

Long Time Behavior of a PDE Model for Invasive Species Control

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Abstract

The Trojan Y Chromosome strategy (TYC) is a theoretical method for eradication of invasive species. It requires constant introduction of artificial individuals into a target population, causing a shift in the sex ratio, that ultimately leads to local extinction. In this work we consider a modified version of the TYC system. We first demonstrate the existence of a unique weak solution to the system. Furthermore, we prove the existence of a compact finite dimensional global attractor for the modified system, in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

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1 Introduction

An invasive species is any species, capable of propagating itself, into a non native environment. Once established, they can be extremely difficult to eradicate or manage [2], [14]. There are numerous cases of harm to the environment and the economy, due to the action of various invasive species [7], [6]. Their effect is considered second only to habitat destruction, [6]. Some well known examples of these include the burmese python in southern regions of the United States, the cane toad in Australia, and the sea lamprey and round goby in the great lakes, in the northern United States. The cane toad was

brought into Australia from Hawaii in 1935, to control the cane beetle. These multiplied rapidly since their introduction, and have negatively affected biodiversity in the region, [18]. The sea lamprey entered the great lakes in the 1800 hundreds, through shipping canals. Their population has since exploded, [19], and these predators have caused a severe decline in lake trout and other lake fish populations, adversely affecting various local fisheries. The burmese python population has been on the rise exponentially, in the Florida glades, since the 1980's. This has been attributed to a number of reasons, such as the exotic pet trade in Florida. As a result, these invaders have had quite a negative impact on the local ecology. Climatic factors might support their spread to a third of the United States, [20]. The US Office of Technology Assessment reported an estimated \$97 billion in agricultural losses caused by 79 exotic species during the period 1906 to 1991. A different study reported a 109-times larger impact at \$120 billion/year by 2004 [12]. Thus, the annual damages caused by invasive species, in the United States alone, could be as much as 45% of the countries gross agricultural production. There have been a number of studies aimed at reducing the effects of such invaders, through genetic means [4],[5]. Despite these studies there have been no conclusive results to eradicate or contain these species in real scenarios.

A strategy for eradication of invasive fish, in which a "Trojan fish" is added to the population was reported by Gutierrez and Teem, [1]. Parshad et al. considered the model proposed in [1], under the inclusion of spatial spread of the species. They showed well posedness and existence of a finite dimensional global attractor for the system, [9], [10]. In the current work we propose a modified form of the earlier model. This strategy is relevant to fish with an XY sex-determination system, in which males carry one X chromosome and one Y chromosome (XY), and females carry two X chromosomes (XX). Note variations in the sex chromosome number can be produced through genetic manipulation. Hormone treatments can be used to reverse the sex, resulting in a feminized (XY) male [4],[8]. Mating between the feminised male(XY) and males (XY), leads half the time to a progeny with two Y chromosomes. We will denote this individual by s , a so called "supermale" (YY). The eradication strategy works as follows. This genetically modified "Trojan" fish, s (YY), bearing two Y chromosomes, is introduced at a rate μ to a target population of an invasive fish species containing normal females and males, denoted as f and m respectively. Mating between the introduced s and the invasive female f , always leads to a male progeny m . This generates a disproportionate number of male fish over time. The higher incidence of males decrease the female to male ratio. Ultimately, the number of f decline to zero, causing local extinction. This simple idea is expressed in figure 1.

We can now model the proposed strategy, via a system of partial differential equations, essentially derived from the pedigree tree in figure 1. We call this

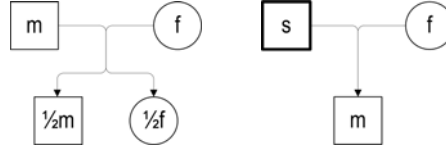


Figure 1: Modified Trojan Y chromosome strategy: Pedigrees of fish in the wild when supermales s have been introduced into the population. The left figure shows mating of a (XX) female (f) and a (XY) male (m). This leads to females half the time, and males half the time. The right figure shows mating of a (XX) female (f) and a (YY) supermale (s). This always leads only to male progeny.

the modified TYC model, and it takes the following form,

$$\frac{\partial f}{\partial t} = D\Delta f + \frac{\beta}{2}fmL - \delta f, \quad f|_{\partial\Omega} = 0. \tag{1}$$

$$\frac{\partial m}{\partial t} = D\Delta m + \left(\frac{\beta}{2}fm + \frac{\beta}{2}fs\right)L - \delta m, \quad m|_{\partial\Omega} = 0. \tag{2}$$

$$\frac{\partial s}{\partial t} = D\Delta s + \left(\frac{f}{1+f}\right)^p \mu(s) - \delta s, \quad \nabla s \cdot n|_{\partial\Omega} = 0. \tag{3}$$

