

Diffusive Wave Approximation to the Shallow Water Equations: Computational Approach

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Abstract

We discuss the use of time adaptivity applied to the one dimensional diffusive wave approximation to the shallow water equations. A simple and computationally economical error estimator is discussed which enables time-step size adaptivity. This robust adaptive time discretization corrects the initial time step size to achieve a user specified bound on the discretization error and allows time step size variations of several orders of magnitude. In particular, in the one dimensional results presented in this work feature a change of four orders of magnitudes for the time step over the entire simulation.

Keywords: time adaptivity, shallow water flow, overland flow

1. Introduction

The diffusive wave approximation of the shallow water equations (DSW) is used to model overland flows such as floods, dam breaks, and flows through vegetated areas. The shallow water equations (SWE) are obtained from the full Navier-Stokes equations by introducing the following simplifying assumption: the vertical momentum scales are small relative to those of the horizontal momentum, that is, due to depth restrictions the velocity structures in the horizontal direction are much larger than the ones in the vertical one. This assumption reduces the vertical momentum equation to a hydrostatic pressure relation, which is integrated in the vertical direction and results in a two dimensional system of equations known as the shallow water equations. Further details and discussion of the scaling assumptions made can be found in [1]. The DSW equation is a further simplification of the shallow water equations. The DSW equation assumes that the horizontal momentum can be linked to the water height by an empirical relation such as Manning's equation [2]. Thus, the DSW reduces to a scalar equation which resembles nonlinear diffusion. While the nonlinearities present challenges, the DSW equation is a simpler framework with which economically simulate shallow flows.

In this paper, we address computational aspects of solving the DSW equation, specifically we focus on time integration and adaptivity. We use the generalized- α method for time integration, which is a second-order accurate method with controlled dissipation for high frequencies [3, 4]. We describe a simple and robust error estimator which can be used to guide time adaptivity first introduced and described in detail in [5]. Time adaptivity in the solution of DSW is useful for two reasons. First, the adaptivity aides in determining the initial time step. After selecting an initial step size, the algorithm will automatically reduce it to acceptable tolerances, relative to the quality of the error

estimator. Second, a significant amount of computation may be avoided by allowing the time step to change, while keeping the error under a user-prescribed tolerance. Our numerical results show that in 1D tests, the time step can change several orders of magnitude over the simulation time. In particular, the simulations presented herein, after the initial time-step size adjustment, the time-step size varies by four orders of magnitude.

2. DSW Equations

As mentioned in the introduction, the DSW strong form is obtained by special assumptions which simplify the shallow water equations, leading to the following initial/boundary value problem on the domain Ω for times $t \in [0, T]$ [6],

$$\begin{cases} \dot{u} - \nabla \cdot (\kappa(u, \nabla u) \nabla u) = f & \text{on } \Omega \times (0, T] \\ u = u_0 & \text{on } \Omega \times \{t = 0\} \\ (\kappa(u, \nabla u) \nabla u) \cdot n = B_N & \text{on } \Gamma_N \times (0, T] \\ u = B_D & \text{on } \Gamma_D \times (0, T] \end{cases} \quad (1)$$

where u is the water height, \dot{u} is its time derivative, f is a forcing function such as rainfall acting as a source or infiltration acting as a sink, u_0 is the initial condition, B_N and B_D are the Neumann and Dirichlet conditions, respectively, and the diffusion coefficient κ is given by

$$\kappa(u, \nabla u) = \frac{(u - z)^{\alpha_M}}{C_f |\nabla u|^{1-\gamma_M}}.$$

The bathymetry is represented by the function z , and α_M and γ_M are constants which specify the empirical method used to obtain the DSW equation. Following [6] we use parameters corresponding to Manning’s formula, $\alpha_M = \frac{5}{3}$ and $\gamma_M = \frac{1}{2}$. The function C_f represents Manning’s coefficient which is also known as a friction coefficient. Typical values are experimentally measured and available in the literature, but for the sake of simplicity we assume $C_f = 1$.

The equation is doubly degenerate in that the diffusion disappears in cases where $u = z$ or when ∇u becomes large such as in regions where the solution represents a wave front. This creates difficulties in developing a numerical solution technique which can handle these difficulties. These properties are discussed in detail in [6].

