Residual extrapolation operators for efficient wavefield construction

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SUMMARY
Solving the wave equation using finite-difference approximations allows for fast extrapolation of the wavefield for modelling, imaging and inversion in complex media. It, however, suffers from dispersion and stability-related limitations that might hamper its efficient or proper application to high frequencies. Spectral-based time extrapolation methods tend to mitigate these problems, but at an additional cost to the extrapolation. I investigate the prospective of using a residual formulation of the spectral approach, along with utilizing Shanks transform-based expansions, that adheres to the residual requirements, to improve accuracy and reduce the cost. Utilizing the fact that spectral methods excel (time steps are allowed to be large) in homogeneous and smooth media, the residual implementation based on velocity perturbation optimizes the use of this feature. Most of the other implementations based on the spectral approach are focussed on reducing cost by reducing the number of inverse Fourier transforms required in every step of the spectral-based implementation. The approach here fixes that by improving the accuracy of each, potentially longer, time step.

Key words: Numerical solutions; Fourier analysis; Numerical approximations and analysis; Body waves; Wave propagation; Acoustic properties.

1 INTRODUCTION
In imaging and inversion applications, and with the recent advances in our computational capacity, short changing the wave equation with high-frequency approximations is no longer necessary, and thus, wavefield time extrapolation methods are now commonly used in practice (Geller & Takeuchi 1998; Kristek et al. 2010). Wave equation solutions based on wavefield time extrapolation are still relatively expensive. In addition, wavefield extrapolation methods, in general, may provide accurate results, but are short on the details, compared to those commonly associated with solutions using wavefield geometrical features (traveltimes).

We typically use second-order, fourth-order or even higher-order finite-difference approximations of the wave equation to extrapolate wavefields. These approximations provide reasonably efficient and flexible extrapolation. They are also very easy to implement and may include initial and boundary conditions, including irregular boundaries. However, such an approach suffers from dispersion and stability-related issues requiring fine space sampling, and consequently small time extrapolation steps. It also does not allow for simple separation of wave modes (solutions) necessary to isolate desired wavefield solutions like the P-wave mode in anisotropic acoustic media (Etgen & Brandsberg-Dahl 2009).

On the other hand, spectral (Tal-Ezer et al. 1987; Reshef et al. 1988), including pseudospectral and analytical, methods for extrapolation have emerged recently to address some of these limitations (Etgen & Brandsberg-Dahl 2009; Chu & Stoffa 2011). Spectral methods extrapolate wavefields by applying the Laplace operator as a phase shift in the Fourier (wavenumber) domain. Thus, the extrapolation is overall dispersion-free and unconditionally stable regardless of the length of the time step. However, large time extrapolation steps in complex media induces inaccuracies in the wavefield solution inducing a smoothing effect to the velocity. Thus, the Fourier implementation poses challenges in handling inhomogeneous media in an efficient way due to the mixed space-wavenumber domain extrapolation operator. These challenges are addressed using smart variable separation methods (Etgen & Brandsberg-Dahl 2009; Zhang & Zhang 2009; Pestana & Stoffa 2010; Fomel et al. 2010), including simple Taylor’s series type expansions of the extrapolator operator (Etgen 1989; Pestana & Stoffa, 2010) to implement pseudospectral methods. Despite the effectiveness of these methods they are not cheap. More importantly they rarely add to our understanding of the subsurface, especially with regard to medium parameter dependency of the extrapolator operator.

Luckily, perturbation theory allows us to divide a complicated problem to a series of simpler problems, where the bulk of the solution can be represented by a few leading terms of the series (Bender & Orszag 1978). Though perturbation theory assumes a constant and independent perturbed parameter (in our case velocity), the small time extrapolation nature of the problem, which is based on a homogeneous medium assumption within the time step (Huygens’s principle), allows us to readily handle any
complexity within the wavelength and the grid spacing involved. Based on this concept, I apply Shank’s transform (Bender & Orszag, 1978)-based expansions (to improve on the Taylor’s series accuracy) that allow for easy separation of the wavefield to a background part and a perturbation part. Though residual operators may be inherently addressed in many implementations, I highlight this aspect here to focus on the value of the velocity perturbation in enhancing performance and accuracy. Later, I apply the approach on the complex BP model.

