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A COMBINED PRECONDITIONING STRATEGY FOR NONSYMMETRIC SYSTEMS*

B. AYUSO DE DIOS†, A. T. BARKER‡, and P. S. VASSILEVSKI‡

Abstract. We present and analyze a class of nonsymmetric preconditioners within a normal (weighted least-squares) matrix form for use in GMRES to solve nonsymmetric matrix problems that typically arise in finite element discretizations. An example of the additive Schwarz method applied to nonsymmetric but definite matrices is presented for which the abstract assumptions are verified. A variable preconditioner, combining the original nonsymmetric one and a weighted least-squares version of it, is shown to be convergent and provides a viable strategy for using nonsymmetric preconditioners in practice. Numerical results are included to assess the theory and the performance of the proposed preconditioners.

Key words. preconditioning, nonsymmetric matrices, normal matrix form, additive Schwarz method

AMS subject classifications. 65F10, 65N20, 65N30

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1. Introduction.

The numerical approximation of most phenomena in science and technology requires the solution of linear or nonlinear algebraic systems. Preconditioning is one of the main techniques that, when combined with a proper iterative method, allows for substantially reducing the cost of solving those systems. Much effort is usually devoted to designing proper preconditioning strategies that allow for efficient and fast solution of the resulting algebraic systems [28, 29]. The development of preconditioners is very often guided by the properties of the underlying problem, and it sometimes can even dictate particular aspects that should be accounted for when devising the numerical discretization of the continuous problem (as, for instance, in [19, 1]).

Even for linear problems, the design and analysis of preconditioners for linear systems is far from complete. For symmetric and coercive problems, a reasonable discretization yields a linear system $A_0\mathbf{x} = \mathbf{b}$ with A_0 symmetric and positive definite (s.p.d.). In such cases, as is well known, the spectral information of the matrix itself dictates completely the convergence of the method. Therefore a preconditioner B_0 would be uniform (and could be turned into an optimal preconditioner) if it captured completely such spectral information, in other words, if B_0 is spectrally equivalent to A_0.

However, for linear systems $A\mathbf{x} = \mathbf{b}$, with nonsymmetric coefficient matrix A, the design of effective preconditioners does not admit a general recipe. Likewise, there is

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†Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, I-40127 Bologna, Italy, and Computer, Electrical, and Mathematical Sciences & Engineering Division (CEMSE), King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia (blanca.ayuso@unibo.it, blanca.ayusodios@kaust.edu.sa).

‡Center for Applied Scientific Computing, Lawrence Livermore National Laboratory, Livermore, CA 94550 (barker29@llnl.gov, panayot@llnl.gov). The work of these authors was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract DE-AC52-07NA27344.

no general iterative solver, and furthermore there is no general theory that could be used to explain the success of a particular preconditioner when it is indeed efficient. In most cases, the spectral information does not provide significant information that could guide the development of any good preconditioner. The field of values has shown some utility in certain circumstances, but has also shown many limitations [15, 27, 23]. At the moment it might seem that each particular problem must be studied separately and a problem-dependent, discretization-dependent preconditioning strategy devised. Even when such preconditioning can be designed, its understanding and analysis are tasks that in most cases are out of reach.

In this paper, we focus on a particular situation where A is nonsymmetric but still positive definite. The motivation and application comes from a nonsymmetric discontinuous Galerkin discretization of an elliptic problem [13]. In [2, 3], additive and multiplicative Schwarz preconditioners were developed for the solution of the resulting algebraic system. In both works, the authors show that the GMRES convergence theory cannot be applied for explaining the convergence since the preconditioned system does not satisfy the *sufficient* conditions required by such theory. However, such discretizations are used in practice and have already shown to have some advantages when approximating advection-diffusion problems [21, 7] and, more recently, in the design of methods for some more complex nonlinear problems [24, 22]. In [10], the authors introduce a solver methodology based on the idea of subspace correction for this type of discretization for elliptic problems (see also [8] for an extension to problems with jump coefficients), providing the analysis of the resulting iterative methods without using any GMRES theory. In this paper we want to examine, in a more general algebraic abstract framework (that in particular will apply to the type of methods discussed above), the issue of providing some convergence theory for a preconditioner based on the classical (but nonsymmetric) Schwarz preconditioner to be used within GMRES. The ultimate goal is to obtain some insight on how to improve and tune the preconditioner.

Here, in a first stage we consider two preconditioners for A: a classical additive Schwarz preconditioner B, which is nonsymmetric, and a symmetric preconditioner Z that basically uses actions of the additive Schwarz preconditioner B and its adjoint. Both will be shown to have their pros and cons. For the former, the nonsymmetry of B and of $B^{-1}A$ precludes developing any theory from which either to extract some a priori information on the convergence or to provide some guidelines on how the preconditioner could be improved or even designed. The latter, while allowing for developing a convergence theory, will be shown to be not the most efficient possible option, although the a priori information on convergence could be of special value depending on the application. Other symmetrizing strategies for classical Schwarz methods, different from the one introduced here, have already been considered in literature by other authors [20].

At first sight, the underlying message that one might obtain from the analysis of this first part is that enforcing the symmetry of the preconditioner for a nonsymmetric matrix might ultimately result in a *wasted effort*. We believe this might be the case in many situations, and we also think it is relevant and important to point it out. At the same time, we do believe that the results obtained for analyzing the preconditioner are of independent interest (also because of its simplicity) and might provide some basis (as has happened already here) or insights for further development of solvers for nonsymmetric systems.

In a second stage of the present paper, we introduce a variable preconditioner

$\mathcal{B}$ that is constructed by considering a linear combination of two given (general) preconditioners B and Z, so that in a sense it tries to integrate and exploit the best of each of them. We describe the construction of this variable preconditioner to be used in GMRES, explaining how the coefficients in its definition are determined at each iteration inside GMRES. We show that from the construction of $\mathcal{B}$ we immediately can deduce (theoretically) a convergence estimate that guarantees better performance of the resulting solver at a fixed iteration. We demonstrate numerically that the new preconditioner outperforms the symmetric preconditioner Z and always converges faster. We also include the construction of a variant of the combined preconditioner, for which one can indeed guarantee faster convergence than for the GMRES using only the Z preconditioner. However, this variant is more expensive, and therefore, although convergence of the whole sequence of iterates can be shown, its advantages and its possible tuning with respect to the original combined preconditioner need further study and will be the subject of future research.

The theory is illustrated with extensive numerical experiments in which we also study the performance of all the considered preconditioners. They are all implemented in parallel to fully take advantage of having considered preconditioners based on additive Schwarz methods. In the numerical tests, we do observe that the combined preconditioner requires fewer GMRES iterations than the classical additive Schwarz preconditioner B to achieve convergence. However, in this particular case, each iteration for the combined preconditioner is more costly, which in the end makes B perform slightly better in terms of execution time. From these observations, it might be inferred that the new combined preconditioner $\mathcal{B}$ might be more competitive in settings where each iteration is expensive, so that the savings in iteration count can make up for the high cost per iteration.

Although we have focused on the nonsymmetric but positive definite case, we believe the ideas presented in the paper might be useful and possibly extended to more complex problems, including the indefinite case. This issue will be the subject of future research.

The idea of using several preconditioners instead of only one with GMRES has already been introduced from different viewpoints (and still different from the one considered here). Saad proposed using a different preconditioner at each GMRES iteration in [25], giving rise to *flexible* GMRES (FGMRES). For an earlier result that uses preconditioners that can change at every iteration step (called variable-step preconditioners), we refer the reader to [6]. More recently, the *multipreconditioned* GMRES (MPGMRES) has been proposed in [17, 18] as a further evolution of the multipreconditioned CG method introduced in [12]. The idea of both works is to enlarge the Krylov spaces over which GMRES minimizes the residual norm by considering an optimal combination of different preconditioners, and therefore using several preconditioners simultaneously at each iteration. The authors have considered in [17] complete (using all possible search directions generated by the different preconditioners) and truncated (where some directions are discarded to reduce the cost) versions of MPGMRES. While the strategy proposed in this paper certainly has similarities with these previous works, it also has many differences, including the motivation. Unlike the present paper, the particular preconditioners used in [17, 18] do not play any particular role. Here, the main motivation comes from the observation that for some discretizations frequently used in applications the simplest possible domain decomposition preconditioner cannot be guaranteed to be convergent. A symmetric version of it, the preconditioner Z, can be shown to converge, but its performance seems to be far from optimal. Therefore, the idea arose of trying to construct an optimal combination of both that could retain the good performance of the nonsymmetric preconditioner and for which some convergence estimate could be given (even if it would be just an upper bound). Here, the focus is on combining in an optimal way very specific preconditioners rather than just any preconditioner. The possible comparison and further combination of both approaches (the combined preconditioned and MPGMRES) are left as a subject of future research.

The outline of the paper is as follows. Section 2 contains a description of the problem and the original motivation. In section 3 we construct the preconditioner Z and present the convergence analysis. The combined preconditioner and its variant are introduced and analyzed in section 4. Finally, in section 5 we consider a particular application and provide numerical experiments that verify the developed theory and assess the performance of the preconditioner.

2. Problem formulation and basic notation. We are interested in preconditioning a given system of linear equations,

$$A\mathbf{x} = \mathbf{b}, \qquad A \in \mathbb{R}^{n\times n}, \quad \mathbf{x}, \mathbf{b} \in \mathbb{R}^n \ , \tag{2.1}$$

where A is nonsymmetric but definite and n is assumed to be large. For the applications we have in mind, A comes from a finite element discretization of some partial differential operator and therefore is sparse and structured. With a small abuse of notation, throughout the paper we will use (the same notation) A to denote both an operator and its matrix representation, since it will be clear from the context in all circumstances.