The weak form for the DSW equation is to find $u \in \mathcal{V}$ such that for every $\forall w \in \mathcal{W}$,

$$B(w, u) = \left(w, \frac{\partial u}{\partial t} \right)_{\Omega} + (\nabla w, \kappa(u, \nabla u) \nabla u)_{\Omega} + (w, f)_{\Omega} = 0 \quad (2)$$

where $(\cdot, \cdot)_{\Omega}$ refers to the L^2 inner product and the trial, \mathcal{V} , and weighting, \mathcal{W} , spaces are appropriately chosen for Eq. 2 to be well defined [6]. A discrete approximation to the solution is obtained constructing a Galerkin approximation appropriately choosing proper subspaces of \mathcal{V}_h and \mathcal{W}_h of \mathcal{V} and \mathcal{W} , respectively [7]. The discrete function space chosen in the numerical example described in section 4 is obtained using 1024 linear elements, while the L^2 inner product over the the domain is approximated using Gaussian quadrature, where four Gauss points per element are used.

3. Time Discretization

We advance in time using the generalized- α method [3, 4]. The method was originally developed in [3] for structural dynamics, which is of second order in time. Subsequently it was extended for first order problems in time in [4]. This time-stepping methodology has been successfully applied to several nonlinear problems such as, turbulent simulations [4, 8], and more recently first time adaptive technique was proposed for the Cahn-Hilliard equations [9] and subsequently used in [10] to model bubble formation and evolution.

The generalized- α method for first order in time problems is stated as follows: given (u_n, \dot{u}_n) , find $(u_{n+1}, \dot{u}_{n+1}, u_{n+\alpha_f}, \dot{u}_{n+\alpha_m})$ such that

$$R(u_{n+\alpha_f}, \dot{u}_{n+\alpha_m}) = 0, \quad (3)$$

$$u_{n+\alpha_f} = u_n + \alpha_f (u_{n+1} - u_n), \quad (4)$$

$$\dot{u}_{n+\alpha_m} = \dot{u}_n + \alpha_m (\dot{u}_{n+1} - \dot{u}_n), \quad (5)$$

$$u_{n+1} = u_n + \Delta t ((1 - \gamma) \dot{u}_n + \gamma \dot{u}_{n+1}), \quad (6)$$

where $\Delta t = t_{n+1} - t_n$ and α_f , α_m , and γ are real-valued parameters. It has been shown in [4] that for a linear model problem, unconditional stability is attained if $\alpha_m \geq \alpha_f \geq \frac{1}{2}$, and second order accuracy can be achieved with $\gamma = \frac{1}{2} + \alpha_m - \alpha_f$. The method can be stated as a one-parameter method, where α_m , α_f and γ can all be expressed in terms of a parameter known as spectral radius, ρ_∞ . We select $\alpha_m = \frac{5}{6}$ and $\alpha_f = \gamma = \frac{2}{3}$ which correspond to $\rho_\infty = \frac{1}{2}$. The generalized- α algorithm is detailed in Alg. 1. The algorithm is a predictor/multi-corrector method where the corrector steps are indicated by a superscript index inside of parenthesis.

Algorithm 1 Generalized- α method

- 1: Compute predictors [4, 9] $u_{n+1}^{(0)} = u_n$ and $\dot{u}_{n+1}^{(0)} = \frac{\gamma-1}{\gamma} \dot{u}_n$
 - 2: $i = 1$
 - 3: **while** $i <$ maximum iterations **do**
 - 4: $u_{n+\alpha_f}^{(i)} = u_n + \alpha_f (u_{n+1}^{(i-1)} - u_n)$
 - 5: $\dot{u}_{n+\alpha_m}^{(i)} = \dot{u}_n + \alpha_m (\dot{u}_{n+1}^{(i-1)} - \dot{u}_n)$
 - 6: $R_{n+1}^{(i)} = R(u_{n+\alpha_f}^{(i)}, \dot{u}_{n+\alpha_m}^{(i)})$
 - 7: $K_{n+1}^{(i)} = \alpha_m \frac{\partial R(u_{n+\alpha_f}^{(i)}, \dot{u}_{n+\alpha_m}^{(i)})}{\partial \dot{u}_{n+\alpha_m}} + \alpha_f \gamma \Delta t \frac{\partial R(u_{n+\alpha_f}^{(i)}, \dot{u}_{n+\alpha_m}^{(i)})}{\partial u_{n+\alpha_f}}$
 - 8: Solve $K_{n+1}^{(i)} \Delta \dot{u}_{n+1}^{(i)} = -R_{n+1}^{(i)}$
 - 9: $\dot{u}_{n+1}^{(i)} = \dot{u}_{n+1}^{(i-1)} + \Delta \dot{u}_{n+1}^{(i)}$
 - 10: $u_{n+1}^{(i)} = u_{n+1}^{(i-1)} + \gamma \Delta t_n \Delta \dot{u}_{n+1}^{(i)}$
 - 11: **if** $\|R_{n+1}^{(i)}\| \leq \epsilon_\alpha \|R_{n+1}^{(0)}\|$ **then**
 - 12: stop
 - 13: **end if**
 - 14: $i = i + 1$
 - 15: **end while**
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3.1. Time adaptivity

