Image denoising using the squared eigenfunctions of the Schrödinger operator

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Abstract

This study introduces a new image denoising method based on the spectral analysis of the semi-classical Schrödinger operator. The noisy image is considered as a potential of the Schrödinger operator, and the denoised image is reconstructed using the discrete spectrum of this operator. First results illustrating the performance of the proposed approach are presented and compared to the singular value decomposition method.

Index Terms – Image denoising, Schrödinger operator, discrete spectrum, squared eigenfunctions.

I. INTRODUCTION

Image denoising is very important for image interpretation and visualization. It is usually a preliminary step in image processing. With the growing production of digital images in various fields, image denoising is an active research topic where the challenge is often to remove the noise without losing pertinent information. Various methods for image denoising have been proposed in the literature [3], [8], [9], [13], [18], [20], [22]. Some of them are based on decomposing the image in a set of functions, which may form a basis or a frame, for instance the wavelet-based methods [8], [13]. It is known that the localization property of the basis or the frame motivates the use of these functions for denoising. In particular this property allows for preserving the edges which is very important in signal denoising applications [21]. Another type of methods is based on partial differential equation (PDE) where the intensity of the illumination on the edges is assumed to vary like a geometric heat flow [3], [18].

Despite the various existing denoising methods and algorithms, the desirable performance level is still not reached. Usually, good denoising performance is achieved but at the expense of loosing some of the image’s characteristics or structures. Then, the performance of a denoising method often depends on the nature of the image and the targeted application. For instance, it happens that good denoising performance are reached where the noise is reduced homogeneously except at the edges which are softened. Sometimes, additional assumptions on the image are required but not always justified.

In this paper, we are interested in a semi-classical signal analysis method (SCSA) which has been proposed in [11], [19] for signal representation, interpretation and analysis. This method has been recently extended to two dimension (2D) for image representation [15], [16]. The main idea of this approach consists in decomposing the image using a set of spatially shifted and localized functions [23]. These functions are given by the squared $L^2$-normalized eigenfunctions associated to the discrete spectrum of the 2D semi-classical Schrödinger operator whose potential is given by the studied image. To reduce the computation complexity and time of the eigenvalues and eigenfunctions of the 2D semi-classical Schrödinger operator, an efficient algorithm has been proposed in [15], [16]. This algorithm splits the 2D semi-classical Schrödinger operator into two one dimensional (1D) operators. Then, the discrete spectrum consisting of the tensor product [6], [14] of the squared $L^2$-normalized eigenfunctions of the 1D operators and negative eigenvalues can be used to decompose, reconstruct and analyze the image.

Motivated by the fact that the $L^2$-normalized eigenfunctions of the Schrödinger operator are localized [23], we propose in this paper to use the 2D SCSA method for image denoising. The main idea is to retain the most significant eigenfunctions and to discard the others which have a significant noise component. We also provide a comparison of the performance of the proposed method to the singular value decomposition method (SVD) [1], [7], [10], [26], which is also based on an eigenvalue decomposition.

The paper is organized as follows. In Section II the definition of the SCSA method in 2D is recalled along with the tensor product based algorithm of the SCSA approach. Then, in Section III, the denoising algorithm is introduced illustrated by some numerical results and followed by the comparison with the SVD method. Finally, a conclusion and some perspectives of our future work are given in Section IV.

II. IMAGE REPRESENTATION USING SQUARED $L^2$-NORMALIZED EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR

A. Definition

In this section, we recall the definition of the 2D SCSA method [15], [16]. Let us consider the following 2D semi-classical Schrödinger operator associated to a potential $V_z$:

$$
\mathcal{H}_0(V_z) = -\frac{h^2}{2}\Delta - V_z,
$$

(1)

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where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2D Laplacian operator, $h \in \mathbb{R}^+_+$ is the semi-classical parameter [4], and $V_2$ is a positive real valued function belonging to $C^\infty(\Omega_2)$ where $\Omega_2 = [a, b] \times [c, d]$ is a bounded open set of $\mathbb{R}^2$.

Inspired form semi-classical properties of the semi-classical Schrödinger operator [12], [17], and the 1D SCSA formula [11], the extension of the SCSA formula in 2D case is given by the following theorem [15], [16].