Here $\Omega \subset R^3$ is a bounded domain. All variables represent density of fish. Also

$$L = 1 - \left(\frac{f + m + s}{K}\right),$$

is a logistic term, where K is the carrying capacity of the ecosystem, D is the diffusivity coefficient, β is a birth coefficient (i.e. what proportion of encounters between males and females result in progeny), and δ is a death coefficient (i.e. what proportion of the population is dying at any given moment). Essentially the change in the density of females is due to spatial spread, mating with normal males, and their dying at death rate δ . The change in the density of males is due to spatial spread, mating with normal females, and mating between normal females and supermales. They also die with death rate δ . The change in density of the supermales is due to spatial spread and death. They cannot be born as a result of normal mating, but need to be introduced into the system at some rate μ . We assume $\mu(s) = \frac{C}{|s|^q}$. The reason is a decaying source, seems more pragmatic, than the constant source considered in earlier works, [9], [10], [11]. Furthermore $\left(\frac{f}{1+f}\right)^p \mu(s)$, is an artificial construct. This creates a dependence of the supermale on the females via the

$\left(\frac{f}{1+f}\right)^p$ term, and essentially when the females, reach extinction $\left(\frac{f}{1+f}\right)^p \rightarrow 0$. Thus $\left(\frac{f}{1+f}\right)^p \mu(s) \rightarrow 0$, and the source is automatically turned of, leading to extinction of the supermales, and then subsequently the males. We assume initial data in $L^2(\Omega)$. We define the phase space for the system as

$$H = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \quad (4)$$

We also define

$$X = H_0^1(\Omega) \times H_0^1(\Omega) \times H^1(\Omega) \quad (5)$$

We also make the assumptions that

$$\frac{1}{|\Omega|}|f|_1 \leq C, \quad \frac{1}{|\Omega|}|m|_1 \leq C, \quad |\mu(s)|_2 \leq C. \quad (6)$$

Note via the restrictions posed by the carrying capacity for the system we have

$$|f|_\infty \leq C, \quad |m|_\infty \leq C. \quad (7)$$

We choose Dirchilet boundary conditions for the f and m variables, and Neumann boundary conditions for the s variable. These facilitate the analysis. The objective of the current manuscript is to show well posedness and existence of a global attractor for the modified model. This will establish the modified TYC model as a sound eradication strategy, and may aid both government and industrial efforts to comabt the menace of invasive species.

2 Well posedness

2.1 A priori estimates for f_n

In order to prove the well posedness we follow the standard approach of projecting onto a finite dimensional subspace, [13]. This reduces the PDE to a finite dimensional system of ODE's. It is on this truncated system that we make a priori estimates. In all estimates that follow through the rest of the manuscript, C is a generic constant, that can change in value from line to line, and sometimes within the same line, if so required. Essentially The truncation for f takes the form,

$$f_n(t) = \sum_{j=1}^n f_{nj}(t)w_j.$$

Here w_j are the eigenfunctions of the negative Laplacian, so $-\Delta w_i = \lambda_i w_i$. A similar truncation can be performed for m , r and s . Thus essentially the following holds $\forall 1 \leq j \leq n$,

$$\frac{\partial f_n}{\partial t} = D\Delta f_n + P_n(F(f_n, m_n, s_n)) - \delta f_n, \tag{8}$$

$$f_n(0) = P_n(f_0).$$

Here P_n is the projection onto the space of the first n eigenvectors. Note in general

$$\langle f_n, P_n(F(f_n)) \rangle = \langle P_n(f_n), F(f_n) \rangle = \langle f_n, F(f_n) \rangle \tag{9}$$

We multiply (8) by f_n and integrate by parts over Ω . We thus obtain

$$\frac{1}{2} \frac{d|f_n|_2^2}{dt} = -D|\nabla f_n|_2^2 + \frac{\beta}{2} \left[\int_{\Omega} m_n f_n^2 \, d\mathbf{x} - \int_{\Omega} m_n f_n^2 \frac{f_n + m_n + s_n}{K} \, d\mathbf{x} \right] - \delta |f_n|_2^2. \tag{10}$$

Via the positivity of f_n , m_n , s_n , and K it follows that

$$\int_{\Omega} m_n f_n^2 \frac{f_n + m_n + s_n}{K} \, d\mathbf{x} \geq \int_{\Omega} m_n f_n^2 \frac{f_n}{K} \, d\mathbf{x}.$$

This estimate is used in (10) to yield

$$\frac{1}{2} \frac{d|f_n|_2^2}{dt} + D|\nabla f_n|_2^2 + \delta |f_n|_2^2 + \frac{\beta}{2K} \int_{\Omega} m_n f_n^3 \, d\mathbf{x} \leq \frac{\beta}{2} \int_{\Omega} m_n f_n^2 \, d\mathbf{x}.$$

