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An algebraic fractional order differentiator for a class of signals satisfying a linear differential equation

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Abstract

This paper aims at designing a digital fractional order differentiator for a class of signals satisfying a linear differential equation to estimate fractional derivatives with an arbitrary order in noisy case, where the input can be unknown or known with noises. Firstly, an integer order differentiator for the input is constructed using a truncated Jacobi orthogonal series expansion. Then, a new algebraic formula for the Riemann-Liouville derivative is derived, which is enlightened by the algebraic parametric method. Secondly, a digital fractional order differentiator is proposed using a numerical integration method in discrete noisy case. Then, the noise error contribution is analyzed, where an error bound useful for the selection of the design parameter is provided. Finally, numerical examples illustrate the accuracy and the robustness of the proposed fractional order differentiator.

Keywords: Fractional order differentiator, Riemann-Liouville derivative, Algebraic parametric method, Modulating functions method, Unknown input, Noise error analysis.

1. Introduction

Fractional calculus has a long history and has been becoming very useful in many scientific and engineering fields, including control, flow propagation, signal processing, electrical networks, and etc. (see, e.g. [1, 2, 3, 4, 5, 6, 7, 8]). An interesting research topic on fractional calculus is related to the estimation of the fractional order derivatives of an unknown signal from its discrete noisy observation. The objective is to design digital fractional order differentiators, which should be robust against noises. Various robust fractional order differentiators have been proposed in the frequency domain (see, e.g. [9, 10]) and in the time domain (see, e.g. [11, 12, 13, 14, 15, 16]). They can be divided into two classes: fractional order model-free differentiators (see, e.g. [9, 10, 11, 12, 13, 14, 15]) and fractional order model-based differentiators (see, e.g. [16]). The first class of fractional order differentiators are obtained by truncating the analytical expression. Hence, this

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generates truncated errors even in noise-free case (see, e.g. [13]). The second class of fractional order differentiators are obtained from the differential equations of considered signals. They do not introduce any truncated errors.

Existing fractional order differentiators are usually extensions of integer order differentiators. Among the existing methods, the recent algebraic parametric method originally introduced by Fliess and Sira-Ramírez for linear identification [17], has been applied to design integer order model-free differentiators (see, e.g. [18, 19, 20, 21, 22, 23]), and integer order model-based differentiators (see, e.g. [24, 25, 26]). The idea of this method is to apply some algebraic operations (such as differentiations and multiplications), in the frequency domain, to the equation of the studied signal. The obtained differentiators are exactly given by algebraic integral formulae in the time domain. It has been shown in [27, 28] that, thanks to the integral formulae, these differentiators exhibit good robustness properties with respect to corrupting noises even if the statistical properties of the noises are unknown. Very recently, the algebraic integer order model-free differentiators have been extended to fractional case [12, 13, 29]. However, the algebraic parametric method has not been applied for fractional order model-based differentiators.

The modulating functions method is another method which has been extended to the fractional case. This method has been introduced by Shinbrot [30]. It gives similar results to the algebraic parametric method but works in the time domain. In [16], generalized modulating functions have been introduced to design fractional order model-based differentiators. However, the generalized modulating functions are more complex to construct than the classical ones.

The aim of this paper is to apply the algebraic parametric method to design a robust fractional order model-based differentiator. Moreover, it will be shown that the proposed differentiator can also be obtained by classical modulating functions without using generalized modulating functions. For this purpose, we will focus on a specific class of signals satisfying a linear differential equation, where the input can be unknown or known with noises.

This paper is organized as follows: definitions and some useful properties of fractional calculus, modulating functions, and Jacobi orthogonal polynomials are recalled in Section 2. The main results are given in Section 3. Firstly, an integer order differentiator for the input is constructed using a truncated Jacobi orthogonal series expansion. Secondly, the algebraic parametric method is applied to express the Riemann-Liouville integrals and derivatives of the considered signal by algebraic formulae in continuous noise-free case. Then, it is shown that these integral formulae can also be obtained using the modulating functions method. Thirdly, a digital fractional order differentiator is introduced in discrete noisy case. Moreover, some error analysis is given. In Section 4, numerical results illustrate the accuracy and the robustness of the proposed fractional order differentiator. Finally, conclusions are outlined in Section 5.

2. Preliminary

2.1. Problem formulation

In this paper, a class of signals satisfying the following differential equation are considered:

$$\forall\, t \in I,\ \sum_{i=0}^{n} a_i\, y^{(i)}(t) = u(t), \tag{1}$$

where $n \in \mathbb{N}^*$, $a_n \in \mathbb{R}^*$, $a_i \in \mathbb{R}$, for $i = 0, \ldots, n-1$, $y \in \mathcal{C}^n(I)$, $u \in \mathcal{C}^n(I)$, and $I = [0, h] \subset \mathbb{R}_+$. If (1) is considered as a linear system with the input u, then y is the corresponding output. The objective of this paper is to estimate the Riemann-Liouville fractional derivatives of the output y in noisy environment, where the input can be unknown or known with noises. For this purpose, some useful tools are recalled in the following subsections.

2.2. Riemann-Liouville integrals and derivatives

Definition 1 *([5] p. 13) Let $\beta \in \mathbb{R}_+^*$, and f be a continuous function defined on $\mathbb{R}$. Then, the β^{th} order Riemann-Liouville fractional integral of f is defined by: $\forall\, t \in \mathbb{R}_+^*$,*

$$\mathrm{J}_t^{\beta} f(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(z) = \int_0^{\infty} \exp(-x)\, x^{z-1}\, dx$ and satisfies $\Gamma(z+1) = z\Gamma(z)$ (see [31] pp. 255-256).

Definition 2 *([4] p. 62) Let $\alpha \in \mathbb{R}_+^*$ with $l-1 \leq \alpha < l$, $l \in \mathbb{N}^*$, and $f \in \mathcal{C}^l(\mathbb{R})$, where $\mathcal{C}^l(\mathbb{R})$ refers to the set of functions being l-times continuously differentiable on $\mathbb{R}$. Then, the α^{th} order Riemann-Liouville fractional derivative of f is defined as follows: $\forall\, t \in \mathbb{R}_+^*$,*

$$\mathrm{D}_t^{\alpha} f(t) := \frac{d^l}{dt^l} \left\{ \mathrm{J}_t^{l-\alpha} f(t) \right\} = \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_0^t (t-\tau)^{l-\alpha-1} f(\tau)\, d\tau. \tag{3}$$

Remark that if $0 < \beta < 1$, then the integral given in (2) is improper. Hence, the β^{th} order Riemann-Liouville integral is defined by an improper integral. Moreover, according to (3) Riemann-Liouville derivatives are also defined by improper integrals, which are the integer order derivatives of the Riemann-Liouville integrals of order smaller than 1.

According to (2) and (3), Riemann-Liouville integrals and derivatives satisfy the following additive index laws.

- Let $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$, $n \in \mathbb{N}$, and $f \in \mathcal{C}^n(\mathbb{R})$. Then, we have ([4] p. 71): $\forall\, t \in \mathbb{R}_+^*$,

$$\frac{d^n}{dt^n} \left\{ \mathrm{J}_t^{\beta} f(t) \right\} = \begin{cases} \mathrm{J}_t^{\beta-n} f(t), & \text{if } \beta > n, \\ \mathrm{D}_t^{n-\beta} f(t), & \text{else.} \end{cases} \tag{4}$$

- Let $\alpha \in \mathbb{R}_+^*$ with $l-1 \leq \alpha < l$, $l \in \mathbb{N}^*$, and $f \in \mathcal{C}^{l+n}(\mathbb{R})$. Then, we have: $\forall\, t \in \mathbb{R}_+^*$,

$$\frac{d^n}{dt^n} \left\{ \mathrm{D}_t^{\alpha} f(t) \right\} = \mathrm{D}_t^{\alpha+n} f(t). \tag{5}$$

In the following theorem, the Leibniz formula for Riemann-Liouville integrals is recalled. The Leibniz formula for Riemann-Liouville derivatives can be found in [5] (p. 33).

Theorem 1 *(**Leibniz formula for Riemann-Liouville integrals** [3] p. 75) Let $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$, and assume that f is continuous on $[0, h]$ with some $h > 0$, and g is analytic on $[0, h]$. Then, the following formula holds: $\forall\, t \in]0, h]$,*

$$\mathrm{J}_t^{\beta}\left[f(t)g(t)\right] = \sum_{i=0}^{\infty} \binom{-\beta}{i} g^{(i)}(t)\, \mathrm{J}_t^{\beta+i} f(t), \tag{6}$$

where $\binom{-\beta}{i} = \dfrac{\Gamma(-\beta+1)}{i!\,\Gamma(-\beta-i+1)}$ is the generalized binomial coefficient.

If g is a polynomial, then the sum with infinite terms in (6) becomes a sum with finite terms. This result is given in the following corollary.

Corollary 1 *Let $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$, $n \in \mathbb{N}$, and f be a continuous function defined on $\mathbb{R}$. Then, the following formulae are obtained: $\forall\, t \in \mathbb{R}_+^*$,*

$$\mathrm{J}_t^{\beta}\left[t^n f(t)\right] = \sum_{i=0}^{n} \binom{-\beta}{i} \frac{n!}{(n-i)!} t^{n-i}\, \mathrm{J}_t^{\beta+i} f(t). \tag{7}$$

$$\mathrm{J}_t^{\beta} f(t) = \frac{1}{t^n}\mathrm{J}_t^{\beta}\left[t^n f(t)\right] - \sum_{i=1}^{n} \binom{-\beta}{i} \frac{n!}{(n-i)!} \frac{1}{t^i} \mathrm{J}_t^{\beta+i} f(t). \tag{8}$$

2.3. Laplace transform of Riemann-Liouville integrals

Let us assume that the Laplace transform of f exists, which is denoted by $\hat{f}$. The Laplace transform of the Riemann-Liouville integral of f is given by (see [4] p. 104): $\forall\, s \in \mathbb{C}$,

$$\mathcal{L}\left\{\mathrm{J}_t^{\beta} f(t)\right\}(s) = \frac{1}{s^{\beta}}\hat{f}(s), \tag{9}$$

where $\beta \in \mathbb{R}_+^*$, and s denotes the variable in the frequency domain. Then, the following lemma is given.

