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Discontinuous Petrov-Galerkin method based on the optimal test space norm for one-dimensional transport problems

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Abstract

We revisit the finite element analysis of convection dominated flow problems within the recently developed Discontinuous Petrov-Galerkin (DPG) variational framework. We demonstrate how test function spaces that guarantee numerical stability can be computed automatically with respect to the so called optimal test space norm by using an element subgrid discretization. This should make the DPG method not only stable but also robust, that is, uniformly stable with respect to the Péclet number in the current application. The effectiveness of the algorithm is demonstrated on two problems for the linear advection-diffusion equation.

Keywords: convection-diffusion, discontinuous Petrov-Galerkin, finite element method

1. Introduction

The success of the traditional Ritz-Bubnov-Galerkin finite element method in most structural problems is based on the so called best approximation property. This means that the difference between the finite element solution and the exact solution becomes minimized with respect to certain norm, often called as the energy norm. The property follows largely from the symmetry of the stiffness matrices that the method produces.

Numerical problems arise when the best approximation property (or the energy norm in the first place) is lost for some reason. This happens, for instance, when the standard Galerkin method is applied to convective transport problems. In these problems, the system matrix associated to convection is not symmetric and numerical solutions tend to show spurious, non-physical oscillations unless the finite element mesh is heavily refined. To save computer resources alternative formulations which avoid oscillations regardless of the mesh size have been developed. The most famous of these is probably the Streamline Upwind Petrov-Galerkin (SUPG) formulation, see [1, 2]. The method belongs to a larger class of stabilized finite element methods reviewed, for example, in [3, 4].

The above mentioned stabilized finite element methods are geared mainly towards the classical h-version of the finite element method. However, a new, relatively general finite element framework has been developed by Demkowicz and Gopalakrishnan in [5, 6]. The proposed variational framework is of discontinuous Petrov-Galerkin type and can be utilized, for example, to symmetrize non-symmetric problems using the concept of optimal test functions similarly as in the early methodology developed by Barrett and Morton in [7].

More precisely, if a fully discontinuous finite element space is used for the test functions, as in the DPG method of Bottasso et al. [8], the optimal test functions can be approximated locally in an enriched finite element space. A similar procedure can also be used to compute local a posteriori error estimates to guide adaptive mesh refinements

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as demonstrated in [9] in the context of convection-dominated diffusion problems and in [10] in the context of shell boundary layers.

In the present work we revisit the analysis of one-dimensional transport problems. In particular, the focus is on the robustness of the DPG method with respect to small diffusion. We complement the earlier works [6, 9] by resolving the stable test function space with respect to the so called optimal, or quasi-optimal, test space norm. The corresponding DPG method is expected to be uniformly stable with respect to vanishing diffusion, but the local problems for the test functions become singularly perturbed. The main contribution of our study is to show how these can be solved approximately by utilizing a carefully designed element subgrid discretization.

Our DPG formulation is valid for the steady convection-diffusion-reaction equation and uses piecewise Bernstein polynomials, or Bézier elements, as basis functions. The motivation for this comes from certain developments in isogeometric analysis (IGA). Isogeometric analysis was introduced by Hughes et al. in [11] and aims at the integration of engineering design and analysis processes. This is achieved by using the functions commonly used for geometry representation in computer aided design (CAD) as a basis for numerical analysis. These functions include non-uniform rational B-splines (NURBS) and their generalization, T-splines, see [12]. Recently it has been shown how so called Bézier extraction can be used to map a Bernstein polynomial basis on Bézier elements to a global NURBS or T-spline basis, see [13, 14].

The paper is structured as follows. We summarize the concept of optimal test functions and the fundamentals of Bézier elements in Section 2. The DPG formulation of the convection-diffusion-reaction problem together with some computational considerations are given in Section 3. Numerical results are presented in Section 4 and the paper is concluded with a summary in Section 5.

