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On the stability of the finite difference based lattice Boltzmann method

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Abstract

This paper is devoted to determining the stability conditions for the finite difference based lattice Boltzmann method (FDLBM). In the current scheme, the 9-bit two-dimensional (D2Q9) model is used and the collision term of the Bhatnagar-Gross-Krook (BGK) is treated implicitly. The implicitness of the numerical scheme is removed by introducing a new distribution function different from that being used. Therefore, a new explicit finite-difference lattice Boltzmann method is obtained. Stability analysis of the resulted explicit scheme is done using Fourier expansion. Then, stability conditions in terms of time and spatial steps, relaxation time and explicitly-implicitly parameter are determined by calculating the eigenvalues of the given difference system. The determined conditions give the ranges of the parameters that have stable solutions.

Keywords: LBM; finite difference LBM; stability anylasis

1. Introduction

The lattice Boltzmann method (LBM) has been suggested as a different approach from the conventional computational fluid dynamics. The LBM becomes an efficient tool for simulating a variety of transport phenomena and fluid dynamics problems [1-4]. The main idea of the LBM is to compute the physical reality of a flow field through a kinetic approach at mesoscopic scale that preserves the hydrodynamic conservation law [5]. Originally, the LBM has been developed from the lattice gas automaton (LGA) [6] model, however, the collision operator was linearized and the single time relaxation (BGK) approximation was adopted [5].

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The lattice Boltzmann equation (LBE) is a discrete scheme of the continuous Boltzmann equation that provides a way to improve the computational efficiency and accuracy of the LBM. Any standard numerical technique can serve the purpose of solving the discrete velocity Boltzmann equation. In order to increase computational efficiency and accuracy by handling nonuniform grids, several traditional numerical methods such as the finite- difference, finite-volume, and finite-element methods have been introduced into LBM [8–12]. In the continuous Boltzmann equation, discretization for the particle velocity can be decoupled from the spatial discretization, since the particle velocity in the Boltzmann equation is independent of the particle position [7]. When the LBM is viewed as a finite-difference method for solving the continuum discrete-velocity Boltzmann equations, it becomes clear that numerical accuracy and stability issues should be addressed [8,13-18].

This paper is organized as follows. In Section 2, the standard two-dimensional 9-bit (D2Q9) lattice Boltzmann method is briefly introduced. Section 3 covers the finite-difference based LBM. The stability analysis of the FDLBM scheme is introduced in Section 4. Finally, the conclusions and future work of the results are drowning in Section 5.

2. The Lattice Boltzmann Method

The lattice Boltzmann model utilizes the distribution function f for the flow, which is calculated by solving the lattice Boltzmann equation. After introducing BGK approximation [5], the general form of the Lattice Boltzmann equation is given by,

$$f_i(x+c_i\Delta t, t+\Delta t) - f_i(x,t) = \frac{\Delta t}{\tau}\left[f_i^{eq}(x,t) - f_i(x,t)\right] \tag{1}$$

The type of the problem that needs to be solved is determined by the definition of the local equilibrium distribution function. The equilibrium distribution function, is defined by,

$$f_i^{eq} = \omega_i \rho\left[1 + 3\frac{c_i \cdot u}{c^2} + \frac{9}{2}\frac{(c_i \cdot u)^2}{c^4} - \frac{3}{2}\frac{u^2}{c^2}\right] \tag{2}$$

where ω_i denotes the equilibrium distribution weight and c_i denotes the discrete velocity in the direction of i. $c = \sqrt{3RT}$ is the speed of lattice, where R is the gas constant. For the 9-bit two-dimensional D2Q9 model (Fig. 1), we have,

$$\omega_i = \begin{cases} 4/9, & i = 0 \\ 1/9, & i = 1,2,3,4 \\ 1/36, & i = 5,6,7,8 \end{cases} \tag{3}$$

and c_i is defined as,

$$c_i = \{c_{ix}, c_{iy}\} = \begin{cases} \{0,0\}, & i = 0 \\ c\{\cos((i-1)\pi/2), \sin((i-1)\pi/2)\}, & i = 1,2,3,4 \\ \sqrt{2}c\{\cos((2i-9)\pi/4), \sin((2i-9)\pi/4)\}, & i = 5,6,7,8 \end{cases} \tag{4}$$