THE SPECTRAL APPROACH

The wave equation in the Fourier space-wavenumber domain reduces to a second-order ordinary differential linear equation in time. For homogeneous acoustic media, it has the following general form:

$$\frac{\partial^2 \hat{P}}{\partial t^2} = \phi^2(\hat{k}) \hat{P}(t, \hat{k}),$$  \hspace{1cm} (1)

where $\hat{P}(t, \hat{k})$ is the wavefield in the wavenumber domain given by $\hat{k} = (k_x, k_y, k_z)$ at time $t$. It is related to the wavefield in the space domain, $P(t, x)$, through the Fourier transform process:

$$P(t, \hat{k}) = \frac{1}{(2 \pi)^3} \int \hat{P}(t, \hat{k}) e^{-i \hat{k} \cdot \hat{x}} d\hat{x},$$  \hspace{1cm} (2)

where $x$ represents the space coordinates $x = (x, y, z)$. A solution to this second-order ordinary differential equation is given by

$$\hat{P}(t, \hat{k}) = \hat{P}(0, \hat{k}) e^{i \phi(\hat{k}) t},$$  \hspace{1cm} (3)

where $t$ is a time step in which we desire a solution to the wave equation with respect to a known solution at time $0$. The $\pm$ reflects the presence of a set of solutions representing ingoing and outgoing waves. In inhomogeneous media, the phase, $\phi$, will depend on position, and thus, we consider a mixed-domain space-wavenumber phase function, $\phi$, for time stepping given by

$$P(t + \Delta t, x) \approx \int \hat{P}(t, \hat{k}) e^{i \phi(\hat{x}, \hat{k}, \Delta t)} d\hat{k}. $$  \hspace{1cm} (4)

This is synonymous to Huygen’s principal in which the secondary source along the wave front is decomposed to its plane wave components prior to applying the appropriate phase shift. In the geometrical approach, the function $\phi(x, k, t)$ appearing in eq. (4) should satisfy the appropriate eikonal equation, which is, in the case of isotropic 3-D media, the dispersion relation

$$\frac{\partial \phi}{\partial z} = \frac{1}{v^2(x, y, z)} \left( \frac{\partial \phi}{\partial t} \right)^2 - \left| \nabla_z \phi \right|^2,$$  \hspace{1cm} (5)

where $v(x, y, z)$ is the velocity of the medium, and $\nabla_z = \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$.

If we consider eq. (4) as an evolution in time, the initial condition for it is

$$\phi(x, k, 0) = k \cdot x,$$  \hspace{1cm} (6)

which simply turns eq. (4) into an inverse Fourier transform from wavenumber to space coordinates. For small steps $\Delta t$, it is appropriate to replace $\phi$ with its Taylor series approximation

$$\phi(x, k, t) \approx k \cdot x + \phi(x, k) t.$$  \hspace{1cm} (7)

The final approximation for the wave extrapolation operator, valid for small $\Delta t$, is

$$P(t \pm \Delta t, x) \approx \int \hat{P}(t, \hat{k}) e^{i \phi(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k},$$  \hspace{1cm} (8)

where the function $\phi(x, k)$ appearing in eq. (8), for isotropic media, is given by

$$\phi(x, k) = v(x)|k| = v(x) \sqrt{k_x^2 + k_y^2 + k_z^2}. $$  \hspace{1cm} (9)

In other words, for a small enough time step, the wave propagation can be considered locally homogenous, and thus, use eq. (3).

THE RESIDUAL FORMULATION

The linear nature of the wave equation allows us to propagate the full wavefield or part of it to allow for the following representation

$$P(t + \Delta t, x) = P_0(t + \Delta t, x) + \Delta P(t + \Delta t, x)$$

$$\approx \int \hat{P}_0(t, \hat{k}) e^{i \phi(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k} + \int \Delta \hat{P}(t, \hat{k}) e^{i \phi(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k},$$  \hspace{1cm} (10)

where $\Delta P(t, \hat{k})$ is part of the wavefield that satisfies the wave equation for the same velocity model but a source based on the perturbation. I refer to this part as the residual wavefield. A perturbation in the velocity model results in a perturbation in the extrapolation operator and more precisely in the phase, $\Delta \phi$.