We now use Young's Inequality to obtain

$$\frac{1}{2} \frac{d|f_n|_2^2}{dt} + D|\nabla f_n|_2^2 + \delta |f_n|_2^2 + \frac{\beta}{2K} \int_{\Omega} m_n f_n^3 \, d\mathbf{x} \leq \frac{\beta}{2K} \int_{\Omega} m_n f_n^3 \, d\mathbf{x} + \frac{\beta K^2}{2} \int_{\Omega} m_n \, d\mathbf{x}.$$

Using

$$|m_n|_1 \leq |m|_1 \leq K|\Omega|,$$

we obtain the following

$$\frac{1}{2} \frac{d|f_n|_2^2}{dt} + D|\nabla f_n|_2^2 + \delta |f_n|_2^2 \leq \frac{\beta K^3}{2} |\Omega|. \tag{11}$$

It follows that

$$\frac{d|f_n|_2^2}{dt} + (CD + \delta) |f_n|_2^2 \leq \beta K^3 |\Omega|.$$

Now we can apply Gronwall's Lemma to yield

$$|f_n(t)|_2^2 \leq e^{-(CD+\delta)t} |f_0|_2^2 + \frac{\beta K^3 |\Omega|}{CD + \delta} \leq C, \quad \forall t \geq 0. \tag{12}$$

On the other hand we can integrate (11) from 0 to T to obtain

$$\frac{1}{2} |f_n(T)|_2^2 + D \int_0^T |\nabla f_n|_2^2 dt + \delta \int_0^T |f_n|_2^2 dt \leq \int_0^T \beta K^3 |\Omega| dt + |f_n(0)|_2^2.$$

This immediately yields

$$\int_0^T |\nabla f_n|_2^2 dt \leq \int_0^T \beta K^3 |\Omega| dt + |f_n(0)|_2^2 \leq \int_0^T \beta K^3 |\Omega| dt + |f(0)|_2^2 \leq C, \tag{13}$$

Thus via (12) and (13) we obtain

$$f_n \in L^\infty(0, T; L^2(\Omega)), \tag{14}$$

and

$$f_n \in L^2(0, T; H_0^1(\Omega)). \tag{15}$$

2.2 Estimate for the time derivative of f_n

We multiply (8) by a $w \in H_0^1(\Omega)$ to yield

$$\left(\frac{\partial f_n}{\partial t}, w \right) = -D \langle \nabla f_n, \nabla w \rangle + \langle F(f_n, m_n, s_n), P_n(w) \rangle - \delta \langle f_n, w \rangle.$$

We estimate the non-linear term as follows

$$\begin{aligned} & (F(f_n), P_n(w)) \\ &= \int_\Omega m_n f_n \left(1 - \frac{f_n + m_n + s_n}{K} \right) P_n(w) \, d\mathbf{x} \\ &\leq \int_\Omega m_n f_n P_n(w) \, d\mathbf{x} \\ &\leq |m_n|_\infty \int_\Omega f_n P_n(w) \, d\mathbf{x} \\ &\leq K |f_n|_4 |P_n(w)|_{\frac{4}{3}} \\ &\leq C |f_n|_4 |w|_{H_0^1}. \end{aligned}$$

This follows via the compact embedding of $H_0^1(\Omega) \hookrightarrow L^{\frac{4}{3}}(\Omega)$. Thus we have

$$\left| \frac{\partial f_n}{\partial t} \right|_{H^{-1}(\Omega)}^2 \leq |f_n|_4^2,$$

integrating both sides of the above in the time interval $[0, T]$ yields

$$\int_0^T \left| \frac{\partial f_n}{\partial t} \right|_{H^{-1}(\Omega)}^2 dt \leq \int_0^T |f_n|_4^2 dt \leq C \int_0^T |\nabla f_n|_2^2 dt \leq C.$$

This follows from the derived estimate via (15) and the compact embedding of $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Thus we obtain

$$\frac{\partial f_n}{\partial t} \in L^2(0, T; H^{-1}(\Omega)). \tag{16}$$

Also note via (14) and (16), and the classical Aubin-Lions lemma, [15], we obtain

$$f_n \in C(0, T; L^2(\Omega)), \tag{17}$$

we can now via (14) and (15) extract a subsequence f_{n_j} such that

$$f_{n_j} \overset{*}{\rightharpoonup} f \text{ in } L^\infty(0, T; L^2(\Omega))$$

$$f_{n_j} \rightharpoonup f \text{ in } L^2(0, T; H_0^1(\Omega))$$

$$f_{n_j} \rightarrow f \text{ in } L^2(0, T; L^2(\Omega)).$$

The convergence in the last equation follows via the compact embedding of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

2.3 A priori estimates for s_n

We proceed as earlier,

$$s_n(t) = \sum_{j=1}^n f_{n_j}(t) w_j.$$

Thus essentially the following holds $\forall 1 \leq j \leq n$,

$$\frac{\partial s_n}{\partial t} = D\Delta s_n + \left(\frac{f}{1+f} \right)^p \frac{C}{|s_n|^q} - \delta s_n, \tag{18}$$

$$s_n(0) = P_n(s_0).$$

We multiply (18) by s_n and integrate by parts over Ω . We thus obtain

$$\begin{aligned} & \frac{1}{2} \frac{d|s_n|_2^2}{dt} + D |\nabla s_n|_2^2 + \delta |s_n|_2^2 \\ &= \int_{\Omega} \frac{C s_n}{|s_n|^q} \left(\frac{f}{1+f} \right)^p d\mathbf{x} \end{aligned} \tag{19}$$

