

Global Games with Noisy Sharing of Information

Behrouz Touri and Jeff Shamma

Abstract—We provide a framework for the study of global games with noisy sharing of information. In contrast to the previous works where it is shown that an intuitive threshold policy is an equilibrium for such games, we show that noisy sharing of information leads to non-existence of such an equilibrium. We also investigate the group best-response dynamics of two groups of agents sharing the same information to threshold policies based on each group’s observation and show the convergence of such dynamics.

I. INTRODUCTION

Global games are examples of games with incomplete information that were used to model pricing debt, currency crises, and mathematical modeling of political opposition [5], [6], [3], [4]. In all these applications, each agent observes an economical or social fundamental through a noisy channel and their decision affect the payoff of all players.

Global games were originally introduced in [1] where two agents observe noisy variants of an underlying signal θ and they decide to take a binary action. Both the unknown parameter θ and the decision of the other player influence the payoff of each agent. Therefore, the decision of each player depends on what the other player think she will do which depends what each player thinks the other player thinks what she thinks, and so on so forth. Therefore, such a decision making scenario can be arbitrarily complex and intricate. Since the original work of [1], many works have studied global games and their applications in the study of socio-economical behaviors [5], [6], [3], [4]. The survey paper [7] contains an extensive overview of the results established in this regard. Recently, [2] proposed and studied global games with perfect sharing of information among agents.

This work is supported in part by ARO-MURI W911NF-12-1-0509.

Behrouz Touri is with the Department of Electrical, Computer, and Energy Engineering, University of Colorado, Boulder, email: behrouz.touri@coloradu.edu. Jeff Shamma is with the School of Electrical and Computer Engineering, Georgia Institute of Technology and King Abdullah University of Science and Technology (KAUST), email: shamma@gatech.edu, jeff.shamma@kaust.edu.sa.

An interesting feature of many variants of the global games is that often there exists an *intuitive* equilibrium for such games. By an intuitive equilibrium, we mean a threshold policy based on the expected value of the underlying signal (fundamental) given each agent’s observations. This work is an attempt to understand global games under more complex information structure and existence of intuitive equilibrium as well as robustness of the existing results in terms of information available to each agent. We indeed show that such results are not robust to sharing of information: if an agent shares her information, through a noisy channel, no intuitive equilibrium will exist. We also investigate the case of perfect sharing of information among two groups of agents and study the group best-response dynamics for those games.

The structure of this paper is as follows: in Section II, we review the basic setting of global games and in Section III, we propose the extension of this setting under arbitrary information structure. In Section IV, we prove non-existence of linear threshold equilibria for global games with noisy sharing of information and in particular, an intuitive equilibrium. Then, we study global games between two groups of agents and group best-response dynamics in the simple case that agents within each group share the same observation and show the convergence of such best-response dynamics to a unique threshold policy. We conclude our discussion in Section VI.

Notation: We denote the set of real numbers by \mathbb{R} . We use 1_S for the indicator function on a set S , i.e. $1_S(\alpha) = 1$ if $\alpha \in S$ and $1_S(\alpha) = 0$ for $\alpha \notin S$. We denote the Gaussian distribution with mean μ and variance σ^2 by $\mathcal{N}(\mu, \sigma^2)$.

II. GLOBAL GAMES

In this section, we present the framework of our study. In the simple case of global games, we have n agents and each of the agents decide to either take the risky action ($a_i = 1$) or the safe action ($a_i = 0$). The payoff of an agent taking the safe action is zero and the payoff of an agent taking the risky action is $\sum_{i=1}^n a_i - \theta$ for some θ . One can think of θ as

the strength of a political regime and $\sum_{i=1}^n a_i$ as the number of people taking the risky action against the regime. One challenging aspect of analyzing global games is that the agents do not observe θ directly and they observe a noisy version of θ . In that regard, a standard assumption is that agent $i \in [n]$ observes $x_i = \theta + \xi_i$ where $\{\xi_1, \dots, \xi_n\}$ are independently and identically distributed $\mathcal{N}(0, \sigma^2)$ Gaussian random variables and θ has some distribution. Although it is rather a non-standard assumption, a common assumption in this context is that θ is uniformly distributed over \mathbb{R} . See [7] and the references therein for discussions on that. It is shown that for the case of θ with the uniform prior, there is a symmetric threshold policy on x_i s which is a Bayesian Nash equilibrium for these games. In other words, there exists a threshold value $t \in \mathbb{R}$ such that for $x_i \leq t$, agent i chooses to take risky action ($a_i = 1$) and for $x_i > t$, she takes the safe action and such an action profile leads to an equilibrium. Such a result is intuitively clear: the higher the value x_i is, the higher value might θ have and hence, the riskier taking the risky action is. Here, an important fact is that $x_i = \mathbb{E}[\theta | x_i]$ which means that in such equilibrium each agent should compare her *expected strength of regime given her observation* to a threshold and take a proper action accordingly. Such an intuitive result is extended to other other cases of global games. For example, it is shown that if θ is a $\mathcal{N}(y, \tau^2)$ Gaussian and y is the public knowledge, still there is an equilibrium with threshold policy on $\mathbb{E}[\theta | x_i]$ for such games [7].

