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Sequential Optimization of Paths in Directed Graphs Relative to Different Cost Functions

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Abstract

This paper is devoted to the consideration of an algorithm for sequential optimization of paths in directed graphs relative to different cost functions. The considered algorithm is based on an extension of dynamic programming which allows to represent the initial set of paths and the set of optimal paths after each application of optimization procedure in the form of a directed acyclic graph.

Keywords: optimal path, dynamic programming, cost function, sequential optimization

1. Introduction

In this paper, we present an algorithm for sequential optimization of paths in a directed graph G relative to different cost functions. Let w_1, w_2 be weight functions which correspond to each edge nonnegative real weights, and p be a directed path in G. We consider two types of cost functions: ψ_{w_1} where $\psi_{w_1}(p)$ is equal to the sum of weights of edges in p, and φ_{w_2} where $\varphi_{w_1}(p)$ is equal to the minimum weight of some edge in p. We study the problem of minimization for ψ_{w_1} and the problem of maximization for φ_{w_2}.

Let us consider an interpretation of G as a network of roads. In this case, each vertex can be interpreted as an intersection point, and each edge as a road connecting two intersection points. If weight (the value of weight function w) is the length of a road, then $\psi_w(p)$ is the length of the path p. If weight is the fee we need to pay for using the road, then $\psi_w(p)$ is the total fee of the path. In such a way, we can also consider the minimization of the risk, or the expected time. Furthermore, if weight is the constraint on the height, width or weight of a vehicle, then $\varphi_w(p)$ is the strongest constraint on the height, width or weight of vehicle in the path p.

Note that we can consider another interpretation of G as a computer network. The first type of the cost functions ψ_w can be used for minimization of delay or packet loss rate, and the second type of the cost functions φ_w can be used for maximization of bit rate, for example.

The problem of optimization of paths in G relative to ψ_w is very well known. There are algorithms based on *greedy approach* such as Dijkstra's algorithm [1], algorithms based on *dynamic programming* such as Bellman-Ford algorithm [2, 3], and Floyd-Warshal algorithm [4, 5]. Cherkassky et al. discuss these and many other algorithms in [6].

The novelty of this work is connected with the following: we represent all initial paths from G (satisfying conditions (a) and (b) from Sect. 2) by a *directed acyclic graph* (DAG) Γ_0 and apply to this graph a procedure of

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optimization relative to a cost function. As a result we have a subgraph Γ_1 of Γ_0, which represents all optimal paths from Γ_0. We can apply to Γ_1 the procedure of optimization relative to another cost function, etc.

The considered approach is an extension of dynamic programming. Similar techniques have been used exclusively for optimization of decision trees relative to different cost functions [7, 8]. Similar work on other combinatorial optimization problem with respect to sequential optimization can be found in [9, 10, 11].

This paper consists of nine sections. In Section 2, we provide preliminary concepts. In Section 3, we give a description of the set of initial paths. We present the procedures of optimization relative to two cost functions in Section 4. In Section 5, we extend the optimization procedures for application of different cost functions sequentially. We provide an example in Section 6. In Section 7, we discuss the computational complexity of our procedures and we conclude the paper in Section 8. The last section, Section 9 is devoted for reference.

2. Preliminaries

Let $G = (V, E)$ be a directed graph without loops and multiple edges. Let s, t be two vertices in G such that there exists a directed path from s to t. We call s the *source* vertex, and t the *target* vertex.

We consider the problem of finding optimal paths in the graph from s to t. The optimization criteria are defined by means of cost functions.

We consider two types of cost functions. Each *cost function* assigns a *cost* to each directed path in G. Let w be a *weight function* that assigns to each edge (v_i, v_j) of G a weight $w(v_i, v_j)$, which is a nonnegative real number.

We consider a directed path $p = v_1, v_2, \dots, v_m, v_{m+1}$ in G. A cost function of the first type ψ_w is defined as follows: $\psi_w(p) = \sum_{i=1}^{m} w(v_i, v_{i+1})$. If $m = 0$ then $\psi_w(p) = 0$. We aim to minimize the value of this function. A cost function of the second type φ_w is defined as follows: $\varphi_w(p) = \min\{w(v_1, v_2), ..., w(v_m, v_{m+1})\}$. If $m = 0$ then $\varphi_w(p) = +\infty$. We aim to maximize the value of this function.