We also assume that A is ill-conditioned and that therefore a good preconditioner B is required to solve efficiently system (2.1) by an iterative method. A simple option is to construct such B as the classical additive Schwarz preconditioner coming from A. More precisely, we denote by I_k, $k = 1, \ldots, N_s$, a set of rectangular matrices, such that I_k extends a local vector $\mathbf{v}_k$ to a global vector $I_k\mathbf{v}_k$ with zero entries outside its index set. Also, let I_c be an interpolation matrix that maps a coarse vector $\mathbf{v}_0 = \mathbf{v}_c$ to a global vector $I_c\mathbf{v}_c$. Then the additive Schwarz preconditioner exploits the local matrices $A_k = I_k^T A I_k$, principal submatrices of A, and the coarse matrix A_c defined as $A_c = I_c^T A I_c$. The inverse of the additive Schwarz preconditioner B takes the following familiar form:

$$B^{-1} = I_c A_c^{-1} I_c^T + \sum_{k=1}^{N_s} I_k A_k^{-1} I_k^T \ . \tag{2.2}$$

Obviously, since A is nonsymmetric, the resulting additive Schwarz preconditioner B is also nonsymmetric. Therefore, for the solution of the resulting preconditioned system $AB^{-1}\mathbf{u} = \mathbf{b}, B^{-1}\mathbf{u} = \mathbf{x}$, one has to use any of the iterative methods for nonsymmetric systems, such as the *generalized minimal residual* (GMRES) method. For analyzing the convergence of the resulting iterative method (for the preconditioned system) one has to resort to one of the available and nonoptimal GMRES theories. In the domain decomposition framework, the GMRES convergence theory of Eisenstat, Elman, and Schultz [14] is generally used. In particular, to derive (a priori) any conclusion on the performance of the preconditioner B, this theory requires some control on the coercivity of AB^{-1} (in some inner product). Therefore, at least in theory, using B directly as a preconditioner for A might not be successful.

Still, we would like to utilize the actions of B^{-1} to define a preconditioner, say Z, for A for which some bounds on the rate of convergence can be a priori determined.

In the next section, we show how such a preconditioner Z can be constructed (and analyzed) by exploiting the fact that although A is nonsymmetric, it is positive definite in some inner product. We also compare numerically, in section 5, the performance of the constructed preconditioner Z with the original nonsymmetric additive Schwarz preconditioner B. As we will show, even if a theory can be developed for Z, it might not be the most efficient option.

We now state our basic assumption regarding the matrix A. More specifically, we assume that there is an s.p.d. matrix A_0 such that A and A_0 are related by the following basic assumption.

ASSUMPTION (H0). *Let $A \in \mathbb{R}^{n\times n}$ be nonsymmetric but definite, and let $A_0 \in \mathbb{R}^{n\times n}$ be s.p.d. We say that the pair of matrices $(A\,,\,A_0)$ satisfies Assumption* (H0) *with constants (c_0, c_1) if they do satisfy the following coercivity and boundedness estimates:*

$$\mathbf{v}^T A\mathbf{v} \geq c_0\ \mathbf{v}^T A_0\mathbf{v} \quad \forall\, \mathbf{v} \in \mathbb{R}^n\ , \tag{2.3}$$

$$\mathbf{w}^T A\mathbf{v} \leq c_1\ \sqrt{\mathbf{v}^T A_0\mathbf{v}}\sqrt{\mathbf{w}^T A_0\mathbf{w}} \quad \forall\, \mathbf{v},\mathbf{w}\ \in\ \mathbb{R}^n\ . \tag{2.4}$$

Apart from the above assumptions, the matrix A we consider is assumed to be highly nonnormal, which precludes the use of iterative algorithms and GMRES convergence theories that exploit the eigenvalue information.

3. An abstract result.

In this section we present the construction and give the analysis of a preconditioner for A that basically only uses the actions of the additive Schwarz method. We start by proving two lemmas that will be required for our subsequent analysis and derivation.

The next lemma shows that for any pair of matrices (A, A_0) satisfying (H0) with constants (c_0, c_1), the corresponding pair (A^{-1}, A_0^{-1}) (consisting of their inverses) also satisfies (H0) with constants (c_3, c_4) that depend only on c_0 and c_1.

LEMMA 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be nonsymmetric but definite, and let $A_0 \in \mathbb{R}^{n\times n}$ be s.p.d. Let (A, A_0) be a pair of matrices that satisfies Assumption* (H0) *with constants (c_0, c_1) (in particular, $A \in \mathbb{R}^{n\times n}$ is nonsymmetric but definite and $A_0 \in \mathbb{R}^{n\times n}$ is s.p.d.). Then the pair (A^{-1}, A_0^{-1}) also satisfies Assumption* (H0) *with constants $\left(\frac{c_0}{c_1^2}, c_0^{-1}\right)$; that is,*

$$\mathbf{v}^T A^{-1}\mathbf{v} \geq \frac{c_0}{c_1^2}\ \mathbf{v}^T A_0^{-1}\mathbf{v} \quad \forall\, \mathbf{v} \in \mathbb{R}^n, \tag{3.1}$$

$$\mathbf{w}^T A^{-1}\mathbf{v} \leq \frac{1}{c_0}\ \sqrt{\mathbf{v}^T A_0^{-1}\mathbf{v}}\sqrt{\mathbf{w}^T A_0^{-1}\mathbf{w}} \quad \forall\, \mathbf{v},\mathbf{w} \in \mathbb{R}^n. \tag{3.2}$$

Proof. We first show the boundedness estimate (3.2). We define the matrix $Y := A_0^{-\frac{1}{2}} A A_0^{-\frac{1}{2}}$. Then (2.3) and (2.4) imply (or read) that Y satisfies

$$\mathbf{v}^T Y\mathbf{v} \geq c_0\ \|\mathbf{v}\|^2 \qquad \forall\, \mathbf{v} \in \mathbb{R}^n\ , \tag{3.3}$$

$$\mathbf{w}^T Y\mathbf{v} \leq\ c_1\|\mathbf{v}\|\|\mathbf{w}\| \qquad \forall\, \mathbf{v},\mathbf{w}\ \in \mathbb{R}^n\ . \tag{3.4}$$

The positivity (3.3) of Y guarantees the existence of Y^{-1}, and so taking $\mathbf{v} := Y^{-1}\mathbf{w}$ in (3.3) and using the symmetry of the standard ℓ^2-inner product of two vectors together with the Cauchy–Schwarz inequality, we find

$$c_0\ \|Y^{-1}\mathbf{w}\|^2 \leq \mathbf{w}^T Y^{-T}\mathbf{w} = \mathbf{w}^T Y^{-1}\mathbf{w} \leq \|\mathbf{w}\|\|Y^{-1}\mathbf{w}\|,$$

which shows that $\|Y^{-1}\mathbf{w}\| \le \frac{1}{c_0}\,\|\mathbf{w}\|$, that is, the boundedness of Y^{-1} in the ℓ^2-norm:

$$\|Y^{-1}\| \le \frac{1}{c_0}. \tag{3.5}$$

In other words we have shown that

$$\mathbf{w}^T A_0^{\frac{1}{2}} A^{-1} A_0^{\frac{1}{2}} \mathbf{v} = \mathbf{w}^T Y^{-1}\mathbf{v} \le \frac{1}{c_0}\,\|\mathbf{v}\|\|\mathbf{w}\| \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n\ .$$

Now setting in the above equation $\mathbf{v} := A_0^{-\frac{1}{2}}\mathbf{v}$ and $\mathbf{w} := A_0^{-\frac{1}{2}}\mathbf{w}$, we reach the desired boundedness estimate (3.2) for A^{-1} in terms of A_0^{-1}.

The positivity estimate (3.1) can be shown as follows. On the one hand, the boundedness (3.4) of Y with $\mathbf{v} = \mathbf{v}$ and $\mathbf{w} = Y^{-1}\mathbf{v}$ gives

$$\|\mathbf{v}\|^2 = \mathbf{v}^T Y (Y^{-1}\mathbf{v}) \le c_1\,\|Y^{-1}\mathbf{v}\|\|\mathbf{v}\| \qquad \forall\, \mathbf{v} \in \mathbb{R}^n\ ,$$

which readily implies

$$\|Y^{-1}\mathbf{v}\| \ge \frac{1}{c_1}\|\mathbf{v}\| \qquad \forall\, \mathbf{v} \in \mathbb{R}^n\ . \tag{3.6}$$

On the other hand, using the positivity estimate (3.3) of Y, we have

$$\mathbf{v}^T Y^{-1}\mathbf{v} = (Y^{-1}\mathbf{v})^T Y^T (Y^{-1}\mathbf{v}) = (Y^{-1}\mathbf{v})^T Y (Y^{-1}\mathbf{v}) \ge c_0\,\|Y^{-1}\mathbf{v}\|^2 \quad \forall\, \mathbf{v} \in \mathbb{R}^n\ .$$

Then the above relation together with estimate (3.6) gives the following positivity estimate for Y^{-1} :

$$\mathbf{v}^T Y^{-1}\mathbf{v} \ge \frac{c_0}{c_1^2}\,\|\mathbf{v}\|^2 \quad \forall\, \mathbf{v} \in \mathbb{R}^n\ .$$

Now, setting in the above estimate $\mathbf{v} := A_0^{-\frac{1}{2}}\mathbf{v}$, we obtain the coercivity relation (3.1) and conclude the proof. ◻

Next, let B_0 denote the s.p.d. additive Schwarz preconditioner of A_0, whose inverse is defined as

$$B_0^{-1} = I_c A_c^{(0)^{-1}} I_c^T + \sum_{k=1}^{N_s} I_k A_k^{(0)^{-1}} I_k^T\ . \tag{3.7}$$

Note that since (A, A_0) satisfy Assumption (H0) with constants (c_0, c_1), this immediately implies that for each $k = 1, \ldots, N_s$ the family of pairs $(A_k, A_k^{(0)})$ with matrices defined by

$$A_k := I_k^T A I_k \quad \text{and} \quad A_k^{(0)} := I_k^T A_0 I_k, \qquad k = 1, \ldots, N_s\ ,$$

also satisfies Assumption (H0) with the same constants. The same is also true for the *coarse* pair of matrices $(A_c, A_c^{(0)})$, where $A_c = I_c^T A I_c$ and $A_c^{(0)} = I_c^T A_0 I_c$. Then, applying Lemma 3.1 to each of these pairs, we have that the corresponding pair of their respective inverses and hence the pair with the product matrices $\left(I_k A_k^{-1} I_k^T\ ,\ I_k A_k^{(0)^{-1}} I_k^T\right)$ satisfies (H0) with constants $(c_0 c_1^{-2}, c_0^{-1})$ (i.e., (3.1) and (3.2)). The latter implies that the inverses of the additive Schwarz preconditioners B^{-1} (as defined in (2.2)) and B_0^{-1} (as defined in (3.7)) are related in the same way (as their individual terms $I_k A_k^{-1} I_k^T$ and $I_k A_k^{(0)^{-1}} I_k^T$). That is, (B^{-1}, B_0^{-1}) also satisfy (H0) with constants $(c_0 c_1^{-2}, c_0^{-1})$. Applying once more Lemma 3.1, we straightaway deduce that the pair (B, B_0) also satisfies (H0), now with constants $(c_0^3 c_1^{-2}, c_1^2 c_0^{-1})$.