**Theorem 2.1:** [15], [16] Let $V_2$ be a positive real valued $C^\infty$ function on $\Omega_2$ considered as potential of the Schrödinger operator (1). Then, for any pair $(\Lambda_2, \lambda)$ such that $\Lambda_2 \subset \Omega_2$ is compact and

\[
\left\{ \begin{array}{l}
\lambda < \inf(-V_2(a, c), -V_2(b, d)), \\
V_2(\Lambda_2) \subset (-\lambda, +\infty), \\
\lambda \text{ is not a critical value of } -V_2,
\end{array} \right.
\]

and, uniformly for $(x, y) \in \Lambda_2$, we have

\[
V_2(x, y) = -\lambda + \lim_{h \to 0} \left( \frac{\hbar^2}{L_{2, \gamma}^k} \sum_{k=1}^{K_h^k} (\lambda - \mu_{k,h})^\gamma \psi_{k,h}^2(x, y) \right)^{\frac{1}{\gamma+1}},
\]

where $\gamma \in \mathbb{R}^+_+$, and $L_{2, \gamma}^k$ is the suitable universal semi-classical constant given by:

\[
L_{2, \gamma}^k = \frac{1}{2\pi \Gamma(\gamma + 1)}
\]

$\Gamma$ refers to the standard Gamma function.

Moreover, $\mu_{k,h}$ and $\psi_{k,h}$ denote the negative eigenvalues with $\mu_{1,h} < \cdots < \mu_{K_h^h,h} < \lambda$, $K_h^h$ is a finite number of negative eigenvalues smaller than $\lambda$, and associated $L^2$-normalized eigenvectors of the operator $\mathcal{H}_h(V_2)$ such that:

\[
\mathcal{H}_h(V_2) \psi_{k,h} = \mu_{k,h} \psi_{k,h}, \quad k = 1, \ldots, K_h^h.
\]

The proof of this Theorem is obtained using a generalization to the 2D case of Theorem 4.1 proposed by Helffer and Lalleg in [11], which uses an extension of Karadzhov’s theorem on the spectral function [17], and the connection of the Riesz means with Leib-Thirring conjecture proposed by Helffer and Robert in [12], (see [15], [16] for more details).

**B. Tensor product based algorithm**

The detailed description of the considered image representation algorithm can be found in [15], [16]. In this section, we denote $I$ an image on space of square matrices $\mathcal{M}_{N \times N}(\mathbb{R}^+)$.

The discretization of the eigenvalue problem (5) is given by the following eigenvalue problem for matrix:

\[
\mathcal{H}_h(I) \psi_{k,h}[i,j] = \mu_{k,h} \psi_{k,h}[i,j],
\]

where $\mu_{k,h}$ and $\psi_{k,h}$, for $k = 1, \ldots, K_h^h$ with $K_h^h < N \times N$, refer to the negative eigenvalues with $\mu_{1,h} < \cdots < \mu_{K_h^h,h} < \lambda$ and associated $L^2$-normalized eigenvectors respectively of the 2D discretized semi-classical Schrödinger operator $\mathcal{H}_h$ and $i,j = 1, \ldots, N$ refer to the $i^{th}$ row and $j^{th}$ column of the matrix respectively.

To solve the 2D eigenvalue problem (6), the idea consists in solving 1D eigenvalue problems and then combining the results using the tensor product as it is often done in image processing [6], [14] (see [5], [21] for concrete examples). This means that the problem can be solved rows by rows and columns by columns which simplifies the computations in terms of complexity and computation time and allows the use of parallel computing.

Hence, in discrete case, the 1D operators are given respectively by:

\[
\begin{align*}
A_{i,h}(I[i,:]) &= -h^2 D_2 - \text{diag} \left( \frac{1}{2} I[i,:] \right), \\
B_{j,h}(I[:,j]) &= -h^2 D_2 - \text{diag} \left( \frac{1}{2} I[:,j] \right),
\end{align*}
\]

where $D_2$ is a second order differentiation matrix obtained by using the Fourier pseudo-spectral method [2], [24], diag $\left( \frac{1}{2} I[i,:] \right)$ and diag $\left( \frac{1}{2} I[:,j] \right)$ are the diagonal matrices of the 1D signal.