The macroscopic density ρ and momentum ρu (u is the macroscopic velocity) are calculated from,

$$\rho = \sum_i f_i, \rho u = \sum_i c_i f_i \tag{5}$$

The Navier–Stokes equation may be recovered from the evolution LBM equation using a Chapman–Enskog expansion for the density distribution function. Therefore, the single relaxation time for hydrodynamics τ in terms of the kinematic viscosity ν may be given as,

$$\tau = \left(\frac{\nu}{c^2 \Delta t} + \frac{1}{2} \right) \tag{6}$$

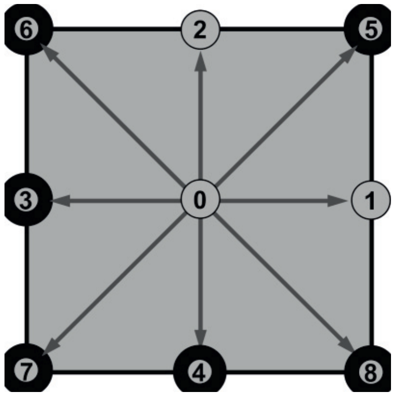

Fig. 1: Two-dimensional 9-bit (D2Q9) LBM model

3. The Finite Difference-Based Lattice Boltzmann Method

Firstly, let us rewrite the LBE (1) in a continuum Boltzmann equation form as follows,

$$\frac{\partial f_i}{\partial t} + c \cdot \nabla f_i = \frac{1}{\tau}\left[f_i^{eq}(x,t) - f_i(x,t) \right], \tag{7}$$

In order to invoking non-uniform mesh and to improve the numerical stability of the LBM, the above equations may be discretized using any standard numerical scheme. In this work, we use a finite difference-based LBGK scheme proposed by Guo and Zhao [8]. Firstly, let us discretize with respect to time, Eq. (7) becomes,

$$f_i^{n+1} - f_i^n + \Delta t \left(c_{ix} \frac{\partial f_i^n}{\partial x} + c_{iy} \frac{\partial f_i^n}{\partial y} \right) = -w\left[(1-\theta)\left(f_i - f_i^{eq}\right)^n + \theta\left(f_i - f_i^{eq}\right)^{n+1} \right] \tag{8}$$

where $w = \Delta t / \tau$ and θ is a parameter controlling the explicitly of the collision operator. A fully explicit collision scheme is obtained if $\theta = 0$ and a fully implicit collision scheme is obtained if $\theta = 1$. Mixed explicit-implicit scheme can be obtained by changing the parameter θ value between 0 and 1, i.e. $0 \le \theta \le 1$. Guo and Zhao [8] reported that for a single LBE the collision term reaches the second-order accuracy in time only at $\theta = 0.5$. In order to eliminate the implicitness of the above scheme, we introduce the following new distribution function,

$$F_i = f_i + w\theta\left(f_i - f_i^{eq}\right) \tag{9}$$

The corresponding equilibrium distribution function is given by, $F_i^{eq} = f_i^{eq}$. The macroscopic density and velocity may be rewritten as,

$$\rho = \sum_i F_i, \ \rho u = \sum_i c_i F_i \tag{10}$$

Eq. (8) can be rewritten in terms of the new distribution functions as follows,

$$F_i^{n+1} + \frac{\Delta t}{1 + w\theta}\left(c_{ix} \frac{\partial \hat{F}_i^n}{\partial x} + c_{iy} \frac{\partial \hat{F}_i^n}{\partial y} \right) = -\frac{1 - w(1-\theta)}{1 + w\theta} \hat{F}_i^n + w(1-\theta) f_i^{eq,n} \tag{11}$$

where $\hat{F}_i = F_i + w\theta f_i^{eq}$.