Now, since B_0 is the classical s.p.d. additive Schwarz preconditioner for the s.p.d., A_0, B_0, and A_0 can be shown to be spectrally equivalent: there exist $\gamma_0, \gamma_1 > 0$ such that

$$\gamma_0 \mathbf{v}^T B_0 \mathbf{v} \le \mathbf{v}^T A_0 \mathbf{v} \le \gamma_1 \mathbf{v}^T B_0 \mathbf{v} \quad \forall\ \mathbf{v} \in \mathbb{R}^n\ , \tag{3.8}$$

where the constants γ_0 and γ_1 might depend on the parameters of the discretization and the problem. B_0 would be optimal if neither γ_0 nor γ_1 depend on the discretization parameters (or size of the system n).

Using this extra information, it is straightforward to deduce that the pair (B, A_0) also satisfies (H0) with constants (β_0, β_1) that depend only on c_0, c_1, γ_0, and γ_1. All these observations are summarized in the following lemma.

LEMMA 3.2. *Let (A, A_0) satisfy Assumption* (H0) *with constants (c_0, c_1). Let B be the additive Schwarz preconditioner of A (defined through* (2.2)*), and let B_0 be the corresponding s.p.d. additive Schwarz preconditioner of A_0 (defined through* (3.7)*) and assume B_0 is such that*

$$\gamma_0 \mathbf{v}^T B_0 \mathbf{v} \le \mathbf{v}^T A_0 \mathbf{v} \le \gamma_1 \mathbf{v}^T B_0 \mathbf{v} \quad \forall\ \mathbf{v} \in \mathbb{R}^n$$

for some $\gamma_0, \gamma_1 > 0$. Then the pair (B, A_0) also satisfies (H0) *with constants (β_0, β_1):*

$$\mathbf{v}^T B \mathbf{v} \ge \beta_0\ \mathbf{v}^T A_0 \mathbf{v} \quad \forall \mathbf{v} \in \mathbb{R}^n \tag{3.9}$$

and

$$\mathbf{w}^T B \mathbf{v} \le \beta_1\ \sqrt{\mathbf{v}^T A_0 \mathbf{v}} \sqrt{\mathbf{w}^T A_0 \mathbf{w}} \quad \forall \mathbf{v},\ \mathbf{w}. \tag{3.10}$$

The constants β_0 and β_1 are given by

$$\beta_0 = \frac{c_0^3}{c_1^2 \gamma_1}, \qquad \beta_1 = \frac{c_1^2}{c_0 \gamma_0}\ . \tag{3.11}$$

We note that the same results, (3.9)–(3.10)*, hold for B replaced with B^T.*

With all of these relations at hand, we define the s.p.d. matrix

$$Z := B A_0^{-1} B^T\ , \tag{3.12}$$

which can be used as a preconditioner for A in GMRES. Observe that the actions of Z^{-1} involve actions of both B^{-1} and B^{-T} as well as multiplications with A_0 (not A_0^{-1}). Therefore, the preconditioner Z is computationally feasible.

We next prove the main result of the section, which guarantees that the preconditioned GMRES method for A with the s.p.d. preconditioner $Z = B A_0^{-1} B^T$ will be convergent with bounds depending only on the constants involved in relations between A and Z.

THEOREM 3.3. *Let (A, A_0) satisfy Assumption* (H0) *with constants (c_0, c_1), and let $B \in \mathbb{R}^{n \times n}$ be the additive Schwarz preconditioner for A, whose inverse is defined through* (2.2)*. Let $Z := B A_0^{-1} B^T$ be a preconditioner for A. Then the pair (A, Z) also satisfies* (H0) *with constants (α_0, α_1) defined by*

$$\alpha_0 = \frac{c_0}{\beta_1} = \frac{c_0^2}{c_1^2} \cdot \gamma_0, \qquad \alpha_1 = \frac{c_1}{\beta_0} = \frac{c_1^2}{c_0^3 \gamma_1}\ . \tag{3.13}$$

Furthermore, the preconditioned GMRES method for A with the s.p.d. preconditioner Z converges with bounds

$$\|\mathbf{r}_m\|_{Z^{-1}} = \|\overline{\mathbf{r}}_m\|_Z \leq \left(1 - \left(\frac{\alpha_0}{\alpha_1}\right)^2\right)^{\left(\frac{m}{2}\right)}, \quad \|\overline{\mathbf{r}}_0\|_Z = \left(1 - \left(\frac{\alpha_0}{\alpha_1}\right)^2\right)^{\left(\frac{m}{2}\right)} \|\mathbf{r}_0\|_{Z^{-1}}, \tag{3.14}$$

where $\overline{\mathbf{r}}_m = Z^{-1}\mathbf{r}_m = Z^{-1}(\mathbf{b}-A\mathbf{x}_m)$ is the preconditioned residual at the mth iteration with $\overline{\mathbf{r}}_0 = Z^{-1}\mathbf{r}_0 := Z^{-1}(\mathbf{b} - A\mathbf{x}_0)$; $\|\cdot\|_Z$ and $\|\cdot\|_{Z^{-1}}$ are the inner product norms induced by the s.p.d. matrices Z and Z^{-1}, respectively.

If we use GMRES with right preconditioning in the standard norm, i.e., $\|\mathbf{v}\| = \sqrt{\mathbf{v}^T\mathbf{v}}$, the following estimate holds:

$$\|\mathbf{r}_m\| \leq \left(\|Z\|\|Z^{-1}\|\right)^{\frac{1}{2}} \left(1 - \left(\frac{\alpha_0}{\alpha_1}\right)^2\right)^{\left(\frac{m}{2}\right)} \|\mathbf{r}_0\|. \tag{3.15}$$

Note that in our case $cond(Z) = \|Z\|\|Z^{-1}\| \simeq cond(A_0) = \mathcal{O}(h^{-2})$.

Proof. From Lemma 3.2, we know that (B, A_0) satisfy (H0) with (β_0, β_1). In particular, the relations (3.9)–(3.10) (used for B^T) show that $X := A_0^{-\frac{1}{2}} B^T A_0^{-\frac{1}{2}}$ is well-conditioned. More precisely, we have

$$\beta_0 \|\mathbf{v}\|^2 \leq \|X\mathbf{v}\|^2 \leq \beta_1 \|\mathbf{v}\|^2 \quad \forall\, \mathbf{v} \in \mathbb{R}^n .$$

That is, the s.p.d. matrix X^TX is well-conditioned. The coercivity of A in terms of A_0 expressed in (2.3) and X^TX being well-conditioned (or bounded) imply that $A_0^{-\frac{1}{2}} A A_0^{-\frac{1}{2}}$ is coercive also in terms of X^TX:

$$\mathbf{v}^T A_0^{-\frac{1}{2}} A A_0^{-\frac{1}{2}} \mathbf{v} \geq c_0 \|\mathbf{v}\|^2 \geq \frac{c_0}{\beta_1} \mathbf{v}^T\, X^TX\mathbf{v} \quad \forall\, \mathbf{v} \in \mathbb{R}^n .$$

Hence, A is coercive in terms of $A_0^{\frac{1}{2}} X^T X A_0^{\frac{1}{2}} = BA_0^{-1}B^T = Z$, which is the first desired result.

Similarly, the boundedness of A in terms of A_0, expressed in (2.4), and X^TX being well-conditioned (or coercive) imply that $A_0^{-\frac{1}{2}} A A_0^{-\frac{1}{2}}$ is bounded also in terms of X^TX:

$$\mathbf{w}^T A_0^{-\frac{1}{2}} A A_0^{-\frac{1}{2}} \mathbf{v} \leq c_1 \|\mathbf{w}\|\|\mathbf{v}\| \leq \frac{c_1}{\beta_0} \sqrt{\mathbf{w}^T X^T X \mathbf{w}}\, \sqrt{\mathbf{v}^T X^T X \mathbf{v}} \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n ,$$

which is equivalent to saying that A is bounded in terms of $A_0^{\frac{1}{2}} X^T X A_0^{\frac{1}{2}} = BA_0^{-1}B^T = Z$. This completes the proof that the pair (A, Z) verifies Assumption (H0) with constants (α_0, α_1) as defined in (3.13).

Since our preconditioner Z is s.p.d., we can apply the (unpreconditioned) GMRES in the standard vector norm to the symmetrically transformed problem $\overline{A}\overline{\mathbf{x}} = \overline{\mathbf{b}}$, where $\overline{A} = Z^{-\frac{1}{2}} A Z^{-\frac{1}{2}}$, $\overline{\mathbf{b}} = Z^{-\frac{1}{2}}\mathbf{b}$, and $\overline{\mathbf{x}} = Z^{\frac{1}{2}}\mathbf{x}$. After rewriting the resulting algorithm in the original quantities it is clear that the resulting method will minimize the (true) residual in the $\|.\|_{Z^{-1}}$-norm. A standard application of the GMRES convergence theory [14] gives (3.14).

