus obtain a coupled pressure equation. Once the pressure p_h^{i+1} is obtained, we can calculate $\mathbf{u}_a^{DG,i+1} \cdot \mathbf{n}_{\gamma}$ by (7.4) and update the saturation S_h^{i+1} by (7.2), which are the same to the IMPES-DG scheme.

Finally, we give some remarks and further developments on the proposed IMPES-DG scheme and improved IMPES-DG scheme as below.

- For the fully implicit time scheme applied to (3.12) and (3.13), we can construct the iterative methods based on the IMPES-DG scheme and improved IMPES-DG scheme, respectively, for solving the resulting nonlinear system. We refer to [23, 25] for similar iterative IMPES methods.
- The IMPES-DG scheme and improved IMPES-DG scheme allow us to use the multiscale time steps for pressures and saturations in different regions; see the detail in [9] for example.
- In future work, we will study the convergence of the IMPES-DG scheme and improved IMPES-DG scheme, especially the error estimates including the approximate time errors.

8. Conclusions. We have developed a class of DG methods with interior penalties for incompressible two-phase flow in porous media with capillary pressures. The proposed methods incorporate the capillary pressures in the pressure equation instead of saturation equation. Differently from the conventional separate convergence analysis for the pressure and the saturation, we introduce a coupling error analysis approach, and based on this approach, we have obtained stability and a priori hp error estimates of the schemes in $L^2(H^1)$ for the pressure and in $L^\infty(L^2)$ and $L^2(H^1)$ for the saturation. We also construct a continuous map to prove the global existence of the discrete solutions and present two time discretization schemes for effectively computing the discrete solutions.

Appendix A. Estimates of G_p . In Appendix A, we use $\{\epsilon_i^p\}_{i \geq 1}$ to indicate a sequence of small positive constants. The first term of G_p is estimated as

$$\begin{aligned} \sum_{E \in \mathcal{E}_h} \int_E \lambda_t(S_h) \mathbf{K} \nabla E_p^I \cdot \nabla E_p^A &\leq \sum_{E \in \mathcal{E}_h} \left(\epsilon_1^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_{0,E}^2 + C \|\nabla E_p^I\|_{0,E}^2 \right) \\ &\leq \epsilon_1^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-2}}, \end{aligned}$$

where we have used the boundedness of $\lambda_t(S_h)$. The second term of G_p is bounded as

$$\begin{aligned} & - \sum_{E \in \mathcal{E}_h} \int_E (\lambda_t(S) - \lambda_t(S_h)) \mathbf{K} \nabla p \cdot \nabla E_p^A \\ & \leq \sum_{E \in \mathcal{E}_h} \|\nabla p\|_{0,\infty,E} \|\mathbf{K}^{1/2} \nabla E_p^A\|_{0,E} \|\lambda_t(S) - \lambda_t(S_h)\|_{0,E} \\ & \leq C \sum_{E \in \mathcal{E}_h} \|\nabla p\|_{0,\infty,E} \|\mathbf{K}^{1/2} \nabla E_p^A\|_{0,E} \|E_S\|_{0,E} \\ & \leq \sum_{E \in \mathcal{E}_h} \left(\epsilon_2^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_{0,E} + C \|\nabla p\|_{0,\infty,E} \|E_S\|_{0,E}^2 \right) \\ & \leq \epsilon_2^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + C \|E_S^A\|_0^2 + C \|S\|_s^2 \frac{h^{\min(2r+2, 2s)}}{r^{2s}}, \end{aligned}$$

where we have used the assumption on the essential boundedness of ∇p and the first constant C results from the Lipschitz continuity constant of λ_t . The third term of G_p is estimated as