2. Preliminaries

2.1. The concept of optimal test functions

Consider an abstract variational problem

$$\text{Find } \mathbf{u} \in \mathcal{U} \quad \text{such that} \quad \mathcal{B}(\mathbf{u}, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathcal{V} \tag{1}$$

where $\mathcal{U}$ and $\mathcal{V}$ are two real Hilbert spaces. For well-posed variational problems, where the bilinear form $\mathcal{B}(\mathbf{u}, \mathbf{v})$ is continuous and satisfies the inf-sup condition, the expression

$$|||\mathbf{u}||| = \sup_{\mathbf{v}\in\mathcal{V}} \frac{\mathcal{B}(\mathbf{u}, \mathbf{v})}{||\mathbf{v}||_{\mathcal{V}}} \tag{2}$$

is a norm in $\mathcal{U}$:

$$\gamma||\mathbf{u}||_{\mathcal{U}} \leq |||\mathbf{u}||| \leq C||\mathbf{u}||_{\mathcal{U}} \quad \text{for all } \mathbf{u} \in \mathcal{U}$$

A finite-dimensional subspace $\mathcal{U}_n \subset \mathcal{U}$ can be equipped with the best approximation property in the norm (2) by defining the corresponding *optimal test space* as $\mathcal{V}_n = \mathbf{T}(\mathcal{U}_n)$, where $\mathbf{T} : \mathcal{U} \to \mathcal{V}$ is the *trial-to-test operator* defined as

$$(\mathbf{T}\mathbf{u}, \delta\mathbf{v})_{\mathcal{V}} = \mathcal{B}(\mathbf{u}, \delta\mathbf{v}) \quad \text{for all } \delta\mathbf{v} \in \mathcal{V} \tag{3}$$

More precisely, each $\mathbf{v}_n \in \mathcal{V}_n$ is of the form $\mathbf{v}_n = \mathbf{T}\mathbf{w}_n$ for some $\mathbf{w} \in \mathcal{U}_n$ by construction so that the variational problem restricted to the subspaces $\mathcal{U}_n$ and $\mathcal{V}_n$ becomes

$$(\mathbf{T}\mathbf{u}_n, \mathbf{T}\mathbf{w}_n)_{\mathcal{V}} = \mathcal{L}(\mathbf{T}\mathbf{w}_n) \quad \text{for all } \mathbf{w}_n \in \mathcal{U}_n \tag{4}$$

This defines $\mathbf{u}_n$ as the best approximation of $\mathbf{u}$ in the norm $||\mathbf{T}(\cdot)||_{\mathcal{V}} = |||\cdot|||$ with the associated system matrix being symmetric (and positive definite).

It is important to note that the characteristics of the *generalized energy norm* (2) depend on the constant in the inf-sup condition and may deteriorate as $\gamma \to 0$. However, this can be avoided if

$$||\mathbf{v}||_{\mathcal{V},\text{opt}} = \sup_{\mathbf{u}\in\mathcal{U}} \frac{\mathcal{B}(\mathbf{u}, \mathbf{v})}{||\mathbf{u}||_{\mathcal{U}}} \tag{5}$$

can be chosen as the norm on $\mathcal{V}$. Namely, the general theory of bounded below operators implies that then $|||\cdot||| = \|\cdot\|_{\mathcal{U}}$ provided that the subspace

$$\{\mathbf{u} \in \mathcal{U} : \quad \mathcal{B}(\mathbf{u}, \mathbf{v}) = 0 \quad \text{for all } \mathbf{v} \in \mathcal{V}\}$$

is trivial, see e.g. [15, 16, 10, 17].

The use of the *optimal test space norm* (5) in actual computations is particularly attractive when the variational problem involves dependency on a crucial parameter, such as the Péclet number in convection-dominated flow problems, since the best approximation property is attained in a norm which is independent of the parameter value. Nevertheless, the parametric dependence surfaces then in the computation of the optimal test space. The novelty of the DPG approach lies in the fact that once the test function space $\mathcal{V}$ is made fully discontinuous, the variational problem (3) can be attacked locally within each element. Moreover, these problems are always symmetric and can be solved approximately using the standard Galerkin method.

