

GENERALIZED MULTISCALE FINITE ELEMENT METHODS FOR WAVE PROPAGATION IN HETEROGENEOUS MEDIA*

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Abstract. Numerical modeling of wave propagation in heterogeneous media is important in many applications. Due to their complex nature, direct numerical simulations on the fine grid are prohibitively expensive. It is therefore important to develop efficient and accurate methods that allow the use of coarse grids. In this paper, we present a multiscale finite element method for wave propagation on a coarse grid. The proposed method is based on the generalized multiscale finite element method (GMsFEM) (see [Y. Efendiev, J. Galvis, and T. Hou, *J. Comput. Phys.*, 251 (2012), pp. 116–135]). To construct multiscale basis functions, we start with two snapshot spaces in each coarse-grid block, where one represents the degrees of freedom on the boundary and the other represents the degrees of freedom in the interior. We use local spectral problems to identify important modes in each snapshot space. These local spectral problems are different from each other and their formulations are based on the analysis. To the best of knowledge, this is the first time that multiple snapshot spaces and multiple spectral problems are used and necessary for efficient computations. Using the dominant modes from local spectral problems, multiscale basis functions are constructed to represent the solution space locally within each coarse block. These multiscale basis functions are coupled via the symmetric interior penalty discontinuous Galerkin method which provides a block diagonal mass matrix and, consequently, results in fast computations in an explicit time discretization. Our methods' stability and spectral convergence are rigorously analyzed. Numerical examples are presented to show our methods' performance. We also test oversampling strategies. In particular, we discuss how the modes from different snapshot spaces can affect the proposed methods' accuracy.

Key words. wave propagation, heterogeneous media, multiscale finite element method, model reduction

AMS subject classifications. 65M60, 65M12

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1. Introduction. Numerical modeling of wave propagation is important in many applications that include geophysics, material science, and so on. For example, in geophysics applications, wave propagation simulations play an important role in determining subsurface properties [34, 33, 30, 23, 31, 24]. These approaches include finite difference methods, finite element methods, and spectral methods that use polynomials basis [11, 19, 28, 18, 25, 20, 22, 21, 26, 35, 36, 37]. While these methods have different strengths and weaknesses, all of them tend to have limitations associated with discretization, especially in three-dimensional applications as frequency content of the simulated wavefield increases. Though the solutions to the wave equation have been shown to be accurate when the grid is fine enough [12], the practical limitations in discretization caused by limitations in computational power restrict this accuracy. An example of an application where this may be important is in the modeling of

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fractured media, where establishing reliable and accurate relationships between the properties of reflected seismic wavefields and variations in the density, orientation, and compliance of fractures may help provide important constraints for hydrocarbon production. While more general finite element and spectral element methods may be able to address some problems by adapting grids to conform to heterogeneous structures, there are basic limitations associated with representing fine-scale features, and there is therefore a need to find approaches that reliably and accurately incorporate fine-scale features in a coarsely gridded model.

In this paper, we present a multiscale finite element method for wave propagation simulations on a coarse grid. The proposed method is based on the generalized multiscale finite element method (GMsFEM) which was proposed in [13] and couples multiscale basis functions via a discontinuous Galerkin coupling (cf. [14]). To construct multiscale basis functions, we start with two snapshot spaces in each coarse-grid block where one represents the degrees of freedom on the coarse grid's boundary and the other represents the degrees of freedom in the interior. We use local spectral problems to identify important modes in each snapshot space. These local spectral problems are constructed so that the errors of the solutions have the required spectral convergence. We remark that the computations of the snapshot spaces and solutions of spectral problems are offline steps, namely, they can be done before the simulations of the wave equations and can be used for multiple simulations. Moreover, the computations are independent across coarse grids, and hence they can be done in parallel. Once local basis functions are identified, we couple these basis functions via the interior penalty discontinuous Galerkin method [17, 29]. The resulting problem is a low dimensional ordinary differential system in time with block diagonal mass matrix and can be discretized by any classical explicit method, giving a very efficient time-marching scheme. The efficiency of the time stepping can be further enhanced by parallelization since the mass matrix is block diagonal and the solution on each coarse grid can be updated independently.

Because these basis functions are discontinuous, the interior penalty discontinuous Galerkin (IPDG) method, for example [17, 10, 29], is an appropriate choice for solving the time-dependent partial differential equation. It generally yields a block diagonal mass matrix; hence the time stepping is very efficient. The staggered discontinuous Galerkin methods [3, 4] have been recently developed for the accurate wave simulations. By using a carefully chosen staggered grid, the resulting method is also energy conserving. Moreover, it is proved that (see [1, 6]) such a method gives smaller dispersion errors, and therefore it is superior for the wave propagation. The staggered idea has also been extended to other problems; see, for example, [7, 8, 9, 5]. Recently, we have used standard MsFEM basis within staggered methods [2, 16]. These methods allow some limited upscaling and provide energy conserving numerical methods on staggered grids. In this paper, our goal is to construct a systematic enrichment by appropriately choosing snapshot spaces and corresponding local spectral problems on nonstaggered grids for the second order wave equation. The construction of GMsFEM for the first order wave equation on staggered grids will be reported in the future.

We will focus our discussions on two-dimensional problems. The methodology can be applied to three-dimensional problems without significant conceptual difference. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of two dimensions. The paper's aim is to develop a new multiscale method for the following wave equation:

$$(1) \quad \frac{\partial^2 u}{\partial t^2} = \nabla \cdot (a \nabla u) + f \quad \text{in} \quad [0, T] \times \Omega$$

with the homogeneous Dirichlet boundary condition $u = 0$ on $[0, T] \times \partial\Omega$. The extension to other boundary conditions will be reported in a forthcoming paper. The function $f(x, t)$ is a given source. The problem (1) is supplemented with the following initial conditions:

$$u(x, 0) = g_0(x), \quad u_t(x, 0) = g_1(x).$$

We assume that the coefficient $a(x)$ is highly oscillatory, representing the complicated model in which the waves are simulated. It is well-known that solving (1) by standard methods requires a very fine mesh, which is computationally prohibited. Thus a coarse grid solution strategy is needed. Next we present our fine scale solver. The fine scale solution is considered as the exact solution when we discuss the convergence of our multiscale method in the following sections. We assume that the domain Ω is partitioned by a set of rectangles, called fine mesh, with maximum side length $h > 0$. We denote the resulting mesh by \mathcal{T}^h and the set of all edges and vertices by \mathcal{E}^h and \mathcal{N}^h , respectively. We assume that the fine-mesh discretization of the wave equation provides an accurate approximation of the solution. The fine scale solver is the standard conforming bilinear finite element method. Let V_h be the standard conforming piecewise bilinear finite element space. We find $u_h \in V_h$ such that

$$(2) \quad \left(\frac{\partial^2 u_h}{\partial t^2}, v \right) + a(u_h, v) = (f, v) \quad \forall v \in V_h,$$

where the bilinear form a is defined by

$$(3) \quad a(u, v) = \int_{\Omega} a \nabla u \cdot \nabla v \quad \forall u, v \in V_h$$

and (\cdot, \cdot) represents the standard L^2 inner product defined on Ω .

The numerical results are presented for several representative examples. We investigate the GMsFEM's accuracy and, in particular, how choosing modes from different snapshot spaces can affect the accuracy. Our numerical results show that choosing the basis functions from interior modes can improve the accuracy of GMsFEM substantially for wave equations. These results differ from those we observe for flow equations [13].

The paper is organized as follows. In section 2, we will present the new multiscale method. Numerical results are shown in section 3. Stability and spectral convergence of the semidiscrete scheme are proved in section 4. In section 5, the convergence of the fully discrete scheme is also proved. Finally, conclusions are presented.

2. The generalized multiscale finite element method. In this section, we will give a detailed description of our new GMsFEM. The method gives a numerical solver on a coarse grid, providing an efficient way to simulate waves in complicated media. As we will discuss next, the local basis functions are obtained via the solutions of some local spectral problems which are used to obtain the most dominant modes. These modes form the basis functions of our MsFEM.

We introduce a coarse mesh that consists of union of connected fine-mesh grid blocks which is denoted by \mathcal{T}^H and the set of all edges by \mathcal{E}^H . We denote the size of the coarse mesh by H . Even though it is convenient to choose rectangular coarse grid blocks, the shapes of the coarse grid blocks can be quite general and our analysis can be applied without the assumption of rectangular coarse grid blocks.

For each coarse grid block K , we define $\partial\mathcal{T}^h(K)$ as the space of continuous piecewise linear functions defined on ∂K with respect to the fine mesh. We remark that,

for a coarse grid edge $e \in \mathcal{E}^H$ that is shared by two coarse grid blocks K_1 and K_2 , the values of the two functions in $\partial\mathcal{T}^h(K_1)$ and $\partial\mathcal{T}^h(K_2)$ on e are in general different. The union of all $\partial\mathcal{T}^h(K)$ is denoted by $\partial\mathcal{T}^h$. Moreover, we define $H^1(\mathcal{T}^H)$ as the space of functions whose restrictions on K belongs to $H^1(K)$.

2.1. Global IPDG solver. We will apply the standard symmetric IPDG approach to solve (1) on the coarse grid \mathcal{T}^H . The method follows the standard framework as discussed in [17, 29], but the finite element space will be replaced by the space spanned by our multiscale basis functions. We emphasize that the use of the IPDG approach is an example of the global coupling of our local multiscale basis functions, and other choices of coarse grid methods are equally good. The key to our proposed method's success of is the construction of our local multiscale basis functions.

First, we introduce some notation. For each interior coarse edge $e \in \mathcal{E}^H$, we let K^- and K^+ be the two coarse grid blocks having the common coarse edge e . Then we define the average and the jump operators, respectively, by

$$\begin{aligned}\{v\}_e &= \frac{v^+ + v^-}{2}, \\ [u]_e &= u^+ - u^-, \end{aligned}$$

where $u^\pm = u|_{K^\pm}$ and we have assumed that the normal vector on e is pointing from K^+ to K^- . For each coarse edge e that lies on the boundary of Ω , we define

$$\{v\}_e = v, \quad [u]_e = u$$

assuming the unit normal vector on e is pointing outside the domain. Let V_H be a finite dimensional function space which consists of functions that are smooth on each coarse grid block but are in general discontinuous across coarse grid edges. We can then state the IPDG method as follows: find $u_H(t, \cdot) \in V_H$ such that

$$(4) \quad \left(\frac{\partial^2 u_H}{\partial t^2}, v \right) + a_{DG}(u_H, v) = l(v) \quad \forall v \in V_H,$$

where the bilinear form $a_{DG}(u, v)$ and the linear functional $l(v)$ are defined by

$$\begin{aligned} a_{DG}(u, v) &= \sum_{K \in \mathcal{T}^H} \int_K a \nabla u \cdot \nabla v \\ &\quad + \sum_{e \in \mathcal{E}^H} \left(- \int_e \{a \nabla u \cdot n\}_e [v]_e - \int_e \{a \nabla v \cdot n\}_e [u]_e + \frac{\gamma}{h} \int_e a [u]_e [v]_e \right) \\ l(v) &= (f, v), \end{aligned}$$

where $\gamma > 0$ is a penalty parameter and n denotes the unit normal vector on e . The initial conditions for the problem (4) are defined by $u_H(0) = P_H(g_0)$ and $(u_H)_t(0) = P_H(g_1)$, where P_H is the L^2 -projection operator into V_H .