In this work, we discretize the convection term of the above equation using a second-order upwind finite-difference scheme to discretize the spatial derivatives of $\hat{F}_i$ in x and y directions as follow,

$$\frac{\partial \hat{F}_i}{\partial x} = \begin{cases} \dfrac{\hat{F}(j,k)-\hat{F}(j-1,k)}{\Delta x} + \dfrac{\hat{F}(j,k)-2\hat{F}(j-1,k)+\hat{F}(j-2,k)}{2\Delta x}, & c_{ix} > 0 \\ \dfrac{\hat{F}(j+1,k)-\hat{F}(j,k)}{\Delta x} - \dfrac{\hat{F}(j,k)-2\hat{F}(j+1,k)+\hat{F}(j+2,k)}{2\Delta x}, & c_{ix} < 0 \end{cases} \tag{12}$$

and

$$\frac{\partial \hat{F}_i}{\partial y} = \begin{cases} \dfrac{\hat{F}(j,k)-\hat{F}(j,k-1)}{\Delta y} + \dfrac{\hat{F}(j,k)-2\hat{F}(j,k-1)+\hat{F}(j,k-2)}{2\Delta y}, & c_{iy} > 0 \\ \dfrac{\hat{F}(j,k+1)-\hat{F}(j,k)}{\Delta y} - \dfrac{\hat{F}(j,k)-2\hat{F}(j,k+1)+\hat{F}(j,k+2)}{2\Delta y}, & c_{iy} < 0 \end{cases} \tag{13}$$

where j, k are the indices in x and y directions, respectively.

4. Stability Analysis of the FDLBM Scheme

In this section, the stability conditions of the finite difference-based lattice Boltzmann method (FDLBM) scheme are determined. The FDLBM scheme introduced above is an explicit procedure, so it is worth to investigate the largest time-step consistent with the stability. Before going to the stability analysis, it is convenient to rewrite Eq. (11) as follows,

$$\hat{F}_i^{n+1} = -\frac{\Delta t}{1+w\theta}\left(c_{ix}\frac{\partial \hat{F}_i^n}{\partial x} + c_{iy}\frac{\partial \hat{F}_i^n}{\partial y} \right) - \frac{1-w(1-\theta)}{1+w\theta}\hat{F}_i^n + w(1-\theta) f_i^{eq,n} + w\theta f_i^{eq,n+1} \tag{14}$$

Mei and Shyy [12] proposed to calculate $f_i^{eq,n+1}$ at a new time level using a linear extrapolation scheme from the two previous time levels as follows,

$$f_i^{eq,n+1} = 2 f_i^{eq,n} - f_i^{eq,n-1} \tag{15}$$

Substituting from Eq. (15) into Eq. (14) one gets,

$$\hat{F}_i^{n+1} = -\frac{\Delta t}{1+w\theta}\left(c_{ix}\frac{\partial \hat{F}_i^n}{\partial x} + c_{iy}\frac{\partial \hat{F}_i^n}{\partial y} \right) - \frac{1-w(1-\theta)}{1+w\theta}\hat{F}_i^n + w(1+\theta) f_i^{eq,n} - w\theta f_i^{eq,n-1} \tag{16}$$

At an arbitrary time the general terms of the Fourier expansion for $\hat{F}_i$ becomes, $\hat{F}_i = \Psi_i(t)e^{i'(ax+by)}$, where $i' = \sqrt{-1}$, a and b are constants. Denoting the value of Ψ at any time step and after one time-step it is denoted by Ψ'. Similarly, $f_i^{eq,n}$ and $f_i^{eq,n-1}$ may be expressed in terms of Fourier expansion $f_i^{eq} = \psi_i(t)e^{i'(ax+by)}$, however, the function $\psi_i(t)$ may be treated as a constant because they are known from the previous steps. Thus, $f_i^{eq,n} = C_{i1}e^{i'(ax+by)}$ and $f_i^{eq,n-1} = C_{i2}e^{i'(ax+by)}$, where C_{i1} and C_{i2}. Substituting these expressions into Eq. (16), we get different expressions for the convection term based on the values of lattice velocities c_{ix} and c_{iy},