Lemma 1 *Let $\beta \in \mathbb{R}_+^*$, $n \in \mathbb{N}$, and f be a continuous function defined on $\mathbb{R}$. Then, the following formula is got: $\forall\, t \in \mathbb{R}_+^*$,*

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}\hat{f}^{(n)}(s)\right\}(t) = (-1)^n \mathrm{J}_t^{\beta}\left[t^n f(t)\right]. \tag{10}$$

Proof. First, the following formula is useful ([31] p. 1020): $\forall\, t \in \mathbb{R}_+^*$,

$$\mathcal{L}^{-1}\left\{\hat{f}^{(n)}(s)\right\}(t) = (-1)^n t^n f(t). \tag{11}$$

Then, the proof can be completed using (9) and (11). □

2.4. Modulating functions

The classical modulating functions are recalled as follows.

Definition 3 *Let $[a,b] \subset \mathbb{R}$, $n \in \mathbb{N}$, and g be a function satisfying the following properties:*

$(P_1):\ g \in \mathcal{C}^{n+1}([a,b])$;

$(P_2):\ g^{(i)}(a) = 0$, *for* $i = 0, 1, \ldots, n$;

$(P_3):\ g^{(i)}(b) = 0$, *for* $i = 0, 1, \ldots, n$.

Then, g is called the n^{th} order modulating function on $[a,b]$ (see, [35]).

The generalized integration by parts is a crucial tool in the application of modulating functions method. This result is recalled in the following lemma which can be obtained by recursively applying the classical integration by parts method.

Lemma 2 *Let $f \in \mathcal{C}^l(\mathbb{R})$ and $g \in \mathcal{C}^m(\mathbb{R})$, where $l, m \in \mathbb{N}^*$ with $m \leq l$. Then, for any interval $[a,b] \subset \mathbb{R}$, we have:*

$$\int_a^b g(t)\, f^{(l)}(t)\, dt = (-1)^m \int_a^b g^{(m)}(t)\, f^{(l-m)}(t)\, dt + \sum_{k=0}^{m-1} (-1)^k \left[g^{(k)}(t) f^{(l-1-k)}(t) \right]_{t=a}^{t=b}. \tag{12}$$

Consequently, if g is a $(m-1)^{th}$ order modulating function on $[a,b]$, all the boundary values in (12) can be eliminated.

2.5. Jacobi orthogonal polynomials

The n^{th} ($n \in \mathbb{N}$) order shifted Jacobi orthogonal polynomial defined on $[0,1]$ is given as follows (see [31] p. 775): $\forall \tau \in [0,1]$,

$$P_n^{(\mu,\kappa)}(\tau) = \sum_{j=0}^{n} \binom{n+\mu}{j} \binom{n+\kappa}{n-j} (\tau - 1)^{n-j}\, \tau^j, \tag{13}$$

where $\mu, \kappa \in]-1, +\infty[$. Let f and g be two functions belonging to $\mathcal{C}([0,1])$, then the scalar product $\langle \cdot, \cdot \rangle_{\mu,\kappa}$ of these functions is defined by (see [31] p. 774):

$$\langle f(\cdot), g(\cdot) \rangle_{\mu,\kappa} = \int_0^1 w_{\mu,\kappa}(\tau)\, f(\tau)\, g(\tau)\, d\tau, \tag{14}$$

where $w_{\mu,\kappa}(\tau) = (1-\tau)^{\mu} \tau^{\kappa}$ is the associated weight function. Thus, the associated norm denoted by $\| \cdot \|_{\mu,\kappa}$ is obtained:

$$\| P_n^{(\mu,\kappa)} \|_{\mu,\kappa}^2 = \frac{\Gamma(\mu+n+1)\, \Gamma(\kappa+n+1)}{\Gamma(\mu+\kappa+n+1)\, \Gamma(n+1)\, (2n+\mu+\kappa+1)}. \tag{15}$$

The n^{th} order derivative of the weight function $w_{\mu+n,\kappa+n}$ is given by the following Rodrigues formula (see [32] p. 67): $\forall\, \tau \in [0,1]$,

$$\frac{d^n}{d\tau^n}\left\{w_{\mu+n,\kappa+n}(\tau)\right\} = (-1)^n n!\, w_{\mu,\kappa}(\tau) P_n^{(\mu,\kappa)}(\tau). \tag{16}$$

Consequently, it can be verified that $w_{\mu+n,\kappa+n}$ is a $(n-1)^{th}$ order modulating function on $[0,1]$.

Finally, if $f \in \mathcal{C}([a,b])$ with $[a,b] \subset \mathbb{R}$ and $T = b-a$, then f can be expressed by the following Jacobi orthogonal series on $[a,b]$: $\forall\, \tau \in [0,1]$,

$$f(b-T\tau) = \sum_{j=0}^{+\infty} \frac{\left\langle P_j^{(\mu,\kappa)}(\cdot), f(b-T\cdot)\right\rangle_{\mu,\kappa}}{\|P_j^{(\mu,\kappa)}\|_{\mu,\kappa}^2}\, P_j^{(\mu,\kappa)}(\tau). \tag{17}$$

3. Main results

In this section, an integer order differentiator and a fractional order differentiator are constructed for the input and the output defined in (1), respectively. The latter can be obtained using either the algebraic parametric method or the modulating functions method. Then, the proposed two differentiators are studied in discrete noisy case, and some error analysis is provided.

3.1. Integer order differentiator of input in continuous noise-free case

In this subsection, the input u defined in (1) is assumed to be unknown, whose integer order derivatives will be estimated. The idea is to locally estimate the derivatives of u by a polynomial on a small sliding window, which is obtained using a truncated Jacobi orthogonal series expansion. A similar idea has been applied to design the algebraic integer order model-free differentiators (see, e.g. [18, 19, 20, 21, 22, 23]). Based on this idea, the following proposition is given.

Proposition 1 *If the input u defined in (1) is unknown, then $\forall\, t \in [T,h]$ with $0 < T < h$, the q^{th} ($q \in \mathbb{N}$) order derivative of u can be locally estimated by the following N^{th} ($N \in \mathbb{N}$) order polynomial on $[t-T,t]$: $\forall\, \tau \in [0,1]$,*

$$D_{T,\mu,\kappa,N,n}^{(q)} u(t-T\tau) := \sum_{j=0}^{N} \lambda_{T,q,n,j}^{(\mu,\kappa)}\, P_j^{(\mu^*,\kappa^*)}(\tau), \tag{18}$$

where $n \in \mathbb{N}^*$, $\mu,\kappa \in]-1,+\infty[$, $\mu^* = \mu+q+n$, $\kappa^* = \kappa+q+n$, and for $j = 0,\ldots,N$,

$$\lambda_{T,q,n,j}^{(\mu,\kappa)} = \frac{1}{\|P_j^{(\mu^*,\kappa^*)}\|_{\mu^*,\kappa^*}^2} \sum_{i=0}^{n} a_i \frac{(j+i+q)!}{(-T)^{i+q} j!} \left\langle P_j^{(\mu+n-i,\kappa+n-i)}(\cdot), y(t-T\cdot)\right\rangle_{\mu+n-i,\kappa+n-i}. \tag{19}$$

Proof. Two steps are needed in this proof.

• **Step 1. Polynomial approximation:** Consider the Jacobi orthogonal series of $u^{(q)}$ on $[t-T,t]$, which can be given by (17). By taking the $N+1$ first terms in this series, the following polynomial is obtained: $\forall\, \tau \in [0,1]$,

$$u_e^{(q)}(t-T\tau) := \sum_{j=0}^{N} \lambda_{T,q,n,j}^{(\mu,\kappa)}\, P_j^{(\mu^*,\kappa^*)}(\tau), \tag{20}$$

where $\lambda_{T,q,n,j}^{(\mu,\kappa)} = \frac{\left\langle P_j^{(\mu^*,\kappa^*)}(\cdot),u^{(q)}(t-T\cdot)\right\rangle_{\mu^*,\kappa^*}}{\|P_j^{(\mu^*,\kappa^*)}\|^2_{\mu^*,\kappa^*}}$ with $\mu^* = \mu+q+n$ and $\kappa^* = \kappa+q+n$.

• **Step 2. Calculation of coefficients:** Using (1) and (14), it yields: for $j = 0,\ldots,N$,

$$\|P_j^{(\mu^*,\kappa^*)}\|^2_{\mu^*,\kappa^*}\, \lambda_{T,q,n,j}^{(\mu,\kappa)} = \sum_{i=0}^{n} a_i \int_0^1 w_{\mu^*,\kappa^*}(\tau) P_j^{(\mu^*,\kappa^*)}(\tau)\, y^{(i+q)}(t-T\tau)d\tau. \tag{21}$$

By applying the Rodrigues formula given in (16) and the generalized integration by parts formula given in (12), (21) becomes:

$$\begin{aligned}
&\|P_j^{(\mu^*,\kappa^*)}\|^2_{\mu^*,\kappa^*}\, \lambda_{T,q,n,j}^{(\mu,\kappa)}\\
=&\sum_{i=0}^{n} a_i \int_0^1 \frac{(-1)^j}{j!}\frac{d^j}{d\tau^j}\left\{w_{\mu^*+j,\kappa^*+j}(\tau)\right\} y^{(i+q)}(t-T\tau)d\tau\\
=&\sum_{i=0}^{n} \frac{a_i}{T^{i+q}} \int_0^1 \frac{(-1)^j}{j!}\frac{d^{j+i+q}}{d\tau^{j+i+q}}\left\{w_{\mu^*+j,\kappa^*+j}(\tau)\right\} y(t-T\tau)d\tau\\
=&\sum_{i=0}^{n} \frac{a_i}{(-T)^{i+q}}\frac{(j+i+q)!}{j!} \int_0^1 w_{\mu+n-i,\kappa+n-i}(\tau) P_j^{(\mu+n-i,\kappa+n-i)}(\tau) y(t-T\tau)d\tau,
\end{aligned} \tag{22}$$

where all the derivative values of y at t and $t-T$ are eliminated, since w_{μ^*+j,κ^*+j} is a $(j+n+q-1)^{th}$ order modulating function on $[0,1]$. $\square$

Finally, $\forall\, t \in [T,h]$, an estimated value of $u^{(q)}(t)$ can be obtained on each sliding window $[t-T,t]$ by fixing the value of τ in Proposition 1. If $\tau = 0$, the value of $u^{(q)}(t)$ is estimated by $D_{T,\mu,\kappa,N,n}^{(q)}u(t)$. If $\tau \neq 0$, the value of $u(t)$ is estimated by $D_{T,\mu,\kappa,N,n}^{(q)}u(t-T\tau)$ which is the estimated value of $u(t-T\tau)$. In this case, such choice of the value of τ produces a time-delay of value $T\tau$. The choice of τ will be discussed later.