If we were to apply GMRES to $A\mathbf{x} = \mathbf{b}$ using, e.g., right preconditioning (cf. [26, section 3.2.2]), then estimate (3.15) is easily seen since

$$\begin{aligned}
\|\mathbf{r}_m\| &= \min\left\{\|p_m(AZ^{-1})\mathbf{r}_0\| : \ p_m(0) = 1,\ p_m \text{ polynomial of degree} \le m\right\}\\
&\le \|Z^{\frac{1}{2}}\| \ \min\left\{\|Z^{-\frac{1}{2}}p_m(AZ^{-1})\mathbf{r}_0\| : \ p_m(0) = 1,\ p_m \text{ polynomial of degree} \le m\right\}\\
&= \|Z^{\frac{1}{2}}\| \ \min\left\{\|p_m(Z^{-\frac{1}{2}}AZ^{-\frac{1}{2}})(Z^{-\frac{1}{2}}\mathbf{r}_0)\| : \ p_m(0) = 1,\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. p_m \text{ polynomial of degree} \le m\right\}\\
&\le \|Z^{\frac{1}{2}}\| \left(1 - \left(\frac{\alpha_0}{\alpha_1}\right)^2\right)^{\left(\frac{m}{2}\right)} \|Z^{-\frac{1}{2}}\mathbf{r}_0\|\\
&\le \|Z^{\frac{1}{2}}\| \, \|Z^{-\frac{1}{2}}\| \left(1 - \left(\frac{\alpha_0}{\alpha_1}\right)^2\right)^{\left(\frac{m}{2}\right)} \|\mathbf{r}_0\|.
\end{aligned}$$

Thus the proof of the theorem is complete. ◻

3.1. Another auxiliary s.p.d. preconditioner W.

To close the section we define another s.p.d. preconditioner, which we shall denote by W, and which uses also only actions of B^{-1} and B^{-T}:

$$W^{-1} = \frac{1}{2}(B^{-1} + B^{-T}). \tag{3.16}$$

Observe that estimates (3.9) and (3.10) from Lemma 3.2 show that the pairs (B, A_0) and (B^T, A_0) satisfy (H0) with constants (β_0, β_1) as given in (3.11). Therefore, both pairs satisfy the assumptions of Lemma 3.1, from which we deduce that the pairs (B^{-1}, A_0^{-1}) and (B^{-T}, A_0^{-T}) are also related through (H0) with constants $(\frac{\beta_0}{\beta_1^2}, \beta_0^{-1})$. This implies in particular that the pair (W, A_0) exhibits property (H0), which means that the preconditioner W is s.p.d. and is spectrally equivalent to A_0.

Arguing then as in the second part of the proof of Theorem 3.3 but with W in place of Z, one can guarantee that the GMRES method applied to A with W as an s.p.d. preconditioner will be convergent. We omit the details for brevity.

4. A combined preconditioner.

In this section, we introduce another preconditioner which in a sense combines the best of both preconditioners B and $Z = BA_0^{-1}B^T$. We define its inverse, $\mathcal{B}^{-1}$, by forming the linear combination

$$\mathcal{B}^{-1} = B^{-1} + \sigma Z^{-1}.$$

The parameter $\sigma \in \mathbb{R}$ is allowed to change from iteration to iteration inside the GMRES iterative solver. Therefore, $\mathcal{B}$ can be regarded as a variable-step (or flexible) preconditioner.

Observe that for $\sigma \ge 0$, by virtue of the analysis of the previous section, the pair $(\mathcal{B}^{-1}, A_0^{-1})$ verifies Assumption (H0); i.e., $\mathcal{B}^{-1}$ is coercive and bounded in the A_0^{-1}-norm. We now describe the (practical) construction of the variable-step preconditioner $\mathcal{B}^{-1}$, but considering a more general form,

$$\mathcal{B}^{-1} = \alpha B^{-1} + \sigma Z^{-1}, \tag{4.1}$$

without assuming the coefficients α and σ have nonnegative sign. We note that since the coefficients α and σ will be allowed to vary from one step to the next, so will $\mathcal{B}^{-1}$; certainly $\mathcal{B}^{-1}$ depends on the iteration m. However, to avoid complicated notation and to ease the presentation, we do not make explicit reference to the iteration m in the notation of $\mathcal{B}^{-1}$ and α and σ, since at all times it will be clear and no confusion should arise.

$\mathcal{B}$ depends on the parameters α and σ and varies from one step to the next. We suggest writing $\mathcal{B}_m$ instead to show the explicit dependence on the iteration count, and the same for the parameters α and σ that define $\mathcal{B}$.

We first discuss the construction of the combined preconditioner and provide a first convergence result for $\mathcal{B}^{-1}$ which asserts faster convergence within GMRES than the one obtained with preconditioner Z^{-1} at a fixed iteration. In section 4.3 we present a variant of the combined preconditioner for which one can indeed guarantee convergence with respect to the GMRES method using only Z^{-1} (at the whole sequence of iterates rather that at a fixed one).

4.1. Construction of the variable preconditioner. We consider the linear system of equations (2.1) that we solve by the preconditioned GMRES method with preconditioner $\mathcal{B}^{-1}$ as defined in (4.1). We now explain how the coefficients are α and σ set inside the GMRES iteration. Let $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ and $\|\cdot\|_* = \sqrt{(.,.)_*}$ be two inner product norms, to be specified and chosen later, and whose role will become clear in the process.

For $m \geq 0$, we denote by $\mathbf{x}_m$ the mth iterate and by $\mathbf{r}_m = \mathbf{b} - A\mathbf{x}_m$ the residual. At the $(m+1)$th iteration of GMRES, we construct the new search direction $\mathbf{d}_{m+1}$ based not only on the previous search directions $\{\mathbf{d}_j\}_{j=0}^m$ but also on the two preconditioned residuals $B^{-1}\mathbf{r}_m$ and $Z^{-1}\mathbf{r}_m$ as follows:

$$\beta_{m+1}\mathbf{d}_{m+1} = \beta_{m+\frac{1}{3}} B^{-1}\mathbf{r}_m + \beta_{m+\frac{2}{3}} Z^{-1}\mathbf{r}_m + \sum_{j=0}^{m} \beta_j \mathbf{d}_j. \tag{4.2}$$

Here, the coefficients β_j, $j = 0, 1, \ldots, m$, are chosen such that

$$(\mathbf{d}_{m+1}, \mathbf{d}_j)_* = 0 \quad \text{for} \quad j < m+1 \qquad \text{and} \qquad \|\mathbf{d}_{m+1}\|_* = \sqrt{(\mathbf{d}_{m+1},\ \mathbf{d}_{m+1})_*} = 1.$$

It is clear, then, that the coefficients $\beta_{m+\frac{s}{3}}$, $s = 1, 2$, can be considered arbitrary parameters at this point. For any such fixed pair in GMRES, the next iterate $\mathbf{x}_{m+1}$ is then computed by minimizing the residual

$$\|\mathbf{b} - A\mathbf{x}_{m+1}\| = \left\| \mathbf{b} - A\left(\mathbf{x}_m + \sum_{j=0}^{m+1} \alpha_j \mathbf{d}_j\right)\right\| \mapsto \min$$

with respect to the coefficients $\{\alpha_j\}_{j=0}^{m+1}$. Notice that out of the two coefficients $\beta_{m+\frac{s}{3}}$, $s = 1, 2$, only their ratio,

$$\sigma = \sigma_{m+1} \equiv \frac{\beta_{m+\frac{2}{3}}}{\beta_{m+\frac{1}{3}}},$$

can be considered a free parameter (the rest is compensated by the α_{m+1} coefficient).

In practice, we proceed as follows. At step $m+1$, based on the previous search directions $\{\mathbf{d}_j\}_{j=0}^m$ and the preconditioned residuals $B^{-1}\mathbf{r}_m$ and $Z^{-1}\mathbf{r}_m$, we have to solve the minimization problem

$$\left\| \mathbf{b} - A\left(\mathbf{x}_m + \sum_{j=0}^{m} \alpha_j \mathbf{d}_j + \alpha_{m+\frac{1}{3}} B^{-1}\mathbf{r}_m + \alpha_{m+\frac{2}{3}} Z^{-1}\mathbf{r}_m\right)\right\| \mapsto \min\ , \tag{4.3}$$

with respect to the coefficients $\{\alpha_j\}_{j=0}^m$ and $\alpha_{m+\frac{s}{3}}$, $s = 1, 2$. As we show next the solution of such a minimization problem can be reduced to the solution of a two-by-two system, by choosing appropriately the inner product $(\cdot,\cdot)_*$.

Consider the quadratic functional $\mathcal{J}(\boldsymbol{\alpha})$ (as a function of the coefficients $\boldsymbol{\alpha} = (\alpha_r)$)

$$\mathcal{J}(\boldsymbol{\alpha}) \equiv \left\| \mathbf{r}_m - A\left(\sum_{j=0}^m \alpha_j \mathbf{d}_j + \alpha_{m+\frac{1}{3}} B^{-1}\mathbf{r}_m + \alpha_{m+\frac{2}{3}} Z^{-1}\mathbf{r}_m\right)\right\|^2 .$$

Then it is obvious that the minimization problem (4.3) reduces to minimizing the functional with respect to the coefficients $\boldsymbol{\alpha} = (\alpha_r)$. We now set the inner product $(\cdot,\cdot)_* = (A(\cdot), A(\cdot))$, which is equivalent to assuming that the search directions $\{\mathbf{d}_j\}$ are $(A(.), A(.))$ orthogonal. Then we have

$$0 = \frac{1}{2}\frac{\partial \mathcal{J}}{\partial \alpha_j} = \alpha_j - (\mathbf{r}_m - \alpha_{m+\frac{1}{3}} AB^{-1}\mathbf{r}_m - \alpha_{m+\frac{2}{3}} AZ^{-1}\mathbf{r}_m,\ A\mathbf{d}_j), \quad j \le m,$$

which gives

$$\alpha_j = (\mathbf{r}_m,\ A\mathbf{d}_j) - \alpha_{m+\frac{1}{3}}(AB^{-1}\mathbf{r}_m,\ A\mathbf{d}_j) - \alpha_{m+\frac{2}{3}}(AZ^{-1}\mathbf{r}_m,\ A\mathbf{d}_j), \quad j \le m. \tag{4.4}$$