III. GLOBAL GAMES WITH INFORMATION STRUCTURE

This work is motivated by understanding the role of information sharing in global games and the structure of emerging equilibria. Here, we discuss the setting under which we study global games with arbitrary information available to each agent. Let (Ω, \mathcal{F}, P) be a probability space. Let $\theta : \Omega \rightarrow \mathbb{R}$ be a measurable function, perhaps with an improper uniform distribution on \mathbb{R} (i.e. unsigned measure with infinite mass). We assume that agent i has access to some signals related to θ which is represented by a random vector $\mathbf{s}_i : \Omega \rightarrow \mathbb{R}^{d_i}$, where d_i is a positive integer. We refer to \mathbf{s}_i as the *private information* or simply *information* of agent i ¹. We refer to a measurable

¹A more general formulation of global games with information structure is to assume that agent i 's information belongs to a σ -algebra $\mathcal{F}_i \subseteq \mathcal{F}$. Our formulation here is perhaps more intuitive and serves well for the purpose of this study.

function $A_i : \mathbb{R}^{d_i} \rightarrow \{0, 1\}$ which maps agent i 's private information to an action as a *strategy*. For any $\omega \in \Omega$, and a given strategy profile $\mathbf{A} = (A_1, \dots, A_n)$ of the n players, let the payoff of agent i be:

$$u_i(\mathbf{A})(\omega) = \left(\sum_{j=1}^n A_j(\mathbf{s}_j(\omega)) - \theta(\omega) \right) A_i(\mathbf{s}_i(\omega)).$$

Note that for an outcome ω , $\sum_{j=1}^n A_j(\mathbf{s}_j(\omega))$ is the number of agents taking the risky action, an agent taking the safe action ($A_i(\mathbf{s}_i(\omega)) = 0$) receives payoff zero, and an agent taking the risky action receives the payoff $A_j(\mathbf{s}_j(\omega)) - \theta(\omega)$.

A. Structure of an Equilibrium

In the subsequent discussions, we are interested in the Bayesian Nash equilibria of global games with an information structure. To introduce this concept, let $\mathbf{A} = (A_1, \dots, A_n)$ be the strategy of n agents and let \mathbf{A}^{-i} be the vector which contains the strategies of all agents except the i th agent's strategy. The best-response of agent i to the strategy \mathbf{A}^{-i} of other players conditioned on her information \mathbf{s}_i is:

$$BR(\mathbf{A}^{-i}) = \begin{cases} 1 & \text{if } 1 + \mathbb{E} \left[\sum_{j \neq i}^n A_j \mid \mathbf{s}_i \right] \geq \mathbb{E}[\theta \mid \mathbf{s}_i] \\ 0 & \text{if } 1 + \mathbb{E} \left[\sum_{j \neq i}^n A_j \mid \mathbf{s}_i \right] < \mathbb{E}[\theta \mid \mathbf{s}_i] \end{cases} \quad (1)$$

We say that $\mathbf{A} = (A_1, \dots, A_n)$ is a Bayesian Nash equilibrium, or simply an equilibrium, if $A_i = BR(\mathbf{A}^{-i})$ for $i = 1, \dots, n$. Throughout the present work, many of the properties and characterizations, such as the equilibrium characterization in Eq. (1) can hold “almost surely” (or with probability one) but for the sake of simplicity and conciseness, we avoid using the term almost surely throughout the following discussion.