3.2. Fractional order differentiator of output in continuous noise-free case

In this subsection, a fractional order differentiator of the output defined in (1) is constructed in frequency and time domains by applying the algebraic parametric method and the modulating function method, respectively.

3.2.1. Algebraic parametric method (Frequency domain)

As mentioned previously, the Riemann-Liouville derivatives of y and its Riemann-Liouville integrals of order smaller than 1 are defined by improper integrals. In order to overcome this problem, the algebraic parametric method is applied in the following theorem to express these Riemann-Liouville derivatives and integrals of y by new algebraic formulae in continuous noise-free case.

Theorem 2 *Let y be a signal satisfying (1), then its Riemann-Liouville integrals of order smaller than 1 can be given as follows: $\forall\, t \in]0,h]$,*

$$\mathrm{J}_t^{\beta} y(t) = \int_0^t g_{\beta,n,m}(\tau)\, u(\tau)\, d\tau + \int_0^t p_{\beta,n,m}(\tau)\, y(\tau)\, d\tau, \tag{23}$$

with

$$\int_0^t g_{\beta,n,m}(\tau)\, u(\tau)\, d\tau = \frac{1}{a_n} \sum_{k=0}^{m} \binom{-n-\beta}{k} \frac{m!}{(m-k)!} \frac{1}{t^k} \mathrm{J}_t^{n+\beta+k} u(t), \tag{24}$$

$$\begin{aligned}\int_0^t p_{\beta,n,m}(\tau)\, y(\tau)\, d\tau = &-\sum_{k=1}^{m} \binom{-\beta}{k} \frac{m!}{(m-k)!} \frac{1}{t^k} \mathrm{J}_t^{\beta+k} y(t) \\ &- \sum_{j=1}^{n} \sum_{i=0}^{j} \frac{a_{n-i}}{a_n} \binom{m}{j-i} \frac{(-1)^{i-j}\,(n-i)!}{(n-j)!} \sum_{k=0}^{m-j+i} \binom{-j-\beta}{k} \frac{(m-j+i)!}{(m-j+i-k)!} \frac{1}{t^{j-i+k}} \mathrm{J}_t^{\beta+j+k} y(t),\end{aligned} \tag{25}$$

where $0 < \beta < 1$, and $n \leq m \in \mathbb{N}$. Moreover, let $\alpha = l - \beta \in \mathbb{R}_+ \setminus \mathbb{N}$ with $l \in \mathbb{N}^*$, then the Riemann-Liouville derivatives of y can be given using a recursive way by (23) and the following formula: $\forall\, t \in]0,h]$, for $l' = 1, \cdots, l$,

$$\mathrm{D}_t^{l'-\beta} y(t) = \frac{d^{l'}}{dt^{l'}} \int_0^t g_{\beta,n,m}(\tau)\, u(\tau)\, d\tau + \frac{d^{l'}}{dt^{l'}} \int_0^t p_{\beta,n,m}(\tau)\, y(\tau)\, d\tau, \tag{26}$$

with

$$\begin{aligned}\frac{d^{l'}}{dt^{l'}} \int_0^t g_{\beta,n,m}(\tau)\, u(\tau)\, d\tau = &\frac{1}{a_n} \frac{d^{l'}}{dt^{l'}} \left\{ \mathrm{J}_t^{n+\beta} u(t) \right\} \\ &+ \frac{1}{a_n} \sum_{k=1}^{m} \binom{-n-\beta}{k} \frac{m!}{(m-k)!} \sum_{k'=0}^{l'} \binom{l'}{k'} \frac{(-1)^{k'}(k'+k-1)!}{(k-1)!} \frac{1}{t^{k+k'}} \frac{d^{l'-k'}}{dt^{l'-k'}} \left\{ \mathrm{J}_t^{\beta+n+k} u(t) \right\},\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{d^{l'}}{dt^{l'}} \int_0^t p_{\beta,n,m}(\tau)\, y(\tau)\, d\tau = &-\sum_{j=1}^{n} \frac{a_{n-j}}{a_n} \frac{d^{l'}}{dt^{l'}} \left\{ \mathrm{J}_t^{\beta+j} y(t) \right\} \\ &- \sum_{k=1}^{m} \binom{-\beta}{k} \frac{m!}{(m-k)!} \sum_{k'=0}^{l'} \binom{l'}{k'} \frac{(-1)^{k'}(k'+k-1)!}{(k-1)!} \frac{1}{t^{k+k'}} \frac{d^{l'-k'}}{dt^{l'-k'}} \left\{ \mathrm{J}_t^{\beta+k} y(t) \right\} \\ &- \sum_{j=1}^{n} \frac{a_{n-j}}{a_n} \sum_{k=1}^{m} \binom{-j-\beta}{k} \frac{m!}{(m-k)!} \sum_{k'=0}^{l'} \binom{l'}{k'} \frac{(-1)^{k'}(k'+k-1)!}{(k-1)!} \frac{1}{t^{k+k'}} \frac{d^{l'-k'}}{dt^{l'-k'}} \left\{ \mathrm{J}_t^{\beta+j+k} y(t) \right\} \\ &- \sum_{j=1}^{n} \sum_{i=0}^{j-1} \frac{a_{n-i}}{a_n} \binom{m}{j-i} \frac{(-1)^{i-j}\,(n-i)!}{(n-j)!} \sum_{k=0}^{m-j+i} \binom{-j-\beta}{k} \frac{(m-j+i)!}{(m-j+i-k)!} \times \\ &\quad \sum_{k'=0}^{l'} \binom{l'}{k'} \frac{(-1)^{k'}(k'+j-i+k-1)!}{(j-i+k-1)!} \frac{1}{t^{j-i+k+k'}} \frac{d^{l'-k'}}{dt^{l'-k'}} \left\{ \mathrm{J}_t^{\beta+j+k} y(t) \right\},\end{aligned} \tag{28}$$

where $\frac{d^{l'}}{dt^{l'}} \left\{ \mathrm{J}_t^{\beta+j} y(t) \right\}$, $\frac{d^{l'-k'}}{dt^{l'-k'}} \left\{ \mathrm{J}_t^{\beta+k} y(t) \right\}$, $\frac{d^{l'-k'}}{dt^{l'-k'}} \left\{ \mathrm{J}_t^{\beta+j+k} y(t) \right\}$, $\frac{d^{l'}}{dt^{l'}} \left\{ \mathrm{J}_t^{n+\beta} u(t) \right\}$, and $\frac{d^{l'-k'}}{dt^{l'-k'}} \left\{ \mathrm{J}_t^{\beta+n+k} u(t) \right\}$ are given by (4).

Proof. In this proof, the following steps are needed when applying the algebraic parametric method.

• **Step 1. Laplace transform:** Applying the Laplace transform to (1) gives:

$$\sum_{i=0}^{n} a_i\, s^i \hat{y}(s) + \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} s^j\, y^{(i-1-j)}(0) = \hat{u}(s), \tag{29}$$

where $\hat{y}$ (resp. $\hat{u}$) is the Laplace transform of $\bar{y}$ (resp. $\bar{u}$), and $\bar{y}$ (resp. $\bar{u}$) is defined on $\mathbb{R}$ in such a way that $\bar{y}(t) = y(t)$ (resp. $\bar{u}(t) = u(t)$) if $t \in I$, and $\bar{y}(t) = 0$ (resp. $\bar{u}(t) = 0$), else.

• **Step 2. Elimination of initial conditions:** Applying m ($m \geq n$) times differentiation with respect to s allows us to eliminate the unknown initial conditions in (29):

$$\sum_{i=0}^{n} a_i \frac{d^m}{ds^m} \left\{ s^i \hat{y}(s) \right\} = \hat{u}^{(m)}(s). \tag{30}$$

Then, the classical Leibniz formula is applied. It yields:

$$\sum_{i=0}^{n} a_i \sum_{j=0}^{i} \binom{m}{j} \frac{i!}{(i-j)!} s^{i-j} \hat{y}^{(m-j)}(s) = \hat{u}^{(m)}(s). \tag{31}$$

• **Step 3. Division by $s^{n+\beta}$ with $0 < \beta < 1$:** In order to apply Lemma 1 to obtain integral formulae when returning into the time domain, the powers of s should be strictly negative in (31). Hence, dividing (31) by $s^{n+\beta}$ gives:

$$\sum_{i=0}^{n} a_i \sum_{j=0}^{i} \binom{m}{j} \frac{i!}{(i-j)!} \frac{1}{s^{n+\beta-i+j}} \hat{y}^{(m-j)}(s) = \frac{1}{s^{n+\beta}} \hat{u}^{(m)}(s). \tag{32}$$

All the terms in the double sums in (32) can be written in the following matrix:

$$\begin{pmatrix} a_n \frac{1}{s^\beta} \hat{y}^{(m)}(s) & a_{n-1} \frac{1}{s^{\beta+1}} \hat{y}^{(m)}(s) & \cdots & a_0 \frac{1}{s^{\beta+n}} \hat{y}^{(m)}(s) \\ a_n \frac{mn}{s^{\beta+1}} \hat{y}^{(m-1)}(s) & a_{n-1} \frac{m(n-1)}{s^{\beta+2}} \hat{y}^{(m-1)}(s) & \cdots & 0 \\ \vdots & \vdots & \iddots & \vdots \\ a_n \binom{m}{n-1} \frac{n!}{s^{\beta+n-1}} \hat{y}^{(m-n+1)}(s) & 0 & \cdots & 0 \\ a_n \binom{m}{n} \frac{n!}{s^{\beta+n}} \hat{y}^{(m-n)}(s) & 0 & \cdots & 0 \end{pmatrix}. \tag{33}$$