Let p be a directed path in G from s to t. One can show that there exists a directed path p' from s to t in G, which can be obtained from p by removal of some vertices and edges such that it satisfies the following conditions:

(a) The length of p' is at most $|V| - 1$,
(b) Both s and t appear exactly once in p',
(c) $\psi_w(p') \leq \psi_w(p)$,
(d) $\varphi_w(p') \geq \varphi_w(p)$.

Therefore, later on we will focus on the study of paths in G from s to t satisfying the conditions (a) and (b). Furthermore, We denote by $P(G)$ the set of all such paths.

3. Description of Paths

In this section, we introduce a directed acyclic graph $\Gamma_0 = \Gamma_0(G)$, which represents the set of directed paths $P(G)$ from s to t satisfying conditions (a) and (b).

Let $V = \{v_1, \dots, v_n\}$, where $v_1 = s$ and $v_n = t$. The set of vertices of Γ_0 is divided into n layers. The first layer contains the only vertex v_1^1. The last layer contains the only vertex v_n^n. For $k = 2, \dots, n-1$ the k-th layer contains $n-2$ vertices $v_2^k, \dots, v_{n-1}^k$.

Let $i, j \in \{2, \dots, n-1\}$, we have an edge from v_1^1 to v_j^2 if and only if $(v_1, v_j) \in E$. For $k = 2, \dots, n-2$, we have an edge from v_i^k to v_j^{k+1} if and only if $(v_i, v_j) \in E$.

Let $2 \leq j \leq n-1$ and $2 \leq k \leq n-1$. Then, we have an edge from v_j^k to v_n^n if and only if $(v_j, v_n) \in E$. We have an edge from v_1^1 to v_n^n if and only if $(v_1, v_n) \in E$. There are no other edges in Γ_0.

A weight function w on G can be extended to Γ_0 in a natural way: $w(v_i^{k_1}, v_j^{k_2}) = w(v_i, v_j)$ if $(v_i^{k_1}, v_j^{k_2})$ is an edge in Γ_0.

We denote by $P(\Gamma_0)$ the set of all directed paths in Γ_0 from v_1^1 to v_n^n. It is clear that there is one-to-one mapping from the set $P(G)$ to the set $P(\Gamma_0)$. The path $v_1, v_{j_2}, \dots, v_{j_t}, v_n$ from $P(G)$ corresponds to the path $v_1^1, v_{j_2}^2, \dots, v_{j_t}^t, v_n^n$ from $P(\Gamma_0)$.

4. Procedures of Optimization

Let Γ be a subgraph of Γ_0 which can be obtained from Γ_0 by removal of some vertices and edges such that there is at least one directed path from v_1^1 to v_n^n. We denote by $P(\Gamma)$ the set of all such paths. We consider procedures of optimization of paths in $P(\Gamma)$ relative to ψ_w and φ_w.

4.1. Optimization Relative to ψ_w

We remove from Γ all vertices (and their incident edges) such that there is no directed paths from v_1^1 to the considered vertex. We denote by Γ' the obtained subgraph of Γ.

For each vertex v_j^k of Γ' we label it with $\Psi_w(v_j^k)$ which is the minimum cost of a directed path in Γ' from v_1^1 to v_j^k relative to ψ_w and remove some edges incoming to v_j^k. We label v_1^1 with $\Psi(v_1^1) = 0$. Let $v_j^k \neq v_1^1$ and $v_{i_1}^{k_1}, \ldots, v_{i_r}^{k_r}$ be all vertices in Γ' such that from each of which there is an edge to v_j^k. Then,

$$\Psi_w(v_j^k) = \min_{1 \le t \le r} \left\{ w(v_{i_t}^{k_t}, v_j^k) + \Psi_w(v_{i_t}^{k_t}) \right\}. \tag{1}$$

We remove from Γ' all edges $(v_{i_t}^{k_t}, v_j^k)$ incoming to v_j^k such that

$$w(v_{i_t}^{k_t}, v_j^k) + \Psi_w(v_{i_t}^{k_t}) > \Psi_w(v_j^k).$$

We denote the obtained subgraph of Γ_0 by Γ_{ψ_w}. It is clear that there is at least one path in Γ_{ψ_w} from v_1^1 to v_n^n. We denote by $P(\Gamma_{\psi_w})$ all such paths.