The main purpose of this study is to investigate the structure of equilibrium in global games, especially existence of a *threshold policy* equilibrium, i.e. an equilibrium of the form $(1_{f_1(\mathbf{s}_1) \leq t_1}, \dots, 1_{f_n(\mathbf{s}_n) \leq t_n})$ for functions $f_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}$ and threshold values $t_1, \dots, t_n \in \mathbb{R}$. We say that a threshold equilibrium is linear if $f_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}$ is a linear functional for all $i \in \{1, \dots, n\}$. The intuitive meaning of a threshold policy is clear: each agent evaluates the simple function f_i of her observations and compares it to a threshold value if it is less than a threshold, she takes the risky action, and if it is above a threshold value, she takes the safe action.

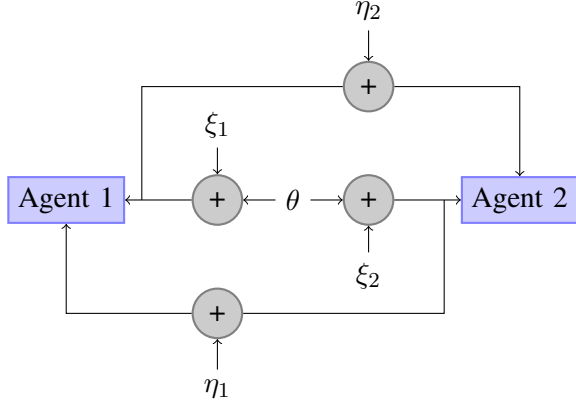


Fig. 1. Global Games with Noisy Sharing of Information

IV. NON-EXISTENCE OF THRESHOLD POLICIES FOR CORRELATED SIGNALS

As discussed in Section II, for several cases of global games, there exists a threshold equilibrium based on individual agent's expected value of θ given her private information. In this section, we show that such an intuitive strategy does not lead to an equilibrium if agents share their information.

Let us discuss the setting and the structure of information sharing for this game. Suppose that we have two agents and assume that θ has a uniform distribution over \mathbb{R} . Suppose that agent i observes $x_i = \theta + \xi_i$ for $i = 1, 2$, where ξ_1, ξ_2 are independent $\mathcal{N}(0, \sigma^2)$ Gaussian random variables. In addition, suppose that agent 1 also observes a noisy variation of agent 2's private information x_2 , i.e. $y_1 = x_2 + \eta_1$ where η_1 is a $\mathcal{N}(0, \tau^2)$ Gaussian random variable and similarly, agent 2 also receives $y_2 = x_1 + \eta_2$ with η_2 being a $\mathcal{N}(0, \tau^2)$ Gaussian random variable. We assume $\theta, \xi_1, \xi_2, \eta_1, \eta_2$ are independent. See Fig. 1 for illustration of the information available to each agent. In this case, the information (private signal) of agent i is the vector (x_i, y_i) . For simplicity, we refer to this game as global games with noisy sharing of information. Let $\theta(x, y) = \mathbb{E}[\theta \mid x_i = x, y_i = y]$ be the expected value of θ given agent i 's observation (x, y) .

The main result of this section is that there is no threshold-policy equilibrium based on $\mathbb{E}[\theta \mid x, y] = \bar{\theta}(x, y)$.

Theorem 1: For $\tau^2 \in (0, \infty)$, there is no equilibrium with threshold policy on the expected value of θ given each agent's information, i.e. there do not exist threshold values $t_1, t_2 \in \mathbb{R}$ such that $(1_{\bar{\theta}(x_1, y_1) \leq t_1}, 1_{\bar{\theta}(x_2, y_2) \leq t_2})$ is an equilibrium for global games with noisy sharing of information.

To prove this, we make use of the following properties which follow from the properties of multivariate Gaussian random vectors and their conditional expectations (see Appendix for the proof).

Lemma 1: Let $\theta, x_1, x_2, y_1, y_2$ be as defined above and let $\bar{\theta}(x, y) = \frac{(\sigma^2 + \tau^2)x + \sigma^2 y}{2\sigma^2 + \tau^2}$. Then,

a. We have

$$\bar{\theta}(x, y) = \mathbb{E}[\theta \mid x_1 = x, y_1 = y].$$

b. (x_2, y_2) conditioned on $(x_1 = x, y_1 = y)$ is a Gaussian random vector with mean:

$$\mu = \begin{pmatrix} \bar{\theta}(x, y) + \frac{\sigma^2(y - \bar{\theta}(x, y))}{\sigma^2 + \tau^2} \\ x \end{pmatrix},$$

and covariance

$$\Sigma^2 = \begin{pmatrix} \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2} & 0 \\ 0 & \tau^2 \end{pmatrix}.$$

c. $\theta(x_2, y_2)$ conditioned on $(x_1 = x, y_1 = y)$ is a Gaussian random variable with mean

$$\begin{aligned} & \mathbb{E}[\bar{\theta}(x_2, y_2) \mid x_1 = x, y_1 = y] \\ &= \frac{\tau^2}{2\sigma^2 + \tau^2} \bar{\theta}(x, y) + \frac{\sigma^2}{2\sigma^2 + \tau^2} (x + y). \end{aligned} \quad (2)$$