Then, by grouping all the terms containing the same order of s in (33), (32) becomes:

$$\sum_{j=0}^{n} \sum_{i=0}^{j} a_{n-i} \binom{m}{j-i} \frac{(n-i)!}{(n-j)!} \frac{1}{s^{j+\beta}} \hat{y}^{(m-j+i)}(s) = \frac{1}{s^{n+\beta}} \hat{u}^{(m)}(s). \tag{34}$$

Hence, it yields:

$$a_n \frac{1}{s^\beta} \hat{y}^{(m)}(s) + \sum_{j=1}^{n} \sum_{i=0}^{j} a_{n-i} \binom{m}{j-i} \frac{(n-i)!}{(n-j)!} \frac{1}{s^{j+\beta}} \hat{y}^{(m-j+i)}(s) = \frac{1}{s^{n+\beta}} \hat{u}^{(m)}(s). \tag{35}$$

• **Step 4. Inverse of Laplace transform:** When returning into the time domain by applying the inverse of the Laplace transform, using Lemma 1 gives: $\forall\, t \in]0, h]$,

$$a_n \mathrm{J}_t^\beta \left[t^m y(t)\right] + \sum_{j=1}^{n} \sum_{i=0}^{j} a_{n-i} \binom{m}{j-i} \frac{(n-i)!}{(n-j)!} (-1)^{i-j} \mathrm{J}_t^{j+\beta} \left[t^{m-j+i} y(t)\right] = \mathrm{J}_t^{n+\beta} \left[t^m u(t)\right]. \tag{36}$$

Then, it yields:

$$\mathrm{J}_t^{\beta}\left[t^m y(t)\right] = \frac{1}{a_n}\mathrm{J}_t^{n+\beta}\left[t^m u(t)\right] - \sum_{j=1}^{n}\sum_{i=0}^{j}\frac{a_{n-i}}{a_n}\binom{m}{j-i}\frac{(n-i)!}{(n-j)!}(-1)^{i-j}\mathrm{J}_t^{j+\beta}\left[t^{m-j+i}y(t)\right]. \quad (37)$$

Hence, (23) can be obtained using Corollary 1.

• **Step 5. Successive differentiations:** By successively taking differentiations to (23), the $(l'-\beta)^{th}$ order Riemann-Liouville derivatives of y can be recursively given by: for $l' = 1, \cdots, l$,

$$\frac{d^{l'}}{dt^{l'}}\left\{\mathrm{J}_t^{\beta}y(t)\right\} = \frac{1}{a_n}\sum_{k=0}^{m}\binom{-n-\beta}{k}\frac{m!}{(m-k)!}\frac{d^{l'}}{dt^{l'}}\left\{\frac{1}{t^k}\mathrm{J}_t^{n+\beta+k}u(t)\right\} - \sum_{k=1}^{m}\binom{-\beta}{k}\frac{m!}{(m-k)!}\frac{d^{l'}}{dt^{l'}}\left\{\frac{1}{t^k}\mathrm{J}_t^{\beta+k}y(t)\right\}$$
$$- \sum_{j=1}^{n}\sum_{i=0}^{j}\frac{a_{n-i}}{a_n}\binom{m}{j-i}\frac{(-1)^{i-j}(n-i)!}{(n-j)!}\sum_{k=0}^{m-j+i}\binom{-j-\beta}{k}\frac{(m-j+i)!}{(m-j+i-k)!}\frac{d^{l'}}{dt^{l'}}\left\{\frac{1}{t^{j-i+k}}\mathrm{J}_t^{j+\beta+k}y(t)\right\}. \quad (38)$$

Finally, the proof can be completed by applying the classical Leibniz formula. □

Remark 1 *According to the proof of Theorem 2, in order to obtain* $\mathrm{J}_t^{\beta}y(t)$ *the following differential operator is constructed in the frequency domain:*

$$\Pi^{n+\beta,m} = \frac{1}{s^{n+\beta}}\cdot\frac{d^m}{ds^m}, \quad (39)$$

where $n \leq m \in \mathbb{N}$*, and* $0 < \beta < 1$*. The role of the design parameter* m *will be studied later.*

Although the formulae proposed in Theorem 2 seem to be complicated, they are not difficult to apply. Indeed, these formulae are only sums of Riemann-Liouville integrals and derivatives of y and u. An algorithm of the following form is required to apply Theorem 2:

Function $[\mathrm{D}_t^{\alpha}y(t)]$ **= Algebraic Differentiator**$([a_0, \ldots, a_n], t, u, y, \alpha, m)$,

where an algorithm for realizing (2) is used. In order to calculate $\mathrm{D}_t^{\alpha}y(t)$, the required orders of Riemann-Liouville integrals and derivatives of y and u are given in Table 1, by admitting $\mathrm{D}_t^{\alpha}y(t) = \mathrm{J}_t^{-\alpha}y(t)$. For instance, according to (23)-(25), $\mathrm{D}_t^{-\beta}y(t) = \mathrm{J}_t^{\beta}y(t)$ $(0 < \beta < 1)$ is given by the Riemann-Liouville integrals of y (resp. of u) with orders varying from $\beta+1$ to $\beta+n+m$ (resp. from $\beta+n$ to $\beta+n+m$), which are proper integrals. Then, $\mathrm{D}_t^{1-\beta}y(t) = \mathrm{J}_t^{\beta-1}y(t)$ is given by taking the integer order derivatives of these proper integrals involving y and u in (26). Hence, the Riemann-Liouville integrals of y (resp. of u) with orders varying from β to $\beta+n+m-1$ (resp. from $\beta+n-1$ to $\beta+n+m-1$) are required, where $\mathrm{J}_t^{\beta}y(t)$ is calculated previously. Consequently, $\mathrm{D}_t^{\alpha}y(t)$ $(\alpha = l-\beta)$ is given by a recursive way using $\mathrm{J}_t^{\beta}y(t)$ and $\mathrm{D}_t^{l'-\beta}y(t)$, for $l' = 1, \cdots, l-1$, where the integrals involving y are always proper. Thus, the algebraic formula proposed for $\mathrm{D}_t^{\alpha}y(t)$ in (26) can be written in the following form:

$$\mathrm{D}_t^{\alpha}y(t) = \int_0^t q_{\alpha,n,m}(\tau)\, y(\tau)\, d\tau + U_{\alpha,n,m}(t), \quad (40)$$

where $q_{\alpha,n,m}$ denotes the corresponding function in the integral of y, and $U_{\alpha,n,m}$ denotes the part involving u. According to Table 1, the following cases should be considered:

• If $n \geq l+1$, $U_{\alpha,n,m}(t)$ only contains the Riemann-Liouville integrals of u of order larger than 1. Hence, $\mathrm{D}_t^{\alpha} y(t)$ is given by proper integrals.

• If $n = l$, $U_{\alpha,n,m}(t)$ contains the Riemann-Liouville integral $\mathrm{J}_t^{\beta} u(t)$ of order smaller than 1. However, $\mathrm{D}_t^{\alpha} y(t)$ is still given by integrals.

• If $n \leq l-1$, $U_{\alpha,n,m}(t)$ contains the Riemann-Liouville derivatives of u. In order to avoid calculating these Riemann-Liouville derivatives, the differential equation given in (1) should be differentiated $(l-n)$ times. Then, according to previous analysis, $\mathrm{D}_t^{\alpha} y(t)$ can be given integrals, where $u^{(l-n)}$ should be estimated using Proposition 1.

3.2.2. Modulating functions method (Time domain)

The modulating functions method has been applied to design fractional order model-based differentiators in [16], where generalized modulating functions were used. In this subsection, the classical modulating functions method is considered without any generalized modulating function.

The following proposition shows that the results in Theorem 2 can also be obtained by applying the modulating functions method.

Proposition 2 *The integral formulae given in (23) and (26) can also be obtained by applying the modulating functions method to (1) with the following $(n-1)^{th}$ order modulating function on $[0,t]$: $\forall\, \tau \in [0,t]$,*

$$g(\tau) = \frac{(t-\tau)^{n+\beta-1}\tau^m}{\Gamma(n+\beta)}, \tag{41}$$

where $n \leq m \in \mathbb{N}$, and $0 < \beta < 1$.

Proof. When applying the modulating functions method, the following steps are needed.

• **Step 1. Multiplication and integration:** Multiplying the modulating function g to (1) and integrating from 0 to t gives: $\forall\, t \in]0,h]$,

$$\sum_{i=0}^{n} a_i \frac{1}{\Gamma(n+\beta)} \int_0^t (t-\tau)^{n+\beta-1}\tau^m y^{(i)}(\tau)\, d\tau = \frac{1}{\Gamma(n+\beta)} \int_0^t (t-\tau)^{n+\beta-1}\tau^m u(\tau)\, d\tau. \tag{42}$$

• **Step 2. Integration by parts:** By successively applying the integration by parts formula to the left side of (42), the following equation can be got:

$$\sum_{i=0}^{n} \frac{a_i}{\Gamma(n+\beta)} \int_0^t (t-\tau)^{n+\beta-1}\tau^m y^{(i)}(\tau)\, d\tau = \sum_{i=0}^{n} \frac{(-1)^i a_i}{\Gamma(n+\beta)} \int_0^t \frac{d^i}{d\tau^i}\left\{(t-\tau)^{n+\beta-1}\tau^m\right\} y(\tau)\, d\tau, \tag{43}$$

where the boundary conditions on $\tau = 0$ and $\tau = t$ are eliminated thanks to the properties of the modulating function g. Then, applying the classical Leibniz formula gives:

$$\frac{d^i}{d\tau^i}\left\{(t-\tau)^{n+\beta-1}\tau^m\right\} = \sum_{j=0}^{i} \binom{i}{j} \frac{(-1)^{i-j}\,\Gamma(n+\beta)}{\Gamma(n+\beta-i+j)} (t-\tau)^{n+\beta-1-i+j} \frac{m!}{(m-j)!}\tau^{m-j}. \tag{44}$$