Thus,

$$P(t + \Delta t, x) \approx \int \hat{P}(t, \hat{k}) e^{i \phi_0(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k} + \int \Delta \hat{P}(t, \hat{k}) e^{i \phi(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k},$$  \hspace{1cm} (11)

where $\phi_0$ is the phase corresponding to the background medium (velocity $v_0$). Taking $\Delta \phi$ to be small, we can expand the exponential in eq. (11) that includes $\Delta \phi$ using any one of the many series or rational approximations, some will be discussed in the next section, but most admit the following form:

$$e^{i \Delta \phi} = 1 + F(\Delta \phi \Delta t),$$  \hspace{1cm} (12)

where $F$ includes the expansion terms as a function $\Delta \phi \Delta t$, using any approach including Taylor’s series (Etgen, 1989). Inserting eq. (12) into eq. (11) yields,

$$P(t + \Delta t, x) \approx \int \hat{P}(t, \hat{k}) e^{i \phi_0(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k} + \int \Delta \hat{P}(t, \hat{k}) F(\Delta \phi \Delta t) e^{i \phi(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k}.$$  \hspace{1cm} (13)

Combining formulae (10) and (13), I obtain a standalone residual wavefield extrapolation:

$$\Delta P(t + \Delta t, x) \approx \int \Delta \hat{P}(t, \hat{k}) e^{i \phi_0(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k} + \int \hat{P}(t, \hat{k}) F(\Delta \phi \Delta t) e^{i \phi(\hat{x}, \hat{k}, \Delta t) + i \hat{k} \cdot \hat{x}} d\hat{k}.$$  \hspace{1cm} (14)

where $\Delta P = P_0 + \Delta P$. Note that the second term on the right-hand side need to be computed only when $\Delta \phi \neq 0$. On the other hand, the first term propagates the residual wavefield using the background, possibly homogenous, model represented by $\phi_0$. Though perturbation theory assumes a constant and independent velocity perturbation variable, considering the time step nature of this implementation the impact of these limitations are heavily mitigated to allow for a Huygen’s principle type wave propagation.

If the background wavefield is obtained for a homogeneous medium, which can be done analytically, the residual wavefield will
approximately address the inhomogeneity. Up to this point there is no apparent advantage to extrapolating the residual operator instead of the full one. In the Appendix the residual development for the second order finite difference approximation in time case is shown.

THE EXPANSION

To derive the residual exponential operator using a form that allows us to separate the background contribution from the residual one, we utilize Taylor’s series expansion (to the sixth order) of the residual exponential extrapolation operator (Etgen, 1989), as follows

\[ F = e^{i\Delta t \phi} - 1 = i \Delta t \phi - \frac{\Delta t^2 \phi^2}{2} - \frac{1}{6} i \Delta t^3 \phi^3 + \frac{\Delta t^4 \phi^4}{24} + \frac{1}{120} i \Delta t^5 \phi^5 - \frac{\Delta t^6 \phi^6}{720} + O \left( \Delta \phi^7 \right) . \]  

(15)

Truncating \( F \) to only the first term amounts to a conventional split-step type implementation (Pestana & Stoffa, 2010), I call it the linear approximation. Using this polynomial-based expansion allows us to readily separate the space dependence part (residual velocity) from the wavenumber part (the Laplace operator), in a form commonly referred to as the pseudospectral implementation (Reshef al., 1988). In other words, we can factor out the space dependence (the velocity) from the extrapolation integral, and thus, admit an inverse Fourier-type integral. In any case, the accuracy of the residual extrapolation relies on the order of the expansion (with each term requiring an inverse Fourier transform), the amount of velocity perturbation and the time step of the extrapolation.

