Supplemental Material for "Elastic metamaterials with simultaneously negative effective shear modulus and mass density"

Ying Wu[1,2], Yun Lai[1,3] and Zhao-Qing Zhang[1*]

[1] Department of Physics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
[2] Division of Mathematical and Computer Sciences and Engineering, King Abdullah University of Science and Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia
[3] Department of Physics, Soochow University, 1 Shizi Street, Suzhou 215006, People's Republic of China

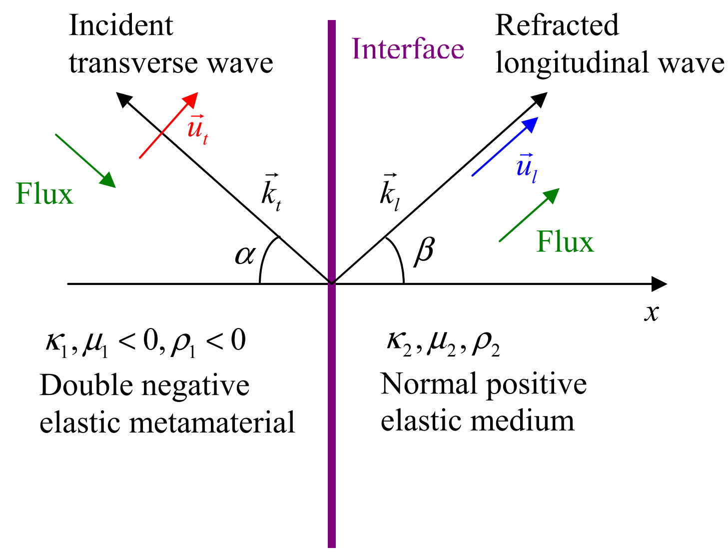

Fig. S1. A schematic graph of *s-p* mode conversion during the negative refraction on the interface of a double-negative elastic metamaterial of Case 1 below.

There can be four cases of total *s-p* or *p-s* mode conversions on the interface of a double-negative metamaterial and a normal solid under negative refraction:

Case 1: Incident shear waves from a double negative metamaterial.
Case 2: Incident longitudinal waves from a double negative metamaterial.
Case 3: Incident shear waves from a normal solid.
Case 4: Incident longitudinal waves from a normal solid.

It is obvious that Case 3 can be obtained from the time reversal of Case 2 and Case 4 can be obtained from the time reversal of Case 1. Below we derive the general conditions of total mode conversions under negative refraction.

For s waves, the displacement is $\vec{u}_t = u_t\left(\sin\alpha\hat{x} + \cos\alpha\hat{y}\right)e^{ik_t(-\cos\alpha x + \sin\alpha y)}$

For p waves, the displacement is $\vec{u}_l = u_l\left(\cos\beta\hat{x} + \sin\beta\hat{y}\right)e^{ik_l(\cos\beta x + \sin\beta y)}$

The displacements must match on the interface $x = 0$, i.e.,

$$\begin{aligned} u_t \sin\alpha e^{ik_t \sin\alpha y} &= u_l \cos\beta e^{ik_l \sin\beta y}, \\ u_t \cos\alpha e^{ik_t \sin\alpha y} &= u_l \sin\beta e^{ik_l \sin\beta y}. \end{aligned} \tag{S1}$$

From Eq. (S1), we find $u_t \sin\alpha = u_l \cos\beta$ and $u_t \cos\alpha = u_l \sin\beta$, which means that $u_l = u_t$ and $\alpha + \beta = \pi/2$. In the case of normal refraction, for incident angle $0 < \alpha < \pi/2$ we have $\beta < 0$, thus $\alpha + \beta < \pi/2$, the displacements are not possible to match on the interface. Total conversion is only possible in the case of negative refraction.

From Eq. (S1), we also find $k_t \sin\alpha = k_l \sin\beta$, which means the wave vector parallel to the interface must conserve. Substituting $\alpha + \beta = \pi/2$, we obtain $k_t \sin\alpha = k_l \cos\alpha$.

From $k_t = \left|\frac{\omega}{v_t}\right| = \frac{\omega}{\sqrt{\mu_1/\rho_1}}$ and $k_l = \left|\frac{\omega}{v_l}\right| = \frac{\omega}{\sqrt{(\kappa_2 + \mu_2)/\rho_2}}$, we find

$$\tan\alpha = \frac{k_l}{k_t} = \frac{\sqrt{\mu_1/\rho_1}}{\sqrt{(\kappa_2 + \mu_2)/\rho_2}}.$$

We also need to consider the matching condition of stresses.