Now, setting the partial derivatives of $\mathcal{J}$ with respect to $\alpha_{m+\frac{s}{3}}$ to zero, we get

$$\begin{aligned}
&\alpha_{m+\frac{1}{3}}\|AB^{-1}\mathbf{r}_m\|^2 - \left(\mathbf{r}_m - A\left(\sum_{j=0}^m \alpha_j\mathbf{d}_j + \alpha_{m+\frac{2}{3}}Z^{-1}\mathbf{r}_m\right),\ AB^{-1}\mathbf{r}_m\right) = 0,\\
&\alpha_{m+\frac{2}{3}}\|AZ^{-1}\mathbf{r}_m\|^2 - \left(\mathbf{r}_m - A\left(\sum_{j=0}^m \alpha_j\mathbf{d}_j + \alpha_{m+\frac{1}{3}}B^{-1}\mathbf{r}_m\right),\ AZ^{-1}\mathbf{r}_m\right) = 0.
\end{aligned} \tag{4.5}$$

Substituting α_j from (4.4) into (4.5), we end up with a system of two equations where the only two unknowns are the coefficients $\alpha_{m+\frac{s}{3}}$ with $s = 1, 2$:

(4.6)

- $$\begin{aligned}
&\alpha_{m+\frac{1}{3}}\left(\|AB^{-1}\mathbf{r}_m\|^2 - \sum_j (AB^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)^2\right)\\
&\quad + \alpha_{m+\frac{2}{3}}\left((AZ^{-1}\mathbf{r}_m,\ AB^{-1}\mathbf{r}_m) - \sum_j (AZ^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)(AB^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)\right)\\
&= (\mathbf{r}_m,\ AB^{-1}\mathbf{r}_m) - \sum_j (\mathbf{r}_m,\ A\mathbf{d}_j)(AB^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)\\
&= \left(\mathbf{r}_m,\ AB^{-1}\mathbf{r}_m - \sum_j (AB^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)A\mathbf{d}_j\right),
\end{aligned}$$
- $$\begin{aligned}
&\alpha_{m+\frac{1}{3}}\left((AZ^{-1}\mathbf{r}_m,\ AB^{-1}\mathbf{r}_m) - \sum_j (AB^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)(AZ^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)\right)\\
&\quad + \alpha_{m+\frac{2}{3}}\left(\|AZ^{-1}\mathbf{r}_m\|^2 - \sum_j (AZ^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)^2\right)\\
&= (\mathbf{r}_m,\ AZ^{-1}\mathbf{r}_m) - \sum_j (\mathbf{r}_m,\ A\mathbf{d}_j)(AZ^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)\\
&= \left(\mathbf{r}_m,\ AZ^{-1}\mathbf{r}_m - \sum_j (AZ^{-1}\mathbf{r}_m,\ A\mathbf{d}_j)A\mathbf{d}_j\right).
\end{aligned}$$

To show the solvability of the above system for $\alpha_{m+\frac{s}{3}}$ with $s = 1, 2$ (which will imply that the variable-step preconditioner $\mathcal{B}^{-1}$ is well defined), we use the following lemma.

LEMMA 4.1. *Let* $(H, (.,.))$ *be a Hilbert space with inner product* $(\cdot,\cdot)$, *and let* $\mathbf{h}, \mathbf{f}, \mathbf{g} \in H$. *Let* S *be a finite-dimensional subspace of* H *spanned by an orthonormal system* $\{\mathbf{p}_j\}_{j=1}^m$, *i.e.,* $(\mathbf{p}_i, \mathbf{p}_j) = \delta_{i,j}$. *Let* $\pi = \pi_S : H \longrightarrow S$ *be the orthogonal projection on* S, *with respect to the inner product* $(\cdot,\cdot)$. *Then the best approximation to* $\mathbf{h}$ *from elements from* S *augmented by the two vectors* $\mathbf{f}$ *and* $\mathbf{g}$ *is given as the solution of the least-squares (or minimization) problem*

$$\min_{\substack{\alpha_r \\ r=\frac{1}{3},\frac{2}{3},1,\ldots,m}} \left\| \mathbf{h} - \alpha_{\frac{1}{3}}\mathbf{f} - \alpha_{\frac{2}{3}}\mathbf{g} - \sum_j \alpha_j \mathbf{p}_j \right\| \mapsto \min \tag{4.7}$$

over the coefficients $\{\alpha_r\}$, $r = \frac{1}{3}, \frac{2}{3}, 1, \ldots, m$. *Solving problem* (4.7) *is equivalent to solving the two-by-two system*

$$\begin{pmatrix} \|(I-\pi)\mathbf{f}\|^2 & ((I-\pi)\mathbf{f},\ (I-\pi)\mathbf{g}) \\ ((I-\pi)\mathbf{f},\ (I-\pi)\mathbf{g}) & \|(I-\pi)\mathbf{g}\|^2 \end{pmatrix} \cdot \begin{bmatrix} \alpha_{\frac{1}{3}} \\ \alpha_{\frac{2}{3}} \end{bmatrix} = \begin{bmatrix} (\mathbf{h},\ (I-\pi)\mathbf{f}) \\ (\mathbf{h},\ (I-\pi)\mathbf{g}) \end{bmatrix}, \tag{4.8}$$

which has a unique solution provided $(I-\pi)\mathbf{f}$ *and* $(I-\pi)\mathbf{g}$ *are linearly independent. If* $(I-\pi)\mathbf{f}$ *and* $(I-\pi)\mathbf{g}$ *are linearly dependent, there is also a solution, since the right-hand side in* (4.8) *is compatible. More specifically, the component*

$$\mathbf{h}^{\perp} = \alpha_{\frac{1}{3}}(I-\pi)\mathbf{f} + \alpha_{\frac{2}{3}}(I-\pi)\mathbf{g}$$

is unique including the case of linear dependent components $(I-\pi)\mathbf{f}$ *and* $(I-\pi)\mathbf{g}$ *(even when* $(I-\pi)\mathbf{f} = (I-\pi)\mathbf{g} = 0$*).*

The remaining coefficients $\{\alpha_j\}$ *are computed from* $\pi(\mathbf{h} - \alpha_{\frac{1}{3}}\mathbf{f} - \alpha_{\frac{2}{3}}\mathbf{g}) = \sum_j \alpha_j \mathbf{p}_j$, *that is,*

$$\alpha_j = (\mathbf{h} - \alpha_{\frac{1}{3}}\mathbf{f} - \alpha_{\frac{2}{3}}\mathbf{g},\ \mathbf{p}_j).$$

Proof. It is clear that the least-squares problem (4.7) reduces to finding the best approximation to $\mathbf{h}$ from the space spanned by the two vectors $(I-\pi)\mathbf{f}$ and $(I-\pi)\mathbf{g}$. Indeed, we can rewrite (4.7) as

$$\left\| (I-\pi)\mathbf{h} - \alpha_{\frac{1}{3}}(I-\pi)\mathbf{f} - \alpha_{\frac{2}{3}}(I-\pi)\mathbf{g} - \sum_j {\alpha'}_j \mathbf{p}_j \right\| \mapsto \min.$$

Since the last component $\sum_j {\alpha'}_j \mathbf{p}_j$ is orthogonal to $(I-\pi)\big(\mathbf{h} - \alpha_{\frac{1}{3}}\mathbf{f} - \alpha_{\frac{2}{3}}\mathbf{g}\big)$, the above minimum equals

$$\begin{aligned}
&\min_{\alpha_{\frac{1}{3}},\,\alpha_{\frac{2}{3}}} \min_{\alpha'_j} \left\| (I-\pi)\mathbf{h} - \alpha_{\frac{1}{3}}(I-\pi)\mathbf{f} - \alpha_{\frac{2}{3}}(I-\pi)\mathbf{g} - \sum_j {\alpha'}_j \mathbf{p}_j \right\| \\
&= \min_{\alpha_{\frac{1}{3}},\,\alpha_{\frac{2}{3}}} \min_{\alpha'_j} \left(\|(I-\pi)\mathbf{h} - \alpha_{\frac{1}{3}}(I-\pi)\mathbf{f} - \alpha_{\frac{2}{3}}(I-\pi)\mathbf{g}\|^2 + \|\sum_j {\alpha'}_j \mathbf{p}_j\|^2 \right)^{\frac{1}{2}} \\
&= \min_{\alpha_{\frac{1}{3}},\,\alpha_{\frac{2}{3}}} \|(I-\pi)\mathbf{h} - \alpha_{\frac{1}{3}}(I-\pi)\mathbf{f} - \alpha_{\frac{2}{3}}(I-\pi)\mathbf{g}\| \\
&= \min_{\alpha_{\frac{1}{3}},\,\alpha_{\frac{2}{3}}} \left(\|\mathbf{h} - \alpha_{\frac{1}{3}}(I-\pi)\mathbf{f} - \alpha_{\frac{2}{3}}(I-\pi)\mathbf{g}\|^2 - \|\pi\mathbf{h}\|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The last problem leads exactly to the Gram system (4.8). We note that

$$\min_{\alpha_{\frac{1}{3}},\,\alpha_{\frac{2}{3}}} \|\mathbf{h} - \alpha_{\frac{1}{3}}(I-\pi)\mathbf{f} - \alpha_{\frac{2}{3}}(I-\pi)\mathbf{g}\| = \|\mathbf{h} - \mathbf{h}^{\perp}\|,$$

where $\mathbf{h}^{\perp}$ is the unique best approximation to $\mathbf{h}$ from the space spanned by the vectors $(I-\pi)\mathbf{f}$ and $(I-\pi)\mathbf{g}$. The latter space has dimension zero, one, or two. This completes the proof. ∎

We now apply the last lemma to our case to show that the system (4.6) has a solution and hence $\mathcal{B}^{-1}$ is well defined. We set $\mathbf{h} = A^{-1}\mathbf{r}_m$, $\mathbf{f} = B^{-1}\mathbf{r}_m$, $\mathbf{g} = Z^{-1}\mathbf{r}_m$, and $\{\mathbf{p}_j\} = \{\mathbf{d}_j\}_{j=0}^m$ for the vector space with inner product $(\cdot,\ \cdot)_* = (A(\cdot),\ A(\cdot))$. Then using that the $\{\mathbf{d}_j\}$ are $(\cdot,\ \cdot)_*$-orthonormal, we conclude by applying Lemma 4.1 that the system (4.6) is solvable.