Using the above properties we can prove Theorem 1:

Proof: Suppose that $(1_{\bar{\theta}(x_1, y_1) \leq t_1}, 1_{\bar{\theta}(x_2, y_2) \leq t_2})$ is an equilibrium for some $t_1, t_2 \in \mathbb{R}$. By Lemma 1-c, $\bar{\theta}(x_2, y_2)$ is a Gaussian random variable with mean (2). Therefore, one can find $x, y \in \mathbb{R}$ such that $\bar{\theta}(x, y) \leq t_1$ and $\mathbb{E}[\bar{\theta}(x_2, y_2) \mid x_1 = x, y_1 = y]$ is an arbitrarily large value. But the variance of $\theta(x_2, y_2)$ given x_1, y_1 is a constant function of σ^2 and τ^2 . Therefore, using Chebyshev's inequality, for the given threshold values t_1, t_2 and an arbitrary $\epsilon > 0$, one can find x, y such that $\bar{\theta}(x, y) \leq t_1$ and

$$P(\theta(x_2, y_2) \leq t_2 \mid x_1 = x, y_1 = y) \leq \epsilon.$$

But by the structure of equilibrium (1), we should have

$$\begin{aligned} \epsilon + 1 &\geq P(\theta(x_2, y_2) \leq t_2 \mid x_1 = x, y_1 = y) + 1 \\ &\geq t_1 = \bar{\theta}(x, y) = \mathbb{E}[\theta \mid x_1 = x, y_1 = y]. \end{aligned}$$

But this inequality holds for for any $\epsilon > 0$ and hence, $t_1 \leq 1$. Note that for any t_1, t , the set

$$\begin{aligned} E_{t_1, t} &= \{(x, y) \mid \bar{\theta}(x, y) \leq t_1, \\ & \mathbb{E}[\bar{\theta}(x_2, y_2) \mid x_1 = x, y_1 = y] - \bar{\theta}(x, y) \geq t\}, \end{aligned}$$

has a non-zero Lebesgue measure and hence, $P((x_1, y_1) \in E_{t_1, t}) > 0$.

On the other hand, using the same argument for any $\epsilon > 0$, one can find x, y such that $\bar{\theta}(x, y) = t_1 + 1 > t_1$ and

$$\mathbb{E}[\bar{\theta}(x_2, y_2) \mid x_1 = x, y_1 = y],$$

is arbitrary small and hence,

$$\begin{aligned} t_1 + \epsilon &\geq \bar{\theta}(x_1, y_1) \\ &> P(\theta(x_2, y_2) \leq s_2 \mid x_1 = x, y_1 = y) + 1 \\ &\geq 1 + (1 - \epsilon) = 2 - \epsilon, \end{aligned}$$

for any $t_2 \in \mathbb{R}$. Since, this holds for any $\epsilon > 0$, it follows that $t_1 \geq 2$ which contradicts $t_1 \leq 1$. Therefore, such a threshold equilibrium does not exist. ■

Note that $\bar{\theta}(x, y)$ is a linear function of x, y . In fact, using Theorem 1, we can generalize this result and show that no linear threshold equilibrium exists for those games.

Theorem 2: Global games with noisy sharing of information do not admit any linear threshold equilibrium.