2.2. Bézier elements

The $p + 1$ Bernstein basis polynomials of degree p are defined for $x \in [0, 1]$ as

$$B_{i,p}(x) = \binom{p}{i} x^i (1 - x)^{p-i}, \quad i = 0, 1, \ldots, p$$

These constitute a basis of the polynomials of degree p, are pointwise non-negative and form a partition of unity, just to list a few properties relevant to numerical analysis. The motivation for performing finite element computations using this basis comes from the fact that a piecewise Bernstein polynomial basis can be mapped onto a B-spline basis by invoking the Bézier extraction operator, see [13]. This transformation enables the representation of a *NURBS* or a T-spline by using a set of Bézier elements.

The continuity between neighbouring elements in the mesh is controlled by the multiplicity of knots in the refined knot vector. For instance, for a univariate B-spline curve of polynomial degree p, the repetition of all knots until they have a multiplicity of $p + 1$ yields a discontinuous piecewise Bernstein polynomial representation of the curve.

3. Steady convective-diffusive-reactive transport problem in one space dimension

3.1. Strong form

We shall consider a stationary convection-diffusion-reaction problem written in the conservative form as

$$(-\varepsilon u'(x) + a(x)u(x))' + s(x)u(x) = f(x), \quad x \in I \tag{6}$$

where $I = (a, b)$ is a bounded interval, $\varepsilon > 0$ is the (constant) diffusion coefficient, $a(x)$ and $s(x)$ are smoothly varying coefficients with $|a(x)| \sim 1$ representing the convection velocity and the reaction rate, respectively, and $f(x)$ is the source term.

In the present work we consider mainly Dirichlet boundary conditions, that is, we assume that the values of the solution $u(x)$ are prescribed at both endpoints of the interval

$$u(a) = u_a, \quad u(b) = u_b$$

The treatment of other boundary conditions is also briefly addressed.

3.2. Ultra-weak variational form

The first step in the finite element analysis of any boundary value problem is a weak formulation. Denoting the total flux by $\sigma(x) = -\varepsilon u'(x) + a(x)u(x)$, the convection-diffusion equation (6) can be written as a system of first order differential equations as

$$\begin{cases} \sigma(x) + \varepsilon u'(x) - a(x)u(x) = 0 \\ \sigma'(x) + s(x)u(x) = f(x) \end{cases}, \quad x \in I$$

Integration of these equation by parts over a single element $K = (x_{k-1}, x_k)$ in an arbitrary partition

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

gives

$$\begin{aligned} \int_{x_{k-1}}^{x_k} \sigma\tau \,\mathrm{d}x - \varepsilon \int_{x_{k-1}}^{x_k} u\tau' \,\mathrm{d}x + \varepsilon u\tau\Big|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} au\tau \,\mathrm{d}x &= 0, \qquad \text{for all } \tau \in H^1(x_{k-1}, x_k) \\ - \int_{x_{k-1}}^{x_k} \sigma v' \,\mathrm{d}x + \sigma v\Big|_{x_{k-1}}^{x_k} + \int_{x_{k-1}}^{x_k} suv \,\mathrm{d}x &= \int_{x_{k-1}}^{x_k} fv \,\mathrm{d}x, \quad \text{for all } v \in H^1(x_{k-1}, x_k) \end{aligned}$$

where $H^1(x_{k-1}, x_k)$ is the standard Sobolev space consisting of square-summable functions with square-summable derivatives over (x_{k-1}, x_k).

Upon declaring the values of εu and σ at the nodes as independent unknowns and denoting them by $\widehat{\varepsilon u}$ and $\hat{\sigma}$, the problem can be formulated in the following ultra-weak variational form corresponding to (1):

$$\text{Find } \mathbf{u} = (\sigma, u, \hat{\sigma}, \widehat{\varepsilon u}) \in \mathcal{U} \text{ such that } \quad \mathcal{B}(\mathbf{u}, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \quad \text{for all } \mathbf{v} = (\tau, v) \in \mathcal{V}$$

where the bilinear form and the load functional are given by

$$\begin{aligned} \mathcal{B}(\mathbf{u}, \mathbf{v}) &= \sum_{k=1}^{N} \left\{ \int_{x_{k-1}}^{x_k} [\sigma(\tau - v') - u(\varepsilon\tau' + a\tau - sv)] \,\mathrm{d}x + (\hat{\sigma} v + \widehat{\varepsilon u}\tau)\Big|_{x_{k-1}}^{x_k} \right\}, \\ \mathcal{L}(\mathbf{v}) &= \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} fv \,\mathrm{d}x \end{aligned}$$