Let $T > 0$ be a fixed time and $\Delta t = T/N$ be the time step size. The time discretization is done in the standard way; we find $u_H^{n+1} \in V_H$ such that

$$(5) \quad (u_H^{n+1}, v) = 2(u_H^n, v) - (u_H^{n-1}, v) - \Delta t^2 \left(a_{DG}(u_H^n, v) - l(v) \right) \quad \forall v \in V_H$$

in each time step. Throughout the paper, the notation u^n represents the value of the function u at time t_n . The initial conditions are obtained as follows:

$$u_H^0 = P_H(g_0),$$

$$u_H^1 = u_H^0 + \Delta t P_H(g_1) + \frac{\Delta t^2}{2} \tilde{v},$$

where $\tilde{v} \in V_H$ is defined by

$$(\tilde{v}, v) = (f(0), v) - a_{DG}(g_0, v) \quad \forall v \in V_H.$$

2.2. Multiscale basis functions. In this section, we will give the definition of the space V_H . We will construct V_H as a sum of two spaces V_H^1 and V_H^2 , namely,

$$V_H = V_H^1 + V_H^2.$$

The above spaces are constructed elementwise on the coarse grid. Given a coarse-grid block K , the restrictions of V_H, V_H^1 , and V_H^2 on K are denoted by $V_H(K), V_H^1(K)$, and $V_H^2(K)$, respectively, with $V_H(K) = V_H^1(K) + V_H^2(K)$. In the following paragraphs, we will give the definitions of $V_H^1(K)$ and $V_H^2(K)$. Conceptually, the space $V_H^1(K)$ contains functions that are obtained from minimum energy extensions of all possible boundary functions on ∂K . We use this space for the degrees of freedom on coarse-grid block boundary. The space $V_H^2(K)$ contains all possible functions in the fine-grid level that are zero on coarse-grid block boundary. We use this space for the degrees of freedom interior to coarse-grid blocks. Moreover, the choices of the above spaces give the required spectral convergence.

On the other hand, the restriction of the conforming space V_h on K is denoted by $V_h(K)$. We also denote the space of functions in $V_h(K)$ with zero boundary values on ∂K by $V_h^0(K)$.

Definition of $V_H^1(K)$. To define $V_H^1(K)$, we first define a space $\tilde{V}_H^1(K)$. For each fine grid node x_i on the boundary of K , we find $\tilde{w}_{i,K} \in V_h(K)$ by solving

$$(6) \quad \int_K a \nabla \tilde{w}_{i,K} \cdot \nabla v = 0 \quad \forall v \in V_h^0(K)$$

with boundary condition $\tilde{w}_{i,K} = 1$ at x_i and $\tilde{w}_{i,K} = 0$ at the other grid points on the boundary of K . The functions $\tilde{w}_{i,K}$ defined above are the a -harmonic extensions of the unit basis functions of $\partial \mathcal{T}^h(K)$. We let n be the number of these a -harmonic extensions and define

$$\tilde{V}_H^1(K) = \text{span}\{\tilde{w}_{1,K}, \dots, \tilde{w}_{n,K}\}.$$

We remark that n is the number of boundary grid points on ∂K , and its value changes with K . In our numerical simulations, we do not need to use all of these basis functions and use a local spectral problem in the space of snapshots to identify multiscale basis functions by choosing dominant modes. We use E to denote the sum of the reciprocal of the eigenvalues of the local spectral problem. We will choose the eigenfunctions corresponding to small eigenvalues so that the sum of the reciprocals of these small eigenvalues is a large percentage of E . The use of eigenfunctions corresponding to small eigenvalues suggests that we use the coarse component in $\tilde{V}_H^1(K)$ as the approximation space.

Local spectral problem on $\tilde{V}_H^1(K)$. The spectral problem we propose is

$$(7) \quad \int_K a \nabla w_\mu \cdot \nabla v = \frac{\mu}{H} \int_{\partial K} w_\mu v \quad \forall v \in \tilde{V}_H^1(K).$$

The scalar μ is the eigenvalue and w_μ is the corresponding eigenfunction in the space $\tilde{V}_H^1(K)$. Since the space $\tilde{V}_H^1(K)$ has dimension n , a set of n eigenvalues μ_i with corresponding n eigenfunctions $w_{i,K}$ can be computed, where $i = 1, 2, \dots, n$. We assume that the eigenvalues are ordered so that $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \dots \leq \mu_n$, and that the corresponding eigenfunctions are normalized with respect to the L^2 -norm on ∂K . We denote the total energy on the coarse grid block K by E_K which is defined by $E_K = \sum_{i=2}^n \mu_i^{-1}$. We can then choose the first p eigenvalues so that the sum $\sum_{i=2}^p \mu_i^{-1}$ is a portion of the total energy E_K . Note that we can take different p for different coarse grid blocks K . Finally, we define

$$V_H^1(K) = \text{span}\{w_{1,K}, \dots, w_{p,K}\}.$$

Clearly, $V_H^1(K) \subset \tilde{V}_H^1(K)$ and $V_H^1(K)$ has a smaller dimension $p < n$. We remark that the above eigenvalue problem (7), which is the eigenvalue problem for the Dirichlet-to-Neumann map, has also been used for a coarse space construction for some domain decomposition methods in [27]. In particular, eigenfunctions corresponding to small eigenvalues are taken as basis functions for the coarse space.

Given a function $w \in V_H^1(K)$, we can define a discrete normal flux $a \frac{\partial w}{\partial n} \in \partial \mathcal{T}^h(K)$ on ∂K by

$$(8) \quad \int_{\partial K} a \frac{\partial w}{\partial n} v = \int_K a \nabla w \cdot \nabla \hat{v}, \quad v \in \partial \mathcal{T}^h(K),$$

where \hat{v} is any a -harmonic extension of v in K . This is well-defined since w is also obtained by an a -harmonic extension.

Definition of $V_H^2(K)$ and spectral problem on $V_h^0(K)$. To define the space $V_H^2(K)$, we will use another spectral problem to identify the important modes. The proposed eigenvalue problem has the following form: find $z_\lambda \in V_h^0(K)$ and a scalar λ such that

$$(9) \quad \int_K a \nabla z_\lambda \cdot \nabla v = \frac{\lambda}{H^2} \int_K z_\lambda v \quad \forall v \in V_h^0(K).$$

A set of eigenvalues $\{\lambda_{i,K}\}$ and their corresponding eigenfunctions $\{z_{i,K}\}$ can be computed. Assume that the eigenvalues are ordered so that $\lambda_{1,K} \leq \lambda_{2,K} \leq \dots$ and the corresponding eigenfunctions are normalized with respect to the L^2 -norm on K . In practice, for each coarse grid block, we can take the first m eigenfunctions, and the space $V_H^2(K)$ is spanned by these functions, that is,

$$V_H^2(K) = \text{span}\{z_{1,K}, \dots, z_{m,K}\}.$$

In principle, one can choose different numbers of eigenfunctions for the space $V_H^2(K)$ for different coarse grid blocks. Nevertheless, our numerical results show that only the first few eigenfunctions are enough to obtain a reasonable accuracy.

Quasi orthogonality of $V_H^1(K)$ and $V_H^2(K)$. Finally, we point out the following quasi-orthogonality condition which will be used in our analysis. For any $v \in V_H^1(K)$ and $u \in V_H^2(K)$, we conclude by (6) that

$$(10) \quad \int_K a \nabla v \cdot \nabla u = 0.$$

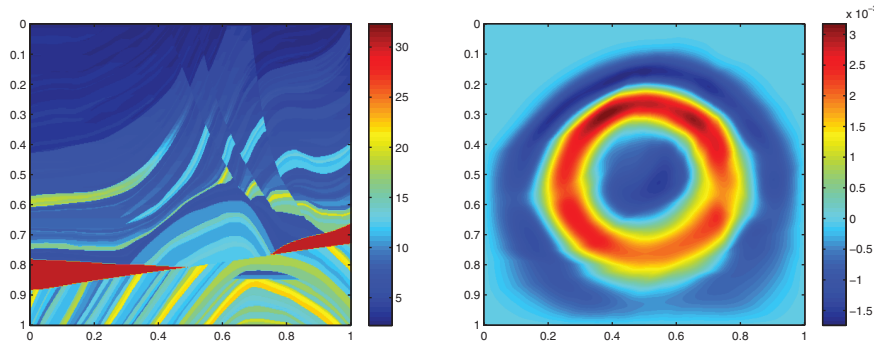


FIG. 1. Left: a subset of the Marmousi model. Right: fine grid solution.

3. Numerical results. In this section, we will present some numerical examples to show the performance of our multiscale method. The media that we will consider is a heterogeneous field which is a modified Marmousi model. (See the left plot of Figure 1.) We have also considered more regular periodic highly heterogeneous fields and observed similar results. We will compare both the accuracy and efficiency of our method with the direct fine scale simulation defined in (2). To compare the accuracy, we will use the error quantities

$$e_2 = \frac{\|u_H - u_h\|_{L^2(\Omega)}}{\|u_h\|_{L^2(\Omega)}}, \quad \bar{e}_2 = \frac{\sqrt{\sum_{K \in \mathcal{T}^H} |\int_K u_H - \int_K u_h|^2}}{\sqrt{\sum_{K \in \mathcal{T}^H} |\int_K u_h|^2}},$$

$$e_{H^1} = \frac{\|\nabla(u_H - u_h)\|_{L^2(\Omega)}}{\|\nabla u_h\|_{L^2(\Omega)}},$$

which are the relative L^2 norm error, the relative L^2 norm error for coarse grid averages, and the relative L^2 norm error of the gradient. We will also consider the jump error on coarse grid edges defined by

$$e_{Jump} = \sum_{e \in \mathcal{E}^H} \int_e [u_H]_e^2.$$

Moreover, we let t_{off} be the time needed for offline computations and t_{on} be the online computational time. These quantities are used to compare the efficiency of our method with direct fine scale simulation. To perform a fair comparison, we will use the same time step size for both of our GMsFEM and the fine scale method, since we only consider spatial upscaling in this paper. However, we note that multiscale basis functions can be used for different source terms and boundary conditions which will provide a substantial computational saving. Furthermore, we will take $\gamma = 2$ and $\Omega = [0, 1]^2$ for all of our examples. The initial conditions g_0 and g_1 are zero. Throughout the paper, all computational times are measured in seconds.

The Ricker wavelet with frequency $f_0 = 20$

$$f(x, y, t) = (10)^2 e^{-10^2((x-0.5)^2 + (y-0.5)^2)} (1 - 2\pi^2 f_0^2 (t - 2/f_0)^2) e^{-\pi^2 f_0^2 (t - 2/f_0)^2}$$

is used as the source term. We will compute the solution at time $T = 0.4$. The coarse mesh size is taken as $H = 1/16$. Each coarse grid block is divided into a 32×32 grid,

TABLE 1
Errors for various choices of energy for the space V_H^1 .

Energy	Number of basis	e_2	$\overline{e_2}$	e_{H^1}	e_{Jump}	μ_{\min}
75%	24-29	0.0423	0.0312	0.1542	4.7304e-04	1.9414
80%	33-40	0.0392	0.0274	0.1486	3.0671e-04	2.9992

TABLE 2
Offline and online computational times.

Energy	t_{off}	t_{on}
75%	326.83	18.21
80%	1019.06	32.43

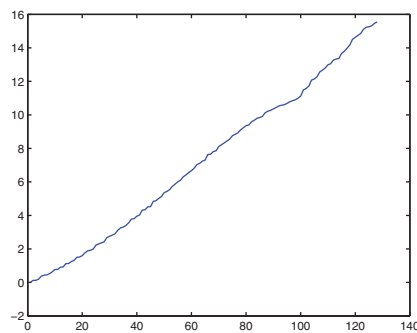


FIG. 2. Eigenvalues for the space V_H^1 .

that is, $n = 32$. Thus, the fine mesh size $h = 1/512$, and there are totally 128 and 961 local basis functions in the space $V_H^1(K)$ and $V_H^2(K)$, respectively, on each coarse grid block. The time step size for both GMsFEM and the fine grid solver is taken as $\Delta t = h/80$ in order to meet the stability requirement and the computation time for fine grid solution is 55.06. We will compare the accuracy and efficiency of our method using the solution computed at the time $T = 0.2$, which is shown in the right figure of Figure 1.