The use of Holder and Young’s inequality yield

$$\begin{aligned} & \frac{1}{2} \frac{d|s_n|_2^2}{dt} + D |\nabla s_n|_2^2 + \delta |s_n|_2^2 \\ & \leq \delta |s_n|_2^2 + C|\mu(s)|_2^2 \end{aligned} \tag{20}$$

This immediately yields via Gronwall’s inequality,

$$|s_n(t)|_2^2 \leq e^{-CD t} |s_n(0)|_2^2 + C \tag{21}$$

Also integration of (20) in the time interval $[0, T]$ before using the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ yields

$$\int_0^T |\nabla s_n|_2^2 dt \leq \int_0^T C dt + |s_n(0)|_2^2 \leq \int_0^T C dt + C_1 \leq CT + C_1, \tag{22}$$

These estimates immediately yield,

$$s_n \in L^\infty(0, T; L^2(\Omega)), \tag{23}$$

and

$$s_n \in L^2(0, T; H_0^1(\Omega)). \tag{24}$$

We next multiply (3) by $|s_n|^p$ and integrate by parts over Ω to obtain

$$\frac{1}{q+1} \frac{d}{dt} |s_n|_{q+1}^{q+1} + \delta |s_n|_{q+1}^{q+1} + q \int_{\Omega} |\nabla s_n|^2 \frac{1}{|s_n|^{q+1}} d\mathbf{x} = \int_{\Omega} \left(\frac{f_n}{1+f_n} \right)^p |s_n|^{q-p} d\mathbf{x}, \tag{25}$$

therefore by Holder’s inequality,

$$\frac{1}{q+1} \frac{d}{dt} |s_n|_{q+1}^{q+1} \leq \int_{\Omega} |s_n|^{q-p} d\mathbf{x} \leq (|s_n|_{q+1}^{q+1})^{\frac{q-p}{q+1}}, \tag{26}$$

Also note that for a general power n we have,

$$\frac{d}{dt} (|s_n|_{q+1}^{q+1})^n \leq n (|s_n|_{q+1}^{q+1})^{n-1} (1+q) (|s_n|_{q+1}^{q+1})^{\frac{q-p}{q+1}}. \tag{27}$$

Choosing $n = \frac{1+p}{1+q}$, we obtain

$$\left(\frac{d}{dt} |s_n|_{q+1}^{q+1} \right)^{\frac{1+p}{1+q}} \leq 1+p, \tag{28}$$

integrating (28) in time interval $[0, t]$ yields

$$(|s_n|_{q+1}) \leq ((1+p)t + |s_n(0)|_{q+1})^{\frac{1}{1+p}} \leq \left((1+p)T + |\Omega|^{\frac{1}{q+1}} M \right)^{\frac{1}{1+p}}, \tag{29}$$

Here we assume $0 < s(0) < M$. Taking the limit as $q \rightarrow \infty$ yields

$$|s_n|_{L^\infty(\Omega \times T)} \leq ((1+p)T + M)^{\frac{1}{1+p}}, \tag{30}$$

Thus it follows that,

$$s_n \in L^\infty(0, T; L^\infty(\Omega)), \tag{31}$$

2.4 Estimate for the time derivative of s

We take the partial w.r.t time of (18) to yield

$$\begin{aligned} & \frac{\partial^2 s_n}{\partial t^2} \\ = & D_3 \Delta \frac{\partial s_n}{\partial t} + \left(\frac{f_n}{1+f_n} \right)^p \mu(s) - \delta \frac{\partial s_n}{\partial t} \\ & + \left(\frac{-q}{|s_n|^{q+1}} \right) \left(\frac{f_n}{1+f_n} \right)^p + \left(\frac{1}{|s_n|^{q+1}} \right) \left(p \frac{f_n}{1+f_n} \right)^{p-1} \left(\frac{1}{1+f_n} \right)^2, \end{aligned} \tag{32}$$

we multiply through by $\frac{\partial s_n}{\partial t}$ to yield and integrate by parts over Ω to yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left| \frac{\partial s_n}{\partial t} \right|_2^2 + \delta \left| \frac{\partial s_n}{\partial t} \right|_2^2 + D \left| \frac{\partial s_n}{\partial t} \right|_2^2 \\ \leq & \int_\Omega \frac{\partial s_n}{\partial t} \left(\frac{1}{|s_n|^{q+1}} \right) \left(p \frac{f_n}{1+f_n} \right)^{p-1} \left(\frac{1}{1+f_n} \right)^2 \end{aligned} \tag{33}$$