and the spaces are defined by

$$\begin{aligned} \mathcal{U} &= L_2(a, b) \times L_2(a, b) \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \\ \mathcal{V} &= W_N \times W_N \end{aligned}$$

where

$$W_N = \{v : \ v\,|_{(x_{k-1}, x_k)} \in H^1(x_{k-1}, x_k), \ 1 \le k \le N\}$$

In the present formulation, boundary conditions can be enforced by prescribing either the value of $\widehat{\varepsilon u}$ or the total flux $\hat{\sigma}$ at the end points of the interval. These are the customary boundary conditions for conservation laws. Boundary conditions for the diffusive flux could be prescribed most easily by leaving the convective flux out of the definition of σ. This would lead us to a slightly different variational formulation which has been studied in the earlier works [6, 9].

The optimal test space norm

Selecting the standard norm

$$||\mathbf{u}||^2_{\mathcal{U}} = \int_a^b (\sigma^2 + u^2) \,\mathrm{d}x + \sum_{k=0}^{N} (\hat{\sigma}(x_k)^2 + \widehat{\varepsilon u}(x_k)^2) \tag{7}$$

on the space $\mathcal{U}$, it is relatively easy to see that the norm (5) squared becomes

$$||\mathbf{v}||^2_{\mathcal{V},\mathrm{opt}} = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [(\tau - v')^2 + (\varepsilon\tau' + a\tau - sv)^2] \,\mathrm{d}x + \sum_{k=1}^{n-1} \left([\![\tau]\!]_k^2 + [\![v]\!]_k^2 \right) + \tau(x_0)^2 + \tau(x_N)^2 + v(x_0)^2 + v(x_N)^2$$

where $[\![q]\!]_j = q(x_j^-) - q(x_j^+)$ denotes the jump of q at the (interior) node x_j.

Dropping the jump and boundary terms and adding an integral contribution of v^2 in order to redeem positive-definiteness leads us to the *quasi-optimal test space norm*

$$||\mathbf{v}||^2_{\mathcal{V}} = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [(\tau - v')^2 + (\varepsilon\tau' + a\tau - sv)^2 + v^2] \,\mathrm{d}x \tag{8}$$

that can be used to determine the corresponding test space *locally*. Notice that this change anyway deflects the generalized energy norm $|||\cdot|||$ slightly from the standard norm defined in Eq. (7).

Figure 1: Subgrid within each element (boundary layer mesh).

3.3. Computational considerations

In our computations we employ discontinuous trial spaces consisting of piecewise Bernstein polynomial of degree p which is the same for all elements in the mesh. As in the earlier studies [6, 9, 10, 16], the optimal test functions are computed approximately in an *enriched finite element space*. We rely here on a spectral approximation by raising the local element order from p to $p_e = p + 3$ but take into account the singular perturbation character of the problem as follows.

When (8) is employed in (3), the corresponding Euler-Lagrange equations on (x_{k-1}, x_k), $k = 1, \ldots, N$ take the form

$$\begin{cases} -\varepsilon^2\tau''(x) + [1 + a^2]\tau(x) - v'(x) = \sigma(x) + \varepsilon u'(x) - au(x) \\ -v''(x) + v(x) + \tau'(x) = \sigma'(x) \end{cases} \quad \& \quad \{\text{boundary conditions}\}$$

where, for the sake of simplicity, we have assumed that the reaction rate vanishes, $s(x) \equiv 0$, and that the advection is constant, $a(x) \equiv a$. Examination of this system reveals that it is of reaction-diffusion type and may therefore feature a boundary layer of width $O(\varepsilon)$ at $x = x_{k-1}, x_k$. According to [18] such problems can be approximated reliably with the p-version of FEM provided that a boundary layer elements of width $p_e\varepsilon$ are added in the vicinity of the end points. Following these guidelines, we make use of a three-element subgrid within each element to approximate the test functions, see Fig. 1.