Hence, according to (2) and using (42), (44), it yields:

$$\sum_{i=0}^{n} a_i \sum_{j=0}^{i} \frac{(-1)^j \, i! \, m!}{j! \, (i-j)! \, (m-j)!} \mathrm{J}_t^{n+\beta-i+j} \left[t^{m-j} y(t)\right] = \mathrm{J}_t^{n+\beta} \left[t^m u(t)\right]. \tag{45}$$

Finally, on the one hand, if the Laplace transform is applied to (45), then (32) can be obtained using (9). On the other hand, if the inverse of the Laplace transform is applied to (32), then (45) can be obtained. □

Remark 2 *According to Proposition 2, there is an equivalence between the algebraic parametric method and the modulating functions method. Indeed, according to Lemma 1, the differential operator* $\Pi^{n+\beta,m}$ *defined in the frequency domain in Remark 1 is associated with the following operator in the time domain:*

$$\hat{\Pi}^{n+\beta,m} : f \longrightarrow (-1)^m \mathrm{J}_t^{n+\beta} \left[t^m f(t)\right] = \frac{(-1)^m}{\Gamma(n+\beta)} \int_0^t (t-\tau)^{n+\beta-1} \tau^m f(\tau) d\tau, \tag{46}$$

where $g(\tau) = \frac{(t-\tau)^{n+\beta-1}\tau^m}{\Gamma(n+\beta)}$ *is the modulating function used in Proposition 2.*

It seems easier to obtain the integral formulae given in (23) and (26) using the modulating function method, without working in the frequency domain. However, the construction of the used modulating function is inspired by the algebraic parametric method. In general, when tackling a complex problem, such as the problems on fractional derivatives, we can first be inspired by the algebraic parametric method by working in the frequency domain (see, e.g. [36]).

According to Theorem 2, the Riemann-Liouville derivatives of y are exactly given by algebraic formulae. In the next subsection, a digital fractional order differentiator is introduced in discrete noisy case. Moreover, some error analysis is given.

3.3. Digital differentiators in noisy case

From now on, let y^{ϖ} be a discrete noisy observation of y on $I = [0, h] \subset \mathbb{R}_+$:

$$y^{\varpi}(t_i) = y(t_i) + \varpi(t_i), \tag{47}$$

where $t_i = iT_s$, for $i = 0, 1, \ldots, M$, with an equidistant sampling period $T_s = \frac{h}{M}$. Moreover, the noise $\{\varpi(t), t \in I\}$ is assumed to be a continuous stochastic process satisfying the following conditions:

(C_1) : for any $s, t \in I$, $s \neq t$, $\varpi(s)$ and $\varpi(t)$ are independent;

(C_2) : the mean value function of $\{\varpi(t), t \in I\}$ denoted by $\mathbb{E}[\cdot]$ is equal to zero;

(C_3) : the variance function of $\{\varpi(t), t \in I\}$ denoted by $\mathrm{Var}[\cdot]$ is bounded on I, *i.e.* $\exists\, \delta \in \mathbb{R}_+, \forall\, t \in I, \mathrm{Var}[\varpi(t)] \leq \delta$.

Note that a zero-mean white Gaussian noise satisfies these conditions.

3.4. Digital integer order differentiator of input in noisy case

If the input defined in (1) is unknown, the integer order differentiator constructed in Proposition 1 should be applied in the discrete noisy case. For this purpose, the integral in (19) needs to be approximated using a numerical integration method, where y is replaced by its observation y^{ϖ}. Thus, the following digital integer order differentiator of the input is obtained:

$$D^{(q),\varpi}_{T,\mu,\kappa,N,n} u(t_i - T\tau) := \sum_{j=0}^{N} \lambda^{(\mu,\kappa),\varpi}_{T,q,n,j} \, P_j^{(\mu+q+n,\kappa+q+n)}(\tau), \tag{48}$$

where $\tau \in [0,1]$, $\lambda^{(\mu,\kappa),\varpi}_{T,q,n,j}$ is the numerical approximation of $\lambda^{(\mu,\kappa)}_{T,q,n,j}$ using y^{ϖ}. Consequently, the digital differentiator $D^{(q),\varpi}_{T,\mu,\kappa,N,n,\tau} u(t_i)$ contains three sources of errors:

- the truncated term error in the Jacobi orthogonal series of $u^{(q)}$;
- the noise error contribution due to the noisy output;
- the numerical error due to the used numerical integration method.

Error analysis for these errors can be given in a similar way as done for the algebraic integer order model-free differentiators (see [18, 21, 22, 23] for more details). According to [18, 21], if the value of τ is taken as the smallest root of $P_{N+1}^{(\mu+q+n,\kappa+q+n)}$, the truncated term error and the noisy error contribution can be significantly reduced by admitting a time-delay.

3.5. Digital fractional order differentiator of output in noisy case

According to (40), the formula of $\mathrm{D}_t^{\alpha} y(t)$ contains two parts: the one given by a proper integral of y, and the other one involving u. In order to estimate the Riemann-Liouville derivatives of y in the discrete noisy case, $\mathrm{D}_t^{\alpha} y(t)$ is approximated by the following digital fractional order differentiator: for $i = 1, \ldots, M$,

$$D^{\alpha}_{n,m} y^{\varpi}(t_i) := T_s \sum_{j=0}^{i} w_j \, q_{\alpha,n,m}(t_j) \, y^{\varpi}(t_j) + \tilde{U}_{\alpha,n,m}(t_i), \tag{49}$$

where $w_i \in \mathbb{R}_+$ are the weights of a given numerical integration method. According to previous study in Subsection 3.2.1, $\tilde{U}_{\alpha,n,m}(t_i)$ is calculated in the following way:

• If $n \geq l+1$, $\tilde{U}_{\alpha,n,m}(t_i)$ is calculated using a numerical integration method.

• If $n = l$, the Riemann-Liouville integral $\mathrm{J}_t^{\beta} u(t)$ $(0 < \beta < 1)$ is numerically calculated in $\tilde{U}_{\alpha,n,m}(t_i)$ using the Grünwald-Letnikov scheme, which will be recalled later.

• If $n \leq l-1$, the differential equation given in (1) should be differentiated $(l-n)$ times. Then, the digital differentiator $D^{(q),\varpi}_{T,\mu,\kappa,N,n} u(t_i - T\tau)$ given in (48) is used to estimate $u^{(l-n)}$, whenever u is unknown or known with noises.

Consequently, the digital fractional order differentiator $D^{\alpha}_{n,m} y^{\varpi}(t_i)$ can be used to estimate the Riemann-Liouville fractional derivatives with an arbitrary order for the signals satisfying (1). It contains three sources of errors:

- the numerical error due to the numerical integration method used to approximate the proper integral involving y;
- the noise error contribution of the following form:

$$e_{n,m}^{\alpha,\varpi}(t_i) := T_s \sum_{j=0}^{i} w_j \, q_{\alpha,n,m}(t_j) \, \varpi(t_j); \tag{50}$$

- the error in $\tilde{U}_{\alpha,n,m}(t_i)$, which can contain the numerical error, the noise error contribution and the truncated term error produced in $D_{T,\mu,\kappa,N,n}^{(q),\varpi} u(t_i - T\tau)$.

It is well known that the numerical error converges to zero when $T_s \to 0$ (see, e.g. [37]). In the following proposition, the convergence in mean square of the noise error contribution is studied when $T_s \to 0$.

Proposition 3 *Let $\{\varpi(t_i)\}$ be a sequence of $\{\varpi(t), t \in I\}$ with an equidistant sampling period T_s, where $\{\varpi(t), t \in I\}$ is a continuous stochastic process satisfying conditions $(C_1) - (C_3)$. Then, the following limit holds:*

$$\mathbb{E}\left[\left(e_{n,m}^{\alpha,\varpi}(t_i)\right)^2\right] \xrightarrow[T_s \to 0]{} 0, \tag{51}$$

where $e_{n,m}^{\alpha,\varpi}(t_i)$ is given by (50).

Proof. Using $(C_1) - (C_2)$ and the properties of the mean value and the variance functions, one obtains:

$$\mathbb{E}[e_{n,m}^{\alpha,\varpi}(t_i)] = 0, \tag{52}$$

$$\mathbb{E}\left[\left(e_{n,m}^{\alpha,\varpi}(t_i)\right)^2\right] = \text{Var}\left[e_{n,m}^{\alpha,\varpi}(t_i)\right] = T_s^2 \sum_{j=0}^{i} w_j^2 \, q_{\alpha,n,m}^2(t_j) \text{Var}\left[\varpi(t_j)\right]. \tag{53}$$

Then, using (C_3) the following inequalities are obtained:

$$\text{Var}\left[e_{n,m}^{\alpha,\varpi}(t_i)\right] \leq T_s \delta \left(T_s \sum_{j=0}^{i} w_j^2 \, q_{\alpha,n,m}^2(t_j) \right) \leq \delta T_s W_i \left(T_s \sum_{j=0}^{i} w_j \, q_{\alpha,n,m}^2(t_j) \right). \tag{54}$$

where $W_i = \max_{0 \leq j \leq i} w_j$. Since all w_j are bounded, W_i is also bounded for any $i \in \mathbb{N}^*$. Moreover, since the integral formulae involving y are proper in Theorem 2, it can be verified that $q_{\alpha,n,m} \in \mathcal{L}^2([0, t_i])$. Hence, the following limit holds:

$$\lim_{T_s \to 0} T_s \sum_{j=0}^{i} w_j \, q_{\alpha,n,m}^2(t_j) = \int_0^{t_i} q_{\alpha,n,m}^2(\tau) \, d\tau < +\infty. \tag{55}$$

Consequently, this proof is completed. □

A similar result has been shown using non-standard analysis in [27, 28]. Consequently, both the numerical error and the noise error contribution in the proposed digital fractional order differentiator can be reduced by decreasing the sampling period T_s. However, if the computations are performed on a finite precision numerical machine, there also are round-off errors (see, e.g. [38]). Hence, the infinite reduction of T_s would not lead to arbitrary reduction of computation errors, since the round-off errors would become too large at some points.