To improve the accuracy of the Taylor’s series expansion with fewer terms, I use Shank’s transform to predict the higher-order behaviour of the series by introducing a denominator. The first-order Shank’s transform (Bender & Orszag, 1978) admits the following form

\[ F = e^{i\Delta t \phi} - 1 = \frac{i \Delta \phi}{1 - \frac{i \Delta \phi}{\Delta t}} . \]  

(16)

This form, unlike Taylor’s series, does not allow us to factor out the velocity from the residual term (\( F \)), requiring a full spectral implementation. We will later address this issue through additional approximations. Pade approximations are also possible (Zhang al., 2004) for the spectral implementations, but they do not admit a series form required by the residual formulation given by eq. (12).

ACCURACY TESTS

Considering an isotropic homogeneous model, I analyse the accuracy of computing the phase of the extrapolation operator using the various approximations of the residual operator considering a background velocity of \( v = 2 \text{ km s}^{-1} \) and initially a velocity perturbation of \( v = 1 \text{ km s}^{-1} \). Despite the relatively large perturbation, I consider a time step of 0.01 s (of course this time step will have to be reduced considerably for extrapolation in complex inhomogeneous media). Fig. 1 demonstrates the accuracy of the approximations given in the previous section, and specifically the accuracy of the Shank’s transform representation (dashed black curves) represented by eq. (16) as a function of radial wavenumber (Fig. 1a) and a function of velocity perturbation (Fig. 1b) for a fixed wavenumber of 10 km\(^{-1}\). Errors increase with an increase in the wavenumber (higher frequency) or an increase in the velocity perturbation. Nevertheless, the increase in errors as a function of velocity perturbation is exponential, which suggests that a residual implementation with small steps in velocity perturbation is worth exploring. Clearly all approximations are extremely accurate for small velocity perturbations or equivalently small time steps, with errors increasing exponentially with an increase in these two variables. Because a larger time step speeds up the extrapolation to the limit that velocity variations allow us, we later focus on the velocity perturbation. The sixth-order Taylor’s series expansion is extremely accurate but it costs a lot. With the smaller time step needed to handle complex velocity models the accuracy difference between the sixth and the first order becomes much smaller. Shank’s transform provided high accuracy, higher than the second order with fewer terms.

However, Shank’s transform does not allow for a straightforward separation of the velocity (space dependency) for a pseudospectral implementation. A simple remedy is provided by replacing \( \phi \) in the denominator of eq. (16) by a space-independent value given by the average velocity perturbation. Considering we are dealing with controllable perturbations, the deviation introduced by this approximation can be small. Fig. 2 demonstrates that changes to the perturbation velocity in the denominator, negatively (Fig. 2a) or positively (Fig. 2b), still yields higher accuracy than the second-order approximation. A constant average velocity for the denominator reduces the number of required inverse Fourier transforms for the perturbation part to one instead of two for the second-order expansion.

![Figure 1](http://gji.oxfordjournals.org/)

**Figure 1.** Per cent errors in the phase of the extrapolator operator given by the first-order (solid black curve) and second-order (solid grey curve) expansions, first-order Shank’s transform 16 (dashed black curve), and sixth-order Taylor’s series expansion (dashed grey curve, almost zero), as a function of wavenumber for a perturbation \( \Delta v = 1 \text{ km s}^{-1} \) (a) and perturbation velocity for \( k_r = 10 \text{ km}^{-1} \) (b). The model has a velocity of 2 km s\(^{-1}\) and a time step of 0.01 s. 