$$\begin{aligned} & \sum_{\gamma \in \Gamma_h} \int_\gamma \langle (\lambda_t(S) - \lambda_t(S_h)) \mathbf{K} \nabla p \cdot \mathbf{n}_\gamma \rangle [E_p^A] \\ & \leq \frac{C}{\epsilon_3^p} \sum_{E \in \mathcal{E}_h} \frac{h_E}{\sigma_0 r^2} \|\nabla p\|_{0,\infty,\partial E}^2 \|\lambda_t(S) - \lambda_t(S_h)\|_{0,\partial E}^2 + \epsilon_3^p \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_\gamma}{h_\gamma} \int_\gamma [E_p^A]^2 \\ & \leq C \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} \|\nabla p\|_{0,\infty,\partial E}^2 \|E_S\|_{0,\partial E}^2 + \epsilon_3^p J_{0,D}^\sigma(E_p^A, E_p^A) \\ & \leq C \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} (\|E_S^A\|_{0,\partial E}^2 + \|E_S^I\|_{0,\partial E}^2) + \epsilon_3^p J_{0,D}^\sigma(E_p^A, E_p^A) \\ & \leq C \|E_S^A\|_0^2 + \epsilon_3^p J_{0,D}^\sigma(E_p^A, E_p^A) + C \|S\|_s^2 \frac{h^{\min(2r+2, 2s)}}{r^{2s+1}}, \end{aligned}$$

where the upper boundedness of \mathbf{K} and the assumption on the essential boundedness of ∇p have been applied, and here we choose the penalty parameters $\sigma_\gamma \geq \sigma_0 > 0$. Here, we have also used the inverse inequality and hp -approximation results. The fourth term consists of two pieces,

$$\sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma \rangle [E_p^A] = \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla (E_p^A - E_p^I) \cdot \mathbf{n}_\gamma \rangle [E_p^A].$$

Let the penalty parameter σ_0 be taken to be sufficiently large, then by the Cauchy–Schwarz inequality and inverse inequality we have

$$\begin{aligned} & \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla E_p^A \cdot \mathbf{n}_\gamma \rangle [E_p^A] \\ & \leq C \sum_{E \in \mathcal{E}_h} \frac{h_E}{\sigma_0 r^2} \|\mathbf{K}^{1/2} \nabla E_p^A\|_{0,\partial E}^2 + \epsilon_4^p \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_\gamma}{h_\gamma} \int_\gamma [E_p^A]^2 \\ & \leq \sum_{E \in \mathcal{E}_h} \frac{C}{\sigma_0} \|\mathbf{K}^{1/2} \nabla E_p^A\|_{0,E}^2 + \frac{1}{2} \epsilon_4^p J_{0,D}^\sigma(E_p^A, E_p^A) \\ & \leq \epsilon_5^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + \frac{1}{2} \epsilon_4^p J_{0,D}^\sigma(E_p^A, E_p^A). \end{aligned}$$

Furthermore, the Cauchy–Schwarz inequality and approximation results yield that

$$\begin{aligned} \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla E_p^I \cdot \mathbf{n}_\gamma \rangle [E_p^A] & \leq C \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} \|\nabla E_p^I\|_{0,\partial E}^2 + \frac{1}{2} \epsilon_4^p J_{0,D}^\sigma(E_p^A, E_p^A) \\ & \leq \frac{1}{2} \epsilon_4^p J_{0,D}^\sigma(E_p^A, E_p^A) + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-1}}. \end{aligned}$$

Therefore, the fourth term of G_p can be estimated as

$$\begin{aligned} \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma \rangle [E_p^A] & \leq \epsilon_5^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + \epsilon_4^p J_{0,D}^\sigma(E_p^A, E_p^A) \\ & \quad + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-1}}. \end{aligned}$$

For the fifth term of G_p , we have a similar result,

$$\begin{aligned} \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma \langle \lambda_t(S_B) \mathbf{K} \nabla E_p \cdot \mathbf{n}_\gamma \rangle E_p^A & \leq \epsilon_6^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + \epsilon_7^p J_{0,D}^\sigma(E_p^A, E_p^A) \\ & \quad + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-1}}. \end{aligned}$$