In Tables 1 and 2, we present the errors and computational times for the case with $m = 1$, that is, we only use the first eigenfunction in the space V_H^2 . We see that if we use 80% of the total energy, the number of basis functions is between 33 and 40 on each coarse grid while the computational time for the offline procedure is 1019.06 and the time for online computations is 32.43. Note that the online computational time is about 59% of that of the online computational time of the direct fine grid simulation. The relative L^2 error and the relative error for cell averages are only 3.92% and 2.74%, respectively. In addition, the relative error for the gradient is 14.86% and the jump error is 0.003. When 75% of the total energy is used, the number of basis functions is reduced to a number between 24 and 29, while the computational time for the offline procedures is 326.83 and the time for online computations is 18.21. The time for the online computation is 33% of the time required for direct fine grid simulation. The relative L^2 error and the relative error for cell averages are increased slightly to 4.23% and 3.12%, respectively. In Table 1, we also present the values of μ_{\min} for the space V_H^1 . Moreover, the eigenvalues are shown in Figure 2. The numerical solutions for these cases are shown in Figure 3. We note that the error decay is not fast mostly due to the error contribution because of the modes corresponding to the interior. Even

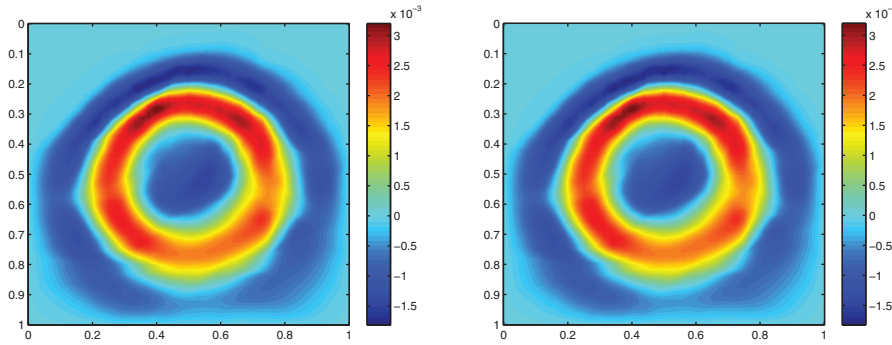


FIG. 3. Left: 75% energy. Right: 80% energy.

TABLE 3

Errors for various number of eigenfunctions in V_H^2 for using 75% energy in V_H^1 .

m	e_2	\bar{e}_2	e_{H^1}	ϵ_{Jump}	λ_{\min}
1	0.0423	0.0312	0.1542	4.7304e-04	3.4805e+04
2	0.0352	0.0259	0.1346	4.7030e-04	3.4873e+04
3	0.0227	0.0187	0.0945	4.5931e-04	5.5906e+04
5	0.0193	0.0163	0.0833	4.5910e-04	6.9650e+04

TABLE 4

Offline and online computational times for various number of eigenfunctions in V_H^2 for using 75% energy in V_H^1 .

m	t_{off}	t_{on}
1	326.83	18.21
2	368.89	18.64
3	405.73	19.88
5	528.47	25.96

though the error between the GMsFEM solution and the solution computed using the entire snapshot space V_H^1 is very small, the overall error between the GMsFEM solution and the fine-scale solution may not be small because we have only used one basis function in V_H^2 . Next, we will add more basis functions from V_H^2 and compare the errors.

Next, we will investigate the use of more eigenfunctions in the space V_H^2 that will allow reducing the overall error. To do so, we consider the first case where 75% energy in the space V_H^1 is used and we consider using various number of eigenfunctions in V_H^2 . The errors and computational times are shown in Tables 3 and 4. In general, we obtain better numerical approximations as more eigenfunctions are used. When two eigenfunctions are used (this corresponds to using less than 3% of the total local degrees of freedom in constructing all GMsFEM basis functions), the relative error is 3.52% and the online computational time is 18.64. When five eigenfunctions are used, the relative error is 1.93% and the online computational time is 18.21. Thus, we see that adding a few eigenfunctions in the space V_H^2 will improve the multiscale solution. This indicates that for the multiscale wave simulations, the modes that represent the interior nodes can improve the accuracy of the method and play an important role in obtaining an accurate solution. We also report the largest eigenvalue used in Table 3. Furthermore, for the use of the central frequency $f_0 = 20$, 75% energy in the space V_H^1

TABLE 5
Simulation results with one basis function in V_H^2 .

Energy	Number of basis	e_2	$\overline{e_2}$	e_{H^1}	e_{Jump}	μ_{\min}
73%	24-30	0.0673	0.0583	0.1866	5.2038e-04	1.6755
79%	33-40	0.0640	0.0548	0.1827	3.4797e-04	2.5681
84%	45-55	0.0626	0.0534	0.1809	2.6388e-04	3.7918

TABLE 6
Errors and computational times for various number of eigenfunctions in V_H^2 for using $E = 73\%$.

m	e_2	$\overline{e_2}$	e_{H^1}	e_{Jump}	λ_{\min}
1	0.0673	0.0583	0.1866	5.2038e-04	3.4805e+04
2	0.0596	0.0524	0.1666	5.1865e-04	3.4873e+04
3	0.0488	0.0449	0.1332	5.0929e-04	5.5906e+04
5	0.0449	0.0419	0.1220	5.0793e-04	6.9650e+04

and $m = 5$ in the space V_H^2 , there are about 35 basis functions per coarse-grid block and the number of points per wavelength is approximately 30, where the number of points per wavelength is computed as $2\pi\sqrt{N_{\text{total}}}/f_0$ and N_{total} is the total number of basis functions. On the other hand, since the fine grid solution is computed on a 512×512 fine grid, the number of points per wavelength is about 160. Therefore, the fine grid solution has sufficient resolution.

We would like to remark that the computational gain will be higher when implicit methods are used or we employ finer grids to resolve the problem. In the latter case, the CPU time for coarse-grid simulations will not change.

3.1. The use of oversampling. In this section, we present the performance of the method when the basis functions in the space V_H^1 are obtained by oversampling. We consider the previous example. The oversampling technique is used and the harmonic extension problems are solved on enlarged coarse grids, which are obtained by extending the original coarse grids by $H/16$ on each side. The results for using one basis functions in V_H^2 and various number of basis functions in V_H^1 are shown in Table 5. Moreover, we compute the errors using 73% energy for V_H^1 and various number of basis functions in V_H^2 . The results are presented in Table 6. We observe that there is no improvement in this case. This is due to the error from the modes representing internal nodes.

3.2. Dispersion error. In this subsection, we will consider $a = 1$, $f = 0$, numerically compute the dispersion error of our GMsFEM, and compare the results with the use of polynomial basis on finer grids. We will use the eigenvalue method to numerically compute the dispersion error; see [6, 32] for details. Notice that the plane waves $e^{i(\omega t - k \cdot x)}$ are solutions of the wave equation (1) when $\omega^2 = |k|^2$. We will compute the numerical wave number ω_H using the eigenvalue method and compute the error $e_\omega = |\omega_H - \omega|/|\omega|$. Furthermore, we will compute the dispersion error when our spectral basis functions are replaced by bilinear functions. The purpose of this is to compare the dispersion error of our GMsFEM and that of DG method with polynomial basis.

In Tables 7 and 8, we present our results for the choice of $k = (100, 100)$. In Table 7, we see clearly that we obtain the expected second order convergence for bilinear DG method when the mesh size H is reduced. In addition, when $H = 1/512$, which is also equal to our fine mesh size, the relative dispersion error e_ω is about 0.3%.

TABLE 7

Dispersion error for piecewise bilinear DG method for various choices of coarse mesh size.

H	Number of unknowns	e_ω
1/64	16384	1.9178e-1
1/128	65536	5.0112e-2
1/256	262144	1.2624e-2
1/512	1048576	3.1616e-3

TABLE 8

Dispersion error for GMsFEM with $H = 1/16$ for various choices of local dimensions.

Dimension of $V_H^1(K)$	Dimension of $V_H^2(K)$	Number of unknowns	e_ω
1	4	1280	2.2738e-3
2	4	1536	2.3905e-3
3	4	1792	2.3684e-3
4	4	2048	6.3731e-4

On the other hand, the results for GMsFEM are shown in Table 8 for coarse mesh size $H = 1/16$. We see that, in all cases shown, the dispersion errors of GMsFEM are smaller than that of the piecewise bilinear DG method. In addition, we report the number of unknowns in the global system (that is, the number coarse grid blocks multiplied by the number of local basis functions) for both the bilinear DG method and the GMsFEM. We see that the number of unknowns required for GMsFEM is much smaller than that of piecewise bilinear DG method.

4. Stability and convergence. In this section, we will prove the stability and convergence of the GMsFEM constructed in section 2. We will first state and prove some preliminary results and then prove the main convergence theorem for the semidiscrete scheme (4).

4.1. Preliminaries. Before we analyze the convergence of our GMsFEM, we first prove some basic results. To do so, we introduce some notation and state the assumptions required in our analysis. For functions $u, v \in H^1(\mathcal{T}^H)$, we define the bilinear form $a(\cdot, \cdot)$ by

$$a(u, v) = \sum_{K \in \mathcal{T}^H} \int_K a \nabla u \cdot \nabla v.$$

Moreover, for any function $u \in H^1(\mathcal{T}^H)$, we define the a -norm by

$$\|u\|_a = \left(a(u, u) + \frac{\gamma}{h} \sum_{e \in \mathcal{E}^H} \|a^{\frac{1}{2}}[u]_e\|_{L^2(e)}^2 \right)^{\frac{1}{2}}$$

and the a -seminorm by

$$|u|_a = a(u, u)^{\frac{1}{2}}.$$

Furthermore, the broken H^1 -norm for $u \in H^1(\mathcal{T}^H)$ is defined as

$$\|u\|_{H^1(\mathcal{T}^H)} = \left(\sum_{K \in \mathcal{T}^H} |u|_{H^1(K)}^2 + \frac{\gamma}{h} \sum_{e \in \mathcal{E}^H} \|[u]_e\|_{L^2(e)}^2 \right)^{\frac{1}{2}}.$$

Assumption 1. The function $a(x)$ is bounded, that is, there exist positive numbers a_0 and a_1 such that

$$a_0 \leq a(x) \leq a_1 \quad \forall x \in \Omega.$$

This assumption implies that the norms $\|\cdot\|_a$ and $\|\cdot\|_{H^1(\mathcal{T}^H)}$ are equivalent.

In the following, we will describe the consistency of the method (4). We define the consistency error by

$$(11) \quad R_{u_h}(v) = l(v) - \left(\frac{\partial^2 u_h}{\partial t^2}, v \right) - a_{DG}(u_h, v) \quad \forall v \in V_H,$$

where u_h is the fine grid finite element solution defined in (2). Clearly, we have

$$(12) \quad R_{u_h}(v) = 0 \quad \forall v \in V_H^2$$

since $V_H^2 \subset V_h$. Thus, we only need to estimate $R_{u_h}(v)$ for $v \in V_H^1$. The following lemma states that the method (4) is consistent with the fine grid solution defined by (2). The proof will be presented in the appendix.