**Figure 2.** Per cent errors in the phase of the extrapolator operator given by the first-order (solid black curve) and second-order (solid grey curve) expansions, first-order Shank’s transform 16 (dashed black curve), and sixth-order Taylor’s series expansion (dashed grey curve, almost zero), as a function of velocity perturbation for a fixed wavenumber of 10 km\(^{-1}\) (a) and a function of perturbation velocity for a fixed wavenumber of 10 km\(^{-1}\) (b). The model has a velocity of 2 km s\(^{-1}\) and a time step of 0.01 s.
and for
\[ n \Delta \phi = e^{i \Delta t \Delta \phi_0} \prod_{i=1}^{n} e^{i \Delta t \Delta \phi_i} \]
\[ = e^{i \Delta t \Delta \phi_0} \prod_{i=1}^{n} (1 + i \Delta t \Delta \phi_i) \]
\[ = e^{i \Delta t \Delta \phi_0} \left( 1 + i \Delta t \left( \sum_{i=1}^{n} \Delta \phi_i \right) - \Delta t^2 \left( \sum_{i \neq j} \Delta \phi_i \Delta \phi_j \right) \right) \]
\[ - i \Delta t^3 \sum_{i \neq j \neq k} \Delta \phi_i \Delta \phi_j \Delta \phi_k + R(\Delta t, n) \right), \] (17)
where \( \Delta \phi_i = \Delta \phi_0 [k] \), \( n \) is the number of velocity discretization terms \( (n > 2) \) and \( \prod \) stands for the product symbol. The function \( R \) contains the other terms of the expansion. For \( n = 3 \), \( R = 0 \) and for \( n > 3 \), \( R \) contains terms up to the order \( \Delta t^4 \). For \( n = 2 \), only the first two terms of the expansion remain. In this case, \( \Delta v_1 = \Delta v_2 = \frac{1}{2} \Delta v \), and thus
\[ F \approx i \Delta t \Delta \phi - \Delta t^2 \Delta \phi^2. \] (18)

Compared to the Taylor’s series expansion (15), we are missing the division over two in the second term. A similar result is obtainable by dividing the time step to artificially smaller ones, but the velocity implementation allows for more options as velocity may vary with position.

We now test the accuracy of this new implementation. To reduce the number of inverse Fourier transforms required I will investigate the cases where \( n = 2 \) and \( n = 4 \). For \( n = 4 \), we also truncate the product expansion back to two terms stopping at \( \Delta t^2 \), and thus only consider the first three terms in brackets in eq. (17), which is equivalent to a second-order approximation. Fig. 3(a) shows the per cent error in the phase operator as a function of wavenumber. Among the second-order equivalent (in cost) implementations, the truncated four-step \((n = 4)\) implementation (dashed grey curve) has the highest accuracy. This fact is appreciated even more in Fig. 3(b) where the errors in the operators are given as a function of the velocity perturbation.

**THE TIME STEP SIZE COMPONENT**

For the isotropic acoustic phase shift operator, there is a clear trade-off between velocity (or velocity perturbation) and the time step. The spectral implementation, unlike the conventional space one, is
shows the part of the model with a salt body. Vertical grid size \( \Delta x \) of the operator is 3, which was necessary to obtain a phase accuracy given by eq. (8) is shown in Fig. 6(a). The rank of the separable operators (space and wavenumber components) is 12.5 m and the time step is 1 ms. Fig. 5 shows a portion of BP 2004 synthetic velocity model.

Figure 4. Amplitude of the exact phase operator (solid curve), equal 1, compared to that extracted using the second-order Taylor’s series expansion in the residual velocity (dashed curve), for (a) \( \Delta t = 0.004 s \), (b) \( \Delta t = 0.01 s \) and (c) \( \Delta t = 0.02 s \). The velocity is equal to 2 km s\(^{-1}\) and \( \Delta t \approx 1 \) km s\(^{-1}\).

Figure 5. Portion of BP 2004 synthetic velocity model.

stable regardless of the time step length. However, if the medium is inhomogeneous, the accuracy can suffer from a large time step that does not honour the rapid change in velocity. This can be improved by considering higher-order terms of the expansion in eq. (7). Here we focus on methodologies that will improve the accuracy of implementing eqs (15) and (16), or in other words, the accuracy of a pseudospectral implementation. In this case, and as the expansions are based on a small \( \Delta \phi \), which is proportional to \( \Delta x \Delta t \), using smaller \( \Delta x \Delta t \) perturbations by, for example, dividing the perturbation into smaller pieces as we saw in the previous section, will allow us to use a larger time step while maintaining the accuracy. Of course, this suggestion does not address the inhomogeneity issue. More perturbations will require more inverse Fourier transform or can be formed in a polynomial series similar to eq. (17) and settle for the leading terms.

An expansion of the exponential phase operator as a function of small velocity and traveltime steps induces an amplitude component to the unitary amplitude nature of the conventional phase operator. This amplitude component depends on the expansion parameters. As such, Fig. 4 shows the amplitude of the phase operator for the second-order approximation. We have to use a time step of 0.02 s (Fig. 4c), which is large, to see some errors especially in the high wavenumber components. Shank’s transform provides practically exact amplitudes.