Then the associated spectral problems are given by:

\[
\begin{align*}
A_{i,h}(I[i,:]) \psi_{i,n,h} &= \kappa_{i,n,h} \psi_{i,n,h}, \\
B_{j,h}(I[:,j]) \psi_{j,m,h} &= \rho_{j,m,h} \psi_{j,m,h}.
\end{align*}
\]
In particular, for the pixel \([i,j]\), the eigenvalue problem (9) (resp. (10)) is solved, and then all the negative eigenvalues \(\kappa_{i,n,h}\) (resp. \(\rho_{j,m,h}\)) and the \(j^{th}\) (resp. \(i^{th}\)) associated \(l^2\)-normalized eigenvectors \(\varphi_{i,n,h}\) for \(n = 1, \cdots, N^\lambda_h\) (resp. \(\varphi_{j,m,h}\) for \(m = 1, \cdots, M^\lambda_h\)) are taken. Hence, we obtain,

\[
\varphi_{i,n,h}[j] \phi_{j,m,h}[i] = \psi_{k,h}[i,j],
\]

and

\[
\kappa_{i,n,h} + \rho_{j,m,h} = \mu_{k,h}.
\]

Then, based on Theorem 2.1, the reconstruction of the image is done pixel by pixel as it is often the case in image processing as follows,

**Definition 2.2** [15], [16] Let \(I \in \mathcal{M}_{N \times N}(\mathbb{R}_+)\) be a positive real valued square matrix. Then, the approximation by the SCSA method of \(I\) is defined by the following formula: \(\forall (i,j) \in \{1, 2, \cdots, N\}^2\),

\[
I_{h,\gamma,\lambda}[i,j] = -\lambda + \left(\frac{\hbar^2}{L^2} \sum_{n=1}^{N^\lambda_h} \sum_{m=1}^{M^\lambda_h} (\lambda - (\kappa_{i,n,h} + \rho_{j,m,h})) \varphi_{i,n,h}^2[j] \phi_{j,m,h}^2[i]\right)^{\frac{1}{\gamma}},
\]

where \(h \in \mathbb{R}_+, \gamma \in \mathbb{R}_+, \lambda \in \mathbb{R}_+, \) and \(L_{2,\gamma}\), known as the suitable universal semi-classical constant, is given by (4).

Moreover, \(\kappa_{i,n,h}\) (resp. \(\rho_{j,m,h}\)) are the negative eigenvalues of the one dimensional semi-classical Schrödinger operator given by (7), (resp. (8)) with \(\kappa_{i,1,h} < \cdots < \kappa_{i,N^\lambda_h,h} < \lambda\) (resp. \(\rho_{j,1,h} < \cdots < \rho_{j,M^\lambda_h,h} < \lambda\)), \(N^\lambda_h\) (resp. \(M^\lambda_h\)) is the number of the negative eigenvalues smaller than \(\lambda\), and \(\varphi_{i,n,h}\) (resp. \(\phi_{j,m,h}\)) are the associated \(l^2\)-normalized eigenvectors such that:

\[
A_{i,j}(I[i,\cdot]) \varphi_{i,n,h} = \kappa_{i,n,h} \varphi_{i,n,h}, \quad n = 1, \cdots, N^\lambda_h
\]

and

\[
\text{(resp. } B_{j,i}(I[\cdot,j]) \phi_{j,m,h} = \rho_{j,m,h} \phi_{j,m,h}, \quad m = 1, \cdots, M^\lambda_h\text{)}
\]

### III. IMAGE DENOISING BASED ON 2D SQUARED \(L^2\)-NORMALIZED EIGENFUNCTIONS

#### A. Denoising algorithm

Along with the numerical validation that have been performed in [15], [16], Theorem 2.1 shows the importance of the parameter \(h\) in the representation of the image. Indeed the number of the negative eigenvalues \(N^\lambda_h \times M^\lambda_h\) increases when \(h\) decreases. However, in practice, it has been shown that good results are obtained with a smaller number of eigenvalues (i.e.; for \(h\) large enough) [19], this comes from the localization property of the eigenfunctions [23] and the significant information that they contain. In fact the first eigenfunction gives a good localization of the most significant intensity in the image, the second for the two peaks that follow the largest intensity, then the last eigenfunctions present several oscillations, which mainly contain the image’s details [15]. This property motivates the use of this method for image denoising. Indeed the idea consists in retaining the most significant eigenfunctions belonging to the noisy image and discard those representing the noise. Instead of a naive truncation of the sum (formula (13)), which will lead to neglect important details in the image, we propose to reduce the number of oscillations by increasing the value of the semi-classical parameter \(h\) (i.e reducing the number of the negative eigenvalues).