LEMMA 1. *Let u_h and u be the finite element solution defined in (2) and the exact solution of the wave propagation problem (1), respectively. Assume that $u, u_t \in L^\infty(0, T; H^2(\Omega))$ and that $u \in W^{3,1}(0, T; H^1(\Omega))$. We have*

$$(13) \quad |R_{u_h}(v)| \leq C(u)h\|v\|_a, \quad v \in V_H^1,$$

where $C(u)$ is a constant which depends on the solution u but independent of the fine mesh size h . This inequality gives the consistency of our method.

Next we will prove that the bilinear form a_{DG} satisfies the following coercivity and continuity conditions for suitably chosen penalty parameter $\gamma > 0$.

LEMMA 2. *Let $\gamma \geq \Lambda a_1^2 a_0^{-2}$, where Λ is a uniform constant defined in (17). Then we have*

$$(14) \quad \frac{1}{2} \|v\|_a^2 \leq a_{DG}(v, v) \quad \forall v \in V_H$$

and

$$(15) \quad a_{DG}(u, v) \leq 2\|u\|_a\|v\|_a \quad \forall u, v \in V_H.$$

Proof. By the definition of a_{DG} , we have

$$a_{DG}(v, v) = \sum_{K \in \mathcal{T}^H} \int_K a \nabla v \cdot \nabla v + \sum_{e \in \mathcal{E}^H} \left(-2 \int_e \{a \nabla v \cdot n\}_e [v]_e + \frac{\gamma}{h} \int_e a [v]_e^2 \right),$$

and by the definition of the a -norm, we have

$$(16) \quad a_{DG}(v, v) = \|v\|_a^2 - 2 \sum_{e \in \mathcal{E}^H} \left(\int_e \{a \nabla v \cdot n\}_e [v]_e \right).$$

Using the Cauchy–Schwarz inequality, we obtain

$$2 \sum_{e \in \mathcal{E}^H} \left(\int_e \{a \nabla v \cdot n\}_e [v]_e \right) \leq \frac{2h}{\gamma a_0} \sum_{e \in \mathcal{E}^H} \int_e \{a \nabla v \cdot n\}_e^2 + \frac{a_0}{2} \sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e [v]_e^2,$$

which implies

$$2 \sum_{e \in \mathcal{E}^H} \left(\int_e \{a \nabla v \cdot n\}_e [v]_e \right) \leq \frac{h}{\gamma a_0} \sum_{K \in \mathcal{T}^H} \int_{\partial K} (a \nabla v \cdot n)^2 + \frac{1}{2} \sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e a [v]_e^2.$$

Since v is a piecewise linear function, there is a uniform constant $\Lambda > 0$ such that

$$(17) \quad 2h \sum_{K \in \mathcal{T}^H} \int_{\partial K} (a \nabla v \cdot n)^2 \leq \Lambda a_1^2 a_0^{-1} |v|_a^2.$$

So we have

$$2 \sum_{e \in \mathcal{E}^H} \left(\int_e \{a \nabla v \cdot n\}_e [v]_e \right) \leq \frac{\Lambda a_1^2}{2\gamma a_0^2} |v|_a^2 + \frac{1}{2} \sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e a [v]_e^2.$$

Therefore, from (16), we obtain

$$\frac{1}{2} \|v\|_a^2 \leq a_{DG}(v, v) \quad \forall v \in V_H$$

if we take $\gamma \geq \Lambda a_1^2 a_0^{-2}$. Thus, we have proved (14).

We can prove (15) in a similar way. By the definition of a_{DG} , we have

$$\begin{aligned} |a_{DG}(u, v)| &= \left| \sum_{K \in \mathcal{T}^H} \int_K a \nabla u \cdot \nabla v + \sum_{e \in \mathcal{E}^H} \left(- \int_e \{a \nabla u \cdot n\}_e \cdot [v]_e \right. \right. \\ &\quad \left. \left. - \int_e \{a \nabla v \cdot n\}_e \cdot [u]_e + \frac{\gamma}{h} \int_e a [u]_e \cdot [v]_e \right) \right| \\ &\leq I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \sum_{K \in \mathcal{T}^H} \left| \int_K a \nabla u \cdot \nabla v \right|, & I_2 &= \sum_{e \in \mathcal{E}^H} \left| \int_e \{a \nabla u \cdot n\}_e \cdot [v]_e \right|, \\ I_3 &= \sum_{e \in \mathcal{E}^H} \left| \int_e \{a \nabla v \cdot n\}_e \cdot [u]_e \right|, & I_4 &= \sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e |a [u]_e \cdot [v]_e|. \end{aligned}$$

First, we note that I_1 and I_4 can be estimated easily as follows:

$$\begin{aligned} I_1 &\leq \left(\sum_{K \in \mathcal{T}^H} \int_K a |\nabla u|^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}^H} \int_K a |\nabla v|^2 \right)^{\frac{1}{2}}, \\ I_4 &\leq \left(\sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e a [u]_e^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e a [v]_e^2 \right)^{\frac{1}{2}}. \end{aligned}$$

For I_2 , we can estimate as follows:

$$\begin{aligned} I_2 &\leq \left(\frac{h}{\gamma} \sum_{e \in \mathcal{E}^H} \int_e \{a \nabla u \cdot n\}_e^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e [v]_e^2 \right)^{\frac{1}{2}} \\ &\leq \left(\frac{h}{\gamma a_0} \sum_{K \in \mathcal{T}^H} \int_{\partial K} (a \nabla u \cdot n)^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e a [v]_e^2 \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\Lambda a_1^2}{\gamma a_0^2} \right)^{\frac{1}{2}} |u|_a \left(\sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e [v]_e^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The same idea can be applied to estimate I_3 to obtain

$$I_3 \leq \left(\frac{\Lambda a_1^2}{\gamma a_0^2} \right)^{\frac{1}{2}} |v|_a \left(\sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e a [u]_e^2 \right)^{\frac{1}{2}}.$$

Finally, combining the above estimates, we have

$$|a_{DG}(u, v)| \leq 2 \|u\|_a \|v\|_a$$

when $\gamma \geq \Lambda a_1^2 a_0^{-2}$. \square

Next, we will prove the convergence of the semidiscrete scheme (4). First, we define the following error quantities. Let

$$(18) \quad \eta = u_h - w_H, \quad \xi = u_H - w_H, \quad \text{and} \quad \varepsilon = u_h - u_H,$$

where $w_H \in V_H$ is defined by solving the following elliptic projection problem:

$$(19) \quad a_{DG}(w_H, v) = a_{DG}(u_h, v) + R_{u_h}(v) \quad \forall v \in V_H.$$

Notice that ε is the difference between the multiscale solution u_H and the fine grid finite element solution u_h . Moreover, η measures the difference between the fine grid solution u_h as its projection w_H . In the following, we will prove estimates for ε . First, we let

$$\|\varepsilon\|_{L^\infty([0, T]; L^2(\Omega))} = \max_{0 \leq t \leq T} \|\varepsilon\|_{L^2(\Omega)} \quad \text{and} \quad \|\varepsilon\|_{L^\infty([0, T]; a)} = \max_{0 \leq t \leq T} \|\varepsilon\|_a.$$

Then we will prove the following two inequalities, which estimate the error for the solution ε by the error for the projection η and the initial errors \mathcal{I}_1 and \mathcal{I}_2 , which are defined in the statements of the theorems.

THEOREM 1. *Let ε, η , and ξ be the error quantities defined in (18). Then we have the following error bound:*

$$(20) \quad \begin{aligned} &\|\varepsilon_t\|_{L^\infty([0, T]; L^2(\Omega))} + \|\varepsilon\|_{L^\infty([0, T]; a)} \\ &\leq C \left(\|\eta_t\|_{L^\infty([0, T]; L^2(\Omega))} + \|\eta\|_{L^\infty([0, T]; H^1(\mathcal{T}^H))} + \|\eta_{tt}\|_{L^1([0, T]; L^2(\Omega))} + \mathcal{I}_1 \right), \end{aligned}$$

where $\mathcal{I}_1 = \|\xi_t(0)\|_{L^2(\Omega)} + \|\xi(0)\|_{H^1(\mathcal{T}^H)}$.

Proof. First, using (4) and the definition of ξ , we have

$$(\xi_{tt}, v) + a_{DG}(\xi, v) = (f, v) - ((w_H)_{tt}, v) - a_{DG}(w_H, v).$$

Then by (13), we have

$$(21) \quad (\xi_{tt}, v) + a_{DG}(\xi, v) = (\eta_{tt}, v).$$

Taking $v = \xi_t$ in (21), we have

$$(\xi_{tt}, \xi_t) + a_{DG}(\xi, \xi_t) = (\eta_{tt}, \xi_t),$$

which implies

$$\frac{1}{2} \frac{d}{dt} \left(\|\xi_t\|_{L^2(\Omega)}^2 + a_{DG}(\xi, \xi) \right) \leq \|\eta_{tt}\|_{L^2(\Omega)} \|\xi_t\|_{L^2(\Omega)}.$$

Integrating from $t = 0$ to $t = \tau$, we have

$$\begin{aligned} \|\xi_t(\tau)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\xi(\tau)\|_a^2 &\leq \|\xi_t(0)\|_{L^2(\Omega)}^2 + 2\|\xi(0)\|_a^2 + 2 \int_0^\tau \|\eta_{tt}\|_{L^2(\Omega)} \|\xi_t\|_{L^2(\Omega)} \\ &\leq \|\xi_t(0)\|_{L^2(\Omega)}^2 + 2\|\xi(0)\|_a^2 + 2 \max_{0 \leq t \leq T} \|\xi_t\|_{L^2(\Omega)} \int_0^T \|\eta_{tt}\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\|\xi_t\|_{L^\infty([0,T];L^2(\Omega))}^2 + \|\xi\|_{L^\infty([0,T];a)}^2 \\ &\leq C \left(\|\xi_t(0)\|_{L^2(\Omega)}^2 + \|\xi(0)\|_a^2 + \left(\int_0^T \|\eta_{tt}\|_{L^2(\Omega)} dt \right)^2 \right). \end{aligned}$$

Finally, (20) is proved by noting that $\varepsilon = \eta - \xi$. \square

THEOREM 2. *Let ε, η , and ξ be the error quantities defined in (18). Then we have the following error bound:*

$$(22) \quad \|\varepsilon\|_{L^\infty([0,T];L^2(\Omega))} \leq C \left(\|\eta_t\|_{L^1([0,T],L^2(\Omega))} + \|\eta\|_{L^\infty([0,T];L^2(\Omega))} + \mathcal{I}_2 \right),$$

where $\mathcal{I}_2 = \|\xi(0)\|_{L^2(\Omega)}$.