In order to estimate the sixth term, by the Cauchy–Schwarz inequality, inverse inequality, and approximation results, we first get the estimate

$$\begin{aligned} \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_t(S_h) \mathbf{K} \nabla E_p^A \cdot \mathbf{n}_\gamma \rangle [E_p^I] & \leq \sum_{E \in \mathcal{E}_h} \frac{h_E}{Cr^2} \|\mathbf{K}^{1/2} \nabla E_p^A\|_{0,\partial E}^2 + \sum_{E \in \mathcal{E}_h} \frac{Cr^2}{h_E} \|E_p^I\|_{0,\partial E}^2 \\ & \leq \epsilon_8^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}, \end{aligned}$$

where we have used the boundedness of λ_t again. Similarly to the fourth term, by using $E_p = E_p^A - E_p^I$, we obtain the estimate of the sixth term of G_p as

$$-s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_t(S_h) \mathbf{K} \nabla E_p^A \cdot \mathbf{n}_{\gamma} \rangle [E_p] \leq (\epsilon_8^p + \epsilon_9^p) \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + \epsilon_{10}^p J_{0,D}^{\sigma}(E_p^A, E_p^A) + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}},$$

and furthermore, we get a similar estimate for the seventh term,

$$-s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_t(S_B) \mathbf{K} \nabla E_p^A \cdot \mathbf{n}_{\gamma}) E_p \leq \epsilon_{11}^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + \epsilon_{12}^p J_{0,D}^{\sigma}(E_p^A, E_p^A) + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.$$

The eighth term is estimated by the Cauchy–Schwarz inequality and approximation results as

$$J_{0,D}^{\sigma}(E_p^I, E_p^A) \leq \epsilon_{13}^p J_{0,D}^{\sigma}(E_p^A, E_p^A) + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.$$

For the ninth term, the Cauchy–Schwarz inequality and approximation results give us

$$\sum_{E \in \mathcal{E}_h} \int_E \lambda_c(S_h) \mathbf{K} \nabla E_S^I \cdot \nabla E_p^A \leq \epsilon_{14}^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-2}},$$

where we have used the boundedness of λ_c . Using the Lipschitz continuity of λ_c and the essential boundedness of ∇S on each element, we bound the tenth term as

$$\begin{aligned} & - \sum_{E \in \mathcal{E}_h} \int_E (\lambda_c(S) - \lambda_c(S_h)) \mathbf{K} \nabla S \cdot \nabla E_p^A \\ & \leq C \sum_{E \in \mathcal{E}_h} \|\nabla S\|_{0,\infty,E} \|\mathbf{K}^{1/2} \nabla E_p^A\|_{0,E} \|E_S\|_{0,E} \\ & \leq \epsilon_{15}^p \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + C \|E_S^A\|_0^2 + C \|S\|_s^2 \frac{h^{\min(2r+2, 2s)}}{r^{2s}}. \end{aligned}$$

Since λ_c and \mathbf{K} are bounded, it follows by the Cauchy–Schwarz inequality, inverse inequality, and approximation results that

$$\begin{aligned} & \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle \lambda_c(S_h) \mathbf{K} \nabla E_S \cdot \mathbf{n}_{\gamma} \rangle [E_p^A] \\ & \leq \frac{C}{\sigma_0} \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} \|\nabla E_S\|_{0,\partial E}^2 + \epsilon_{16}^p J_{0,D}^{\sigma}(E_p^A, E_p^A) \\ & \leq \epsilon_{16}^p J_{0,D}^{\sigma}(E_p^A, E_p^A) + \epsilon_{17}^p \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-1}}, \end{aligned}$$

where we choose the penalty parameters σ_0 to be sufficiently large. Applying tech-

niques similar to those used in the twelfth term, we get

$$\begin{aligned} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} \lambda_c(S_B) \mathbf{K} \nabla E_S \cdot \mathbf{n}_{\gamma} E_p^A &\leq \epsilon_{18}^p J_{0,D}^{\sigma}(E_p^A, E_p^A) + \epsilon_{19}^p \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 \\ &\quad + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-1}}. \end{aligned}$$

We use the Lipschitz continuity of λ_c and the the boundedness of ∇S and \mathbf{K} again to estimate the thirteen term:

$$\begin{aligned} \sum_{\gamma \in \Gamma_h} \int_{\gamma} \langle (\lambda_c(S) - \lambda_c(S_h)) \mathbf{K} \nabla S \cdot \mathbf{n}_{\gamma} \rangle [E_p^A] \\ \leq C \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} \|\nabla S\|_{0,\infty,\partial E}^2 \|E_S\|_{0,\partial E}^2 + \epsilon_{20}^p J_{0,D}^{\sigma}(E_p^A, E_p^A) \\ \leq \epsilon_{20}^p J_{0,D}^{\sigma}(E_p^A, E_p^A) + C \|E_S^A\|_0^2 + C \|S\|_s^2 \frac{h^{\min(2r+2, 2s)}}{r^{2s+1}}, \end{aligned}$$

where we have also used the approximation results and inverse inequality. The last term is estimated as

$$-J_0^{\sigma}(p_c(S) - p_c(S_h), E_p^A) \leq a_1 J_0^{\sigma}(E_S^A, E_S^A) + \frac{1}{2} J_0^{\sigma}(E_p^A, E_p^A) + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.$$