For s waves, $\vec{u}_t = u_t\left(\sin\alpha\hat{x} + \cos\alpha\hat{y}\right)e^{ik_t(-\cos\alpha x + \sin\alpha y)}$. Thus, the strains are

$$S_{xx}^t = \frac{\partial u_{tx}}{\partial x} = u_t \sin\alpha\left(-ik_t \cos\alpha\right)e^{ik_t(-\cos\alpha x + \sin\alpha y)} = -iu_t k_t \sin\alpha\cos\alpha e^{ik_t(-\cos\alpha x + \sin\alpha y)},$$

$$S_{yy}^t = \frac{\partial u_{ty}}{\partial y} = u_t \cos\alpha\left(ik_t \sin\alpha\right)e^{ik_t(-\cos\alpha x + \sin\alpha y)} = iu_t k_t \sin\alpha\cos\alpha e^{ik_t(-\cos\alpha x + \sin\alpha y)},$$

$$\begin{aligned} S_{xy}^t &= \frac{1}{2}\left(\frac{\partial u_{tx}}{\partial y} + \frac{\partial u_{ty}}{\partial x}\right) \\ &= \frac{1}{2}\left(u_t \sin\alpha\left(ik_t \sin\alpha\right)e^{ik_t(-\cos\alpha x + \sin\alpha y)} + u_t \cos\alpha\left(-ik_t \cos\alpha\right)e^{ik_t(-\cos\alpha x + \sin\alpha y)}\right) \\ &= \frac{1}{2}\left(iu_t k_t \sin^2\alpha - iu_t k_t \cos^2\alpha\right)e^{ik_t(-\cos\alpha x + \sin\alpha y)}. \end{aligned}$$

And the stress can be written as

$$\sigma_{xx}^t = (\kappa_1+\mu_1) S_{xx}^t + (\kappa_1-\mu_1) S_{yy}^t$$
$$= -(\kappa_1+\mu_1) iu_t k_t \sin\alpha\cos\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)} + (\kappa_1-\mu_1) iu_t k_t \sin\alpha\cos\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)}$$
$$= -2i\mu_1 u_t k_t \sin\alpha\cos\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)}$$
$$= -i\mu_1 u_t k_t \sin 2\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)},$$
$$\sigma_{yy}^t = (\kappa_1+\mu_1) S_{yy}^t + (\kappa_1-\mu_1) S_{xx}^t$$
$$= (\kappa_1+\mu_1) iu_t k_t \sin\alpha\cos\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)} - (\kappa_1-\mu_1) iu_t k_t \sin\alpha\cos\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)}$$
$$= 2i\mu_1 u_t k_t \sin\alpha\cos\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)}$$
$$= i\mu_1 u_t k_t \sin 2\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)},$$
$$\sigma_{xy}^t = 2\mu_1 S_{xy}^t$$
$$= \mu_1 \left(iu_t k_t \sin^2\alpha - iu_t k_t \cos^2\alpha \right) e^{ik_t(-\cos\alpha x+\sin\alpha y)}$$
$$= -i\mu_1 u_t k_t \cos 2\alpha e^{ik_t(-\cos\alpha x+\sin\alpha y)}.$$

For p waves, $\vec{u}_l = u_l \left(\cos\beta\hat{x} + \sin\beta\hat{y}\right) e^{ik_l(\cos\beta x+\sin\beta y)}$. Thus, the strains are

$$S_{xx}^l = \frac{\partial u_{lx}}{\partial x} = u_l \cos\beta\left(ik_l\cos\beta\right) e^{ik_l(\cos\beta x+\sin\beta y)} = iu_l k_l \cos^2\beta e^{ik_l(\cos\beta x+\sin\beta y)},$$
$$S_{yy}^l = \frac{\partial u_{ly}}{\partial y} = u_l \sin\beta\left(ik_l\sin\beta\right) e^{ik_l(\cos\beta x+\sin\beta y)} = iu_l k_l \sin^2\beta e^{ik_l(\cos\beta x+\sin\beta y)},$$
$$S_{xy}^l = \frac{1}{2}\left(\frac{\partial u_{lx}}{\partial y} + \frac{\partial u_{ly}}{\partial x}\right)$$
$$= \frac{1}{2}\left(u_l\cos\beta\left(ik_l\sin\beta\right)e^{ik_l(\cos\beta x+\sin\beta y)} + u_l\sin\beta\left(ik_l\cos\beta\right)e^{ik_l(\cos\beta x+\sin\beta y)}\right)$$
$$= iu_l k_l \sin\beta\cos\beta e^{ik_l(\cos\beta x+\sin\beta y)}.$$

