Supplemental Material for

Statistical Emulation of Climate Model Projections based on Precomputed GCM Runs

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1 Regions used for emulation

See Figure S1.

Figure S1: Regions used for statistical emulations described in the text. Regions are a variant of those described in [2], with some ocean areas further subdivided to capture qualitative differences in transient precipitation response.

2 Details about the statistical fit

We give a more detailed description of how the profile likelihood method has been used to fit temperature. If we have only a single scenario with \( n \) observations, and we call \( \sigma^2_T := \frac{\sigma^2}{1-\phi^2} \), then equation (1) can be reformulated as

\[
T = X(\rho)\beta + \epsilon \quad \epsilon \sim \mathcal{N}(0, \sigma^2_T K(\phi))
\]  

(1)

where \( K(\phi) \) is the correlation matrix of an AR(1) process with parameter \( \phi \) and \( X(\rho) \) is a \( n \times 3 \) matrix whose entries are

\[
X_{t,1} = 1 \\
X_{t,2} = \frac{1}{2} \log[\text{CO}_2](t) + \frac{1}{2} \log[\text{CO}_2](t - 1) \\
X_{t,3} = X_{t,3}(\rho) = (1 - \rho) \sum_{i=2}^{+\infty} \rho^{i-2} \log[\text{CO}_2](t - i)
\]
Dropping the dependence of the design matrix $X$ on $\rho$ to simplify the notation, the loglikelihood for model (1) is

$$l(\rho, \beta, \phi, \sigma_\phi^2 | T, [CO_2]) = -\frac{n}{2} \log(2\pi) - \frac{1}{2} \log |\sigma_\phi^2 K(\phi)|$$

$$-\frac{1}{2\sigma_\phi^2} (T - X\beta)^t K(\phi)^{-1} (T - X\beta).$$  \(2\)

For given values of $\rho$ and $\phi$, the Maximum Likelihood Estimator (MLE) of $\beta$ is

$$\hat{\beta} = (X^t K(\phi)^{-1} X)^{-1} X^t K(\phi)^{-1} T,$$  \(3\)

and the MLE for $\sigma_\phi^2$ is

$$\hat{\sigma}_\phi^2 = \frac{1}{n} (T - X\hat{\beta})^t K(\phi)^{-1} (T - X\hat{\beta}),$$  \(4\)

We can compute $|K(\phi)| = (1 - \phi^2)^{n-1}$, and plugging (3) and (4) into (2) we have, if we call $W = X^t K(\phi)^{-1} X$

$$\tilde{l}(\rho, \phi | T, [CO_2]) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} + \frac{1}{2} \log(1 - \phi^2)$$

$$-\frac{n}{2} \log (T^t (K(\phi)^{-1} - K(\phi)^{-1} X W^{-1} X^t K(\phi)^{-1}) T)$$  \(5\)

which is the profile likelihood (see e.g. [1] ch.4) and is a function only of the parameters $\rho$ and $\phi$ and can be easily and quickly maximized.

If more than one scenario/realization is present, the approach can be easily generalized in those cases where we can assume that all the processes are independent.

3 Parameter estimates with associated variability

Table S1 gives the parameter estimates for the temperature emulator (1) along with their standard errors, based on large sample approximations when the emulator is trained with one realization each of the fast and jump scenario. Specifically, the variability of $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$ is computed conditionally on $(\hat{\rho}, \hat{\phi}, \hat{\sigma}_2)$ via generalized least squares, whereas the variability of $(\hat{\rho}, \hat{\phi})$ is computed by first estimating the Fisher information matrix.
Table S1: Parameter estimates for (1) in all regions, with their standard deviations in parentheses. In several regions (ARO, CNA, NEU, NAS and NNA), $\hat{\rho}$ is numerically equal to 1, which implies that $\sum_{t=2}^{4} w_{t-2} \log[\text{CO}_2(t-i)] \approx 0$ so $\hat{\beta}_2$ is highly unstable.
4 Emulator prediction for different realizations

Figure S2 shows the small variations in predictions one gets by using different realizations of the same scenarios in the training set. The only feature that is noticeably affected is the emulation near the sudden drop in CO$_2$. Note that because all five realizations of the fast and jump are used in this exercise, the five estimated curves are not quite independent because some pairs of curves will have been trained with identical climates during 1870-2010, although for any one curve, the two training runs were selected so that they never shared a restart year.

![Figure S2](https://example.com/s2.png)

Figure S2: An example of temperature emulation for different realizations in the North Pacific West (NPW) region (a-b), and of precipitation emulation in the Equatorial Pacific West (EPW) region (c-d). Panels a and c show the emulated slow scenario and b and d the drop scenario. Each emulator was trained by one realization each of the fast and jump scenarios. The solid lines in different colors represent the emulated value for each of the realization, and the gray lines represent the five realizations for the scenarios. Emulation appears indistinguishable for the slow, while for drop different realizations result in somewhat different estimated values near the drops.
5 Empirical coverages

See Figure S3.

6 Long term predictions

See Figure S4.

References


Figure S3: Empirical coverage (multiplied by 100) of the nominal 95% prediction bands for temperature. The training set is one realization of the fast and jump scenario. Top panel shows the slow and bottom panel the drop scenarios. The coverage for slow is very close to the nominal value, for drop there is a sign of misspecification for some regions in the southern hemisphere, as noticed for the mean emulation.
Figure S4: Training set a fast (until year 2449) and a jump realization (until year 2399). Example of emulation for a very long time scale (until year 4599) for the fast scenario for the SAT region (on the left) and NAU region (on the right). The red line represents the estimated mean and the dashed red lines represent the 95% prediction intervals.