$$(c_x, c_y) = \{(0,0),(1,0),(0,1),(-1,0),(0,-1),(1,1),(-1,1),(-1,-1),(1,-1)\}, \tag{17}$$

one may get,

$$\Psi_i' = -\alpha\left(c_{ix}D_x + c_{iy}D_y\right)\Psi_i - \beta\Psi_i + \gamma C_{i1} - \delta C_{i2} \tag{18}$$

where $\alpha = \dfrac{\Delta t}{1+w\theta}, \beta = \dfrac{1-w(1-\theta)}{1+w\theta}, \gamma = w(1+\theta), \delta = w\theta$, and,

$$D_x = \begin{cases} D_{x1} = \left(\dfrac{3 - 4e^{-i'a\Delta x} + e^{-2i'a\Delta x}}{2\Delta x} \right), & c_{ix} > 0 \\ D_{x2} = \left(\dfrac{-3 + 4e^{i'a\Delta x} + e^{2i'a\Delta x}}{2\Delta x} \right), & c_{ix} < 0 \end{cases}, \quad D_y = \begin{cases} D_{y1} = \left(\dfrac{3 - 4e^{-i'b\Delta y} + e^{-2i'b\Delta y}}{2\Delta y} \right), & c_{iy} > 0 \\ D_{y2} = \left(\dfrac{-3 + 4e^{i'b\Delta y} + e^{2i'b\Delta y}}{2\Delta y} \right), & c_{iy} < 0 \end{cases} \tag{19}$$

Eq. (18) can be rewritten as,

$$\Psi' = A\Psi + B \tag{20}$$

where $\Psi' = [\Psi'_0 \quad \Psi'_1 \quad \cdots \quad \Psi'_8]^T$, $\Psi = [\Psi_0 \quad \Psi_1 \quad \cdots \quad \Psi_8]^T$

$B = [\gamma C_{01} - \delta C_{02} \quad \gamma C_{11} - \delta C_{12} \quad \cdots \quad \gamma C_{81} - \delta C_{82}]^T$ and, $A = diag[\lambda_1, \lambda_2, \ldots, \lambda_9]$.

$$\begin{aligned} &\lambda_1 = -\beta, \lambda_2 = -\alpha D_{x1} - \beta, \lambda_3 = -\alpha D_{y1} - \beta, \lambda_4 = \alpha D_{x2} - \beta, \\ &\lambda_5 = \alpha D_{y2} - \beta, \lambda_6 = -\alpha(D_{x1} + D_{y1}) - \beta, \lambda_7 = \alpha(D_{x2} - D_{y1}) - \beta, \\ &\lambda_8 = \alpha(D_{x2} + D_{y2}) - \beta, \lambda_9 = -\alpha(D_{x1} - D_{y2}) - \beta \end{aligned} \tag{21}$$

D_{x1}, D_{x2}, D_{y1}, and D_{y2} are defined as,

$$D_{x1} = \left(\frac{(1 - \cos a\Delta x(2 - \cos a\Delta x)) + i' \sin a\Delta x(2 - \cos a\Delta x)}{\Delta x} \right),$$

$$D_{y1} = \left(\frac{(1 - \cos b\Delta y(2 - \cos b\Delta y)) + i' \sin b\Delta y(2 - \cos b\Delta y)}{\Delta y} \right),$$

$$D_{x2} = \left(\frac{-2 + \cos a\Delta x(2 + \cos a\Delta x) + i' \sin a\Delta x(2 + \cos a\Delta x)}{\Delta x} \right),$$

$$D_{y2} = \left(\frac{-2 + \cos b\Delta y(2 + \cos b\Delta y) + i' \sin b\Delta y(2 + \cos b\Delta y)}{\Delta y} \right)$$