Summing the above estimates of all terms of G_p , we can see that

$$\begin{aligned} \text{(A.1) } G_p &\leq (\epsilon_1^p + \epsilon_2^p + \epsilon_5^p + \epsilon_6^p + \epsilon_8^p + \epsilon_9^p + \epsilon_{11}^p + \epsilon_{14}^p + \epsilon_{15}^p) \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 \\ &\quad + \left(\frac{1}{2} + \epsilon_3^p + \epsilon_4^p + \epsilon_7^p + \epsilon_{10}^p + \epsilon_{12}^p + \epsilon_{13}^p + \epsilon_{16}^p + \epsilon_{18}^p + \epsilon_{20}^p \right) J_{0,D}^{\sigma}(E_p^A, E_p^A) \\ &\quad + (\epsilon_{17}^p + \epsilon_{19}^p) \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + C \|E_S^A\|_0^2 + a_1 J_0^{\sigma}(E_S^A, E_S^A) \\ &\quad + C (\|p\|_s^2 + \|S\|_s^2) \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}. \end{aligned}$$

Appendix B. Estimates of G_S . We now estimate the terms of G_S , and in Appendix B, let $\{\epsilon_i^s\}_{i \geq 1}$ denote a sequence of small positive constants. Using the Cauchy–Schwarz inequality and approximation results, we can obtain the estimate of the first term as

$$- \sum_{E \in \mathcal{E}_h} \int_E \lambda_n(S_h) \mathbf{K} \nabla E_p^I \cdot \nabla E_S^A \leq \epsilon_1^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + C \|p\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-2}},$$

where the boundedness of λ_n has been used. Using the Lipschitz continuity of $\lambda_n(S)$ and approximation results, we estimate the second term as

$$\begin{aligned} \sum_{E \in \mathcal{E}_h} \int_E (\lambda_n(S) - \lambda_n(S_h)) \mathbf{K} \nabla p \cdot \nabla E_S^A \\ \leq C \sum_{E \in \mathcal{E}_h} \|\nabla p\|_{0,\infty,E} \|E_S\|_{0,E} \|\mathbf{K}^{1/2} \nabla E_S^A\|_{0,E} \\ \leq C \sum_{E \in \mathcal{E}_h} \|E_S\|_{0,E}^2 + \epsilon_2^s \sum_{E \in \mathcal{E}_h} \|\mathbf{K}^{1/2} \nabla E_S^A\|_{0,E}^2 \\ \leq C \|E_S^A\|_0^2 + \epsilon_2^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + C \|S\|_s^2 \frac{h^{\min(2r+2, 2s)}}{r^{2s}}, \end{aligned}$$

where we have also used the boundedness of the absolute permeability \mathbf{K} and ∇p . Applying the inverse inequality and approximation results, we estimate the third term to be

$$\begin{aligned} & - \sum_{\gamma \in \Gamma_h} \int_{\gamma} (f_n(S^*) \langle \lambda_t(S) \mathbf{K} \nabla p \cdot \mathbf{n}_{\gamma} \rangle - f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla p \cdot \mathbf{n}_{\gamma} \rangle) [E_S^A] \\ & \leq C \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} \|\nabla p\|_{0,\infty,\partial E}^2 \|E_S\|_{0,\partial E}^2 + \epsilon_3^s \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_{\gamma}}{h_{\gamma}} \int_{\gamma} [E_S^A]^2 \\ & \leq C \|E_S^A\|_0^2 + \epsilon_3^s J_{0,D}^{\sigma}(E_S^A, E_S^A) + C \|S\|_s^2 \frac{h^{\min(2r+2,2s)}}{r^{2s+1}}, \end{aligned}$$