Once the coefficients $\alpha_{m+\frac{s}{3}}$, $s = 1, 2$, are determined, we compute the new direction $\mathbf{d}_{m+1}$ from

$$\beta_{m+1}\mathbf{d}_{m+1} = \alpha_{m+\frac{1}{3}}B^{-1}\mathbf{r}_m + \alpha_{m+\frac{2}{3}}Z^{-1}\mathbf{r}_m - \sum_{j=0}^{m}\beta_j\mathbf{d}_j$$

by choosing the coefficients $\{\beta_j\}_{j=0}^m$ to satisfy the required orthogonality conditions

$$(\mathbf{d}_{m+1}, \mathbf{d}_j)_* = (A\mathbf{d}_{m+1}, A\mathbf{d}_j) = 0 \quad \text{for } j < m+1,$$

which gives (assuming by induction that $(\mathbf{d}_j,\ \mathbf{d}_k)_* = \delta_{j,k}$)

$$\beta_j = (\alpha_{m+\frac{1}{3}}B^{-1}\mathbf{r}_m + \alpha_{m+\frac{2}{3}}Z^{-1}\mathbf{r}_m,\ \mathbf{d}_j)_* \quad \text{for } j \le m.$$

The last coefficient, β_{m+1}, is computed so that $\|\mathbf{d}_{m+1}\|_* = 1$.

4.2. Convergence. We now establish two results that provide estimates for the convergence of the variable preconditioned GMRES method.

THEOREM 4.2. *Let (A, A_0) satisfy Assumption* (H0) *with constants (c_0, c_1), and let $B \in \mathbb{R}^{n\times n}$ be the additive Schwarz preconditioner for A, whose inverse is defined through* (2.2). *Let $Z := BA_0^{-1}B^T$ be a preconditioner for A. Let $\mathcal{B}$ be the variable-step preconditioner with inverse defined through* (4.1) *with coefficients determined inside the GMRES iteration by minimization of the residual. Then the variable preconditioned GMRES method for A, at every step k, converges at least as fast as performing one iteration of the restarted (at step k) preconditioned GMRES method with preconditioner Z.*

Proof. The proof of the theorem follows by the definition of $\mathcal{B}^{-1}$ (as explained before). From its construction it is straightforward to infer the following comparative convergence estimate:

$$\|\mathbf{r}_{m+1}\| \le \min_{\alpha,\,\sigma} \|\mathbf{r}_m - A(\alpha B^{-1} + \sigma Z^{-1})\mathbf{r}_m\| \le \min_{\sigma} \|\mathbf{r}_m - \sigma A Z^{-1}\mathbf{r}_m\|.$$

Moreover, if we choose $\|\cdot\|$ to be the norm $\|\mathbf{v}\|_{Z^{-1}} = \sqrt{\mathbf{v}^T Z^{-1}\mathbf{v}}$, we can apply the result from Theorem 3.3. We then have

$$\begin{aligned}\|\mathbf{r}_{m+1}\|_{Z^{-1}} &\le \min\nolimits_{\sigma} \|\mathbf{r}_m - \sigma A Z^{-1}\mathbf{r}_m\|_{Z^{-1}} \\ &\le \left[1 - \left(\frac{\alpha_0}{\alpha_1}\right)^2\right]^{\frac{1}{2}} \|\mathbf{r}_m\|_{Z^{-1}} \le \cdots \le \left[1 - \left(\frac{\alpha_0}{\alpha_1}\right)^2\right]^{\frac{m+1}{2}} \|\mathbf{r}_0\|_{Z^{-1}}.\end{aligned}$$

That is, estimate (3.14) is an upper bound for the convergence estimate of the combined preconditioned GMRES method. Therefore, we showed that the rate of convergence of the combined preconditioned GMRES can be bounded at least with the same bound as that of the GMRES method with preconditioner Z only. ∎

We wish to stress that Theorem 4.2 ensures that at *every iteration* the combined preconditioner gives better reduction of the residual than applying one iteration of the restarted GMRES using only one of the preconditioners, Z (or B), and dropping the previous search directions. This is sufficient to show convergence of the combined preconditioned GMRES algorithm since in our case with Z being s.p.d. we can prove that it is convergent (cf. Theorem 3.3).

We close the section by showing (arguing as before) convergence for another combined preconditioner, the performance of which is also demonstrated in the numerical experiments section (section 5).

THEOREM 4.3. *Let* (A, A_0) *satisfy Assumption* (H0) *with constants* (c_0, c_1), *and let* $B \in \mathbb{R}^{n\times n}$ *be the additive Schwarz preconditioner for* A, *whose inverse is defined through* (2.2). *Let* $\mathcal{B}$ *be the variable-step preconditioner with inverse defined through*

$$\mathcal{B}^{-1} = \alpha B^{-1} + \sigma B^{-T}, \tag{4.9}$$

with coefficients determined inside the GMRES iteration by minimization of the residual. Then the variable preconditioned GMRES method for A, *at every step* k, *converges at least as fast as performing one iteration of the restarted (at step* k*) preconditioned GMRES method with preconditioner* W *defined in* (3.16).

Proof. We start by recalling that the GMRES method with the s.p.d. preconditioner W defined in (3.16) is convergent for A. Then, arguing as in the proof of Theorem 4.2, we have

$$\begin{aligned}\|\mathbf{r}_{m+1}\| &\le \min\nolimits_{\alpha,\,\sigma}\ \|\mathbf{r}_m - A(\alpha B^{-1} + \sigma B^{-T})\mathbf{r}_m\| \\ &\le \min\nolimits_{\sigma}\ \|\mathbf{r}_m - \sigma A(\tfrac{1}{2}\left(B^{-1} + B^{-T}\right))\mathbf{r}_m\| \\ &= \min\nolimits_{\sigma}\ \|\mathbf{r}_m - \sigma A W^{-1}\mathbf{r}_m\|.\end{aligned}$$

Since we have convergence for the latter one, the proof is complete. ∎

Notice that, arguing as in the proof of Theorem 4.2, one could straightforwardly establish the bounds

$$\|\mathbf{r}_{m+1}\| \le \min_{\alpha,\,\sigma}\ \|\mathbf{r}_m - A(\alpha B^{-1} + \sigma B^{-T})\mathbf{r}_m\| \le \min_{\sigma}\ \|\mathbf{r}_m - \sigma A B^{-T}\mathbf{r}_m\|,$$

$$\|\mathbf{r}_{m+1}\| \le \min_{\alpha,\,\sigma}\ \|\mathbf{r}_m - A(\alpha B^{-1} + \sigma B^{-T})\mathbf{r}_m\| \le \min_{\alpha}\ \|\mathbf{r}_m - \alpha A B^{-1}\mathbf{r}_m\|.$$

However, they might or might not be overestimates. From them, it is not possible to deduce a rigorous convergence comparison.

4.3. A modified GMRES method with combined preconditioning. Here, we present a modified GMRES method which can be guaranteed to converge better than standard GMRES using preconditioner Z but which is much more expensive than the previous described method.

The possible modification of the algorithm we present now allows for comparing the convergence of the GMRES method using the combined preconditioner and using only one of the preconditioners.

The idea is to incorporate in (4.2) the set of search directions $\{\mathbf{d}_j^Z\}_{j=0}^m$ generated by running separately the GMRES method with the Z preconditioner. Obviously we are increasing the dimension of the Krylov subspaces over which the minimization of the residual norm is done. Notice that the variable-step preconditioner (4.1) corresponds to the case $m_0 = 0$ in the present algorithm.

We start constructing two sets of search directions, the set $\{\mathbf{d}_j\}_{j=1}^m$ and another set $\{\mathbf{d}_j^Z\}_{j=0}^m$ that corresponds to running GMRES using Z as a preconditioner which also produces its residual $\mathbf{r}_m^Z$. Based on $\mathbf{r}_m^Z$ and $\{\mathbf{d}_j^Z\}_{j=0}^m$, we compute $\mathbf{d}_{m+1}^Z$, which is a linear combination of $Z^{-1}\mathbf{r}_m^Z$ and the set $\{\mathbf{d}_j^Z\}_{j=1}^m$. This process is in principle independent of the combined preconditioned GMRES and can be run separately.

At step $m+1$ of the combined method, we construct a new search direction $\mathbf{d}_{m+1}$ as follows:

$$\beta_{m+1}\mathbf{d}_{m+1} = \beta_{m+\frac{1}{3}} B^{-1}\mathbf{r}_m + \beta_{m+\frac{2}{3}} Z^{-1}\mathbf{r}_m + \sum_{j=0}^{m} \beta_j \mathbf{d}_j + \sum_{j=0}^{m} \beta_j^Z \mathbf{d}_j^Z. \tag{4.10}$$

The difference from the previous version (see (4.2)) is that we also use the search directions coming from the GMRES process corresponding to the preconditioner Z.