Proof: Suppose that a global game with noisy sharing of information admits a linear threshold equilibrium $L_1(x_1, y_1) = \alpha_1 x_1 + \beta_1 y_1$ and $L_2(x_2, y_2) = \alpha_2 x_2 + \beta_2 y_2$ with the threshold values $s_1, s_2 \in \mathbb{R}$, respectively. Based on the characterization of an equilibrium in global games with information structure (Eq. (1)), for any x, y such that $L_1(x, y) \leq s_1$, we should have:

$$P(L_2(x_2, y_2) \leq t_2 \mid x, y) \geq \bar{\theta}(x, y) - 1. \quad (3)$$

This implies that $L_1(x, y) = \lambda \bar{\theta}(x, y)$ for some $\lambda \in \mathbb{R}$, otherwise, we can find scalars $x, y \in \mathbb{R}$ such that $L_1(x, y) \leq t_1$ but $\bar{\theta}(x_1, y_1) - 1 \geq 2$ which contradicts (3). Similarly, it follows that $L_2(x, y) = \lambda_2 \bar{\theta}(x, y)$ for some $\lambda_2 \in \mathbb{R}$. But any threshold policy on $\lambda \bar{\theta}$ translates to a threshold policy on $\bar{\theta}$ and by Theorem 1, such a threshold equilibrium does not exist. ■

V. GLOBAL GAMES BETWEEN GROUPS OF AGENTS

Suppose that we have two separated groups of agents S_1, S_2 . Agents in the same group, observe the same signal. In this section, we investigate the structure of threshold policies for this information structure. First, we show that at equilibrium, agents belonging to one group should follow the same strategy. In fact, we show a more general result: in global games with arbitrary information sharing scheme, if two agents have the same information, they should follow the same strategy at an equilibrium. Based on this, we study the group best-response dynamics of each group to the other group's threshold policy.

A. Agents with Same Information

One natural question in the context of global games with information structure is whether same information leads to different strategies and actions or equivalently, can we have contradicting decisions between two agents with similar information in this setting? The answer is negative as shown below.

Lemma 2: Let $\mathbf{s}_i = \mathbf{s}_{i'}$ for some $i \neq i'$ and assume that the strategy profile \mathbf{A} is an equilibrium for a global game with information structure (as described in Section II). Then $A_i = A_{i'}$.

Proof: Suppose that on the set \mathcal{E} , we have $A_i(\mathbf{s}_i(\omega)) = 1$. Then, the other agent's action on this event is definitely less than or equal to 1, i.e. $A_{i'}(\mathbf{s}_{i'} = A_{i'}(\mathbf{s}_i(\omega)) \leq A_i(\mathbf{s}_i(\omega)) = 1$ on \mathcal{E} . Therefore, using (1), on the set \mathcal{E} , we have

$$\begin{aligned} \mathbb{E}[\theta \mid \mathbf{s}_{i'}] &= \mathbb{E}[\theta \mid \mathbf{s}_i] \leq 1 + \mathbb{E}\left[\sum_{j \neq i}^n A_j \mid \mathbf{s}_i\right] \\ &= 1 + A_{i'}(\mathbf{s}_{i'}) + \mathbb{E}\left[\sum_{j \neq i, i'}^n A_j \mid \mathbf{s}_i\right] \\ &\leq 1 + A_i(\mathbf{s}_i) + \mathbb{E}\left[\sum_{j \neq i, i'}^n A_j \mid \mathbf{s}_{i'}\right] \\ &= 1 + \mathbb{E}\left[\sum_{j \neq i'}^n A_j \mid \mathbf{s}_{i'}\right]. \end{aligned}$$

And hence, $A_{i'}(\mathbf{s}_{i'}) = 1$. The same argument holds for the event \mathcal{E}' where $A_{i'}(\mathbf{s}_{i'}) = 1$ and hence, $A_i(\mathbf{s}_i) = A_{i'}(\mathbf{s}_{i'})$. ■

B. Equilibrium Structure in Groups

Consider two groups $S_1 = \{1, \dots, m\}$ and $S_2 = \{m+1, \dots, m+n\}$ of m and n agents, respectively, such that every agent in the same group observes the same signal. More precisely, let $x_\ell = \theta + \xi_\ell$ for $\ell = 1, 2$, where θ is uniformly distributed on \mathbb{R} and ξ_1, ξ_2 are independent $\mathcal{N}(0, \sigma^2)$ Gaussian random variables. We assume that $\mathbf{s}_i = x_\ell$ for all $i \in S_\ell$ and $\ell = 1, 2$. As a result of Lemma 2, in an equilibrium, the agents in the same group should share the same strategy. Motivated by this, in the following discussion, we investigate the group best-response dynamics to the other group's threshold policy. In other words, we study the question of what happens if agents in the same group follow the collective best-response to the agents in the other group *assuming* that all the agents

would commit to such a policy. More precisely, let

$$([A_1]_m, [A_2]_n) = (\underbrace{A_1, \dots, A_1}_{m \text{ times}}, \underbrace{A_2, \dots, A_2}_{n \text{ times}}).$$