**WAVEFIELD EXTRAPOLATION**

Using the low-rank approach of Fomel et al. (2010) to deal with the mixed space-wavenumber domain nature of the phase operator function in eq. (8), we extrapolate waves in inhomogeneous media. The rank of the separable operators (space and wavenumber dependent) control the number of inverse Fourier transforms required in each extrapolation step, and thus, the cost. This is typically dependent on the complexity of the velocity model and the size of the time step used in the extrapolation in relation to the complexity.

We apply the residual extrapolator on the BP (the Oil company) model (Billette & Brandsberg-Dahl, 2004) using a Ricker-wavelet at a point source located near the surface at 33 750 m with a maximum frequency of 50 Hz. The horizontal grid size \( \Delta x \) is 37.5 m, the vertical grid size \( \Delta z \) is 12.5 m and the time step is 1 ms. Fig. 5 shows the part of the model with a salt body.

A snapshot of the wavefield at time 3.2 s extracted from the wavefield extrapolation using the low-rank method applied to the exact operator given by eq. (8) is shown in Fig. 6(a). The rank of the operator is 3, which was necessary to obtain a phase accuracy of \( 10^{-5} \). A rank of 3 implies the need to execute 3 multidimensional inverse Fourier transform every time step. We compare this reference wavefield with that obtained using the residual extrapolator based on a linear approximation of the Taylor’s series expansion in eq. (15) (using only the first term on the right-hand side); a snapshot at 3.2 s shown in Fig. 6(b). The cost of the linear approximation, which includes mainly the cost of a residual implementation from a background wavefield corresponding to a constant velocity of 3 km s\(^{-1}\), is equivalent to two inverse Fourier transform for each time step. The difference plotted at the same scale is shown in Fig. 6(c). The difference between the reference exact and that using Shank’s transform instead (which requires the same number of pseudospectral inverse Fourier transform as the linear) for a residual implementation from the same constant velocity background wavefield as the linear, plotted again at the same scale is shown in Fig. 6(d). Clearly, the difference associated with the Shank’s transform implementation, with the same cost, is lower.

Another display of the difference between the exact operator and the Shank’s transform one plotted at a scale \( 10 \times \) larger than that in Fig. 6(d) is shown in Fig. 7(a). In addition, I show the difference between the exact operator and the linear approximation (shown in Fig. 7 b), and the difference between the exact operator and the modified Shank’s transform one, which allows for a pseudo spectral implementation (Fig. 7c), both plotted also at a \( 10 \times \) gain compared to Figs 6(a)-(d). The errors in the modified Shank’s transform are clearly smaller than the linear approximation. Considering that both extrapolations cost the same complexity.
Figure 6. Snapshots of the wavefield at 3.2 s using the conventional extrapolation operator, eq. (8) (a), using the linear approximation (b), the difference (c) and the difference between the exact operator and the Shank’s transform approximation (d) for the BP model shown in Fig. 5, with a source near the surface, all plotted at the same scale.

in a pseudo spectral implementation, the added accuracy with the modified Shank’s transform implementation is helpful for many applications.

COST AND OPPORTUNITY

Though the emphasis here is on accuracy, a residual implementation has potential cost advantages. Considering a background time extrapolation operator, $\phi_0$, for homogeneous or smoothly inhomogeneous media, the extrapolation of such operators tend to be cheap, as approximations in the spectral implementation requires less inverse Fourier transforms per time step (or, in other words, a lower rank representation, especially for a relatively large time step; for homogeneous media only one inverse Fourier transform is required regardless of the time step). Thus, using the residual implementation given by eq. (14), we need to evaluate the second term, with the extrapolation operator for the more complex medium, only when residuals in the medium exists ($\Delta v \neq 0$). A single combined extrapolation operator will not recognize such simplicity in the model in certain areas (like within a salt body). This opens the door for many smart velocity separation-based operators that can help in optimally improving the speed of the extrapolation. In the BP model example of the previous section, and for the same cost of a pseudospectral implementation of the linear approximation of the perturbation part, the modified Shank’s transform approximation provided higher accuracy.