When the sampling period is set, on the one hand since the integral of y given in the formula of $\mathrm{D}_t^\alpha y(t)$ is proper, the numerical error can be much smaller than the noise error contribution. On the other hand, the noise error contribution $e_{n,m}^{\alpha,\varpi}(t_i)$ satisfying $\mathbb{E}[e_{n,m}^{\alpha,\varpi}(t_i)] = 0$ can be bounded using the Bienaymé-Chebyshev inequality:

$$\forall \gamma \in \mathbb{R}_+^*, \ \Pr\left(\left|e_{n,m}^{\alpha,\varpi}(t_i)\right| < \gamma \left(\mathrm{Var}[e_{n,m}^{\alpha,\varpi}(t_i)]\right)^{\frac{1}{2}}\right) > 1 - \frac{1}{\gamma^2}, \tag{56}$$

i.e. the probability for $\left|e_{n,m}^{\alpha,\varpi}(t_i)\right|$ to be smaller than $\gamma \left(\mathrm{Var}[e_{n,m}^{\alpha,\varpi}(t_i)]\right)^{\frac{1}{2}}$ is larger than $1 - \frac{1}{\gamma^2}$. Thus, the following error bound is deduced from (54) and (56):

$$\left|e_{n,m}^{\alpha,\varpi}(t_i)\right| \overset{p_\gamma}{<} \gamma T_s \left(\delta \sum_{j=0}^{i} w_j^2 \, q_{\alpha,n,m}^2(t_j)\right)^{\frac{1}{2}}, \tag{57}$$

where $a \overset{p_\gamma}{<} b$ means that the probability for a real number b to be larger than another real number a is equal to p_γ with $p_\gamma > 1 - \frac{1}{\gamma^2}$. Remark that the value of p_γ can be given by the probability density function of the considered noise ϖ. In particular, if ϖ is a zero-mean white Gaussian noise, then $e_{n,m}^{\alpha,\varpi}(t_i)$ is a normally distributed random number. Thus, according to the three-sigma rule, we have:

$$\left|e_{n,m}^{\alpha,\varpi}(t_i)\right| \overset{p_\gamma}{\leq} \gamma T_s \left(\delta \sum_{j=0}^{i} w_j^2 \, q_{\alpha,n,m}^2(t_j)\right)^{\frac{1}{2}}, \tag{58}$$

where $p_1 = 68.26\%$, $p_2 = 95.44\%$ and $p_3 = 99.73\%$, for $\gamma = 1, 2, 3$, respectively.

Since the noise error bound obtained in (57) is sharp and it depends on the design parameter m, the study of the influence of m on this error bound is useful to deduce the influence of m on the noise error contribution. Hence, the noise error contribution can be reduced by choosing the value of m which minimizes the noise error bound.

4. Simulation results

In this section, it is assumed that $y^\varpi(t_i) = y(t_i) + \varpi(t_i)$ is a discrete noisy observation defined in (47) with $I = [0, 10]$, where the noise $\{\varpi(t_i)\}$ is simulated from a zero-mean white Gaussian *iid* sequence, and the variance is adjusted such that the signal-to-noise ratio $\mathrm{SNR} = 10\log_{10}\left(\frac{\sum |y^\varpi(t_i)|^2}{\sum |\varpi(t_i)|_2^2}\right)$ is equal to 15 dB.

In noise-free case, the Riemann-Liouville derivatives of y can be numerically approximated by the well-known Grünwald-Letnikov scheme (see [8] pp. 217-218):

$$\mathrm{D}_{t_i}^\alpha y(t_i) \approx \frac{1}{T_s^\alpha} \sum_{j=0}^{i} w_j^{(\alpha)} y(t_i - jT_s), \tag{59}$$

where the coefficients $w_j^{(\alpha)}$ are recursively calculated with the following formula:

$$\begin{cases} w_0^{(\alpha)} = 1, & \text{if } j = 0, \\ w_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)}, & \text{else.} \end{cases}$$

The Grünwald-Letnikov scheme can help us to numerically calculate Riemann-Liouville derivatives, when the analytical derivatives are unknown. Moreover, this scheme can also be used to numerically calculate Riemann-Liouville integrals by taking negative values of α in (59). However, this scheme is not robust against noises.

The aim of this section is to show the accuracy and the robustness of the digital fractional order differentiator given in (49), where the trapezoidal numerical integration method is considered. In the following examples, y is assumed to be a sinusoidal signal and the output of a linear system, respectively.

Example 1. Sinusoidal signal

In this example, the following sinusoidal signal is considered:

$$\forall\, t \in I,\ y(t) = A \sin(\omega t + \phi), \tag{60}$$

where $A = 3$, $\omega = 5$, and $\phi = 0.5$. Hence, y satisfies the following harmonic oscillator equation:

$$\forall\, t \in I,\ \ddot{y}(t) + \omega^2 y(t) = 0. \tag{61}$$

The Riemann-Liouville derivatives of y can be given using the expansion of y and the linearity of the Riemann-Liouville derivatives ([4], p. 91):

$$\forall\, t \in I,\ \mathrm{D}_t^{\alpha} y(t) = A \cos\phi\, \mathrm{D}_t^{\alpha} \sin(\omega t) + A \sin\phi\, \mathrm{D}_t^{\alpha} \cos(\omega t) \tag{62}$$

with

$$\mathrm{D}_t^{\alpha} \sin(\omega t) = \frac{\omega\, t^{1-\alpha}}{\Gamma(2-\alpha)}\, {}_1\mathrm{F}_2 \left(1; \frac{1}{2}(2-\alpha), \frac{1}{2}(3-\alpha); -\frac{1}{4}\omega^2 t^2\right), \tag{63}$$

$$\mathrm{D}_t^{\alpha} \cos(\omega t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\, {}_1\mathrm{F}_2 \left(1; \frac{1}{2}(1-\alpha), \frac{1}{2}(2-\alpha); -\frac{1}{4}\omega^2 t^2\right), \tag{64}$$

where ${}_1\mathrm{F}_2$ is the generalized hypergeometric function (see [3] p. 83, p. 303 and pp. 317-318).

The discrete signal y and its noisy observation y^{ϖ} are shown in Fig. 1(a), where $T_s = 0.01$ and $\mathrm{Var}[\varpi(t_i)] = 0.3778^2$. The Riemann-Liouville derivatives $\mathrm{D}_{t_i}^{0.5} y(t_i)$ and $\mathrm{D}_{t_i}^{1.5} y(t_i)$ for $t_i \in [0.5, 10]$, will be estimated by applying the proposed digital fractional differentiator to Eq. (61), where the frequency ω is assumed to be unknown. The relative estimation error for ω^2 is shown in Fig. 1(b), where ω^2 is estimated by applying the algorithm used in [16].

Firstly, the choice of the value of the parameter m is studied. For this purpose, the noise error bound obtained in (58) is used by taking $\gamma = 2$. The variation of this noise error bound with respect to m is shown in Fig. 1(c), where $\alpha = 0.5$. As shown in Fig. 1(c), the noise error bound is increasing (resp. decreasing) with respect to m when $t_i \in [0.5, 1.73]$ (resp. $t_i \in [1.74, 10]$). Thus, the influence of m on the noise error contribution can be deduced. Hence, in order to reduce the noise error contribution, the following values of m are chosen: $m = 6$ for $t_i \in [0.5, 1.73]$, and $m = 16$ for $t_i \in [1.74, 10]$. Similarly, the variation of the noise error bound can also be studied in the case where $\alpha = 1.5$. Then, the same values of m are chosen: $m = 6$ for $t_i \in [0.5, 1.73]$, and $m = 16$ for $t_i \in [1.74, 10]$. This study is done by considering Eq. (61). Similar study can be done, if the equation $\dddot{y}(t) + \omega^2 \dot{y}(t) = 0$ is considered. The obtained variation of the noise error bound with respect to m is shown in Fig. 1(d). By comparing Fig. 1(c) to Fig. 1(d), it can be seen that although the noise error bound is increasing with t when t becomes large, the noise error bound can be reduced by differentiating Eq. (61).

Secondly, using these values of m, the estimated derivatives and the corresponding absolute estimation errors are shown in Fig. 2. According to the study in Subsection 3.3, these estimation errors contain the noise error contributions and the numerical errors due to the trapezoidal numerical integration method. The error bounds for the noise error contributions are shown in Fig. 2(c) – Fig. 2(d), and the absolute numerical errors are shown in Fig. 3, where the unknown frequency is estimated in noise-free case. It can be concluded that the numerical errors are much smaller than the noise error contributions.

Finally, in order to compare to the Grünwald-Letnikov scheme, the absolute numerical errors for the estimations obtained by the Grünwald-Letnikov scheme are shown in Fig. 3.

Example 2. Output of a linear system

In this example, the following linear system is considered: $\forall\, t \in I$,

$$\begin{cases} a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = u(t), \\ \dot{y}(0) = y(0) = 0, \end{cases} \tag{65}$$

where $a_2 = a_1 = a_0 = 1$, and the input is given by $u(t) = \sin(3t)$.

In this example, the Riemann-Liouville derivative $\mathrm{D}_{t_i}^{0.5} y(t_i)$ for $t_i \in [0.5, 10]$ will be estimated by the proposed digital fractional differentiator in the following situation:

- the discrete output is numerically approximated by the finite difference scheme from the following model: $a_2 \ddot{y}(t_i) + a_1 \dot{y}(t_i) + a_0 y(t_i) + 0.5\varpi_3(t_i) = u(t_i)$, where $0.5\varpi_3(t_i)$ is a model error,
- the parameters a_j for $j = 0, 1, 2$, are known with errors: $\tilde{a}_j(t_i) = a_j + 0.02\varpi_j(t_i)$,
- the input is known with noises: $u^{\varpi}(t_i) = u(t_i) + 0.126\varpi_4(t_i)$ (SNR $= 15\,$dB), or the input is unknown,

where $\{\varpi_j(t_i)\}$ for $j = 0, 1, 2, 3, 4$ are simulated from zero-mean white Gaussian *iid* sequences with $\mathrm{Var}[\varpi_j(t_i)] = 1$. The original output y and the discrete noisy output y^{ϖ} are shown in Fig. 4(a), where $\mathrm{Var}[\varpi(t_i)] = 0.0204^2$ with $T_s = 0.001$.