where the Lipschitz continuity of λ_{α} and f_n has also been used. By the virtue of the boundedness of λ_t and f_n , the inverse inequality, and approximation results, the fourth term is estimated as

$$\begin{aligned} & - \sum_{\gamma \in \Gamma_h} \int_{\gamma} f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla E_p \cdot \mathbf{n}_{\gamma} \rangle [E_S^A] \\ & \leq \frac{C}{\sigma_0} \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} \left(\|\mathbf{K}^{1/2} E_p^A\|_{0,\partial E}^2 + \|\nabla E_p^I\|_{0,\partial E}^2 \right) + \epsilon_4^s \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_{\gamma}}{h_{\gamma}} \int_{\gamma} [E_S^A]^2 \\ & \leq \epsilon_5^s \|\mathbf{K}^{1/2} E_p^A\|_0^2 + \epsilon_4^s J_{0,D}^{\sigma}(E_S^A, E_S^A) + C \|p\|_s^2 \frac{h^{\min(2r,2s-2)}}{r^{2s-1}}, \end{aligned}$$

where we choose the penalty parameters σ_0 to be sufficiently large for the sake of getting a suitable small ϵ_5^s for $\|\mathbf{K}^{1/2} E_p^A\|_0^2$. Similarly, for the fifth term, we have

$$\begin{aligned} & - \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_n(S_B) \mathbf{K} \nabla E_p \cdot \mathbf{n}_{\gamma}) E_S^A \leq \epsilon_6^s \|\mathbf{K}^{1/2} E_p^A\|_0^2 + \epsilon_7^s J_{0,D}^{\sigma}(E_S^A, E_S^A) \\ & \quad + C \|p\|_s^2 \frac{h^{\min(2r,2s-2)}}{r^{2s-1}}. \end{aligned}$$

For the sixth term, we get

$$\begin{aligned} & s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} f_n(S_h^*) \langle \lambda_t(S_h) \mathbf{K} \nabla E_S^A \cdot \mathbf{n}_{\gamma} \rangle [E_p] \\ & \leq \frac{C}{\sigma_0} \sum_{E \in \mathcal{E}_h} \frac{h_E}{r^2} \|\mathbf{K}^{1/2} \nabla E_S^A\|_{0,\partial E}^2 + \epsilon_8^s J_{0,D}^{\sigma}(E_p^A, E_p^A) + C \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_{\gamma}}{h_{\gamma}} \int_{\gamma} [E_p^I]^2 \\ & \leq \epsilon_9^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + \epsilon_8^s J_{0,D}^{\sigma}(E_p^A, E_p^A) + C \sum_{E \in \mathcal{E}_h} \frac{r^2}{h_E} \|E_p^I\|_{0,\partial E}^2 \\ & \leq \epsilon_9^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + \epsilon_8^s J_{0,D}^{\sigma}(E_p^A, E_p^A) + C \|p\|_s^2 \frac{h^{\min(2r,2s-2)}}{r^{2s-3}}. \end{aligned}$$

A similar estimate can be obtained for the seventh term,

$$\begin{aligned} & s_{\text{form}} \sum_{\gamma \in \Gamma_{h,D}} \int_{\gamma} (\lambda_n(S_B) \mathbf{K} \nabla E_S^A \cdot \mathbf{n}_{\gamma}) E_p \leq \epsilon_{10}^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + \epsilon_{11}^s J_{0,D}^{\sigma}(E_p^A, E_p^A) \\ & \quad + C \|p\|_s^2 \frac{h^{\min(2r,2s-2)}}{r^{2s-3}}. \end{aligned}$$

The terms from eight to thirteen of G_S can be estimated by similar techniques. Here we need to use the Lipschitz continuity of $\lambda_c(S)$ and the boundedness of λ_c and ∇S .