The coefficients β_j and β_j^Z are computed from the orthogonality conditions

$$(\mathbf{d}_{m+1}, \mathbf{d}_j)_* = 0 \quad \text{and} \quad (\mathbf{d}_{m+1},\ \mathbf{d}_j^Z)_* = 0 \quad \text{for} \quad j = 0, \dots, m. \tag{4.11}$$

Note that by construction we are ensuring that

$$(\mathbf{d}_j,\ \mathbf{d}_k^Z)_* = 0 \quad \text{for } j > k \geq 0. \tag{4.12}$$

Now, the coefficients $\beta_{\frac{1}{3}}$ and $\beta_{\frac{2}{3}}$ in (4.10) can be computed as in Lemma 4.1, where π is now the $(.,.)_*$-orthogonal projection on the space spanned by the two sets of search directions, $\{\mathbf{d}_j\}_{j=1}^m$ and $\{\mathbf{d}_j^Z\}_{j=0}^m$, and setting $\mathbf{h} = \mathbf{r}_m$. Then the system for the coefficients β_j and β_j^Z, $j \leq m$, (4.11), is solvable and simplifies somewhat using the orthogonality of the two search directions given in (4.11) and (4.12). The next iterate of the method takes the form

$$\begin{aligned}\mathbf{x}_{m+1} &= \mathbf{x}_m + \sum_{j=0}^{m+1} \left(\overline{\alpha}_j \mathbf{d}_j + \overline{\alpha}_j^Z \mathbf{d}_j^Z\right) \\ &= \mathbf{x}_0 + \sum_{j=0}^{m+1} \left(\beta_j \mathbf{d}_j + \beta_j^Z \mathbf{d}_j^Z\right).\end{aligned}$$

The latter equality holds, since by construction $\mathbf{x}_m - \mathbf{x}_0$ is spanned by the previous search directions $\{\mathbf{d}_j\}_{j=0}^m$ and $\{\mathbf{d}_j^Z\}_{j=0}^m$. The coefficients $\{\beta_j\}_{j=0}^m$ and $\{\beta_j^Z\}_{j=0}^m$ are computed by solving the minimization problem

$$\begin{aligned}\|\mathbf{r}_{m+1}\| &= \min_{\{\overline{\alpha}_j\}_{j=0}^m,\ \{\overline{\alpha}_j^Z\}_{j=0}^{m+1}} \left\| \mathbf{r}_m - A\left(\sum_{j=0}^{m+1} \overline{\alpha}_j \mathbf{d}_j + \sum_{j=0}^{m+1} \overline{\alpha}_j^Z \mathbf{d}_j^Z\right)\right\| \\ &= \min_{\{\beta_j\}_{j=0}^m,\ \{\beta_j^Z\}_{j=0}^{m+1}} \left\| \mathbf{r}_0 - A\left(\sum_{j=0}^{m+1} \beta_j \mathbf{d}_j + \sum_{j=0}^{m+1} \beta_j^Z \mathbf{d}_j^Z\right)\right\|.\end{aligned}$$

This immediately shows that

$$\begin{aligned}\|\mathbf{r}_{m+1}\| &= \min_{\{\beta_j,\, \beta_j^Z\}_{j=0}^{m+1}} \left\| \mathbf{r}_0 - A\left(\sum_{j=0}^{m+1} \beta_j \mathbf{d}_j + \sum_{j=0}^{m+1} \beta_j^Z \mathbf{d}_j^Z\right)\right\| \\ &\le \min_{\{\beta_j^Z\}_{j=0}^{m+1},} \left\| \mathbf{r}_0 - A\left(\sum_{j=0}^{m+1} \beta_j^Z \mathbf{d}_j^Z\right)\right\| \\ &= \|\mathbf{r}_{m+1}^Z\|.\end{aligned}$$

At this point, we wish to note that the full version of the modified method might not be very practical for large m since we need $2m$ search directions and require $2m$ applications of the preconditioner Z (in addition to m actions of the preconditioner B). A more practical version of the method might be to use restart for computing the Z-based search directions (or both sets of search directions) and use the above modified method for $m = m_0 \ge 0$ steps. We have focused in the present paper on the (extreme) case $m_0 = 0$, corresponding to the variable-step preconditioner (4.1). The more general case $m_0 > 0$, however, requires a more detailed investigation and is left for future study.

5. Applications and numerical results. In this section we present an application of the results presented in the previous sections, which will allow us to verify the developed theory and assess the performance of the different preconditioners.

The application we consider comes from a nonsymmetric discontinuous Galerkin discretization of an elliptic problem, which was the starting motivation of this work. In [2, 3], additive and multiplicative Schwarz preconditioners were developed for the solution of the above algebraic system. In both works, the authors show that the GMRES convergence theory cannot be applied for explaining the observed convergence since the preconditioned system does not satisfy the *sufficient* conditions for such a theory. Here we aim at comparing the performance of the different preconditioners introduced in the previous sections for such discretizations.

More precisely, we consider the following model problem on the unit square with domain $\Omega = [0, 1]^2$:

$$-\Delta u^* = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega\,,$$

where the right-hand side f is chosen so the exact solution is $u^* = \sin(\pi x)\sin(\pi y)$. We focus on the incomplete interior penalty discontinuous Galerkin (IIPG) [13] discretization of the above model problem, with linear discontinuous finite element space (denoted by V^{DG}) on a shape-regular triangulation of Ω, denoted by $\mathcal{T}_h$. The resulting method reads as follows: Find $u \in V^{DG}$ such that

$$a_h(u, v) = \int_\Omega f v\, dx \quad \forall\, v \in V^{DG}\,.$$

The bilinear form of the IIPG method is given by

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \nabla u \cdot \nabla v\, dx - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla u\}\!\} \cdot [\![v]\!]\; ds + \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|} \int_e [\![u]\!] \cdot [\![v]\!]\; ds \tag{5.1}$$

for all $u, v \in V^{DG}$. Here, $K \in \mathcal{T}_h$ refers to an element of the triangulation, $e \subset \partial K$ denotes an edge of the element, and we have denoted by $\mathcal{E}_h$ the set of all such edges or skeleton of the partition $\mathcal{T}_h$. We have used the standard definition of the average $\{\{.\}\}$ and jump $[\![.]\!]$ operators from [5], and the penalty parameter η is set to 5 in all the experiments. We denote by A_h the matrix representation of the operator associated to the bilinear form (5.1), with standard Lagrange basis functions. As noted in the first section, with a small abuse of notation, we use the same notation A, B^{-1}, Z^{-1}, and so on for operators and for their representations as matrices. Here, $\mathbf{u}$ and $\mathbf{f}$ denote the vector representations (in the same basis) of the solution (that we aim to compute) and the right-hand side. In the end, the solution process amounts to solving the nonsymmetric system

$$A_h \mathbf{u} = \mathbf{f}. \tag{5.2}$$

The preconditioners we use are based on the standard two-level overlapping domain decomposition additive Schwarz preconditioner, which we denote by B^{-1}. To define it, we consider an overlapping partition of Ω into rectangular subdomains Ω_k which overlap each other by an amount equal to the fine discretization size h. Then

$$B^{-1} = I_H A_H^{-1} I_H^T + \sum_{k=1}^{N_s} I_k A_k^{-1} I_k^T. \tag{5.3}$$

Here the A_k operators are restrictions of the original operator A_h to the finite element space V_k that is only supported on Ω_k; that is, they correspond to the bilinear forms,

$$a_k(u, v) = a_h(u, v) \quad \forall\, u, v \in V_k,$$

as in [16, 11]. Since $V_k \subset V_h$, the operators I_k are standard injection. The operator A_H corresponds to the bilinear form (5.1) on a coarser discretization of the original domain Ω, where we label the coarse discretization size H. We assume that the fine mesh is a refinement of the coarse mesh used to represent A_H so that I_H is the natural injection on nested grids. The penalty parameter on the coarse solver is taken to be $5H/h$ in order to account for the difference of scales in the edge lengths in the penalty terms (see [2, 16] for further details). We implement these preconditioners on a parallel machine where each subdomain of the Schwarz preconditioner corresponds to one processor, where this domain decomposition is in principle independent of the coarse space V_H. The subdomains are square or rectangular, so that on four processors the decomposition is two by two and on eight processors it is four by two, and so on for larger processor counts. The tests were run on a commodity Linux cluster using PETSc as the software framework [9] and direct solves for the subdomain problems.

Another preconditioner we consider is

$$Z^{-1} = B^{-T} A_0 B^{-1}, \tag{5.4}$$

as outlined in the analysis above. Here A_0 is a symmetric operator corresponding to the bilinear form

$$a_0(u, v) = \sum_K \int_K \nabla u \cdot \nabla v \, dx + \sum_e \frac{\eta_0}{|e|} \int_e [\![u]\!] \cdot [\![v]\!] \; ds, \tag{5.5}$$

where the penalty parameter $\eta_0 = 5$ is the same as it is in the full bilinear form (5.1).

In what follows we consider four different preconditioning techniques for the nonsymmetric system (5.2), namely,

Table 1
Iterations to convergence for $h = 2^{-7}, H = 2^{-5}$ with a nearly exact coarse solver and no restart.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	Z^{-1}	B^{-1}
4	15	16	38	16
8	15	18	42	18
16	16	18	42	18
32	16	18	44	18
64	15	17	43	17
128	16	18	46	18

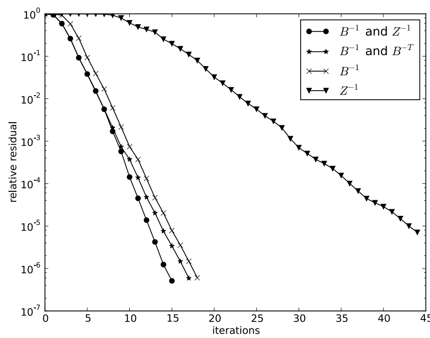

Fig. 1. *Convergence of the relative residual ($\|r_k\|/\|r_0\|$) on a log scale versus iteration number k for the different preconditioning strategies, corresponding to the tests in Table* 1 *with $N_s = 32$.*

1. the standard additive Schwarz preconditioner B^{-1} (5.3) used in a right-preconditioned GMRES algorithm, for comparison with the other options;
2. the preconditioner $Z^{-1} = B^{-T}A_0B^{-1}$ from (3.12) again used in a right-preconditioned GMRES;
3. the variable-step preconditioner GMRES variant (4.1) that uses combinations of B^{-1} and Z^{-1};
4. the variable-step preconditioner GMRES variant (4.9) that uses combinations of B^{-1} and B^{-T}.

We note that in the third case, if we have applied B^{-1} to a vector $\mathbf{u}$, we can construct $Z^{-1}\mathbf{u}$ by applying $B^{-T}A_0$ to save ourselves one preconditioner application. Our comparisons are done with right-preconditioned GMRES because the procedure in section 4.3 minimizes the true residual, as in right-preconditioned GMRES, rather than the preconditioned residual.