Holder and Young’s inequality yield

$$\frac{1}{2} \frac{d}{dt} \left| \frac{\partial s_n}{\partial t} \right|_2^2 + D \left| \frac{\partial s_n}{\partial t} \right|_2^2 \leq \int_{\Omega} \frac{1}{|s_n|^{2q}} \tag{34}$$

Gronwall’s lemma and the integrability condition on the source yield

$$\left| \frac{\partial s_n}{\partial t} \right|_2^2 \leq C \tag{35}$$

Now via the compact embedding of $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ we have

$$\left| \frac{\partial s_n}{\partial t} \right|_{H^{-1}(\Omega)}^2 \leq \left| \frac{\partial s_n}{\partial t} \right|_2^2 \leq C,$$

integrating both sides of the above in the time interval $[0, T]$ yields

$$\int_0^T \left| \frac{\partial s_n}{\partial t} \right|_{H^{-1}(\Omega)}^2 dt \leq C.$$

Thus we obtain

$$\frac{\partial s_n}{\partial t} \in L^2(0, T; H^{-1}(\Omega)).$$

We can now via (24) and (31) extract a subsequence s_{n_j} such that

$$s_{n_j} \overset{*}{\rightharpoonup} s \text{ in } L^\infty(0, T; L^\infty(\Omega))$$

$$s_{n_j} \rightharpoonup s \text{ in } L^2(0, T; H_0^1(\Omega))$$

$$s_{n_j} \rightarrow s \text{ in } L^2(0, T; L^2(\Omega)).$$

The convergence in the last equation follows via the compact embedding of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. We briefly discuss passing to the limit in order to demonstrate existence. Passing to the limit in the nonlinear term follows via a lemma from [10], which we recall

Lemma 2.1 *Consider the non linear terms $F(f_1, m_1, r_1, s_1)$ and $F(f_2, m_2, r_2, s_2)$ for the Trojan Y Chromosome model. The following estimate for their difference holds,*

$$\begin{aligned} & |F(f_1, m_1, r_1, s_1) - F(f_2, m_2, r_2, s_2)|_2 \leq \\ & C (|f_1 - f_2|_2 + |m_1 - m_2|_2 + |s_1 - s_2|_2 + |r_1 - r_2|_2). \end{aligned}$$

An easy modification of this gives the similar result for the nonlinear terms in (1) and (2). Furthermore the nonlinear term in (3) is Lipschitz. We proceed as follows. Consider a $\phi \in C_0^\infty(0, T)$. We multiply (8) by $\phi(t)$ and integrate by parts in time to yield

$$\begin{aligned}
 & - \int_0^T \langle f_{n_j}, \phi'(t)w_j \rangle dt \\
 = & -D \int_0^T \langle \nabla f_{n_j}, \nabla w_j \phi(t) \rangle dt + \int_0^T \langle F(f_{n_j}, m_{n_j}, s_{n_j}), \phi(t)w_j \rangle dt \\
 & - \delta \int_0^T \langle f_{n_j}, \phi(t)w_j \rangle dt. \tag{36}
 \end{aligned}$$

Now taking the limit as $j \rightarrow \infty$ in (36) we obtain

$$\begin{aligned}
 & \lim_{j \rightarrow \infty} \int_0^T \langle f_{n_j}, \phi'(t)w_j \rangle dt + D \int_0^T \langle \nabla f_{n_j}, \nabla w_j \phi(t) \rangle dt \\
 + & \delta \int_0^T \langle f_{n_j}, \phi w_j \rangle dt - \int_0^T \langle F(f_{n_j}, m_{n_j}, s_{n_j}), \phi w_j \rangle dt \\
 = & \int_0^T \langle f, \phi'(t)w_j \rangle dt + D \int_0^T \langle \nabla f, \nabla w_j \phi \rangle dt \\
 + & \delta \int_0^T \langle f, \phi w_j \rangle dt - \int_0^T \langle F(f, m, s), \phi w_j \rangle dt \\
 = & 0.
 \end{aligned}$$

This follows as we have demonstrated

$$f_{n_j} \overset{*}{\rightharpoonup} f \text{ in } L^\infty(0, T; L^2(\Omega))$$

$$f_{n_j} \rightharpoonup f \text{ in } L^2(0, T; H_0^1(\Omega))$$

$$f_{n_j} \rightarrow f \text{ in } L^2(0, T; L^2(\Omega)).$$

and the assumption on the nonlinear term.