Proof. Integrating by parts with respect to time in (21), we have

$$-(\xi_t, v_t) + \partial_t(\xi_t, v) + a_{DG}(\xi, v) = \partial_t(\eta_t, v) - (\eta_t, v_t).$$

Taking $v(x, t) = \int_t^\gamma \xi(x, \tau) d\tau$, we have $v_t = -\xi$ and $v(\gamma) = 0$. So,

$$(\xi_t, \xi) - \partial_t(\xi_t, v) - a_{DG}(v_t, v) = \partial_t(\eta_t, v) + (\eta_t, \xi),$$

which implies that

$$\frac{1}{2} \frac{d}{dt} \|\xi\|_{L^2(\Omega)}^2 - \partial_t(\xi_t, v) - \frac{1}{2} \frac{d}{dt} a_{DG}(v, v) = \partial_t(\eta_t, v) + (\eta_t, \xi).$$

Integrating from $t = 0$ to $t = \gamma$, we have

$$\begin{aligned} &\frac{1}{2} \|\xi(\gamma)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\xi(0)\|_{L^2(\Omega)}^2 \\ &\quad + (\xi_t(0), v(0)) + \frac{1}{2} a_{DG}(v(0), v(0)) = (\eta_t(0), v(0)) + \int_0^\gamma (\eta_t, \xi). \end{aligned}$$

Since $\xi_t - \eta_t = (u_H - u_h)_t$, we obtain

$$(\xi_t(0) - \eta_t(0), v(0)) = ((u_H - u_h)_t(0), v(0)) = 0.$$

Using the coercivity of a_{DG} , we have

$$\begin{aligned} \|\xi(\gamma)\|_{L^2(\Omega)}^2 &\leq \|\xi(0)\|_{L^2(\Omega)}^2 + 2 \int_0^\gamma \|\eta_t\|_{L^2(\Omega)} \|\xi\|_{L^2(\Omega)} \\ &\leq \|\xi(0)\|_{L^2(\Omega)}^2 + 2 \max_{0 \leq t \leq T} \|\xi\|_{L^2(\Omega)} \int_0^T \|\eta_t\|_{L^2(\Omega)}. \end{aligned}$$

Hence (22) is proved by noting that $\varepsilon = \eta - \xi$. \square

From Theorems 1 and 2, we see that, in order to estimate the error $\varepsilon = u_h - u_H$, we will need to find a bound for η given that the initial values $\xi_t(0)$ and $\xi(0)$ are sufficiently accurate.

4.2. Convergence analysis. In this section, we will derive an error bound for $\eta = u_h - w_H$. Notice that, on each coarse grid block K , we can express u_h as

$$u_h = \sum_{i=1}^n c_{i,K} \tilde{w}_{i,K} + \sum_{i=1}^{n_0} d_{i,K} z_{i,K} = u_{1,K} + u_{2,K}$$

for some suitable coefficients $c_{i,K}$ and $d_{i,K}$ determined by a L^2 -type projection, where n_0 is the dimension of $V_h^0(K)$. We write $u_h = u_1 + u_2$ with $u_i|_K = u_{i,K}$ for $i = 1, 2$. Moreover, we recall that $C(u, f)$, defined in (41), is the constant appearing in the consistency error estimate in Lemma 1. In the following theorem, we will give an estimate for the difference between the fine grid solution u_h and the projection of u_h into the coarse space V_H defined in (19). The theorem says that such difference is bounded by a best approximation error $\|u_h - v\|_a$ and a consistency error $hC(u, f)$. We emphasize that, even though the coarse mesh size H is fixed, but the fine mesh size h can be arbitrary small, and hence the consistency error is small compared with the best approximation error $\|u_h - v\|_a$.

THEOREM 3. *Let $w_H \in V_H$ be the solution of (19) and u_h be the solution of (2). Then, for a fixed time $t \in [0, T]$, we have*

$$(23) \quad \|u_h - w_H\|_a \leq C(\|u_h - v\|_a + hC(u, f)) \quad \forall v \in V_H.$$

Proof. By the definition of w_H , we have

$$a_{DG}(w_H, v) = a_{DG}(u_h, v) + R_{u_h}(v) \quad \forall v \in V_H.$$

So, we have

$$a_{DG}(w_H - v, w_H - v) = a_{DG}(u_h - v, w_H - v) + R_{u_h}(w_H - v).$$

By (14), (15), and (13), we get

$$\begin{aligned} \|w_H - v\|_a^2 &\leq 2a_{DG}(w_H - v, w_H - v) \\ &= 2a_{DG}(u_h - v, w_H - v) + 2R_{u_h}(w_H - v) \\ &\leq C(\|u_h - v\|_a + hC(u, f))\|w_H - v\|_a. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \|u_h - w_H\|_a &\leq \|u_h - v\|_a + \|w_H - v\|_a \\ &\leq C(\|u_h - v\|_a + hC(u, f)). \quad \square \end{aligned}$$

From the above theorem, we see that the error $\|u_h - w_H\|_a$ is controlled by the quantity $\|u_h - v\|_a$ for an arbitrary choice of the function $v \in V_H$. Thus, to obtain our final error bound, we only need to find a suitable function $v \in V_H$ to approximate the finite element solution u_h . In the following theorem, we will choose a specific v in Theorem 3 and prove the corresponding error estimate.

THEOREM 4. *Let $u_h \in V_h$ be the finite element solution. Then, for a fixed time $t \in [0, T]$, we have*

$$(24) \quad \begin{aligned} &\|u_h - \phi\|_a^2 \\ &\leq \sum_{K \in \mathcal{T}^H} \left(\frac{H}{\mu_{p+1,K}} \left(1 + \frac{2a_1\gamma H}{h\mu_{p+1,K}} \right) \int_{\partial K} \left(a \frac{\partial u_1}{\partial n} \right)^2 + \frac{H^2}{\lambda_{m+1,K}} \|f - (u_h)_{tt}\|_{L^2(K)}^2 \right), \end{aligned}$$

where the function $\phi \in V_H$ is defined as

$$\phi|_K = \sum_{i=1}^p c_{i,K} \tilde{w}_{i,K} + \sum_{i=1}^m d_{i,K} z_{i,K} = \phi_{1,K} + \phi_{2,K}.$$

Proof. For a given coarse grid block K , using the orthogonality condition (10), we have

$$\int_K a|\nabla(u_h - \phi)|^2 = \int_K a|\nabla(u_1 - \phi_1)|^2 + \int_K a|\nabla(u_2 - \phi_2)|^2,$$

which implies

$$\|u_h - \phi\|_a^2 = \|u_1 - \phi_1\|_a^2 + \|u_2 - \phi_2\|_a^2,$$

where we write $\phi = \phi_1 + \phi_2$ and $\phi_i|_K = \phi_{i,K}$ for $i = 1, 2$. We will first estimate $\|u_1 - \phi_1\|_a^2$. By the definition of a -norm, we have

$$(25) \quad \begin{aligned} \|u_1 - \phi_1\|_a^2 &= \sum_{K \in \mathcal{T}^H} \left(\int_K a|\nabla(u_1 - \phi_1)|^2 + \sum_{e \in \mathcal{E}^H} \frac{\gamma}{h} \int_e a|[(u_1 - \phi_1)]_e|^2 \right) \\ &\leq \sum_{K \in \mathcal{T}^H} \left(\int_K a|\nabla(u_1 - \phi_1)|^2 + \frac{2\gamma}{h} \int_{\partial K} a|(u_1 - \phi_1)|^2 \right) \\ &\leq \sum_{K \in \mathcal{T}^H} \left(\int_K a|\nabla(u_1 - \phi_1)|^2 + \frac{2a_1\gamma}{h} \int_{\partial K} |(u_1 - \phi_1)|^2 \right). \end{aligned}$$

Next, we will estimate the right-hand side of (25) for each K .

We note that the eigenvalue problem (7) is motivated by the right-hand side of (25). In particular, based on the right-hand side of (25) we consider

$$(26) \quad \int_K a\nabla w_\mu \cdot \nabla v + \frac{1}{H} \int_{\partial K} w_\mu v = \hat{\mu} \int_K R(w_\mu) \cdot R(v) \quad \forall v \in V_H^1(K),$$

where the choice of R , e.g., $R = \sqrt{a}\nabla w_\mu$, depends on how we would like to bound the error. Indeed, choosing the eigenvectors that correspond to the largest L_K eigenvalues, one can guarantee that the best L_K dimensional space *in the space of snapshots* is given by the first L_K dominant eigenvectors. The choice of $R(\cdot)$ is important and can influence the eigenvalue behavior. For example, the use of oversampling domains both for the snapshot space and the eigenvalue can provide a faster convergence. In this paper, we take

$$R = \sqrt{a}\nabla w_\mu,$$

which allows estimating the right-hand side of (25) by the energy norm. Note that, in (7), we use the smallest eigenvalues to determine the basis functions which is the same as choosing the largest eigenvectors that correspond to the largest eigenvalues of (26) because $\hat{\mu} = 1 + \frac{1}{\mu}$.

Note that the eigenvalue problem (7) is equivalent to

$$a \frac{\partial w_\mu}{\partial n} = \frac{\mu}{H} w_\mu \quad \text{on} \quad \partial K,$$

where the normal flux $a \frac{\partial w_\mu}{\partial n}$ is defined discretely in (8). So, for each K ,

$$\begin{aligned} \int_{\partial K} \left(a \frac{\partial u_{1,K}}{\partial n} \right)^2 &= \int_{\partial K} \left(a \frac{\partial}{\partial n} \left(\sum_{i=1}^n c_{i,K} w_{i,K} \right) \right)^2 \\ (27) \quad &= \int_{\partial K} \left(\sum_{i=1}^n \frac{\mu_{i,K}}{H} c_{i,K} w_{i,K} \right)^2 = \sum_{i=1}^n \left(\frac{\mu_{i,K} c_{i,K}}{H} \right)^2, \end{aligned}$$

where we have used the fact that $\int_{\partial K} w_{i,K} w_{j,K} = \delta_{ij}$. Then, by using the eigenvalue problem defined in (7), we have

$$\frac{1}{h} \int_{\partial K} |(u_{1,K} - \phi_{1,K})|^2 = \frac{1}{h} \sum_{i=p+1}^n c_{i,K}^2 \leq \frac{H^2}{h\mu_{p+1,K}^2} \sum_{i=p+1}^n \left(\frac{\mu_{i,K}}{H} \right)^2 c_{i,K}^2$$

and

$$\int_K a |\nabla(u_{1,K} - \phi_{1,K})|^2 = \sum_{i=p+1}^n \frac{\mu_{i,K}}{H} c_{i,K}^2 \leq \frac{H}{\mu_{p+1,K}} \sum_{i=p+1}^n \left(\frac{\mu_{i,K}}{H} \right)^2 c_{i,K}^2.$$

Note that, by using (27), we have

$$\sum_{i=p+1}^n \left(\frac{\mu_{i,K}}{H} \right)^2 c_{i,K}^2 \leq \sum_{i=1}^n \left(\frac{\mu_{i,K}}{H} \right)^2 c_{i,K}^2 = \int_{\partial K} \left(a \frac{\partial u_{1,K}}{\partial n} \right)^2.$$

Therefore,

$$\begin{aligned} \|u_1 - \phi_1\|_a^2 &\leq \sum_{K \in \mathcal{T}^H} \left(\frac{H}{\mu_{p+1,K}} \left(1 + \frac{2a_1\gamma H}{h\mu_{p+1,K}} \right) \sum_{i=p+1}^n \left(\frac{\mu_{i,K}}{H} \right)^2 c_{i,K}^2 \right) \\ (28) \quad &\leq \sum_{K \in \mathcal{T}^H} \left(\frac{H}{\mu_{p+1,K}} \left(1 + \frac{2a_1\gamma H}{h\mu_{p+1,K}} \right) \int_{\partial K} \left(a \frac{\partial u_1}{\partial n} \right)^2 \right). \end{aligned}$$

Next, we will estimate $|u_2 - \phi_2|_a^2$. Since u_h satisfies

$$\int_K a \nabla u_h \cdot \nabla v = \int_K (f - (u_h)_{tt}) v \quad \forall v \in V_h^0(K).$$

Putting $v = z_{i,K}$, we obtain

$$\frac{\lambda_{i,K}}{H^2} d_{i,K} = \int_K a \nabla u_h \cdot \nabla z_{i,K} = \int_K (f - (u_h)_{tt}) z_{i,K}.$$

We define $f_{i,K} = \int_K (f - (u_h)_{tt}) z_{i,K}$. Then we have $f_{i,K} = \frac{\lambda_{i,K}}{H^2} d_{i,K}$ and

$$\sum_{i=1}^{n_0} f_{i,K}^2 \leq \|f - (u_h)_{tt}\|_{L^2(K)}^2.$$

Hence,

$$\begin{aligned} |u_2 - \phi_2|_a^2 &= \sum_{K \in \mathcal{T}^H} \int_K a |\nabla(u_2 - \phi_2)|^2 \\ &= \sum_{K \in \mathcal{T}^H} \sum_{i \geq m+1} \frac{\lambda_{i,K}}{H^2} d_{i,K}^2 \\ &\leq \sum_{K \in \mathcal{T}^H} \frac{H^2}{\lambda_{m+1,K}} \sum_{i \geq m+1} \frac{\lambda_{i,K}^2}{H^4} d_{i,K}^2 \\ &= \sum_{K \in \mathcal{T}^H} \frac{H^2}{\lambda_{m+1,K}} \sum_{i \geq m+1} f_{i,K}^2 \\ &\leq \sum_{K \in \mathcal{T}^H} \frac{H^2}{\lambda_{m+1,K}} \|f - (u_h)_{tt}\|_{L^2(K)}^2. \quad \square \end{aligned}$$

We note that, by the technique in [14], we can also derive a bound for $\|u_1 - \phi_1\|_a$ as follows:

$$\|u_1 - \phi_1\|_a^2 \leq \sum_{K \in \mathcal{T}^H} \sum_{i \geq p+1} c_{i,K}^2.$$

This bound shows the decay of the error when more basis functions are used.