The number of GMRES iterations necessary to reduce the relative residual by 10^{-6} for our four different preconditioning approaches using various numbers of subdomains N_s for a fixed problem is given in Table 1. Here we solve the coarse problem involving A_H^{-1} to a tolerance of 10^{-10} so that this solve is nearly exact in order to satisfy the theory more closely. The preconditioning techniques are seen to be scalable in the sense that the number of iterations does not increase as N_s increases for all four methods (see [28, Def. 1.3]). The corresponding convergence curves for the case $N_s = 32$ are shown in Figure 1. We show the convergence factors for this test in Table 2, where the convergence factor is defined as

$$\rho = \left(\frac{\|r_k\|}{\|r_0\|}\right)^{1/k}$$

TABLE 2
Convergence factor for $h = 2^{-7}, H = 2^{-5}$ with a nearly exact coarse solver and no restart.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	Z^{-1}	B^{-1}
4	0.38	0.41	0.73	0.41
8	0.40	0.45	0.76	0.45
16	0.40	0.44	0.75	0.44
32	0.40	0.45	0.76	0.45
64	0.39	0.44	0.76	0.44
128	0.39	0.45	0.77	0.45

TABLE 3
Time to solution for $h = 2^{-7}, H = 2^{-5}$ with a nearly exact coarse solver and no restart.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	Z^{-1}	B^{-1}
4	1.74	1.79	3.12	0.84
8	1.39	1.57	3.22	0.74
16	0.70	0.75	1.58	0.34
32	1.66	1.83	4.63	0.94
64	1.44	1.66	4.65	0.88
128	2.70	2.98	8.59	1.76

for the true residuals r_k, where k is the number of iterations.

To get a rough idea of computational cost, we show the time to solution in seconds for the four approaches in Table 3. We conclude that the Z^{-1} preconditioner is not competitive because it is the most expensive in terms of time per iteration, and it also requires the most iterations. For this reason we do not consider it further in these numerical results. The two preconditioning techniques that use the variable-step preconditioner GMRES variant are seen to be effective in convergence rate but to be somewhat more expensive than the classical B^{-1} preconditioner, as we might expect. The timings presented in this table are intended to compare the computational costs of the preconditioning strategies and not to demonstrate high performance on parallel machines, which is not evident unless we consider a much larger problem.

In practice for parallel computing applications the coarse solve in (5.3) would not be done exactly. Another modification that is often made in practice is to restart GMRES after several iterations. In Tables 4, 5, and 6 we repeat the previous experiment where the relative residual tolerance for the coarse solves is set to 10^{-4} and GMRES is restarted every 10 iterations. (These tables should be compared to Tables 1, 2, and 3, respectively.) We see that the convergence behavior is quite similar and the computational cost is lower, suggesting that these common modifications are also useful and effective for our preconditioning strategies. Corresponding convergence curves are shown in Figure 2.

To see how these methods scale to larger problems, we consider a much finer mesh in Tables 7, 8, and 9, while keeping the mesh size for the coarse solve quite coarse. The scalability of the preconditioners in terms of iterations is still present, the preconditioner still performs well, and in these cases we can see fairly good parallel scalability in the sense that for a fixed problem size, doubling the number of processors in the parallel solve roughly cuts the execution time in half for all our preconditioning strategies. Again convergence curves for this case are shown in Figure 3.

In all the results we have presented so far, the variable-step preconditioner GMRES variant has performed slightly better than the classical B^{-1} preconditioner in terms of number of iterations to convergence, but the increased computational cost per it-

Table 4

Iterations to convergence for $h = 2^{-7}, H = 2^{-5}$ with an inexact coarse solver and restarting every 10 iterations.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	B^{-1}
4	15	18	21
8	16	18	20
16	16	18	20
32	16	18	19
64	16	18	19
128	16	18	19

Table 5

Convergence factor for $h = 2^{-7}, H = 2^{-5}$ with an inexact coarse solver and restarting every 10 iterations.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	B^{-1}
4	0.39	0.46	0.51
8	0.41	0.46	0.49
16	0.41	0.46	0.48
32	0.41	0.46	0.48
64	0.39	0.45	0.46
128	0.40	0.46	0.48

Table 6

Time to solution for $h = 2^{-7}, H = 2^{-5}$ with an inexact coarse solver and restarting every 10 iterations.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	B^{-1}
4	1.59	1.73	0.99
8	1.29	1.35	0.72
16	0.59	0.62	0.35
32	0.73	0.79	0.46
64	0.67	0.71	0.44
128	0.89	0.95	0.60

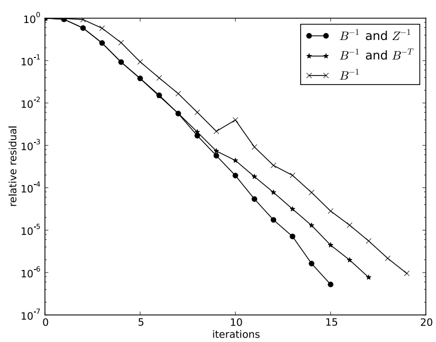

Fig. 2. *Convergence of the relative residual ($\|r_k\|/\|r_0\|$) on a log scale versus iteration number k for the different preconditioning strategies, corresponding to the tests in Table 4 with $N_s = 32$.*

eration of the combined preconditioned GMRES method has ended up making the classical preconditioner perform better in execution time. This suggests that the new method may be competitive in settings where each iteration is very expensive, so that the savings in iteration count can make up for the increased cost per iteration. To

TABLE 7
Iterations to convergence for $h = 2^{-10}, H = 2^{-6}$ with an inexact coarse solver and restarting every 10 iterations.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	B^{-1}
32	30	32	31
64	30	31	32
128	30	31	31
256	30	31	30

TABLE 8
Convergence factor for $h = 2^{-10}, H = 2^{-6}$ with an inexact coarse solver and restarting every 10 iterations.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	B^{-1}
32	0.62	0.64	0.64
64	0.63	0.64	0.64
128	0.62	0.63	0.63
256	0.63	0.63	0.63

TABLE 9
Time to solution for $h = 2^{-10}, H = 2^{-6}$ with an inexact coarse solver and restarting every 10 iterations.

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	B^{-1}
32	94.32	95.55	40.96
64	37.89	37.59	16.66
128	15.55	15.60	7.82
256	7.76	8.81	5.50

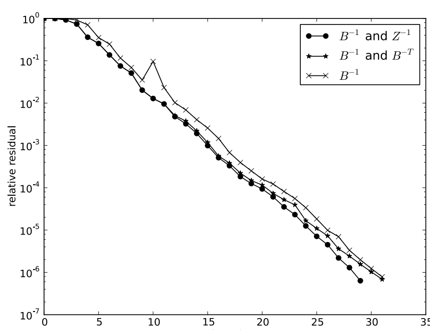

FIG. 3. *Convergence of the relative residual ($\|r_k\|/\|r_0\|$) on a log scale versus iteration number k for the different preconditioning strategies corresponding to the tests in Table 7 with $N_s = 128$.*

examine this setting we consider a problem in Tables 10 and 11 where the coarse grid solve is done on a relatively fine grid and is therefore quite expensive. The results in this somewhat artificial setting do show that the new methods are competitive with the classical preconditioning techniques in terms of computational cost.

As a final test, we consider the variable-step preconditioner GMRES method using (4.1) for solving a nonsymmetric convection-diffusion problem. The model problem is set on $\Omega = [0, 1]^2$ and given by

$$-\epsilon\Delta u^* + \mathbf{b} \cdot \nabla u^* = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega\,.$$

TABLE 10
Iterations to convergence for $h = 2^{-10}, H = 2^{-9}$ with an inexact coarse solver and restarting every 10 *iterations.*

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	B^{-1}
32	19	20	23
64	17	18	22
128	15	17	21
256	15	16	20

TABLE 11
Time to solution for $h = 2^{-10}, H = 2^{-9}$ with an inexact coarse solver and restarting every 10 *iterations.*

N_s	B^{-1} and Z^{-1}	B^{-1} and B^{-T}	B^{-1}
32	277.50	280.69	352.85
64	170.94	187.29	133.73
128	79.01	93.61	99.50
256	40.92	45.41	39.59

TABLE 12
For the convection-diffusion problem, number of iterations to convergence with $h = 2^{-7}, H = 2^{-5}, \epsilon = 0.01$, using a nearly exact coarse solver and no restart.

N_s	B^{-1}	B^{-1} and Z^{-1}
4	18	16
8	19	17
16	19	17
32	20	18
64	20	18
128	20	19

In our tests we take $\mathbf{b} = (2, 1)^T$ and $\epsilon = 0.01$. The data f is chosen so that the exact solution is

$$u^* = \left[x + \frac{e^{b_1 x/\epsilon} - 1}{1 - e^{b_1/\epsilon}} \right] \cdot \left[y + \frac{e^{b_2 y/\epsilon} - 1}{1 - e^{b_2/\epsilon}} \right]$$

which has a boundary layer at the top and right sides of the domain. For this problem we use the symmetric interior penalty Galerkin (SIPG) method [4, 5] for discretizing the elliptic operator and standard upwind for the convection term. The nonsymmetry here comes from the discretization of the convective term.

The results we present are for the combined GMRES with the two preconditioners B^{-1} and Z^{-1}, but for Z^{-1} we need to define a symmetric positive definite auxiliary matrix A_0. To try to include some of the convective terms, we define A_0 to correspond to the bilinear form

$$a_0^c(u, v) = \epsilon a_0(u, v) + \sum_e \int_e |\mathbf{b} \cdot \mathbf{n}| \, [\![u]\!] \cdot [\![v]\!] \, ds,$$

where $a_0(\cdot, \cdot)$ is the symmetrized form previously defined in (5.5).

Results for this example are shown in Tables 12 and 13, where we see that the convergence of the variable-step preconditioner performs better than the usual Schwarz preconditioning. We do not present timing for this example because we used a different computer than for the previous results, but qualitatively the timing comparison is similar to that in Table 3.

TABLE 13

For the convection-diffusion problem, convergence factor with $h = 2^{-7}, H = 2^{-5}, \epsilon = 0.01$, using a nearly exact coarse solver and no restart.

N_s	B^{-1}	B^{-1} and Z^{-1}
4	0.46	0.41
8	0.46	0.43
16	0.47	0.44
32	0.49	0.46
64	0.49	0.46
128	0.49	0.48

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