This implies that we have continuity with respect to w_j . Thus we obtain that for any $v \in H_0^1(\Omega)$ the following holds

$$\begin{aligned}
 & - \int_0^T \langle f, \phi'(t)v \rangle dt + D \int_0^T \langle \nabla f, \nabla v \phi(t) \rangle dt + \delta \int_0^T \langle f, \phi(t)v \rangle dt \\
 = & \int_0^T \langle F(f, m, s), \phi(t)v \rangle dt.
 \end{aligned}$$

This yields the existence of an f such that the following is true in a distributional sense

$$\frac{d}{dt}(f, v) + D \langle \nabla f, \nabla v \rangle + \delta \langle f, v \rangle = \langle F(f, m, s), v \rangle, \quad \forall v \in H_0^1(\Omega) \quad (37)$$

In other words there exists a weak solution f to (1). Also uniqueness follows via a standard argument of taking a difference of two weak solutions, and deriving via an energy method that the difference is 0, yielding uniqueness. The estimates for the other variables is made similarly. See [10] for details of the TYC system. Thus we can state the following result,

Theorem 2.2 *Consider the modified Trojan Y Chromosome model, (1)-(3). There exists a unique weak solution (f, s, m) to the system for initial data in $L^2(\Omega)$, and for any $T > 0$, such that*

$$f \in C([0, T]; L^2(\Omega)) \cap L^\infty([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega)),$$

$$m \in C([0, T]; L^2(\Omega)) \cap L^\infty([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega)),$$

$$s \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^1(\Omega)) \cap L^\infty([0, T]; L^\infty(\Omega)),$$

furthermore (f, s, m) are continuous with respect to initial data.

3 Existence of global attractor

In this section we prove the existence of a compact global attractor for the modified TYC model.

3.1 Preliminaries

Recall the phase space H introduced earlier

$$H = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega).$$

Also

$$X = H_0^1(\Omega) \times H_0^1(\Omega) \times H^1(\Omega),$$

We recap the following definitions

Definition 3.1 Let $\mathcal{A} \subset H$. Then \mathcal{A} is said to be a (H, H) global attractor if the following conditions are satisfied

- i) \mathcal{A} is compact in H .
- ii) \mathcal{A} is invariant, i.e, $S(t)\mathcal{A} = \mathcal{A}$, $t \geq 0$
- iii) If B is bounded in H then

$$\text{dist}_H(S(t)B, \mathcal{A}) \rightarrow 0, t \rightarrow \infty.$$

Compactness is usually hard to prove directly. It usually suffices to prove the property of asymptotic compactness.

Definition 3.2 The semigroup $\{S(t)\}_{t \geq 0}$ associated with a dynamical system is said to be asymptotically compact in $L^2(\Omega)$, if for any $\{f_{0,n}\}_{n=1}^\infty$ bounded in $L^2(\Omega)$, and a sequence of times $\{t_n \rightarrow \infty\}$, $S(t_n)f_{0,n}$ possesses a convergent subsequence in $L^2(\Omega)$.

Remark 3.3 Thus the key ingredients in proving existence of a global attractor are to show existence of bounded absorbing sets in the phase space, followed by demonstrating the asymptotic compactness of the semigroup in question.

3.2 Existence of L^2 absorbing sets for f

For completeness we cover some of the details of obtaining the absorbing sets. We work on f described via (1), the estimates for m follow similarly.

$$\frac{\partial f}{\partial t} = D\Delta f + \frac{\beta}{2}Lfm - \delta f, \quad f|_{\partial\Omega} = 0. \tag{38}$$

We multiply Equation 38 through by f and integrate by parts over Ω to yield

$$\frac{1}{2} \frac{d|f|_2^2}{dt} = -D|\nabla f|_2^2 + \frac{\beta}{2} \left[\int_{\Omega} m f^2 dx - \int_{\Omega} m f^2 \frac{(f+m+s)}{K} dx \right] - \delta |f|_2^2. \tag{39}$$

Via the positivity of f , m , s , and K it follows that

$$\int_{\Omega} m f^2 \frac{(f+m+s)}{K} dx \geq 0. \tag{40}$$

Thus we obtain

$$\frac{1}{2} \frac{d|f|_2^2}{dt} + D|\nabla f|_2^2 + \delta |f|_2^2 + \beta \int_{\Omega} m f^3 dx \leq \frac{\beta}{2} \int_{\Omega} m f^2 dx.$$

We now use Holder's inequality to obtain

$$\frac{1}{2} \frac{d|f|_2^2}{dt} + D|\nabla f|_2^2 + \delta |f|_2^2 \leq \frac{\beta_1 K^2}{2} |f|_1 \tag{41}$$

Poincaré's inequality, and the integrability assumption on f yields

$$\frac{1}{2} \frac{d|f|_2^2}{dt} + \left(\frac{D}{C} + \delta\right) |f|_2^2 \leq \frac{\beta}{2} K^2 C |\Omega|.$$

Application of Gronwall's Lemma yields

$$|f(t)|_2^2 \leq e^{-(DC+2\delta)t} |f_0|_2^2 + \frac{\frac{\beta}{2} K^2 C |\Omega|}{DC + 2\delta}.$$