Time step adaptivity is achieved by a simple error predictor for first order methods [5]. Given a u_n, \dot{u}_n , and Δt_n the solution at t_{n+1} may be computed given the generalized- α algorithm described in Alg. 1. For $\rho_\infty = \frac{1}{2}$, we can then use \dot{u}_{n+1} to make a first order approximation of u_{n+1}

$$\bar{u}_{n+1} = u_n + \Delta t_n \dot{u}_{n+1}. \quad (7)$$

which is obtained by manipulating a first-order Taylor expansion from u_{n+1} to u_n . With this inexpensive approximation we can estimate the error as

$$E_{n+1} = \frac{\|\bar{u}_{n+1} - u_{n+1}\|}{\|u_{n+1}\|} \quad (8)$$

and adapt the time-step size using the typical equation [9],

$$\Delta t_{n+1} = \rho \left(\frac{\epsilon}{E_{n+1}} \right)^{\frac{1}{2}} \Delta t_n. \quad (9)$$

Following [9], the factor of safety ρ was chosen to be 0.9 and the tolerance ϵ to be 10^{-3} . The time step is rejected if $E_{n+1} > \epsilon$ and recomputed once Δt is modified. We added an additional constraint in the implementation of this algorithm which restricts the growth of the time step at a given time,

$$\frac{1}{10} \leq \frac{\Delta t_{n+1}}{\Delta t_n} \leq 5. \quad (10)$$

We found this last constraint to be useful in situations where the step size grows and then suddenly must reduce.

4. Numerical Results

The spatial, linear finite elements, and temporal, generalized- α discretizations were applied to the solution of a 1D problem, which models a fictitious dam break, arrested by a series of two dikes. The bathymetry is shown in Fig. 1(a), indicated by grey shading. The source of water is modeled by a Neumann boundary condition on the left side and a zero Dirichlet condition on the right. This example illustrates how the solution of the DSW can benefit from adaptive time stepping.

Figures 1(a)-(h) are a graphical representation of the solution obtained by our numerical simulation. As time progresses, the downhill flow (Figs. 1(a)-(c)) is arrested by a large dike (Fig. 1(d)) where it pools until the water level overtakes the dike (Fig. 1(e)). A second, smaller dike is reached after a long plain where the flow is arrested again (Figs. 1(f)-(g)) until finally the second dike is also overtaken (Fig. 1(h)).

Figure 2 emphasizes the usefulness of time adaptivity in this problem. The initial time step chosen was $\Delta t = 0.1$ which initially is 3 orders of magnitude too large. The adaptive step process assists in choosing this initial step in that values which are too large are cut to acceptable sizes. In this case the adaptation reduces the time step to $\Delta t = 3 \cdot 10^{-3}$. Alternatively an initial time step which is too small, will quickly grow. Figure 3 shows a comparison between the evolution of two initial step sizes, $\Delta t = 1.0 \cdot 10^{-1}$ and $\Delta t = 1.0 \cdot 10^{-8}$. In both situations, the time-step size evolves over the solution time with good agreement.

From step number 2-50, the time step grows two orders of magnitude where it then remains roughly constant until step 145 (Fig. 1(c)). At this point, the flow is being slowed down by the first dike and begins to pool. The water height is not changing drastically during this phase of the simulation and so the time step grows further until the water begins to flow over the first dike, step number 170. Large and sudden changes in the water height require a smaller step size and so the step is reduced again. As the simulation continues, two other size changes are seen,

1. as the flow hits the bottom of the first dike on the right side (around step number 200)
2. as the flow passes over the second dike (around step number 260).

We observe that time adaptivity is a great advantage when flow must pass sharp obstacles such as these dikes.

While each subfigure is titled with the time at which the solution is displayed, these times do not correspond to real flooding events. The frictional coefficient, C_f , in the DSW equations is taken here to be unity. This choice allows us to focus on the time adaptive scheme where flow velocity is controlled by topographic gradients. Notice that in spite of having a constant C_f the diffusivity is not constant, due to its non-linear dependence on both the water height, u , and its gradient ∇u .

5. Conclusions

We briefly described the diffusive wave approximation to the shallow water equations and presented a numerical strategy for their approximation using a Galerkin finite element procedure for spatial discretization and the generalized- α method for temporal discretization. We discussed a new error estimator which can be used to economically enable time adaptivity. This adaptivity proves useful even in simple 1D problems, while being robust. This work is an initial study on solution strategies for the DSW equation for the modeling of 2D overland flows. In future work, we will both extend the numerical technique to 2D as well as incorporate methods for more accurately determining and utilizing accurate frictional coefficients.