Firstly, the noise error bound obtained in (58) is studied by taking $\gamma = 2$ and $\alpha = 0.5$, whose variation with respect to m is shown in Fig. 4(b). As shown in Fig. 4(b), the noise error bound is increasing with respect to m. Hence, $m = 6$ is taken in order to reduce the noise error contribution due to the noise $\{\varpi(t_i)\}$.

Secondly, $\mathrm{D}_{t_i}^{0.5} y(t_i)$ is estimated with the noisy input u^{ϖ}. The estimated derivative and u^{ϖ} are shown in Fig. 5. Since the analytical Riemann-Liouville derivatives of the output are unknown, the estimation obtained by the Grünwald-Letnikov scheme in the noise-free case is also given to verify the estimation obtained by the proposed fractional order differentiator.

Thirdly, $\mathrm{D}_{t_i}^{0.5} y(t_i)$ is estimated by assuming the input is unknown. Then, the input is estimated by the digital differentiator given in (49) with $\mu = \kappa = 1$, $N = 1$, $n = 2$, $q = 0$, $T = 0.5$ and $\tau = 0.33$ which is the smaller root of $P_{N+1}^{(\mu+q+n,\kappa+q+n)}$. This choice of τ produces a time-delay in the estimation of u, and the shifted estimation is shown in Fig. 6(a). By increasing the value of T, the noise error contribution in the estimation of u can be reduced, however the value of the time-delay will be increased (see [23] for more details). Finally, using the time-delayed input estimation, the obtained estimation of $\mathrm{D}_{t_i}^{0.5} y(t_i)$ is shown in Fig. 6(b).

5. Conclusions

In this paper, a digital fractional order differentiator has been introduced to estimate the Riemann-Liouville derivatives of a class of signals satisfying a linear differential equation in discrete noisy case, where the input can be unknown or known with noises. Firstly, inspired by the ideas of constructing the algebraic integer order model-free differentiators, an integer order differentiator of the input has been constructed using a truncated Jacobi orthogonal series expansion. Secondly, by applying the algebraic parametric method and the Leibniz formula for Riemann-Liouville integrals, the Riemann-Liouville integrals of order smaller than 1 of the considered signal has been expressed by a proper integral formula. Then, a new algebraic formula for the Riemann-Liouville derivatives has been given by taking the integer order derivatives of the obtained proper integral formula for Riemann-Liouville integrals. These formulae do not contain any sources of errors in continuous noise-free case, if the input is exactly known. It has been shown that the proposed formulae can also be obtained using the modulating functions method. Indeed, the differential operator constructed by the algebraic parametric method in the frequency domain is related to a modulating function used in the time domain. Thirdly, a digital fractional order differentiator has been introduced in discrete noisy case, which can be used to estimate the Riemann-Liouville derivatives with an arbitrary order of the considered signal. Then, the convergence in mean square of the noise error contribution due to a class of stochastic processes has been studied when the sampling period tends to zero. Moreover, a noise error bound has been provided, which can help us to choose the design parameter's value in order to reduce the noise error contribution. Finally, two numerical examples have been given to show the accuracy and the robustness of the proposed fractional order differentiator. A comparison to the Grünwald-Letnikov scheme has been given in discrete noise-free case.

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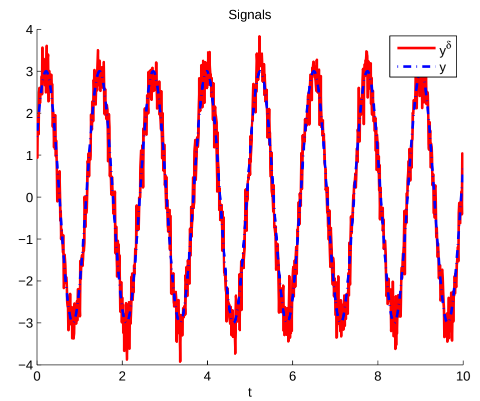
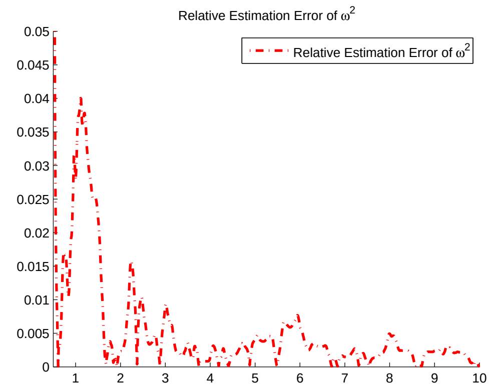
$-\alpha$	required orders for y	required orders for u
$0 < \beta < 1$	$\{\beta+1, \beta+2, \cdots, \beta+n+m\}$	$\{\beta+n, \beta+n+1, \cdots, \beta+n+m\}$
$\beta-1$	$\{\beta, \beta+1, \cdots, \beta+n+m-1\}$	$\{\beta+n-1, \beta+n, \cdots, \beta+n+m-1\}$
$\vdots$	$\vdots$	$\vdots$
$\beta-n+1$	$\{\beta-n+2, \beta-n+3, \cdots, \beta+m+1\}$	$\{\beta+1, \beta+2, \cdots, \beta+m+1\}$
$\beta-n$	$\{\beta-n+1, \beta-n+2, \cdots, \beta+m\}$	$\{\beta, \beta+1, \cdots, \beta+m\}$
$\beta-n-1$	$\{\beta-n, \beta-n+1, \cdots, \beta+m-1\}$	$\{\beta-1, \beta, \cdots, \beta+m-1\}$
$\vdots$	$\vdots$	$\vdots$

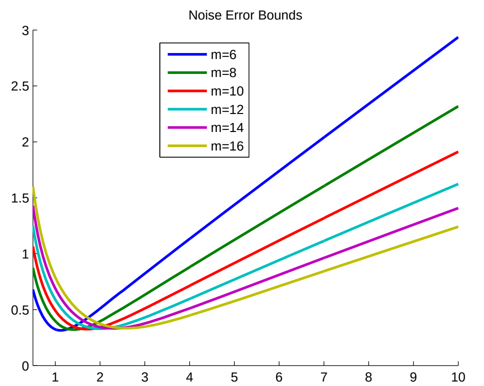
Table 1: Required orders of Riemann-Liouville integrals and derivatives of y and u in the formula of $\mathrm{D}_t^{\alpha} y(t)$.



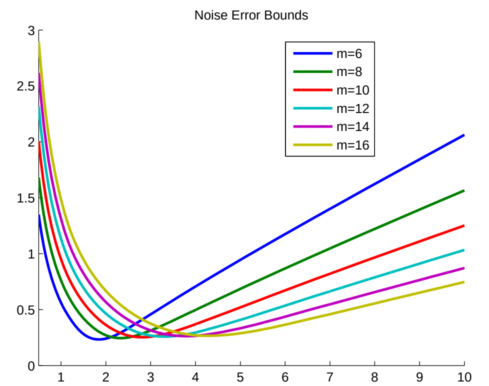
(a) Discrete sinusoidal signal y and its noisy observation.



(b) Relative estimation error for the parameter ω^2.

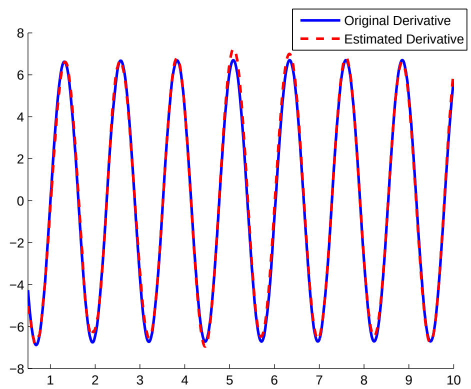


(c) Variation of the noise error bound given in (58) with $\alpha = 0.5$ and $m = 6, \ldots, 16$, in the case $\ddot{y}(t) + \omega^2 y(t) = 0$.

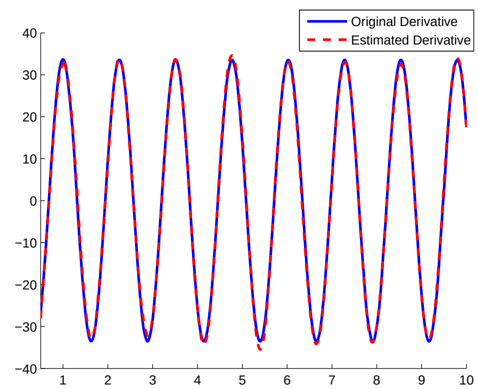


(d) Variation of the noise error bound given in (58) with $\alpha = 0.5$ and $m = 6, \ldots, 16$, in the case $\dddot{y}(t) + \omega^2 \dot{y}(t) = 0$.

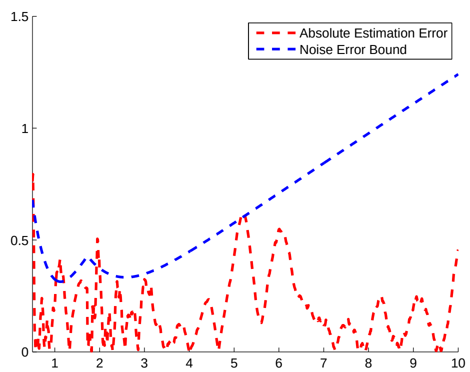
Figure 1: Example 1: Discrete sinusoidal signal $y(t_i) = 3\sin(5t_i + 0.5)$ with $T_s = 10^{-2}$, and its noisy observation corrupted by a zero-mean white Gaussian noise with SNR$= 15\,$dB.



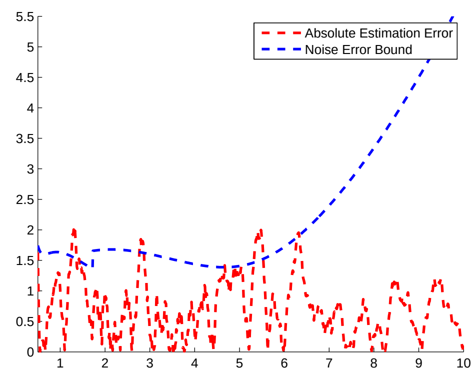
(a) Estimation of the 0.5^{th} order derivative of y.