Seeking the stability of the above system, the moduli of each of the eigenvalues of the coefficients matrix A should be calculated. For a stable system the eigenvalues should be less than or equal to unity. Therefore, the stability conditions are $|\lambda_1| \le 1$, $|\lambda_2| \le 1, \ldots, |\lambda_9| \le 1$, for all a and b. It is notable that the coefficients α, β, γ and δ are positive and real. For the first eigenvalue, we have, $|\lambda_1| = |-\beta| = |\beta| \le 1$. This implies to,

$$\left| 1 - \frac{w}{1 + w\theta} \right| \le 1$$

i.e.,

$$0 \le \frac{w}{1 + w\theta} \le 2$$

Thus, the first stability condition can be finally written as,

$$\Delta t \le \frac{2\tau}{1 - 2\theta} \tag{22}$$

The second eigenvalue is a complex variable its real and imaginary parts are, $\text{Re}\{\lambda_2\} = -\dfrac{\alpha}{\Delta x}(1 - \cos a\Delta x(2 - \cos a\Delta x)) - \beta$ and $\text{Im}\{\lambda_2\} = -\dfrac{\alpha}{\Delta x} \sin a\Delta x(2 - \cos a\Delta x)$, respectively. So, one can calculate, $|\lambda_2|^2 = (\mathrm{Re}\{\lambda_2\})^2 + (\mathrm{Im}\{\lambda_2\})^2$. Representing the eigenvalues using Argand diagram, the maximum values of $|\lambda_2|$ occur when $a\Delta x = r\pi$ and $b\Delta y = s\pi$, where *r* and *s* are positive integers. The values of $|\lambda_2|$ are maximum when Δt is sufficiently large, at both *r* and *s* are odd integers. The maximum value of this value is, $\max\{|\lambda_2|^2\} = \left[4\left(\frac{\alpha}{\Delta x}\right) + \beta\right]^2$. Therefore, the second stability condition $|\lambda_2| \le 1$ becomes, $\frac{4\alpha}{\Delta x} + \beta \le 1$. Substitute α and β in the above equation, we obtain,

$$\Delta t \le \frac{\tau}{(4\tau/\Delta x - \theta - 1)} \tag{23}$$

Eq. (23) represents the second stability condition. Similarly, one can get the third stability condition $|\lambda_3| \le 1$ in the form,

$$\Delta t \le \frac{\tau}{(4\tau/\Delta y - \theta - 1)} \tag{24}$$

Also, the real and imaginary parts of the fourth eigenvalue are, $\frac{\alpha}{\Delta x}(-2 + \cos a\Delta x(2 + \cos a\Delta x)) - \beta$ and $\frac{\alpha}{\Delta x}(\sin a\Delta x(2 + \cos a\Delta x))$, respectively. Therefore, one may calculate $\max\{|\lambda_4|^2\} = \left[3\left(\frac{\alpha}{\Delta x}\right) + \beta\right]^2$. Thus, the fourth stability condition $|\lambda_4| \le 1$ becomes, $\frac{3\alpha}{\Delta x} + \beta \le 1$. Substitute α and β in the above equation, we may get,

$$\Delta t \le \frac{\tau}{(3\tau/\Delta x - \theta - 1)} \tag{25}$$

Eq. (25) represents the fourth stability condition. Similarly, one can get the fifth stability condition $|\lambda_5| \le 1$ as,

$$\Delta t \le \frac{\tau}{(3\tau/\Delta y - \theta - 1)} \tag{26}$$

The real and imaginary parts of the sixth eigenvalue are,

$$\mathrm{Re}\{\lambda_6\} = -\frac{\alpha}{\Delta x}(1 - \cos a\Delta x(2 - \cos a\Delta x)) - \frac{\alpha}{\Delta y}(1 - \cos b\Delta y(2 - \cos b\Delta y)) - \beta$$

$$\mathrm{Im}\{\lambda_6\} = -\frac{\alpha}{\Delta x}\sin a\Delta x(2 - \cos a\Delta x) - \frac{\alpha}{\Delta y}\sin b\Delta y(2 - \cos b\Delta y)$$