Theorem 1. *The set of paths $P(\Gamma_{\psi_w})$ coincides with the set of paths from $P(\Gamma)$ that have minimum cost relative to ψ_w.*

Proof. We denote by $P(\Gamma')$ the set of all directed paths in Γ' from v_1^1 to v_n^n. It is clear that $P(\Gamma') = P(\Gamma)$. Let us prove by induction on vertices in Γ' that for each vertex v_j^k of Γ' the number $\Psi_w(v_j^k)$ is the minimum cost of a directed path in Γ' from v_1^1 to v_j^k relative to ψ_w and the set of directed paths from v_1^1 to v_j^k in Γ_{ψ_w} coincides with the set of directed paths from v_1^1 to v_j^k in Γ' that have minimum cost relative to ψ_w. If $v_j^k = v_1^1$ we have $\Psi_w(v_1^1) = 0$. We have exactly one path from v_1^1 to v_1^1 both in Γ' and Γ_{ψ_w}. The length and the cost of this path are equal to zero. So the considered statement holds for v_1^1. For $v_j^k \neq v_1^1$, let $v_{i_1}^{k_1}, \ldots, v_{i_r}^{k_r}$ be all vertices in Γ' such that from each of which there is an edge to v_j^k. Let us assume that the considered statement holds for $v_{i_1}^{k_1}, \ldots, v_{i_r}^{k_r}$. Since Γ' is a directed acyclic graph, each path from v_1^1 to v_j^k has no v_j^k as an intermediate vertex. From here, (1) and inductive hypothesis it follows that $\Psi_w(v_j^k)$ is the minimum cost of a directed path in Γ' from v_1^1 to v_j^k.

Using (1), inductive hypothesis, and description of optimization procedure we have that each path in Γ_{ψ_w} from v_1^1 to v_j^k has the cost $\Psi_w(v_j^k)$. So the set of directed paths from v_1^1 to v_j^k in Γ_{ψ_w} is a subset of the set of directed paths from v_1^1 to v_j^k in Γ' that have minimum cost relative to ψ_w.

Let p be a path from v_1^1 to v_j^k in Γ' with minimum cost relative to ψ_w. Then for some $t \in \{1, \ldots, r\}$ the path p passes through the vertex $v_{i_t}^{k_t}$. Therefore, $\psi_w(p) = \Psi_(v_j^k) = \psi_w(p') + w(v_{i_t}^{k_t}, v_j^k)$ where p' is the first part of p from v_1^1 to $v_{i_t}^{k_t}$. It is clear that $\psi_w(p') \ge \Psi_w(v_{i_t}^{k_t})$. From here and the description of optimization procedure it follows that the edge $(v_{i_t}^{k_t}, v_j^k)$ belongs to Γ_{Ψ_w}. Let us assume that $\psi_w(p') > \Psi_w(v_{i_t}^{k_t})$. Then $\Psi_w(v_j^k) > \Psi_w(v_{i_t}^{k_t}) + w(v_{i_t}^{k_t}, v_j^k)$ but this is impossible (see (1)), so $\psi_w(p') = \Psi_w(v_{i_t}^{k_t})$. According to inductive hypothesis, p' belongs to the set of paths in Γ_{ψ_w} from v_1^1 to $v_{i_t}^{k_t}$ and therefore p belongs to the set of paths in Γ_{ψ_w} from v_1^1 to v_j^k. Thus the considered statement holds. From this statement (if we set $v_j^k = v_n^n$) it follows the statement of the theorem. □

4.2. Optimization Relative to φ_w

We remove from Γ all vertices (and their incident edges) such that there is no directed path from v_1^1 to the considered vertex. We denote by Γ' the obtained subgraph of Γ.