This implies that there exists a time t_1 defined by

$$t_1 = \max\left(0, \frac{\ln(|f_0|_2^2)}{(DC + 2\delta)}\right)$$

such that for all times $t \geq t_1$ the following estimate holds uniformly

$$|f(t)|_2^2 \leq 1 + \frac{\frac{\beta}{2} K^2 C |\Omega|}{DC + 2\delta} \leq C. \quad (42)$$

We integrate (41) in the time interval $[t_1, t]$ to obtain

$$D \int_{t_1}^t |\nabla f(s)|_2^2 ds \leq \frac{1}{2} |f(t_1)|_2^2 + \int_{t_1}^t \frac{\beta}{2} K^3 |\Omega| ds$$

Thus choosing $t = t_1 + 1$ we obtain

$$\frac{D}{t_1 + 1 - t_1} \int_{t_1}^{t_1+1} |\nabla f(s)|_2^2 ds \leq \frac{1}{2} |f(t_1)|_2^2 + \int_{t_1}^{t_1+1} \frac{\beta}{2} K^3 |\Omega| ds \leq C$$

Thus using a mean value Theorem for integrals we obtain the existence of a time $t_2 \in [t_1, t_1 + 1]$ such that the following estimate holds uniformly

$$|\nabla f(t_2)|_2^2 \leq C$$

3.3 Existence of L^2 absorbing sets for s

The proof of bounded absorbing set for s is similar to deriving a priori estimates on the galerkin truncation, that was performed earlier. We multiply (3) by s and integrate by parts over Ω . We thus obtain

$$\begin{aligned} & \frac{1}{2} \frac{d|s|_2^2}{dt} + D |\nabla s|_2^2 + \delta |s|_2^2 \\ &= \int_{\Omega} \frac{Cs}{|s|^q} \left(\frac{f}{1+f}\right)^p dx \end{aligned} \quad (43)$$

The use of Holder and Young's inequality yield

$$\begin{aligned} & \frac{1}{2} \frac{d|s|_2^2}{dt} + D |\nabla s|_2^2 + \delta |s|_2^2 \\ & \leq \delta |s|_2^2 + C |\mu(s)|_2^2 \end{aligned} \tag{44}$$

This immediately yields via Gronwall's inequality,

$$|s(t)|_2^2 \leq e^{-CD t} |s(0)|_2^2 + C \tag{45}$$

This implies that there exists a time t_3 defined by

$$t_3 = \max \left(0, \frac{\ln(|s_0|_2^2)}{CD} \right)$$

such that for all times $t \geq t_3$ the following estimate holds uniformly

$$|s(t)|_2^2 \leq 1 + C \leq C. \tag{46}$$

Also integration of (43) in the time interval $[t_3, t_3 + 1]$ yields,

$$\int_{t_3}^{t_3+1} |\nabla s|_2^2 dt \leq \int_{t_3}^{t_3+1} (C) dt \leq C \tag{47}$$

These estimates immediately yield the existence of a time $t_4 \in [t_3, t_3 + 1]$ such that

$$|\nabla s(t_4)|_2^2 \leq C \tag{48}$$

3.4 Asymptotic compactness

We multiply Equation (8) by $-\Delta f_n$ and integrate by parts over Ω to obtain

$$\begin{aligned} & \frac{\partial |\nabla f_n|_2^2}{\partial t} + D |\Delta f_n|_2^2 + \delta |\nabla f_n|_2^2 \\ & = \frac{\beta}{2} \left(\int_{\Omega} m_n f_n \Delta f_n \frac{(f_n + m_n + s_n)}{K} dx - \int_{\Omega} m_n f_n \Delta f_n dx \right). \end{aligned} \tag{49}$$

Consider the first integral on the right-hand side of (49),

$$\begin{aligned} & \int_{\Omega} m_n f_n \Delta f_n \frac{(f_n + m_n + s_n)}{K} d\mathbf{x} \\ &= \frac{1}{K} \int_{\Omega} m_n f_n^2 \Delta f_n d\mathbf{x} + \frac{1}{K} \int_{\Omega} m_n^2 f_n \Delta f_n d\mathbf{x} \\ & \quad + \frac{1}{K} \int_{\Omega} m_n s_n f_n \Delta f_n d\mathbf{x}. \end{aligned} \tag{50}$$

An estimate is obtained for each integral on the right-hand side of (50) via Young's inequality. Consider the first integral.

$$\begin{aligned} \frac{1}{K} \int_{\Omega} m_n f_n^2 \Delta f_n d\mathbf{x} &\leq \frac{1}{K} \left(\int_{\Omega} \frac{\beta_1}{D} (m_n f_n^2)^2 d\mathbf{x} + \int_{\Omega} \frac{D}{\beta} (\Delta f_n)^2 d\mathbf{x} \right) \\ &\leq \frac{1}{K} \left(\frac{\beta}{D} K^6 |\Omega| + \frac{D}{\beta} |\Delta f_n|_2^2 \right). \end{aligned} \tag{51}$$