(b) Estimation of the 1.5^{th} order derivative of y.

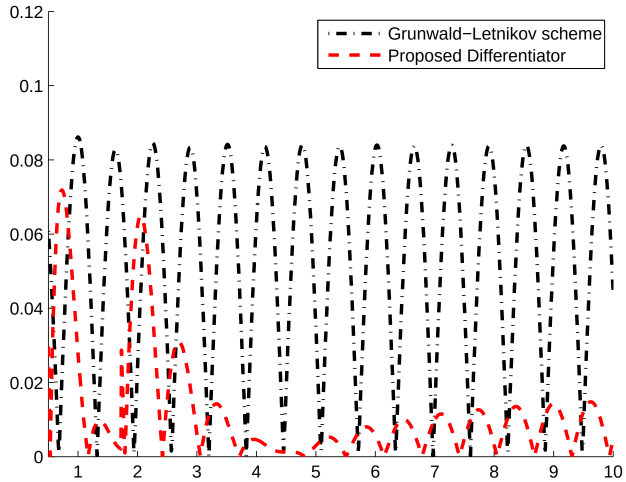


(c) Absolute estimation error for the estimation of the 0.5^{th} order derivative of y, and the corresponding noise error bound.

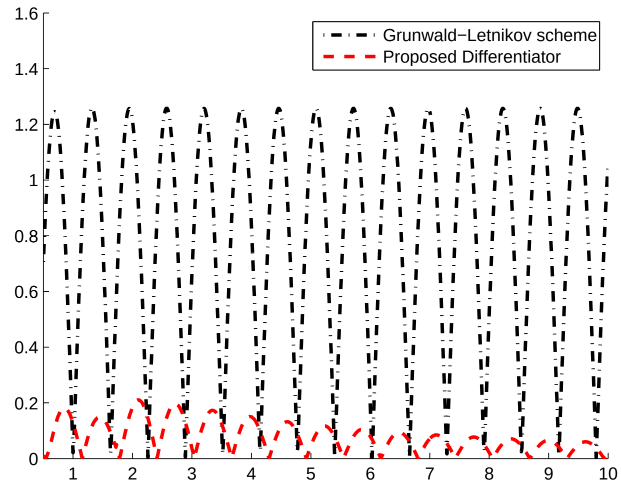


(d) Absolute estimation error for the estimation of the 1.5^{th} order derivative of y, and the corresponding noise error bound.

Figure 2: Estimations of the Riemann-Liouville derivatives of the sinusoidal signal y in the discrete noisy case.

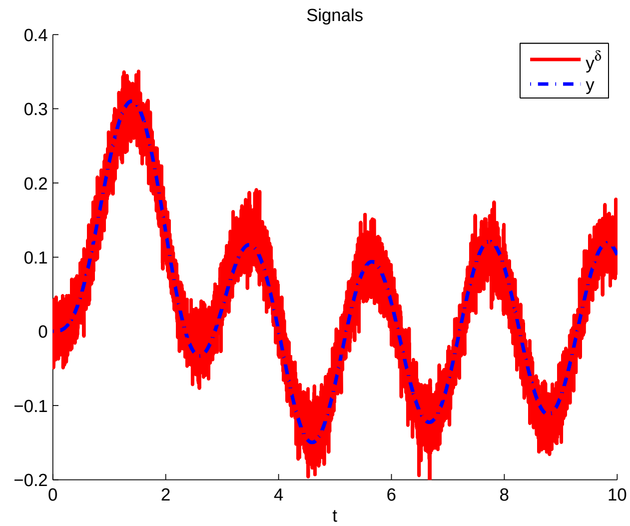


(a) Absolute numerical errors for the estimations of the 0.5^{th} order derivative of y.

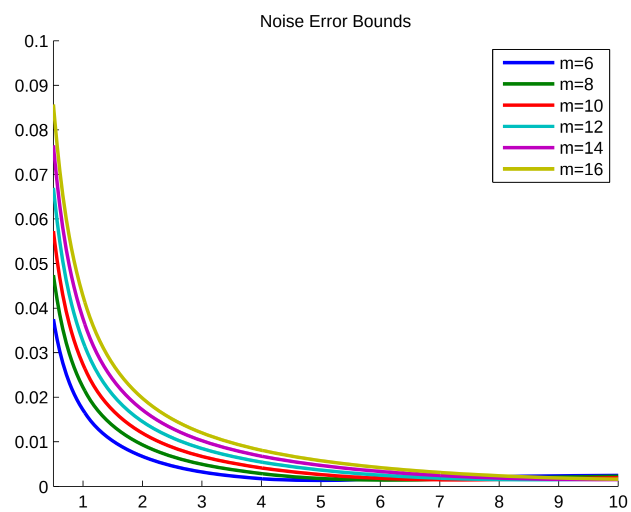


(b) Absolute numerical errors for the estimations of the 1.5^{th} order derivative of y.

Figure 3: Absolute estimation errors obtained by the proposed differentiator and the Grünwald-Letnikov scheme in discrete noise-free case (numerical errors due to the trapezoidal numerical integration method).

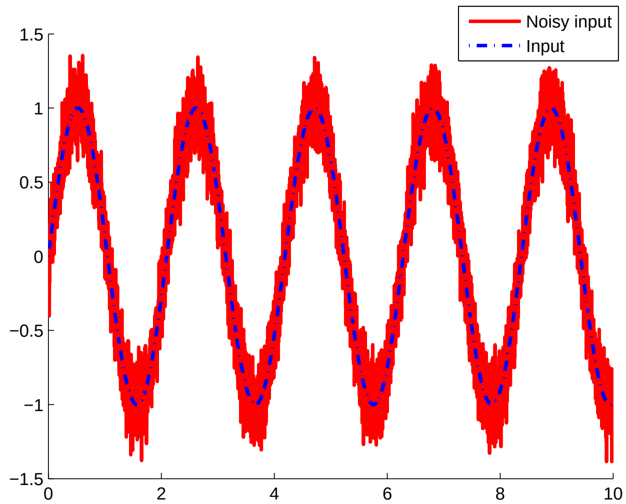


(a) Original output y and the discrete noisy output.

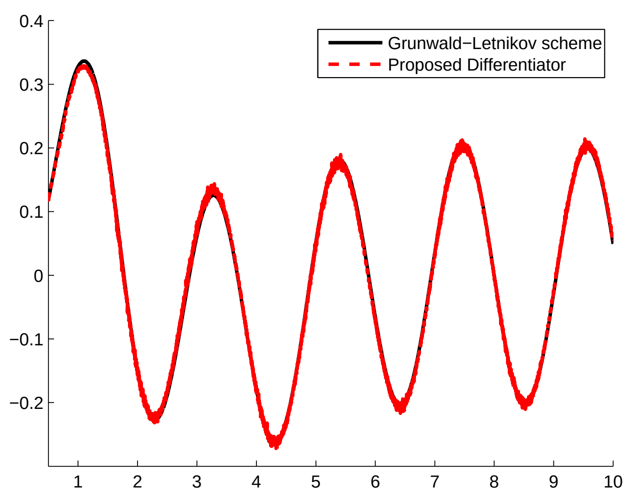


(b) Variation of the noise error bound given in (58) with $\alpha = 0.5$ and $m = 6, 8, \ldots, 16$.

Figure 4: Example 2: Original output of the linear system defined in (65), and the discrete noisy output generated with a model error and corrupted by a zero-mean white Gaussian noise with SNR $= 15\,$dB.

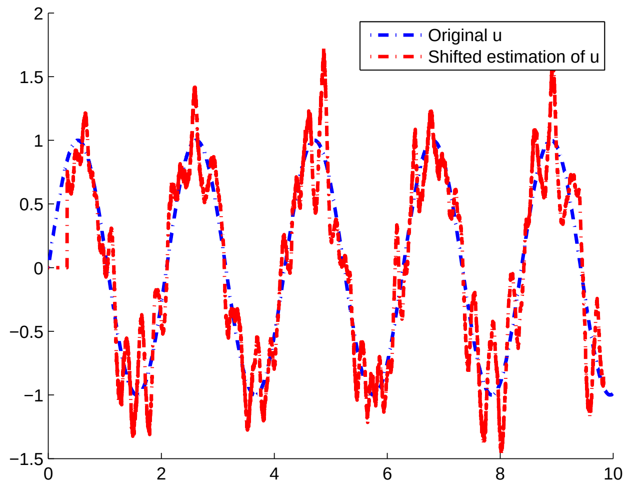


(a) Discrete input and its noisy observation.

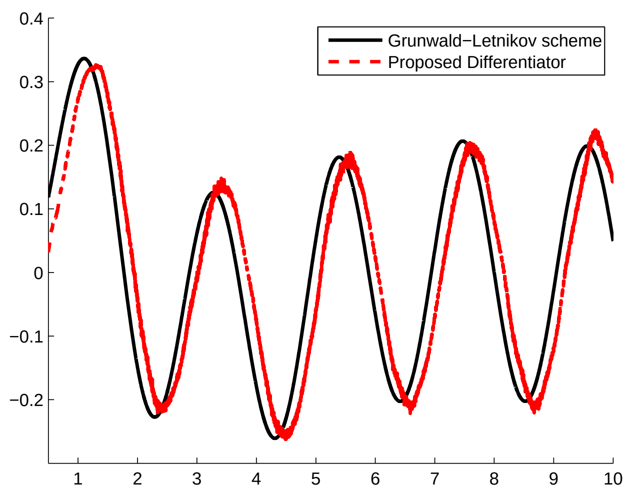


(b) Estimations of the 0.5^{th} order derivative of y.

Figure 5: Estimation of the Riemann-Liouville derivatives of the output y obtained by the proposed differentiator in the discrete noisy case with a noisy input, and the one obtained by the Grünwald-Letnikov scheme in the discrete noise-free case.



(a) Discrete input and its shifted estimation.



(b) Estimations of the 0.5^{th} order derivative of y.

Figure 6: Estimation of the Riemann-Liouville derivatives of the output y obtained by the proposed differentiator in the discrete noisy case with an unknown input, and the one obtained by the Grünwald-Letnikov scheme in the discrete noise-free case.