Therefore, the maximum value of this eigenvalue becomes $\max\{|\lambda_6|^2\} = \left[4\alpha\left(\frac{\Delta x + \Delta y}{\Delta x\Delta y}\right) + \beta\right]^2$. Thus, the sixth stability condition $|\lambda_6| \le 1$ becomes, $4\alpha\left(\frac{\Delta x + \Delta y}{\Delta x\Delta y}\right) + \beta \le 1$, or,

$$\Delta t \le \frac{\tau}{[4\tau\Delta x\Delta y/(\Delta x + \Delta y) - \theta - 1]} \tag{27}$$

The real and imaginary parts of the seventh eigenvalue are given by,

$$\mathrm{Re}\{\lambda_7\} = \frac{\alpha}{\Delta x}\left(-2+\cos a\Delta x(2+\cos a\Delta x)\right) - \frac{\alpha}{\Delta y}\left(1-\cos b\Delta y(2-\cos b\Delta y)\right) - \beta$$

$$\mathrm{Im}\{\lambda_7\} = \frac{\alpha}{\Delta x}\sin a\Delta x\left(2+\cos a\Delta x\right) - \frac{\alpha}{\Delta y}\sin b\Delta y\left(2-\cos b\Delta y\right)$$

And the maximum value of this eigenvalue is, $\max\left\{|\lambda_7|^2\right\} = \left[\alpha\frac{3\Delta y - 4\Delta x}{\Delta x\Delta y} + \beta\right]^2$. Therefore, the seventh stability condition $|\lambda_7| \le 1$ becomes, $\alpha\frac{3\Delta y - 4\Delta x}{\Delta x\Delta y} + \beta \le 1$. Thus,

$$\Delta t \le \frac{\tau}{\left[\tau\Delta x\Delta y/(3\Delta y - 4\Delta x) - \theta - 1\right]} \tag{28}$$

The eighth eigenvalue is also complex its real and imaginary parts are given by,

$$\mathrm{Re}\{\lambda_8\} = \frac{\alpha}{\Delta x}\left(-2+\cos a\Delta x(2+\cos a\Delta x)\right) + \frac{\alpha}{\Delta y}\left(-2+\cos b\Delta y(2+\cos b\Delta y)\right) - \beta$$

$$\mathrm{Im}\{\lambda_8\} = \frac{\alpha}{\Delta x}\sin a\Delta x\left(2+\cos a\Delta x\right) + \frac{\alpha}{\Delta y}\sin b\Delta y\left(2+\cos b\Delta y\right)$$

We get, $\max\left\{|\lambda_8|^2\right\} = \left[3\alpha\frac{\Delta x + \Delta y}{\Delta x\Delta y} + \beta\right]^2$. Therefore, the eighth stability condition $|\lambda_8| \le 1$ becomes, $3\alpha\frac{\Delta x + \Delta y}{\Delta x\Delta y} + \beta \le 1$, therefore,

$$\Delta t \le \frac{\tau}{\left[3\tau\Delta x\Delta y/(\Delta x + \Delta y) - \theta - 1\right]} \tag{29}$$

Finally, the real and imaginary parts of the ninth eigenvalue are,

$$\mathrm{Re}\{\lambda_9\} = -\frac{\alpha}{\Delta x}\left(1-\cos a\Delta x(2-\cos a\Delta x)\right) + \frac{\alpha}{\Delta y}\left(-2+\cos b\Delta y(2+\cos b\Delta y)\right) - \beta$$

$$\mathrm{Im}\{\lambda_9\} = \frac{\alpha}{\Delta x}\sin a\Delta x\left(2-\cos a\Delta x\right) + \frac{\alpha}{\Delta y}\sin b\Delta y\left(2+\cos b\Delta y\right)$$

Therefore, $\max\left\{|\lambda_9|^2\right\} = \left[\alpha\frac{3\Delta x + 4\Delta y}{\Delta x\Delta y} + \beta\right]^2$. The ninth stability condition $|\lambda_9| \le 1$ becomes, $\alpha\frac{3\Delta x + 4\Delta y}{\Delta x\Delta y} + \beta \le 1$, i.e.

$$\Delta t \le \frac{\tau}{\left[\tau\Delta x\Delta y/(3\Delta x + 4\Delta y) - \theta - 1\right]} \tag{30}$$

5. Conclusions and Future Work

From this paper, we may conclude that the FDLB scheme is conditionally stable under the restrictions (22)-(30). The stability conditions are determined in terms of time and spatial steps, relaxation time and explicitly-implicitly parameter. The formulas of these conditions give the ranges of the parameters that have stable solutions. Extension of this analysis to include more complex models such as D3Q19 and D3Q27 or any other model is straightforward. Also, similar analysis of more complex physics models such as thermal and solutal transport can be obtained.