The bound in (24) gives the spectral convergence of our GMsFEM. Notice that the term

$$H \sum_{K \in \mathcal{T}^H} \sum_{\partial K} \left(a \frac{\partial u_1}{\partial n} \right)^2$$

is uniformly bounded and can be considered as a norm for u_1 . Thus, (24) states that the error behaves as $O(\mu_{p+1,K}^{-1} + \lambda_{m+1,K}^{-1})$. We note that the eigenvalues increase (and go to the infinity as the fine mesh size decreases), and thus the error decreases as we increase the coarse space dimension.

Combining the results in Theorems 3 and 4, we obtain

$$\begin{aligned} (29) \quad \|\eta\|_a^2 &\leq C \sum_{K \in \mathcal{T}^H} \left(\frac{H^2}{\lambda_{m+1,K}} \|f - (u_h)_{tt}\|_{L^2(K)}^2 \right. \\ &\quad \left. + \frac{H}{\mu_{p+1,K}} \left(1 + \frac{2a_1 \gamma H}{h \mu_{p+1,K}} \right) \int_{\partial K} \left(a \frac{\partial u_1}{\partial n} \right)^2 \right) + h^2 C(u)^2. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \|\eta_t\|_a^2 \leq C \sum_{K \in \mathcal{T}^H} & \left(\frac{H^2}{\lambda_{m+1,K}} \|f_t - (u_h)_{tt}\|_{L^2(K)}^2 \right. \\ & \left. + \frac{H}{\mu_{p+1,K}} \left(1 + \frac{2a_1\gamma H}{h\mu_{p+1,K}} \right) \int_{\partial K} \left(a \frac{\partial(u_1)_t}{\partial n} \right)^2 \right) + h^2 C(u_t)^2. \end{aligned}$$

Finally, using these bounds for η , as well as the estimates proved in Theorem 1 and Theorem 2, we obtain estimates for the error ε .

5. Convergence of the fully discrete scheme. In this section, we will prove the convergence of the fully discrete scheme (5). To simplify the notation, we define the second order central difference operator δ^2 by

$$\delta^2(u^n) = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}.$$

By the definition of the consistency error (11), at the time t_n we have

$$(30) \quad ((u_h)_{tt}^n, v) + a_{DG}(u_h^n, v) = (f^n, v) - R_{u_h^n}(v).$$

The fully discrete scheme (5) can be written using the operator δ^2 as

$$(31) \quad (\delta^2(u_H^n), v) + a_{DG}(u_H^n, v) = (f^n, v) \quad \text{for } n \geq 1.$$

Moreover, we define

$$(32) \quad r^n = \begin{cases} u_{tt}^n - \delta^2(w_H^n) & \text{for } n \geq 1, \\ \Delta t^{-2}(\xi^1 - \xi^0) & \text{for } n = 0, \end{cases}$$

and

$$R^n = \Delta t \sum_{i=0}^n r^i.$$

In order to prove the convergence for the fully discrete scheme, we first prove the following lemma. The result will be needed in the derivation of an upper bound for the time step size Δt .

LEMMA 3. *There exists a positive constant $\beta(h)$ such that*

$$a_{DG}(v, v) \leq \beta(h)^{-1} \|v\|_{L^2(\Omega)}^2 \quad \forall v \in V_H.$$

Moreover, the constant $\beta(h)$ can be taken as $h^2 a_1^{-1} (24 + 32\sqrt{3\Lambda} + 16\Lambda a_1^2 a_0^{-2})^{-1}$.

Proof. We first note that, if p is a linear function defined on the interval $I = [x_1 - h/2, x_1 + h/2]$, then we have

$$(33) \quad \|p\|_{L^\infty(I)}^2 \leq \frac{4}{h} \|p\|_{L^2(I)}^2,$$

$$(34) \quad \|p\|_{H^1(I)}^2 \leq \frac{12}{h^2} \|p\|_{L^2(I)}^2.$$

Then by the definition of a_{DG} and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} a_{DG}(v, v) &\leq \sum_{K \in \mathcal{T}^H} \int_K a |\nabla v|^2 - 2 \sum_{e \in \mathcal{E}^H} \left(\int_e \{a \nabla v \cdot n\}_e \cdot [v]_e + \frac{\gamma}{h} \int_e a [v]_e^2 \right) \\ &\leq \sum_{K \in \mathcal{T}^H} \int_K a |\nabla v|^2 + 2 \left(\sum_{K \in \mathcal{T}^H} h \|a \nabla v \cdot n_{\partial K}\|_{L^2(\partial K)}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}^H} h^{-1} \|[v]\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \\ &\quad + \frac{\gamma}{h} \sum_{e \in \mathcal{E}^H} \int_e a [v]_e^2. \end{aligned}$$

Then by using the discrete trace inequality (17), $a \leq a_1$, and estimating the jump terms by $L^2(\partial K)$ norms, we have

(35)

$$\begin{aligned} a_{DG}(v, v) &\leq a_1 \left(\sum_{K \in \mathcal{T}^H} \int_K |\nabla v|^2 + 2 \left(\sum_{K \in \mathcal{T}^H} \Lambda \int_K |\nabla v|^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}^H} h^{-1} \|[v]\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{\gamma}{h} \sum_{e \in \mathcal{E}^H} \int_e [v]_e^2 \right) \\ &\leq a_1 \left(\sum_{K \in \mathcal{T}^H} \int_K |\nabla v|^2 + 4 \left(\sum_{K \in \mathcal{T}^H} \Lambda \int_K |\nabla v|^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}^H} h^{-1} \|v\|_{L^2(\partial K)}^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{2\gamma}{h} \sum_{K \in \mathcal{T}^H} \|v\|_{L^2(\partial K)}^2 \right). \end{aligned}$$

Thus, it remains to estimate $\|\nabla v\|_{L^2(K)}$ and $\|v\|_{L^2(\partial K)}$ by the norm $\|v\|_{L^2(K)}$. The techniques used are the same as those used for standard finite element inverse inequalities. We include the derivation below for the explicit expression of $\beta(h)$.

We will estimate the term $\|\nabla v\|_{L^2(K)}$ first. For a given coarse grid block K , we can write it as the union of fine grid blocks $K = \cup_{F \subset K} F$, where we use F to represent a generic fine grid block. Since the fine grid blocks are rectangles, we can write F as a tensor product of two intervals, namely, $F = I_x^F \times I_y^F$. For any $v \in V_H$ we can also write the restriction of v on F as $v(x, y) = v_{F,1}(x)v_{F,2}(y)$.

$$\begin{aligned} \int_K |\nabla v|^2 &= \sum_{F \subset K} \int_F |\nabla v|^2 \\ &= \sum_{F \subset K} \left(h (v'_{F,2})^2 \int_{I_x^F} (v_{F,1}(x))^2 + h (v'_{F,1})^2 \int_{I_y^F} (v_{F,2}(y))^2 \right) \\ &= \sum_{F \subset K} \left(\int_{I_y^F} (v'_{F,2}(y))^2 \int_{I_x^F} (v_{F,1}(x))^2 + \int_{I_x^F} (v'_{F,1}(x))^2 \int_{I_y^F} (v_{F,2}(y))^2 \right). \end{aligned}$$

Then, using (34), we have

$$\begin{aligned} \int_K |\nabla v|^2 &\leq 12h^{-2} \sum_{F \subset K} \left(\int_{I_y^F} (v_{F,2}(y))^2 \int_{I_x^F} (v_{F,1}(x))^2 + \int_{I_x^F} (v_{F,1}(x))^2 \int_{I_y^F} (v_{F,2}(y))^2 \right) \\ &= 24h^{-2} \sum_{F \subset K} \int_F |v|^2. \end{aligned}$$

Next, we estimate the term $\|v\|_{L^2(\partial K)}$. For a generic fine grid cell F , we write $I_x^F = [x_1, x_2]$ and $I_y^F = [y_1, y_2]$. Then, by using (33),

$$\begin{aligned} &\|v\|_{L^2(\partial K)}^2 \\ &= \sum_{F \subset K} \int_{\partial F \cap \partial K} (v_F)^2 \\ &= \sum_{F \subset K} \left(\int_{\partial F \cap (I_x \times \{y_1\})} (v_{F,2}(y_1)v_{F,1}(x))^2 + \int_{\partial F \cap (I_x \times \{y_2\})} (v_{F,2}(y_2)v_{F,1}(x))^2 \right) \\ &\quad + \sum_{F \subset K} \left(\int_{\partial F \cap (\{x_1\} \times I_y)} (v_{F,1}(x_1)v_{F,2}(y))^2 + \int_{\partial F \cap (\{x_2\} \times I_y)} (v_{F,1}(x_2)v_{F,2}(y))^2 \right) \\ &\leq \frac{4}{h} \sum_{F \subset K} \left(\int_{\partial F \cap (I_x \times \{y_1\})} \int_{[y_1, y_1+h]} v_{F,2}(y)^2 (v_{F,1}(x))^2 \right. \\ &\quad \left. + \int_{\partial F \cap (I_x \times \{y_2\})} \int_{[y_2-h, y_2]} v_{F,2}(y)^2 (v_{F,1}(x))^2 \right) \\ &\quad + \frac{4}{h} \sum_{F \subset K} \left(\int_{\partial F \cap (\{x_1\} \times I_y)} \int_{[x_1, x_1+h]} v_{F,1}(x)^2 v_{F,2}(y)^2 \right. \\ &\quad \left. + \int_{\partial F \cap (\{x_2\} \times I_y)} \int_{[x_2-h, x_2]} v_{F,1}(x)^2 v_{F,2}(y)^2 \right) \\ &\leq \frac{4}{h} \sum_{F \subset K} \left(\int_F (v_{F,2}(y)v_{F,1}(x))^2 + \int_F (v_{F,1}(x)v_{F,2}(y))^2 \right) \\ &= \frac{8}{h} \|v\|_{L^2(K)}^2. \end{aligned}$$

Consequently, combining the above results and using (35),

$$a_{DG}(v, v) \leq \frac{a_1}{h^2} \left(24 + 32\sqrt{3\Lambda} + 16\gamma \right) \|v\|_{L^2(\Omega)}^2.$$

Furthermore, using the lower bound of γ , we see that we can take $\beta(h)$ as

$$\beta(h) = h^2 a_1^{-1} (24 + 32\sqrt{3\Lambda} + 16\Lambda a_1^2 a_0^{-2})^{-1}. \quad \square$$

Finally, we will state and prove the convergence of the fully discrete scheme (5).