For each vertex v_j^k of Γ' we label it with $\Phi_w(v_j^k)$ which is the maximum cost of a directed path in Γ' from v_1^1 to v_j^k relative to φ_w. We label v_1^1 with $\Phi_w(v_1^1) = +\infty$. Let $v_j^k \neq v_1^1$ and $v_{i_1}^{k_1}, \ldots, v_{i_r}^{k_r}$ be all vertices in Γ' such that from each of which there is an edge to v_j^k. Then

$$\Phi_w(v_j^k) = \max_{1 \le t \le r} \left\{ \min \left\{ w(v_{i_t}^{k_t}, v_j^k), \Phi_w(v_{i_t}^{k_t}) \right\} \right\}. \tag{2}$$

Let $c = \Phi_w(v_n^n)$. We denote by Γ_{φ_w} the graph obtained from Γ' by removal of all edges $(v_{j_1}^{k_1}, v_{j_2}^{k_2})$ for which $w(v_{j_1}^{k_1}, v_{j_2}^{k_2}) < c$. It is clear that there is at least one path in Γ_{φ_w} from v_1^1 to v_n^n. We denote by $P(\Gamma_{\varphi_w})$ all such paths.

Theorem 2. *The set of paths $P(\Gamma_{\varphi_w})$ coincides with the set of paths from $P(\Gamma)$ that have maximum cost relative to φ_w.*

Proof. We denote by $P(\Gamma')$ the set of all directed paths in Γ' from v_1^1 to v_n^n. It is clear that $P(\Gamma') = P(\Gamma)$. Let us prove by induction on vertices in Γ' that for each vertex v_j^k of Γ' the number $\Phi_w(v_j^k)$ is the maximum cost of a directed path in Γ' from v_1^1 to v_j^k relative to φ_w. If $v_j^k = v_1^1$ we have $\Phi_w(v_1^1) = +\infty$, and the considered statement holds for v_1^1 since there exists a path p from v_1^1 to v_1^1 with length equal to zero and cost $\varphi_w(p)$ equal to $+\infty$. For $v_j^k \neq v_1^1$, let $v_{i_1}^{k_1}, \ldots, v_{i_r}^{k_r}$ be all vertices in Γ' such that from each of which there is an edge to v_j^k. Let us assume that the considered statement holds for $v_{i_1}^{k_1}, \ldots, v_{i_r}^{k_r}$.

Since Γ' is a directed acyclic graph, each path from v_1^1 to v_j^k has no v_j^k as an intermediate vertex. From here, (2), and inductive hypothesis it follows that $\Phi_w(v_j^k)$ is the maximum cost of a directed path in Γ' from v_1^1 to v_j^k.

Let $c = \Phi_w(v_n^n)$ then we know that c is the maximum cost of a path in Γ' from v_1^1 to v_n^n. So each path in Γ' from v_1^1 to v_n^n, that does contain edges with weight less than c, is an optimal path (a path with maximum cost). Also each optimal path in Γ' from v_1^1 to v_n^n does not contain edges whose weight is less than c. From here it follows that the set $P(\Gamma_{\varphi_w})$ coincides with the set of paths from $P(\Gamma)$ that have maximum cost relative to φ_w. □

5. Sequential Optimization

The optimization procedures presented in this paper allow sequential optimization relative to different cost functions. Let us consider one of the possible scenarios.

Let w_1 and w_2 be two weight functions for G. Initially, we get the DAG $\Gamma_0 = \Gamma_0(G)$. We apply the procedure of optimization relative to ψ_{w_1} to the graph Γ_0. As a result we obtain the subgraph $\Gamma = (\Gamma_0)_{\psi_{w_1}}$ of the graph Γ_0. By Theorem 1, the set of directed paths in Γ from v_1^1 to v_n^n coincides with the set of directed paths $\mathcal{D}$ in Γ_0 from v_1^1 to v_n^n that have the minimum cost relative to ψ_{w_1}.

We can apply the procedure of optimization relative to φ_{w_2} to the graph Γ. As a result we obtain the subgraph $\Gamma_{\varphi_{w_2}}$ of the graph Γ_0. By Theorem 2, the set of directed paths $P(\Gamma_{\varphi_{w_2}})$ in $\Gamma_{\varphi_{w_2}}$ from v_1^1 to v_n^n coincides with the set of directed paths from $\mathcal{D}$ that have the maximum cost relative to φ_{w_2}.

We can continue the procedure of optimization relative to different cost functions.

6. Example

We consider a directed graph $G = (V, E)$ (as depicted in Figure 1) and two weight functions w_1 and w_2 for this graph as

$$w_1(e) = 1 \quad \text{for all } e \in E,$$

and

$$\begin{aligned} w_2(s,a) &= 1, \quad w_2(a,t) = 1, \quad w_2(s,b) = 2, \\ w_2(b,t) &= 2, \quad w_2(a,b) = 2, \quad w_2(b,a) = 2. \end{aligned}$$

Initially, we get a DAG $\Gamma_0 = \Gamma_0(G)$ (as depicted in Figure 2). Here we keep the original vertex labels from G and put $'$ and $''$ to represent the layer index.