We explain now how to select the design parameters of this method. On one hand, it has been shown in [11], that the parameter \(\lambda\) gives information on the part of the signal to be reconstructed. In the following, we set \(\lambda\) to zero. On the other hand, the parameter \(\gamma\) is important since its value affects the performance of the reconstruction when the number of eigenvalues is small (i.e \(h\) large enough). In the following, we set \(\gamma = 4\) as described in [15], [16].

The proposed denoising algorithm is summarized in the following:

**Algorithm 1**: Algorithm for image denoising

**Input**: The noisy image  
**Output**: Original (filtered) image

**Step 1**: Initialize \(h\) and \(\gamma\).

**Step 2**: Discretize the 1D Laplace operator \(D_2\).

**Step 3**: Solve 1D eigenvalue problems (9) and (10) (for all rows \(i\) and columns \(j\) with \(i, j = 1, \cdots, N\) respectively).

**Step 4**: Reconstruct the image using formula (13).
B. Numerical examples

Some numerical examples are now presented to illustrate the good performance of the proposed method and its stability. The peak signal-to-noise ratio (PSNR) together with the structural similarity index (SSIM), and the mean structural similarity index (MSSIM) [25] are employed as objective indices to evaluate the image quality of the denoised images. The images are subject to additive Gaussian white noise with zero mean and different level of standard deviation \( \sigma \) (different values of SNR).

Figure 1 shows the original image of Lena\(^1\) and the noisy one respectively. The standard deviation \( \sigma \) is equal to 75 and the corresponding SNR is 11.2 dB. Figure 2b illustrates the denoising of the Lena’s image using the good value of \( h \) which is equal to 1.65 and obtained after trials. However, the use of a value for \( h \) smaller, does not filter completely the noise, but helps to reconstruct the noisy image (see figure 2a), and with a larger value of \( h \), the pertinent informations in the image are lost because significant eigenfunctions are not accurate as we shown in figure 2c.

The histogram of the original image of Lena, the noisy (\( \sigma = 75 \)) and the denoised one are illustrated in figure 3. Figure 3b, which represents the histogram of the noisy image, has the shape of a Gaussian function. Using the 2D SCSA method in the denoising process (figure 3c), the shape of the original image (figure 3a) is obtained even at high level of noise.

C. Comparison to the SVD method

In the following, we compare the proposed method with SVD method [1], [26] in noisy case. Our choice focused on the SVD method since it is one of the most used methods and it is also based on an eigenvalues decomposition which depend on the image. We added Gaussian white noise with zero mean and different levels of standard deviation \( \sigma = \{10, 20, 25, 30, 40, 50, 100\} \) respectively to the original image of Lena. Due to the limitation in space, Table I, shows the PSNR and SSIM results of these two algorithms. It is clear that the denoising using the proposed method is better than the SVD method. Particularly, at high noise level, the proposed method provides the best results even if the image has several textures and it provides better visual quality. The edges and textures of the image are better preserved in the proposed method unlike in the SVD method where they are blurred or lost (see figure 5). With a suitable choice of the semi-classical parameter \( h \), These first results confirm that the proposed method is more robust to additive Gaussian noise.

\(^1\)http://www.ece.rice.edu/ wakin/images/
Fig. 3: The histogram of, (a) Original image of Lena, (b) Noisy image ($\sigma = 75$), (c) Denoised image.

Fig. 4: Zoom on denoising of Lena’s image corrupted by noise with $\sigma = 40$, (a) original image, (b) noisy image.

Fig. 5: Zoom on denoising of Lena’s image corrupted by noise with $\sigma = 40$, (a) proposed method, (b) SVD method [26].

### IV. Conclusion

In this paper, a new method for image denoising has been presented. The proposed method is based on using a tensor product of an adaptive set of spatially shifted and localized function which is given by the squared $L^2$-normalized eigenfunctions associated to the negative eigenvalues of one dimensional semi-classical Schrödinger operator. Thanks to the
localization of the Schrodinger operator eigenfunctions and with an appropriate choice of the semi-classical parameter \( \hbar \), the numerical results demonstrate good performance of the proposed approach comparing to the SVD method in term of PSNR, SSIM, stability and visual quality at different levels of noise. An algorithm for the optimal choice of the semi-classical parameter \( \hbar \) for denoising is currently under consideration along with the comparison to other state-of-the-art methods.

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