THEOREM 5. *Assume that the time step size Δt satisfies the stability condition $\Delta t^2 < 4\beta(h)$. We have*

$$(36) \quad \max_{0 \leq n \leq N} \|\varepsilon^n\|_{L^2(\Omega)} \leq C \left(\|\varepsilon^0\|_{L^2(\Omega)} + \max_{0 \leq n \leq N} \|\eta^n\|_{L^2(\Omega)} + \Delta t \sum_{n=0}^N \|R^n\|_{L^2(\Omega)} \right).$$

Proof. Subtracting (31) by (30), we have

$$(\delta^2(u_H^n) - (u_h)_{tt}^n, v) + a_{DG}(u_H^n - u_h^n, v) = R_{u_h^n}(v) \quad \text{for } n \geq 1,$$

which implies

$$(\delta^2(u_H^n - w_H^n + w_H^n), v) + a_{DG}(u_H^n - u_h^n, v) = ((u_h)_{tt}^n, v) + R_{u_h^n}(v).$$

Using the fact that $\xi = u_H - w_H$ and the definition of the elliptic projection w_H given in (19), we have

$$(\delta^2(\xi^n), v) + a_{DG}(\xi^n, v) = (r^n, v) \quad \text{for } n \geq 1,$$

where r^n is defined in (32). Using the definition of the operator δ^2 , we have

$$\left(\frac{\xi^{n+1} - \xi^n}{\Delta t}, v \right) - \left(\frac{\xi^n - \xi^{n-1}}{\Delta t}, v \right) + \Delta t a_{DG}(\xi^n, v) = \Delta t (r^n, v).$$

Summing the above equations, we get for $n \geq 1$,

$$(37) \quad \left(\frac{\xi^{n+1} - \xi^n}{\Delta t}, v \right) - \left(\frac{\xi^1 - \xi^0}{\Delta t}, v \right) + \Delta t \sum_{i=1}^n a_{DG}(\xi^i, v) = \Delta t \sum_{i=1}^n (r^i, v).$$

To simplify the notation, we define

$$\Xi^n = \Delta t \sum_{i=1}^n \xi^i \quad \text{for } n \geq 1, \quad \text{and} \quad \Xi^0 = 0.$$

Using the above definition and the definition of R^n , we can write (37) as

$$\left(\frac{\xi^{n+1} - \xi^n}{\Delta t}, v \right) + a_{DG}(\Xi^n, v) = (R^n, v), \quad n \geq 1.$$

Substituting $v = \xi^{n+1} + \xi^n$, we have

$$\|\xi^{n+1}\|_{L^2(\Omega)}^2 - \|\xi^n\|_{L^2(\Omega)}^2 + \Delta t a_{DG}(\Xi^n, \xi^{n+1} + \xi^n) = \Delta t (R^n, \xi^{n+1} + \xi^n),$$

and summing for all $n \geq 1$, we have

$$(38) \quad \|\xi^{n+1}\|_{L^2(\Omega)}^2 - \|\xi^1\|_{L^2(\Omega)}^2 + \Delta t \sum_{i=1}^n a_{DG}(\Xi^i, \xi^{i+1} + \xi^i) = \Delta t \sum_{i=1}^n (R^i, \xi^{i+1} + \xi^i).$$

By the definition of Ξ^n , we have $\Xi^{n+1} - \Xi^{n-1} = \Delta t(\xi^{n+1} + \xi^n)$ for $n \geq 1$. So

$$\begin{aligned} \Delta t \sum_{i=1}^n a_{DG}(\Xi^i, \xi^{i+1} + \xi^i) &= \sum_{i=1}^n a_{DG}(\Xi^i, \Xi^{i+1} - \Xi^{i-1}) \\ &= \sum_{i=1}^n a_{DG}(\Xi^i, \Xi^{i+1}) - \sum_{i=0}^{n-1} a_{DG}(\Xi^i, \Xi^{i+1}) \\ &= a_{DG}(\Xi^n, \Xi^{n+1}). \end{aligned}$$

Moreover,

$$\begin{aligned} a_{DG}(\Xi^n, \Xi^{n+1}) &= a_{DG}\left(\frac{\Xi^n + \Xi^{n+1}}{2}, \frac{\Xi^n + \Xi^{n+1}}{2}\right) - a_{DG}\left(\frac{\Xi^n - \Xi^{n+1}}{2}, \frac{\Xi^n - \Xi^{n+1}}{2}\right) \\ &\geq -\frac{\Delta t^2}{4} a_{DG}(\xi^{n+1}, \xi^{n+1}). \end{aligned}$$

So, (38) becomes

$$\|\xi^{n+1}\|_{L^2(\Omega)}^2 - \frac{\Delta t^2}{4} a_{DG}(\xi^{n+1}, \xi^{n+1}) \leq \|\xi^1\|_{L^2(\Omega)}^2 + \Delta t \sum_{i=1}^n (R^i, \xi^{i+1} + \xi^i), \quad n \geq 1.$$

Using the assumption $\Delta t^2 < 4\beta(h)$, we define $C_s = 1 - \frac{\Delta t^2}{4\beta(h)} > 0$. By Lemma 3,

$$\begin{aligned} C_s \|\xi^{n+1}\|_{L^2(\Omega)}^2 &\leq \|\xi^1\|_{L^2(\Omega)}^2 + \Delta t \sum_{i=1}^n (R^i, \xi^{i+1} + \xi^i) \\ &\leq \|\xi^1\|_{L^2(\Omega)}^2 + 2\Delta t \max_{1 \leq i \leq n+1} \{\|\xi^i\|_{L^2(\Omega)}\} \sum_{i=1}^n \|R^i\|_{L^2(\Omega)} \\ &\leq \|\xi^1\|_{L^2(\Omega)}^2 + \frac{C_s}{2} \max_{1 \leq i \leq n+1} \{\|\xi^i\|_{L^2(\Omega)}\}^2 + \frac{2}{C_s} \left(\Delta t \sum_{i=1}^n \|R^i\|_{L^2(\Omega)} \right)^2. \end{aligned}$$

Therefore, we have

$$\max_{1 \leq i \leq n+1} \{\|\xi^i\|_{L^2(\Omega)}\} \leq C \left(\|\xi^1\|_{L^2(\Omega)} + \Delta t \sum_{i=1}^n \|R^i\|_{L^2(\Omega)} \right).$$

Since $\xi^1 = \xi^0 + \Delta t^2 r^0$, we have

$$\max_{1 \leq i \leq n+1} \{\|\xi^i\|_{L^2(\Omega)}\} \leq C \left(\|\xi^0\|_{L^2(\Omega)} + \Delta t^2 \|r^0\|_{L^2(\Omega)} + \Delta t \sum_{i=1}^n \|R^i\|_{L^2(\Omega)} \right),$$

and using the definition of R^0 ,

$$\max_{1 \leq i \leq n+1} \{\|\xi^i\|_{L^2(\Omega)}\} \leq C \left(\|\xi^0\|_{L^2(\Omega)} + \Delta t \sum_{i=0}^n \|R^i\|_{L^2(\Omega)} \right).$$

Thus,

$$\max_{0 \leq i \leq n+1} \{\|\xi^i\|_{L^2(\Omega)}\} \leq C \left(\|\xi^0\|_{L^2(\Omega)} + \Delta t \sum_{i=0}^n \|R^i\|_{L^2(\Omega)} \right).$$

Finally, by using the relation $\varepsilon = \eta - \xi$, we obtain (36). \square

From Theorem 5, we see that the error ε^n of the fully discrete scheme mainly depends on two quantities, which are η^n and R^n . Recall that η^n can be estimated as in (29). Therefore, it remains to get a bound for R^n . To do so, we will prove the following two lemmas for an upper bound of r^n .

LEMMA 4. *We have*

$$\|r^0\|_{L^2(\Omega)} \leq C(\Delta t^{-1}\|\eta_t\|_{L^\infty([0,T];L^2(\Omega))} + \Delta t\|(u_h)_{ttt}\|_{C([0,T];L^2(\Omega))}).$$

Proof. By (32), we have $r^0 = \Delta t^{-2}(\xi^1 - \xi^0)$, and by the definition of u_H^0 , we have

$$(u_H^0 - u^0, v) = 0 \quad \forall v \in V_H.$$

Then using the definitions of ξ^1 and ξ^0 , we have

$$\begin{aligned} (\xi^1 - \xi^0, v) &= (u_H^1 - w_H^1, v) - (u_H^0 - w_H^0, v) \\ &= (u_h^1 - w_H^1, v) + (u_H^1 - u_h^1, v) - (u_h^0 - w_H^0, v) \\ &= ((u_h^1 - u_h^0) - (w_H^1 - w_H^0), v) + (u_H^1 - u_h^1, v). \end{aligned}$$

The first term can be estimated in the following way:

$$\begin{aligned} |((u_h^1 - u_h^0) - (w_H^1 - w_H^0), v)| &\leq \left| \left(\int_0^{t^1} \partial_t(u_h - w_H), v \right) \right| \\ &\leq \Delta t \|\eta_t\|_{L^\infty([0,T];L^2(\Omega))} \|v\|_{L^2(\Omega)}. \end{aligned}$$

To estimate the second term, by the Taylor's expansion we get

$$u_h^1 = u_h^0 + \Delta t (u_h)_t^0 + \frac{\Delta t^2}{2} (u_h)_{tt}^0 + \frac{\Delta t^3}{6} (u_h)_{ttt}(\cdot, s), \quad \text{where } 0 < s < t^1.$$

By the definition of u_H^1 ,

$$(u_H^1, v) = (u_h^1, v) = (u_h^0 + \Delta t (u_h)_t^0 + \frac{\Delta t^2}{2} (\tilde{v}), v).$$

Thus,

$$\begin{aligned} (u_H^1 - u_h^1, v) &= \frac{\Delta t^2}{2} (\tilde{v} - (u_h)_{tt}^0, v) - \frac{\Delta t^3}{6} ((u_h)_{ttt}(\cdot, s), v) \\ &= \frac{\Delta t^2}{2} [(f^0, v) - a(u_h^0, v) + ((u_h)_{tt}^0, v)] - \frac{\Delta t^3}{6} ((u_h)_{ttt}(\cdot, s), v) \\ &= -\frac{\Delta t^3}{6} ((u_h)_{ttt}(\cdot, s), v), \end{aligned}$$

which proves the lemma. \square

LEMMA 5. *For $n \geq 1$, we have*

$$\|r^n\|_{L^2(\Omega)} \leq C \left(\Delta t^{-1} \int_{t_{n-1}}^{t_{n+1}} \|\eta_{tt}(\cdot, \tau)\|_{L^2(\Omega)} + \Delta t \int_{t_{n-1}}^{t_{n+1}} \|(u_h)_{tttt}(\cdot, \tau)\|_{L^2(\Omega)} \right).$$

Proof. By the definition of r^n ,

$$\begin{aligned} \|r^n\|_{L^2(\Omega)} &= \|(u_h)_{tt}^n - \delta^2 w_H^n\|_{L^2(\Omega)} \\ &\leq \|\delta^2 (w_H^n - u_h^n)\|_{L^2(\Omega)} + \|(u_h)_{tt}^n - \delta^2 u_h^n\|_{L^2(\Omega)}. \end{aligned}$$