We say that (A'_1, A'_2) is a group best-response to (A_1, A_2) if $A'_1 = BR((([A_1]_m, [A_2]_n)^{-i})$ for all $i \in \{1, \dots, m\}$ and $A'_2 = BR((([A_1]_m, [A_2]_n)^{-j})$ for all $j \in \{m+1, \dots, m+n\}$. In this case, A'_1 is the strategy that if all the agents in S_1 follow it, it would be the best-response to the agents in the other group when all of them are using A_2 . Based on this, we say that $\{(A_1[k], A_2[k])\}$ is the group best-response dynamics if $(A_1[k], A_2[k])$ is the group best-response to $(A_1[k-1], A_2[k-1])$ for $k > 0$.

Theorem 3: Let $A_1[0] = 1_{x_1 \leq t_1[0]}$ and $A_2[0] = 1_{x_2 \leq t_2[0]}$ for some $t_1[0], t_2[0] \in \mathbb{R}$. Then for the group best-response dynamics $\{(A_1[k], A_2[k])\}$:

- a. There exists a unique threshold $t_\ell[k]$ for $\ell = 1, 2$ and $k > 0$ such that $A_\ell[k] = 1_{x_\ell \leq t_\ell[k]}$ and also

$$\begin{aligned} n\Phi\left(\frac{t_2[k-1] - t_1[k]}{\sqrt{2}\sigma}\right) + m &= t_1[k] \\ m\Phi\left(\frac{t_1[k-1] - t_2[k]}{\sqrt{2}\sigma}\right) + n &= t_2[k], \end{aligned} \quad (4)$$

where Φ is the cumulative distribution function of the normal distribution.

- b. Eq. (4) has a unique fixed point t_1^*, t_2^* satisfying

$$\begin{aligned} (m+n)\Phi\left(\frac{t_2^* - t_1^*}{\sqrt{2}\sigma}\right) &= n - (t_2^* - t_1^*) \\ t_1^* &= n\Phi\left(\frac{t_2^* - t_1^*}{\sqrt{2}\sigma}\right) + m. \end{aligned}$$

- c. For $\ell = 1, 2$ and $k \geq 2$, we have

$$|t_\ell[k] - t_\ell^*| \leq \frac{1}{(1 + \frac{2\sqrt{\pi}\sigma}{m})(1 + \frac{2\sqrt{\pi}\sigma}{n})} |t_\ell[k-2] - t_\ell^*|,$$

and hence, $t_\ell[k]$ converges to t_ℓ^* .

Proof:

- a. Suppose that the strategy of all agents $i = 1, \dots, m$ in S_1 is $1_{x_1 \leq t_1}$. Then, if agents in S_2 observe $x_2 = x$, the payoff of the risky action would be:

$$\begin{aligned} mP(x_1 \leq t_1 \mid x_2 = x) + n - \mathbb{E}[\theta \mid x_2 = x] \\ = mP(x_1 \leq t_1 \mid x_2 = x) + n - x. \end{aligned}$$

But x_1 conditioned on $x_2 = x$ is a Gaussian with mean x and variance $2\sigma^2$. Therefore, agents in S_2 decide to take risky action if $m\Phi(\frac{t_1-x}{\sqrt{2}\sigma}) + n - x \geq 0$. The function $m\Phi(\frac{t_1-x}{\sqrt{2}\sigma}) + n - x$ is a strictly

decreasing function of β (ranging from $+\infty$ to $-\infty$) and hence, has a unique root $\beta(t_1)$ satisfying:

$$m\Phi\left(\frac{t_1 - \beta(t_1)}{\sqrt{2}\sigma}\right) + n = \beta(t_1). \quad (5)$$

As a result, for $x_2 \leq \beta(t_1)$ taking the risky action is beneficial to agents in S_2 and for $x_2 > \beta(t_1)$ taking risky action is not beneficial. Similarly, for the group best-response to the strategy $1_{x_2 \leq t_2}$, the best-response is the unique strategy $1_{x_1 \leq \alpha(t_2)}$ satisfying:

$$n\Phi\left(\frac{t_2 - \alpha(t_2)}{\sqrt{2}\sigma}\right) + m = \alpha(t_2). \quad (6)$$

- b. Let d be the unique solution of the following equation (which exists and is unique because of monotonicity):

$$(m+n)\Phi\left(\frac{d}{\sqrt{2}\sigma}\right) = n - d. \quad (7)$$

It is not hard to check that (t_1^*, t_2^*) defined by $t_1^* = n\Phi(\frac{d}{\sqrt{2}\sigma}) + m$ and $t_2^* = d + t_1^*$ is a fixed point of (4). On the other hand, subtracting (6) and (5) implies that for any fixed point (t_1^*, t_2^*) , $d = t_2^* - t_1^*$ should satisfy (7) and hence, such a fixed point is unique.