We apply the procedure of optimization relative to ψ_{w_1} to the graph Γ_0 and obtain the graph $\Gamma = (\Gamma_0)_{\psi_{w_1}}$ as shown in Figure 3. The set $P(\Gamma)$ contains exactly two paths.

We apply the procedure of optimization to the graph Γ relative to φ_{w_2}. As a result we obtain the subgraph $\Gamma_{\varphi_{w_2}}$ of Γ_0, see Figure 4. The set $P(\Gamma_{\varphi_{w_2}})$ contains exactly one path.

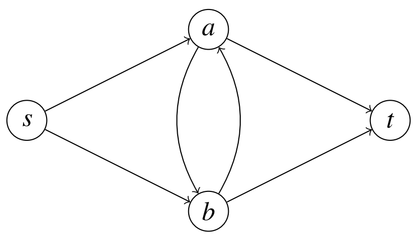

Figure 1: The directed graph G.

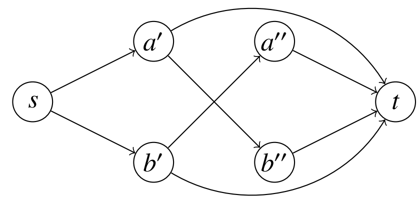

Figure 2: The initial DAG Γ_0.

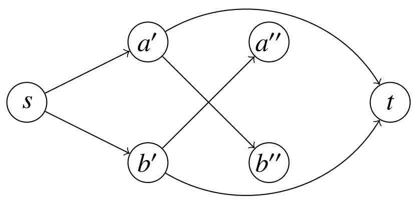

Figure 3: The subgraph $\Gamma = (\Gamma_0)_{\psi_{w_1}}$.

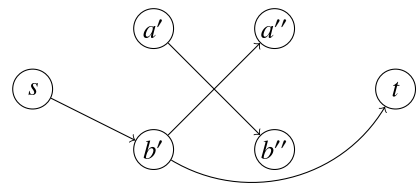

Figure 4: The subgraph $\Gamma_{\varphi_{w_2}}$.

7. Computational Complexity

Let $G = (V, E)$ be a directed graph with $|V| = n$ and $|E| = m$, and let $\Gamma_0 = \Gamma_0(G) = (V_0, E_0)$ be the directed acyclic graph corresponding to G. It is clear that $|V_0| = O(n^2)$ and $|E_0| = O(mn)$. Let $\Gamma = (V_1, E_1)$ be a subgraph of the graph Γ_0 obtained from Γ_0 after some number of applications of the optimization procedures. Let us evaluate the time complexity of optimization procedure applied to Γ relative to ψ_w (to φ_w).

We can find all vertices in Γ such that there is a directed path from v_1^1 to the considered vertex using *breadth-first search* algorithm with the running time $O(|V_1| + |E_1|)$ [12]. It is clear that $|V_1| \leq |V_0|$ and $|E_1| \leq |E_0|$. Therefore, the running time is $O(n^2 + mn)$.

After removal from Γ all vertices (and their incident edges), which are not reachable from v_1^1, we obtain a subgraph Γ' of Γ. According to the description of optimization procedure relative to ψ_w we need to make at most two comparisons and one addition per each edge to obtain the graph Γ_{ψ_w} from Γ'. So we need $O(mn)$ operations of comparison and addition to construct the graph Γ_{ψ_w} from Γ'. Similarly, according to the description of optimization procedure relative to φ_w we need to make at most three comparisons per each edge of Γ' to obtain the graph Γ_{φ_w} from Γ'. So we need $O(mn)$ operations of comparison to construct the graph Γ_{φ_w} from Γ'.

8. Conclusions

In this paper, we study an algorithm for optimization of paths in directed graphs which allows sequential optimization relative to different cost functions. The sets of initial paths and optimal paths are presented in a natural way by acyclic directed graphs. The results about structure of the set of optimal paths and bounds on the time complexity of considered algorithm show that this algorithm can be a useful tool for multicriteria optimization of paths.

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