Using the identity

$$v^{n+1} - 2v^n + v^{n-1} = \Delta t \int_{t_{n-1}}^{t_{n+1}} \left(1 - \frac{|\tau - t_n|}{\Delta t} \right) v_{tt}(\tau) d\tau,$$

the first term can be estimated as follows:

$$\begin{aligned} (\delta^2(w_H^n - u_h^n), v) &= \frac{1}{\Delta t} \int_{t_{n-1}}^{t_{n+1}} \left(1 - \frac{|\tau - t_n|}{\Delta t}\right) (((w_H)_{tt} - (u_h)_{tt}), v)(\tau) d\tau \\ &\leq \frac{1}{\Delta t} \int_{t_{n-1}}^{t_{n+1}} \|\eta_{tt}(\cdot, \tau)\|_{\tilde{L}^2(\Omega)} \|v\|_{\tilde{L}^2(\Omega)} d\tau. \end{aligned}$$

To estimate the term $\|(u_h)_{tt}^n - \delta^2 u_h^n\|_{L^2(\Omega)}$, we use

$$\delta^2 u_h^n = (u_h)_{tt}^n + \frac{1}{6\Delta t^2} \int_{t_{n-1}}^{t_{n+1}} (\Delta t - |\tau - t_n|)^3 (u_h)_{tttt}(\cdot, \tau) d\tau.$$

This implies

$$\|(u_h)_{tt}^n - \delta^2 u_h^n\|_{L^2(\Omega)} \leq \frac{\Delta t}{6} \int_{t_{n-1}}^{t_{n+1}} \|(u_h)_{tttt}(\cdot, \tau)\|_{L^2(\Omega)} d\tau. \quad \square$$

Using the definition of R^n and the above two lemmas, we get

$$\begin{aligned} \|R^n\|_{L^2(\Omega)} &\leq C \left(\int_0^{t_n} \|\eta_{tt}(\cdot, \tau)\|_{L^2(\Omega)} + \|\eta_t\|_{L^\infty([0, T]; L^2(\Omega))} \right. \\ &\quad \left. + \Delta t^2 \int_0^{t_n} \|(u_h)_{tttt}(\cdot, \tau)\|_{L^2(\Omega)} + \Delta t^2 \|(u_h)_{ttt}\|_{C([0, T]; L^2(\Omega))} \right). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \Delta t \sum_{n=0}^N \|R^n\|_{L^2(\Omega)} &\leq 2T \max_{0 \leq n \leq N} \|R^n\|_{L^2(\Omega)} \\ &\leq C \left(\int_0^T \|\eta_{tt}(\cdot, \tau)\|_{L^2(\Omega)} + \|\eta_t\|_{L^\infty([0, T]; L^2(\Omega))} \right. \\ &\quad \left. + \Delta t^2 \int_0^T \|(u_h)_{tttt}(\cdot, \tau)\|_{L^2(\Omega)} + \Delta t^2 \|(u_h)_{ttt}\|_{C([0, T]; L^2(\Omega))} \right). \end{aligned}$$

Combining the estimates of η proved in section 3 and Theorem 5, we obtain the error estimate for the fully discrete scheme (5).

6. Conclusions. In this paper, we present a multiscale simulation method based on GMsFEM for solving the wave equation in heterogeneous media. For the construction of multiscale basis functions, we divide the snapshot space into two spaces. The first snapshot space represents the degrees of the freedom associated with boundary nodes and consists of a -harmonic functions. The second snapshot space represents

the interior degrees of the freedom and consists of all zero Dirichlet vectors. For each snapshot space, we introduce local spectral problems motivated by the analysis presented in the paper. We use these local spectral problems to identify important modes in each of the snapshot spaces. The local spectral problems are designed to achieve a high accuracy and are motivated by the global coupling formulation. The use of multiple snapshot spaces and multiple spectral problems is one of the novelties of this work. Using the dominant modes from local spectral problems, multiscale basis functions are constructed to represent the solution space locally within each coarse block. These multiscale basis functions are coupled via the symmetric interior penalty discontinuous Galerkin method, which provides a block diagonal mass matrix and, consequently, results in fast computations in an explicit time discretization. Numerical examples are presented. In particular, we discuss how the modes from our snapshot spaces can affect the accuracy of the method. Our numerical results show that one can obtain an accurate approximation of the solution with GMsFEM using less than 3% of the total local degrees of freedom. We also test oversampling strategies following [15]. Analysis of the method is presented.

7. Appendix. In this appendix, we will prove Lemma 1. Let $v \in V_H^1$. By assumption, $u \in H^2(\Omega)$ for all time $t \in [0, T]$. Thus $a_{DG}(u, v)$ is well-defined, the normal flux is continuous across coarse edges, and we have

$$(39) \quad \sum_{K \in \mathcal{T}^H} \left(\frac{\partial^2 u}{\partial t^2}, v \right)_{L^2(K)} + a_{DG}(u, v) = \sum_{K \in \mathcal{T}^H} (f, v)_{L^2(K)} \quad \forall v \in H^1(\mathcal{T}^H).$$

Moreover, the following standard finite element error estimate holds:

$$\begin{aligned} \left\| \frac{\partial}{\partial t}(u - u_h) \right\|_{L^2(\Omega)} + |u - u_h|_{H^1(\Omega)} \\ \leq Ch \left(|u|_{L^\infty(0, T; H^2(\Omega))} + |u|_{W^{2,1}(0, T; H^1(\Omega))} \right) \quad \forall t \in [0, T] \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{\partial^2}{\partial t^2}(u - u_h) \right\|_{L^2(\Omega)} + \left| \frac{\partial}{\partial t}(u - u_h) \right|_{H^1(\Omega)} \\ \leq Ch \left(|u_t|_{L^\infty(0, T; H^2(\Omega))} + |u|_{W^{3,1}(0, T; H^1(\Omega))} \right) \quad \forall t \in [0, T]. \end{aligned}$$

We remark that, for simplicity, we assume that the initial conditions belong to the fine space V_h . In the following derivations, we will fix the time variable t . By the definition of the consistency error and (39), we have

$$(40) \quad R_{u_h}(v) = \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_h}{\partial t^2}, v \right) + a_{DG}(u, v) - a_{DG}(u_h, v) \quad \forall v \in H^1(\mathcal{T}^H).$$

Next, we define $v_c \in V_h$ in the following way. For each vertex in the triangulation, the value of v_c is defined as the average value of v at this vertex. Then by direct calculations, we have

$$\sum_{K \in \mathcal{T}^H} |v - v_c|_{H^1(K)}^2 \leq C \frac{1}{h} \sum_{e \in \mathcal{E}^H} \|[v]\|_{L^2(e)}^2$$

and

$$\sum_{K \in \mathcal{T}^H} \|v - v_c\|_{L^2(K)}^2 \leq Ch \sum_{e \in \mathcal{E}^H} \|[v]\|_{L^2(e)}^2.$$

Clearly, we have $[v - v_c]_e = [v]_e$ for all $e \in \mathcal{E}^H$ since $v_c \in C^0(\Omega)$. Therefore, we get

$$\|v - v_c\|_{H^1(\mathcal{T}^H)}^2 \leq C \frac{1}{h} \sum_{e \in \mathcal{E}^H} \|[v]\|_{L^2(e)}^2.$$

By (40) and (2) as well as the fact that $a_{DG}(u_h, v_c) = a(u_h, v_c)$, we have

$$R_{u_h}(v) = \sum_{K \in \mathcal{T}^H} \left(\frac{\partial^2(u - u_h)}{\partial t^2}, v - v_c \right)_{L^2(K)} + a_{DG}(u - u_h, v - v_c).$$

Next, we will estimate the two terms on the right-hand side. For the first term, we have

$$\begin{aligned} & \sum_{K \in \mathcal{T}^H} \left(\frac{\partial^2(u - u_h)}{\partial t^2}, v - v_c \right)_{L^2(K)} \\ & \leq \left\| \frac{\partial^2(u - u_h)}{\partial t^2} \right\|_{L^2(\Omega)} \left(\sum_{K \in \mathcal{T}^H} \|v - v_c\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \\ & \leq Ch \left\| \frac{\partial^2(u - u_h)}{\partial t^2} \right\|_{L^2(\Omega)} \left(\frac{1}{h} \sum_{e \in \mathcal{E}^H} \|[v]\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \\ & \leq Ch^2 \left(|u_t|_{L^\infty(0,T;H^2(\Omega))} + |u|_{W^{3,1}(0,T;H^1(\Omega))} \right) \left(\frac{1}{h} \sum_{e \in \mathcal{E}^H} \|[v]\|_{L^2(e)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

For the second term, by the definition of a_{DG} and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & a_{DG}(u - u_h, v - v_c) \\ & = \sum_{K \in \mathcal{T}^H} \int_K a \nabla(u - u_h) \cdot \nabla(v - v_c) - \sum_{e \in \mathcal{E}^H} \int_e \{a \nabla(u - u_h) \cdot n\} [v] \\ & \leq \sum_{K \in \mathcal{T}^H} a_1 |u - u_h|_{H^1(K)} |v - v_c|_{H^1(K)} \\ & \quad + \left(\sum_{K \in \mathcal{T}^H} \int_{\partial K} (a \nabla(u - u_h) \cdot n)^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}^H} \int_e [v]^2 \right)^{\frac{1}{2}}. \end{aligned}$$

To estimate the flux term above, we let I_K be the standard finite element interpolant.

Then we have

$$\begin{aligned} & \sum_{K \in \mathcal{T}^H} \int_{\partial K} (a \nabla(u - u_h) \cdot n)^2 \\ & \leq 2 \left(\sum_{K \in \mathcal{T}^H} \int_{\partial K} (a \nabla(u - I_K(u)) \cdot n)^2 \right) + 2 \left(\sum_{K \in \mathcal{T}^H} \int_{\partial K} (a \nabla(I_K(u) - u_h) \cdot n)^2 \right) \\ & \leq C a_1 \left(h |u|_{H^2(K)}^2 + \frac{1}{h} |I_K(u) - u_h|_{H^1(K)}^2 \right) \\ & \leq C a_1 \left(h |u|_{H^2(K)}^2 + \frac{1}{h} |I_K(u) - u|_{H^1(K)}^2 + \frac{1}{h} |u - u_h|_{H^1(K)}^2 \right) \\ & \leq C a_1 h |u|_{H^2(K)}^2. \end{aligned}$$

Next we will estimate the term $\sum_{K \in \mathcal{T}^H} a_1 |u - u_h|_{H^1(K)} |v - v_c|_{H^1(K)}$. We have

$$\begin{aligned} & \sum_{K \in \mathcal{T}^H} a_1 |u - u_h|_{H^1(K)} \cdot |v - v_c|_{H^1(K)} \\ & \leq C a_1 \left(\sum_{K \in \mathcal{T}^H} |u - u_h|_{H^1(K)}^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}^H} |v - v_c|_{H^1(K)}^2 \right)^{\frac{1}{2}} \\ & \leq C a_1 \left(\frac{1}{h} \sum_{K \in \mathcal{T}^H} |u - u_h|_{H^1(K)}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}^H} \|[v]\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \\ & \leq C a_1 h |u|_{H^2(\Omega)} \left(\frac{1}{h} \sum_{e \in \mathcal{E}^H} \|[v]\|_{L^2(e)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned} |R_{u_h}(v)| & \leq \frac{C}{a_0} h \left(h (|u_t|_{L^\infty(0,T;H^2(\Omega))} + |u|_{W^{3,1}(0,T;H^1(\Omega))}) \right. \\ & \quad \left. + a_1 |u|_{L^\infty(0,T;H^2(\Omega))} \right) \left(\frac{1}{h} \sum_{e \in \mathcal{E}^H} \|a[v]\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

for all $t \in [0, T]$. Finally, the constant $C(u)$ in the lemma can be chosen as

$$(41) \quad C(u) \approx |u_t|_{L^\infty(0,T;H^2(\Omega))} + |u|_{W^{3,1}(0,T;H^1(\Omega))} + |u|_{L^\infty(0,T;H^2(\Omega))}.$$

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