4. Numerical results

4.1. Simple transport problem

We begin by investigating the homogeneous convection-diffusion equation ($f(x) \equiv s(x) \equiv 0$) on the unit interval $I = (0, 1)$. In this example, the advection $a(x)$ is set to unity and the Dirichlet conditions $u(0) = 1$ and $u(1) = 0$ are specified. The solution reads

$$u(x) = (1 - e^{(x-1)/\varepsilon})/(1 - e^{-1/\varepsilon})$$

and develops a boundary layer of width ε at the outflow boundary $x = 1$.

We have computed the numerical approximation with a mesh of four uniform elements for different polynomial degrees. The results are shown in Fig. 2 for diffusivity $\varepsilon = 0.1$ and $p = 1, 2, 4, 8$. The best-approximation property in the L_2-norm is rather evident. Fig. 3 shows that this holds true also for smaller values of ε (higher Péclet number).

4.2. The Hemker problem

As another example, we study the so called Hemker problem. The problem is formulated in [19, Chapter 2] using the advective form of the convection-diffusion equation:

$$-\varepsilon u''(x) - xu'(x) = \varepsilon\pi^2 \cos(\pi x) + \pi x \sin(\pi x), \quad x \in (-1, 1)$$

The exact solution corresponding to Dirichlet boundary conditions

$$u(-1) = -2, \quad u(1) = 0$$

is

$$u(x) = \cos(\pi x) + \mathrm{erf}(x/\sqrt{2\varepsilon})/\mathrm{erf}(1/\sqrt{2\varepsilon})$$

and features an interior layer at $x = 0$.