- c. Let $\mu(s) = \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{s^2}{4\sigma^2}}$ be the probability density function of a $\mathcal{N}(0, 2\sigma^2)$ Gaussian random variable. If we subtract both sides of (4) from the corresponding fixed point equations, we have

$$\begin{aligned} n \int_{t_2^* - t_1^*}^{t_2[k-1] - t_1[k]} \mu(s) ds &= (t_1[k] - t_1^*), \\ m \int_{t_1^* - t_2^*}^{t_1[k-1] - t_2[k]} \mu(s) ds &= (t_2[k] - t_2^*). \end{aligned}$$

By the mean-value theorem for the integrals, for some $\lambda_1[k]$ between $t_2[k-1] - t_1[k]$ and $t_2^* - t_1^*$ and some $\lambda_2[k]$ between $t_1^* - t_2^*$ and $t_1[k-1] - t_2[k]$, we have:

$$\begin{aligned} n((t_2[k-1] - t_1[k]) - (t_2^* - t_1^*)) \\ \times \mu(\lambda_1[k]) &= (t_1[k] - t_1^*) \\ m((t_2[k] - t_1[k-1]) - (t_1^* - t_2^*)) \\ \times \mu(\lambda_2[k]) &= (t_2[k] - t_2^*) \end{aligned}$$

and hence,

$$\begin{aligned} (t_1[k] - t_1^*) &= \frac{n\mu(\lambda_1[k])}{1 + n\mu(\lambda_1[k])} (t_2[k-1] - t_2^*) \\ (t_2[k] - t_2^*) &= \frac{m\mu(\lambda_2[k])}{1 + m\mu(\lambda_2[k])} (t_1[k-1] - t_1^*). \end{aligned}$$

Using the same equality for the best-response strategies at time $k - 1$ and $k - 2$, we have:

$$\begin{aligned} (t_1[k] - t_1^*) &= \frac{n\mu(\lambda_1[k])}{1 + n\mu(\lambda_1[k])} \frac{m\mu(\lambda_2[k-1])}{1 + m\mu(\lambda_2[k-1])} (t_1[k-2] - t_1^*) \\ (t_2[k] - t_2^*) &= \frac{m\mu(\lambda_2[k])}{1 + m\mu(\lambda_2[k])} \frac{n\mu(\lambda_1[k-1])}{1 + n\mu(\lambda_1[k-1])} (t_2[k-2] - t_2^*). \end{aligned}$$

As a result, for $\ell = 1, 2$, we have

$$|t_\ell[k] - t_\ell^*| \leq \frac{1}{(1 + \frac{2\sqrt{\pi}\sigma}{m})(1 + \frac{2\sqrt{\pi}\sigma}{n})} |t_\ell[k-2] - t_\ell^*|,$$

and hence, $\lim_{k \rightarrow \infty} t_\ell[k] = t_\ell^*$. ■

By the characterization of the fixed point (t_1^*, t_2^*) in Theorem 3-b, the larger group has a higher threshold at equilibrium (i.e. if $n > m$, we have $t_2^* > t_1^*$) which means that for the smaller group, the group best-response dynamics converges to a more conservative strategy.

VI. CONCLUSION AND SUGGESTIONS FOR FURTHER STUDIES

In this work, we investigated the role of information sharing in global games. We showed non-existence of threshold policies on the expected value of the economic fundamental conditioned on agents' information when agents share their information (through a noisy channel). We also studied an extreme case of sharing information when agents in two groups play global games and agents in the same group share the same information. We showed the uniqueness of threshold policies in this case and convergence of the group best-response dynamics.

For the global games with noisy sharing of information, finding a threshold strategy and how it is shaped by the sharing of information is left as an open problem for future studies.