And the stress can be written as

$$\sigma_{xx}^{l} = (\kappa_2 + \mu_2) S_{xx}^{l} + (\kappa_2 - \mu_2) S_{yy}^{l}$$
$$= (\kappa_2 + \mu_2) i u_l k_l \cos^2 \beta e^{ik_l(\cos\beta x + \sin\beta y)} + (\kappa_2 - \mu_2) i u_l k_l \sin^2 \beta e^{ik_l(\cos\beta x + \sin\beta y)}$$
$$= i\Big(\kappa_2 \left(\cos^2 \beta + \sin^2 \beta\right) + \mu_2 \left(\cos^2 \beta - \sin^2 \beta\right)\Big) u_l k_l e^{ik_l(\cos\beta x + \sin\beta y)}$$
$$= i(\kappa_2 + \mu_2 \cos 2\beta) u_l k_l e^{ik_l(\cos\beta x + \sin\beta y)},$$
$$\sigma_{yy}^{l} = (\kappa_2 + \mu_2) S_{yy}^{l} + (\kappa_2 - \mu_2) S_{xx}^{l}$$
$$= (\kappa_2 + \mu_2) i u_l k_l \sin^2 \beta e^{ik_l(\cos\beta x + \sin\beta y)} + (\kappa_2 - \mu_2) i u_l k_l \cos^2 \beta e^{ik_l(\cos\beta x + \sin\beta y)}$$
$$= i\Big(\kappa_2 \left(\cos^2 \beta + \sin^2 \beta\right) + \mu_2 \left(-\cos^2 \beta + \sin^2 \beta\right)\Big) u_l k_l e^{ik_l(\cos\beta x + \sin\beta y)}$$
$$= i(\kappa_2 - \mu_2 \cos 2\beta) u_l k_l e^{ik_l(\cos\beta x + \sin\beta y)},$$
$$\sigma_{xy}^{l} = 2\mu_2 S_{xy}^{l}$$
$$= 2\mu_2 i u_l k_l \sin\beta \cos\beta e^{ik_l(\cos\beta x + \sin\beta y)}$$
$$= i\mu_2 u_l k_l \sin 2\beta e^{ik_l(\cos\beta x + \sin\beta y)}.$$

σ_{xx} and σ_{xy} must also be continuous on the interface $x = 0$, i.e.,

$$\begin{aligned} -i\mu_1 u_t k_t \sin 2\alpha e^{ik_t \sin\alpha y} &= i(\kappa_2 + \mu_2 \cos 2\beta) u_l k_l e^{ik_l \sin\beta y}, \\ -i\mu_1 u_t k_t \cos 2\alpha e^{ik_t \sin\alpha y} &= i\mu_2 u_l k_l \sin 2\beta e^{ik_l \sin\beta y}. \end{aligned} \tag{S2}$$

By substituting $\alpha + \beta = \pi/2$, $u_l = u_t$ and $k_t \sin\alpha = k_l \cos\alpha$ in Eq. (S2), we find $\kappa_2 = \mu_2 \cos 2\alpha - \mu_1 \sin 2\alpha / \tan\alpha$ and $-\mu_1 = \mu_2 \tan 2\alpha \tan\alpha$.

Conclusion: total conversion is possible under the case of negative refraction.

The total conversion matching conditions are:

$$\begin{aligned}
&\alpha+\beta=\pi/2, \\
&\kappa_2=\mu_2\cos 2\alpha-\mu_1\sin 2\alpha/\tan\alpha, \\
&-\mu_1=\mu_2\tan 2\alpha\tan\alpha, \\
&\tan\alpha=\frac{k_l}{k_t}=\frac{\sqrt{\mu_1/\rho_1}}{\sqrt{(\kappa_2+\mu_2)/\rho_2}}.
\end{aligned} \qquad \text{(S3)}$$

Since Case 3 can be obtained from the time reversal of Case 2 and Case 4 can be obtained from the time reversal of Case 1, here we discuss only Cases 1 and 3.

Case 1: we consider a transverse plane wave incident from the left medium of $\mu_1<0,\rho_1<0$ with an incident angle of $0<\alpha<\pi/2$ as shown in Fig. S1. First, we can obtain μ_2 by using $-\mu_1=\mu_2\tan 2\alpha\tan\alpha$. Then, we can obtain κ_2 by using $\kappa_2=\mu_2\cos 2\alpha-\mu_1\sin 2\alpha/\tan\alpha=\mu_2/\cos 2\alpha$. At last, we can obtain ρ_2 by using $\tan\alpha=\frac{k_l}{k_t}=\frac{\sqrt{\mu_1/\rho_1}}{\sqrt{(\kappa_2+\mu_2)/\rho_2}}$. Therefore, μ_2, κ_2 and ρ_2 are all obtained. It is worth mentioning that when $\alpha>\pi/4$, we have $\mu_2<0$, but $\kappa_2>0$ and $\kappa_2+\mu_2=\mu_2(1+\cos 2\alpha)/\cos 2\alpha>0$, which indicates a double positive medium for refracted longitudinal waves on the right (together with $\rho_2>0$). In this case the medium on the right is not a normal solid.

Case 3: we consider a transverse plane wave incident from the left medium of $\mu_1>0,\rho_1>0$ with an incident angle of $0<\alpha<\pi/2$. μ_2, κ_2 and ρ_2 can also be obtained from Eq. (S3). Note that we have $\kappa_2+\mu_2=\mu_2(1+\cos 2\alpha)/\cos 2\alpha<0$ and $\rho_2<0$, which indicate a double negative medium for refracted longitudinal waves on the right.