For the sake of simplicity, we drop the derivations and only list the estimate results:

$$\begin{aligned}
& - \sum_{E \in \mathcal{E}_h} \int_E \lambda_c(S_h) \mathbf{K} \nabla E_S^I \cdot \nabla E_S^A \\
& \leq \epsilon_{12}^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-2}}, \\
& \sum_{E \in \mathcal{E}_h} \int_E (\lambda_c(S) - \lambda_c(S_h)) \mathbf{K} \nabla p \cdot \nabla E_S^A \\
& \leq C \|E_S^A\|_0^2 + \epsilon_{13}^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + C \|S\|_s^2 \frac{h^{\min(2r+2, 2s)}}{r^{2s}}, \\
& - \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_c(S_h) \mathbf{K} \nabla E_S \cdot \mathbf{n}_\gamma \rangle [E_S^A] \\
& \leq \epsilon_{14}^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + \epsilon_{15}^s J_{0,D}^\sigma(E_S^A, E_S^A) + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-1}}, \\
& - \sum_{\gamma \in \Gamma_h} \int_\gamma \langle (\lambda_c(S) - \lambda_c(S_h)) \mathbf{K} \nabla S \cdot \mathbf{n}_\gamma \rangle [E_S^A] \\
& \leq C \|E_S^A\|_0^2 + \epsilon_{16}^s J_{0,D}^\sigma(E_S^A, E_S^A) + C \|S\|_s^2 \frac{h^{\min(2r+2, 2s)}}{r^{2s+1}}, \\
& - \sum_{\gamma \in \Gamma_{h,D}} \int_\gamma (\lambda_c(S_B) \mathbf{K} \nabla E_S \cdot \mathbf{n}_\gamma) E_S^A \\
& \leq \epsilon_{17}^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + \epsilon_{18}^s J_{0,D}^\sigma(E_S^A, E_S^A) + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-1}}, \\
& - s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_\gamma \langle \lambda_n(S_h) \mathbf{K} \nabla (E_p^A - E_S^A) \cdot \mathbf{n}_\gamma \rangle [p_c(S) - p_c(S_h)] \\
& \leq \epsilon_{19}^s \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + \epsilon_{20}^s \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 + \epsilon_{21}^s J_{0,D}^\sigma(E_S^A, E_S^A) + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.
\end{aligned}$$

The rest of the terms are trivially estimated by

$$\begin{aligned}
& (\omega + 1) J_0^\sigma(p_c(S) - p_c(\hat{S}), E_S^A) + J_D^\sigma(E_S^I, E_S^A) \\
& \leq \epsilon_{22}^s J_{0,D}^\sigma(E_S^A, E_S^A) + C \|S\|_s^2 \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.
\end{aligned}$$

Taking into account the boundedness of porosity, and by the Cauchy–Schwarz inequality and approximation results, we can estimate the last term of G_S as

$$\left(\phi \frac{\partial E_S^I}{\partial t}, E_S^A \right) \leq C \|E_S^A\|_0^2 + C \left\| \frac{\partial E_S^I}{\partial t} \right\|_0^2 \leq C \|E_S^A\|_0^2 + C \|S_t\|_{s-1}^2 \frac{h^{\min(2r+2, 2s-2)}}{r^{2s-2}}.$$

Finally, we sum the above estimates of G_S and obtain that

$$\begin{aligned}
\text{(B.1)} \quad G_S & \leq (\epsilon_1^s + \epsilon_2^s + \epsilon_9^s + \epsilon_{10}^s + \epsilon_{12}^s + \epsilon_{13}^s + \epsilon_{14}^s + \epsilon_{17}^s + \epsilon_{20}^s) \|\mathbf{K}^{1/2} \nabla E_S^A\|_0^2 \\
& + (\epsilon_3^s + \epsilon_4^s + \epsilon_7^s + \epsilon_{15}^s + \epsilon_{16}^s + \epsilon_{18}^s + \epsilon_{21}^s + \epsilon_{22}^s) J_{0,D}^\sigma(E_S^A, E_S^A) \\
& + (\epsilon_5^s + \epsilon_6^s + \epsilon_{19}^s) \|\mathbf{K}^{1/2} \nabla E_p^A\|_0^2 + (\epsilon_8^s + \epsilon_{11}^s) J_{0,D}^\sigma(E_p^A, E_p^A) \\
& + C \|E_S^A\|_0^2 + C (\|p\|_s^2 + \|S\|_s^2 + \|S_t\|_{s-1}^2) \frac{h^{\min(2r, 2s-2)}}{r^{2s-3}}.
\end{aligned}$$

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