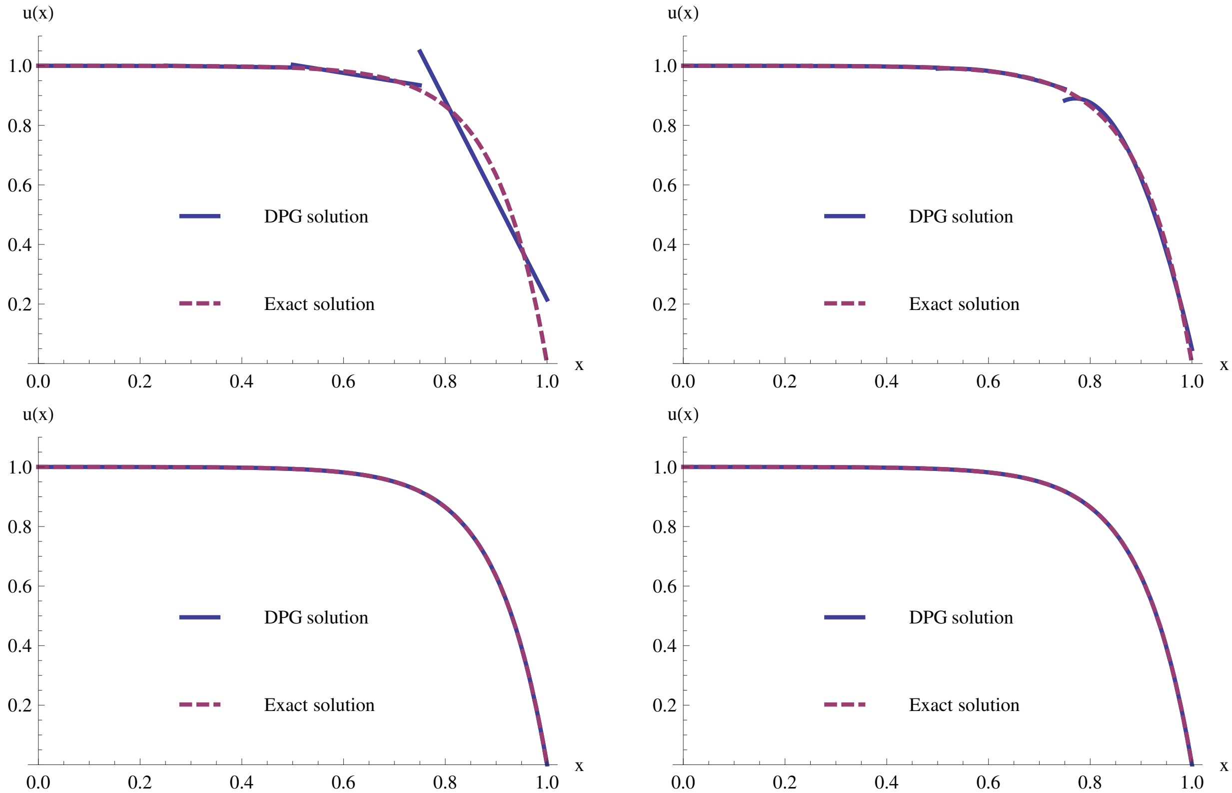


Figure 2: Discontinuous Petrov-Galerkin solution (solid line) of the simple transport problem with $\varepsilon = 0.1$ using four elements with $p = 1, 2, 4, 8$.

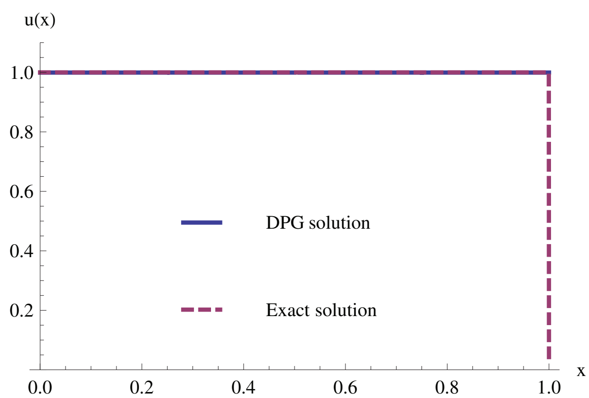


Figure 3: Discontinuous Petrov-Galerkin solution (solid line) of the simple transport problem with $\varepsilon = 10^{-6}$ using four linear elements.