The opportunity for a residual implementation allows us to modify an existing wavefield to correspond to a new velocity model. If the difference between the background velocity model used to compute the original wavefield and the new velocity model is small, the residual admits accurate solutions. Because accuracy reduces exponentially with an increase in the perturbation or the time step (considering the trade-off), we can take smaller velocity perturbation steps to allow for bigger time steps. The truncation of such a velocity-based series expansion, as we saw, yields higher accuracy than the conventional Taylor’s series expansion. To combine this fact with the possibilities we have in discretizing the velocity perturbation from the background velocity opens doors for additional improvements in speed.

CONCLUSIONS

Isolating the residual part of the wavefield in the time extrapolation process allows us to utilize accuracy enhancements methods obtainable from expansions based on small velocity perturbations. Further accuracy is accessible through Shank’s transform, which may admit after a small modification a residual implementation-friendly form. The accuracy of the residual implementation supports its usefulness in wavefield extrapolation speed up.
Residual wavefields

Figure 7. Snapshots of the difference in the wavefield at 3.2 s (a) between the exact operator and the Shank’s transform approximation (same as Fig. 6d multiplied by a factor of 10 to enhance the difference), (b) between the exact operator and linear approximation at the same 10-factor scale and (c) and the Shank’s transform approximation with the constant average velocity used in the denominator to allow for the residual implementation at same factor of 10 enhancement in scale. The cost of implementing the last two approximate operators is the same.

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REFERENCES

EXPANSIONS WITH RESPECT TO THE cos OPERATOR
A second-order time extrapolation can be extracted from the sum of two first-order approximations 8 for a positive and negative time step. After minor algebraic changes, it has the following form:

\[ P(t + \Delta t, \mathbf{x}) + P(t - \Delta t, \mathbf{x}) \approx 2 \int \tilde{P}(t, \mathbf{k}) \cos \{\phi(\mathbf{x}, \mathbf{k}) \Delta t\} e^{\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}. \]  

(A1)
Considering perturbations in the velocity, the phase operator can be written in a separable form as follows:

$$\cos ((\phi_0 + \Delta \phi) \Delta t) = \cos (\phi_0 \Delta t) \cos (\Delta \phi \Delta t) - \sin (\phi_0 \Delta t) \sin (\Delta \phi \Delta t).$$  

(A2)

Consequently, for small $\Delta \phi$ and small $\Delta t$, we obtain

$$\cos (\phi_0 \Delta t + \Delta \phi \Delta t) = \cos (\phi_0 \Delta t) (1 + F) - \sin (\phi_0 \Delta t) E,$$  

where, using Taylor’s series expansion,

$$F = -\frac{\Delta t^2 \Delta \phi^2}{2} + \frac{\Delta t^3 \Delta \phi^4}{24} - \frac{\Delta t^4 \Delta \phi^6}{720} + O (\Delta \phi^7)$$

$$E = \Delta t \Delta \phi - \frac{\Delta t^2 \Delta \phi^3}{6} + \frac{\Delta t^5 \Delta \phi^5}{120} + O (\Delta \phi^7).$$  

(A4)

Again, considering $P = P_0 + \Delta P$, we obtain

$$\Delta P(t + \Delta t, \mathbf{x}) + \Delta P(t - \Delta t, \mathbf{x})$$

$$\approx 2 \int \Delta \tilde{P}(t, \mathbf{k}) (\cos (\phi_0 \Delta t) (1 + F) - \sin (\phi_0 \Delta t) E) e^{i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k}$$
$$+ 2 \int \tilde{P}_0(t, \mathbf{k}) (\cos (\phi_0 \Delta t) F - \sin (\phi_0 \Delta t) E) e^{i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k}$$
$$+ 2 \int \Delta \tilde{P}(t, \mathbf{k}) (\cos (\phi_0 \Delta t) F - \sin (\phi_0 \Delta t) E) e^{i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k},$$  

(A5)

which describes the extrapolation of the residual field, and it includes a term that extrapolates the residual field using the background operator and another containing the interaction between the residual operator and the full wavefield.