VII. APPENDIX

Here, we briefly discuss the derivation of the properties in Lemma 1. To simplify notation, we use $P(h)$ or $E[h]$ instead of $P(H = h)$ or $E[H = h]$ for some random variable (vector) H and a realization h . For the conditional probability of (x_2, y_2) given (x_1, y_1) under uniform distribution assumption for θ , we have:

$$\begin{aligned} P(\theta | x, y) &= \frac{P(x, y | \theta)P(\theta)}{P(x, y)} \\ &= \frac{P(x, y | \theta)P(\theta)}{\int_{-\infty}^{\infty} P(x, y | \lambda)P(\lambda)d\lambda} = \frac{P(x, y | \theta)}{\int_{-\infty}^{\infty} P(x, y | \lambda)d\lambda}. \end{aligned}$$

$$\text{But } P(x, y | \lambda) = \frac{1}{2\pi\sqrt{\sigma^2(\sigma^2 + \tau^2)}} e^{-\frac{(x-\lambda)^2}{2\sigma^2}} e^{-\frac{(y-\lambda)^2}{2(\sigma^2 + \tau^2)}}.$$

Using the properties of exponential integrals and exponential integral tables, it follows that:

$$\int_{-\infty}^{\infty} P(x, y | \lambda)d\lambda = \frac{1}{\sqrt{2\pi(2\sigma^2 + \tau^2)}} e^{-\frac{(x-y)^2}{2(2\sigma^2 + \tau^2)}}.$$

$$\text{Therefore, } E[\theta | x, y] = \frac{(\sigma^2 + \tau^2)x + \sigma^2 y}{2\sigma^2 + \tau^2}.$$

For computing $E[\bar{\theta}(x_2, y_2) | x_1 = x, y_1 = y]$, first let us derive the conditional distribution of x_2, y_2 given x_1, y_1 . For a given θ , (x_1, y_1, x_2, y_2) is a Gaussian random vector with mean μ' and covariance matrix Σ' :

$$\mu' = \begin{pmatrix} \theta \\ \theta \\ \theta \\ \theta \end{pmatrix}, \Sigma' = \begin{pmatrix} \sigma^2 & 0 & 0 & \sigma^2 \\ 0 & \sigma^2 + \tau^2 & \sigma^2 & 0 \\ 0 & \sigma^2 & \sigma^2 & 0 \\ \sigma^2 & 0 & 0 & \sigma^2 + \tau^2 \end{pmatrix}.$$

Therefore, using the properties of multivariate Gaussian vectors (see e.g. [8]), x_2, y_2 given $x_1 = x, y_1 = y$ will be a Gaussian random vector with mean:

$$\mu = \begin{pmatrix} \bar{\theta}(x, y) + \frac{\sigma^2(y - \bar{\theta}(x, y))}{\sigma^2 + \tau^2} \\ x \end{pmatrix}, \sigma = \begin{pmatrix} \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2} & 0 \\ 0 & \tau^2 \end{pmatrix}.$$

As a result, we have:

$$\begin{aligned} E[\theta(x_2, y_2) | x_1 = x, y_1 = y] &= E\left[\frac{(\sigma^2 + \tau^2)x_2 + \sigma^2 y_2}{2\sigma^2 + \tau^2} | x, y\right] \\ &= \frac{\tau^2}{2\sigma^2 + \tau^2} \bar{\theta}(x, y) + \frac{\sigma^2}{2\sigma^2 + \tau^2} (x + y). \end{aligned}$$

REFERENCES

- [1] H. Carlsson and E. van Damme, *Global games and equilibrium selection*, *Econometrica* **61** (1993), 989–1018.
- [2] M. Dahleh, A. Tahbaz-Salehiz, J.N. Tsitsiklis, and S. Zoumpoulis, *Coordination with local information*, July 2013.
- [3] I. Goldstein, *Interdependent banking and currency crises in a model of self-fulfilling beliefs*, Ph.D. thesis, 2000, University of Tel Aviv.
- [4] S. Morris, *Contagion*, *Review of Economic Studies* **67** (2000), 57–78.
- [5] S. Morris and H. S. Shin, *Unique equilibrium in a model of self-fulfilling currency attacks*, *American Economic Review* **88** (1998), 587–597.
- [6] S. Morris and H. S. Shin, *Coordination risk and the price of debt*, (1999b), Available at <http://www.econ.yale.edu/smorris>.
- [7] ———, *Global games: theory and applications*, *Econometric Society Monographs* **35** (2003), 56–114.
- [8] C.E. Rasmussen and C.K.I. Williams, *Gaussian processes for machine learning*, *Adaptive computation and machine learning series*, University Press Group Limited, 2006.