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References

[1] Benzi, R, Succi, S, Vergassola, M, 1992. The lattice Boltzmann equation: theory and applications. *Phys. Rep.* 222, p.145.
[2] Chen, S, Doolen, G D, 1998. Lattice Boltzmann method for fluid flows. *Ann. Rev. Fluid Mech.* 30, p. 329.
[3] Falcucci, G, Ubertini, S, Biscarini, C, Di Francesco, S, Chiappini, D, Palpacelli, S, De Maio, A, Succi, S, 2011. Lattice Boltzmann methods for multiphase flow simulations across scales. *Commun. Comput. Phys.* 9, p. 269.
[4] Chiappini, D, Bella, G, Succi, S, Toschi, F, Ubertini, S, 2010. Improved lattice Boltzmann with- out parasitic currents for Rayleigh-Taylor instability. *Commun. Comput. Phys.* 7, p. 423.
[5] Bhatnagar, PL, Gross EP, Krook, M, 1954. A model for collision processes in gases I: small amplitude processes in charged and neutral one component systems. *Phys. Rev.* 94, p. 511.
[6] Rivet, JP, Boon, JP, *Lattice gas hydrodynamics*. Cambridge University Press; 2000.
[7] Cao NZ, Chen SY, Jin S, Martinez D. 1997. Physical symmetry and lattice symmetry in the lattice Boltzmann method. *Phys Rev E* 55: R21–4.
[8] Guo ZL, Zhao TS, 2003. Explicit finite-difference lattice Boltzmann method for curvilinear coordinates. *Phys Rev E* 67. Art. No. 066709.
[9] Guo ZL, Zhao TS, 2005. Finite-difference based lattice Boltzmann model for dense binary mixture. *Phys Rev E* 71. Art. No. 026701.
[10] Xi HW, Peng GW, Chou SH., 1999. Finite-volume lattice Boltzmann method. *Phys Rev E* 59, p. 6202.
[11] Lee T, Lin CL, 2001. A characteristic Galerkin method for discrete Boltzmann equation. *J Comput Phys* 171, p. 336.
[12] Mei RW, Shyy W, 1998. On the finite difference-based lattice Boltzmann method in curvilinear coordinates. *J Comput Phys* 143, p. 426.
[13] Sterling JD, Chen S. 1996. Stability analysis of lattice Boltzmann methods. *J Comput Phys* 123, p. 196.
[14] El-Amin, MF, Sun S, 2011. A finite difference scheme for double-diffusive unsteady free convection from a curved surface to a saturated porous medium with a non-Newtonian fluid. *Procedia Computer Science* 4, p. 957.
[15] El-Amin, MF, Salama A, Sun S, 2012. A conditionally stable scheme for a transient flow of a non-Newtonian fluid saturating a porous medium. *Procedia Computer Science* 9, p. 651.
[16] El-Amin, MF, Gorla RSR, 2008. Unsteady free convection in a fluid past a vertical plate immersed in a porous medium. *Int J Fluid Mech Res* 35, p. 434.
[17] Ali, MM, El-Amin, MF, 1997. The characteristic indices for linear differential systems with variable coefficients. *Proc 3rd Int Conf Eng Math Phys* (ICEMP) 1, p. 147.
[18] El-Amin, MF, Ebrahiem, NA, 2006. Effects of viscous dissipation on unsteady free convection in a fluid past a vertical plate immersed in a porous medium. *Trans Porous Media* 64, p. 1.