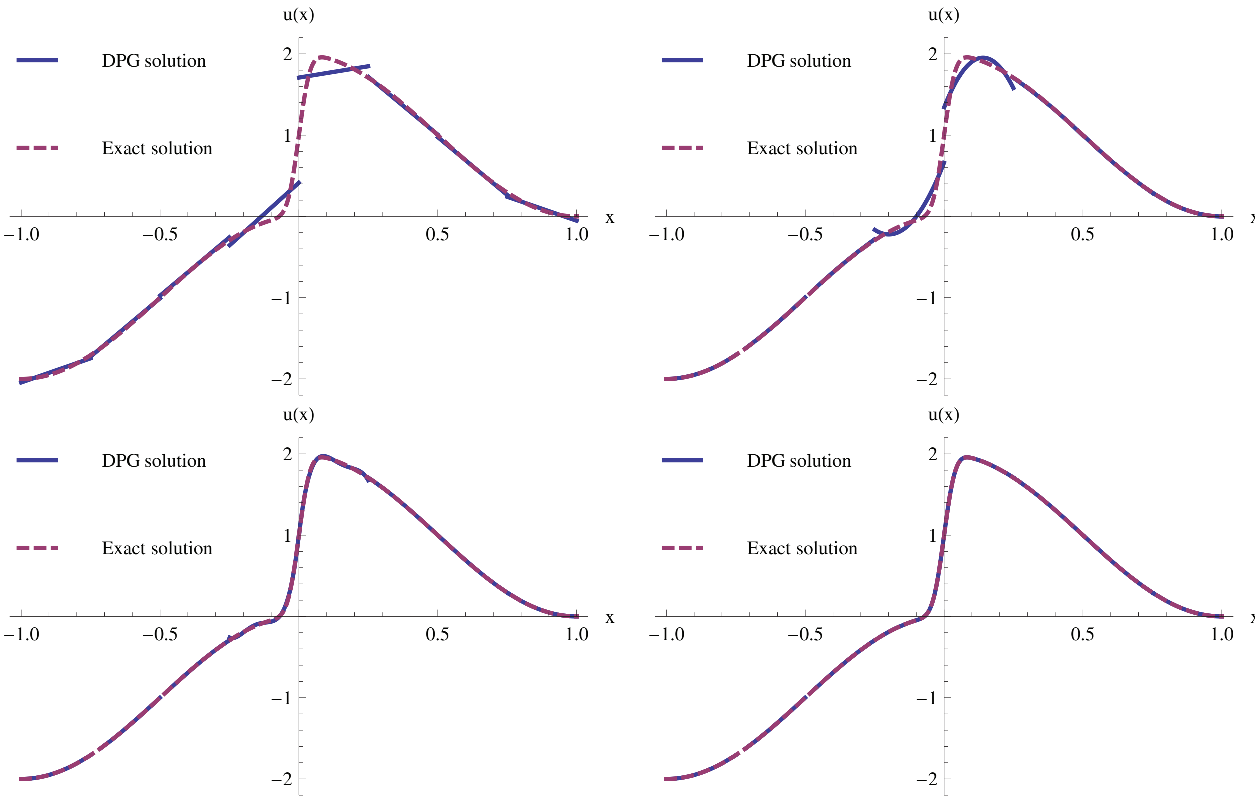


Figure 4: Discontinuous Petrov-Galerkin solution (solid line) of the Hemker problem with $\varepsilon = 0.001$ using eight elements with $p = 1, 2, 4, 8$.

The problem was formulated in the form (6) by defining

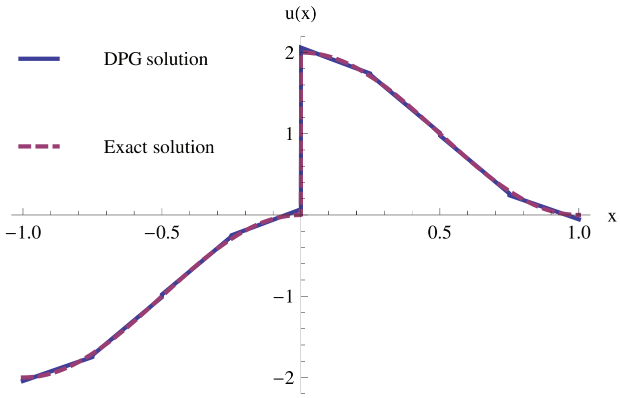
$$a(x) = -x, \quad s(x) = 1, \quad f(x) = \varepsilon\pi^2 \cos(\pi x) + \pi x \sin(\pi x)$$

and was solved numerically for $\varepsilon = 10^{-3}$ by using eight uniform elements with $p = 1, 2, 4, 8$. As Fig. 4 shows, the best-approximation property of the DPG method is not affected by the non-constant convection velocity. Fig. 5 shows that a good approximation is obtained also when $\varepsilon = 10^{-7}$.

It should be noted that the formulation loses its robustness if no inter-element node is located at $x = 0$. This occurs because the subgrid discretization used in the approximation of the test functions does not account for changes of sign of the advection $a(x)$ within an element (x_{k-1}, x_k).

5. Summary

We have performed a numerical study on the approximation of convection-diffusion-reaction problems using the discontinuous Petrov-Galerkin method with Bézier elements. We have constructed a robust algorithm based on the optimal test space norm and a three-element subgrid discretization for resolving the corresponding weighting functions. We will study alternative subgrid discretizations utilizing k-refinements as well as the extension of these techniques to higher space dimensions in future works.



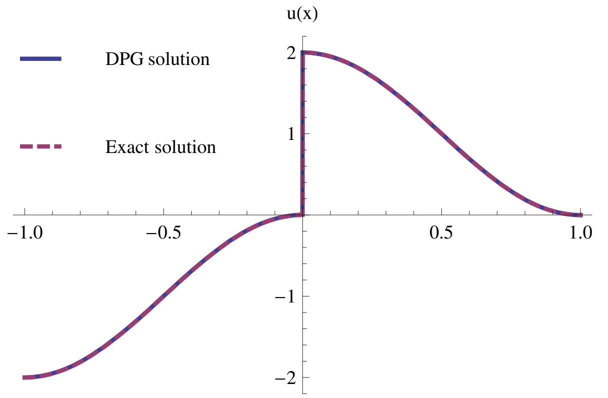


Figure 5: Discontinuous Petrov-Galerkin solution (solid line) of the Hemker problem with $\varepsilon = 10^{-7}$ using eight elements linear and quadratic elements.

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