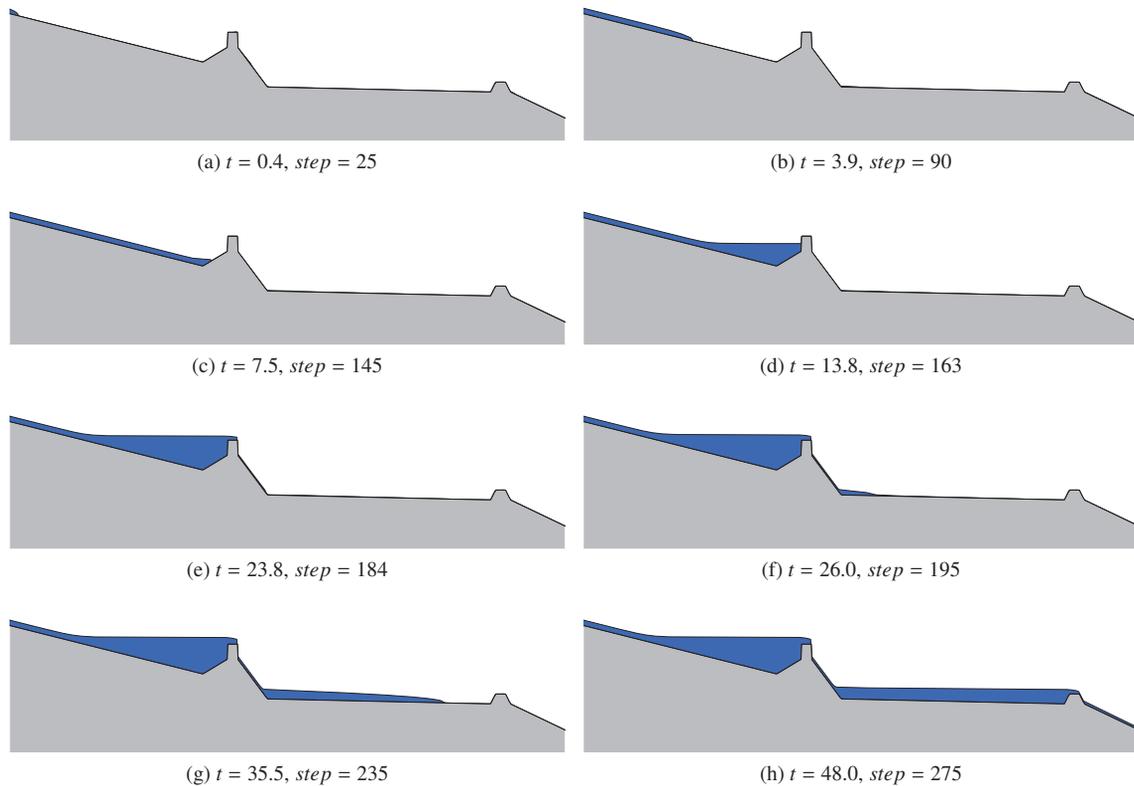


Figure 1: Time sequence of solutions to DSW with bathymetry shown in grey and the water flow shown in blue. This sequence models the arrest of a dam break by a series of dikes.

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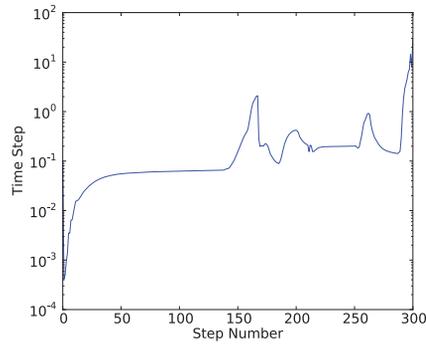


Figure 2: The evolution of the time step growth over the simulation

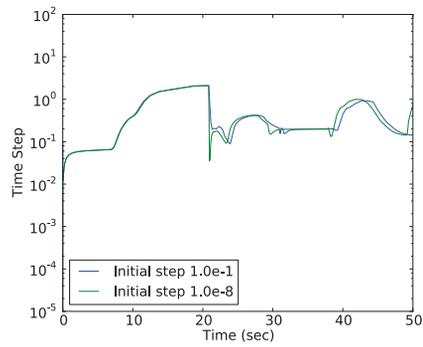


Figure 3: The evolution of the time step growth over the simulation