A similar analysis for the other terms yields

$$\begin{aligned} & \int_{\Omega} m_n f_n \Delta f_n \frac{(f_n + m_n + s_n)}{K} d\mathbf{x} \\ &\leq \frac{4}{K} \left(K^6 |\Omega| \frac{\beta}{D} + \frac{D}{\beta} |\Delta f_n|_2^2 \right) \\ &\leq 4K^5 |\Omega| \frac{\beta}{D} + \frac{D}{\beta} |\Delta f_n|_2^2. \end{aligned} \tag{52}$$

Thus we obtain,

$$\int_{\Omega} m_n f_n \Delta f_n d\mathbf{x} \leq K^4 |\Omega| \frac{\beta}{D} + \frac{D}{\beta} |\Delta f_n|_2^2.$$

Inserting the above estimates into (49) yields

$$\frac{\partial |\nabla f_n|_2^2}{\partial t} + D |\Delta f_n|_2^2 + \delta |\nabla f_n|_2^2 \leq \frac{3}{D} \beta_1^2 K^5 |\Omega| + D |\Delta f_n|_2^2.$$

This yields

$$\frac{\partial |\nabla f_n|_2^2}{\partial t} + \delta |\nabla f_n|_2^2 \leq \frac{3}{D} \beta^2 K^5 |\Omega|.$$

We can now apply the Gronwall Lemma by integrating in the time interval $[t_2, t]$ to yield

$$\begin{aligned} |\nabla f_n(t)|_2^2 &\leq e^{-\delta t} |\nabla f_n(t_2)|_2^2 + \frac{3}{\delta D} \beta^2 K^5 |\Omega| \\ &\leq e^{-\delta t} \frac{\beta}{2D} K^3 |\Omega| + \frac{1}{2D} \left(1 + \frac{\frac{\beta}{2} K^3 |\Omega|}{DC + 2\delta} \right) + \frac{3}{\delta D} \beta^2 K^5 |\Omega| \end{aligned}$$

Therefore consider

$$t_5 = \max \left(0, t_2, \frac{\ln \left(\frac{\beta}{2D} K^3 |\Omega| + \frac{1}{2D} \left(1 + \frac{\frac{\beta}{2} K^3 |\Omega|}{DC + 2\delta} \right) \right)}{\delta} \right).$$

For $t \geq t_5$ we obtain the following uniform estimate

$$|\nabla f_n(t)|_2^2 \leq 1 + \frac{3}{\delta D} \beta^2 K^5 |\Omega| \leq C$$

This finishes the uniform $H_0^1(\Omega)$ estimates for f_n and the $H_0^1(\Omega)$ estimate for m_n follow similarly.

We multiply (3) by $-\Delta s_n$ and integrate by parts over Ω to obtain,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} |\nabla s_n|_2^2 + \delta |\nabla s_n|_2^2 + D |\Delta s_n|_2^2 \\ &\leq \int_{\Omega} \frac{\partial s_n}{\partial t} \left(\frac{1}{|s_n|^{q+1}} \right) \left(p \frac{f_n}{1+f_n} \right)^{p-1} \left(\frac{1}{1+f_n} \right)^2. \end{aligned} \tag{53}$$

Cauchy Schwartz and Holder's inequality, followed via the use of Gronwall's lemma in the time interval $[t_4, t]$ yield,

$$|\nabla s_n(t)|_2^2 \leq e^{-CDt} |\nabla s_n(t_4)|_2^2 + C. \tag{54}$$

Thus there exists a time $t_4^* = \max \left(0, t_4, \ln \left(\frac{|\nabla s_n(t_4)|_2^2}{CD} \right) \right)$, such that for $t > t_4^*$ we obtain,

$$|\nabla s_n(t)|_2^2 \leq C. \tag{55}$$

Assume the compactification time for ∇m , is t_3^m . We set

$$t_3^* = \max(t_3, t_3^m, t_4^*)$$

We can now state the following lemma,

Lemma 3.4 *Let f, m, s , be solutions to the modified Trojan Y Chromosome system with $f_0, m_0, s_0 \in L^2(\Omega)$. There exists a time t_3^* , and a constant C independent of time and the index n , and depending only on D, K, β, δ and μ , such that for any $t > t_3^*$ the following uniform a priori estimates hold,*

$$|\nabla f_n|_2^2 \leq C, \tag{56}$$

$$|\nabla m_n|_2^2 \leq C, \tag{57}$$

$$|\nabla s_n|_2^2 \leq C, \tag{58}$$

The above uniform estimates enable us to extract subsequences f_{n_j}, m_{n_j} and s_{n_j} such that

$$f_{n_j} \rightharpoonup f \text{ in } H_0^1(\Omega),$$

$$m_{n_j} \rightharpoonup m \text{ in } H_0^1(\Omega),$$

$$s_{n_j} \rightharpoonup s \text{ in } H^1(\Omega).$$

However via the compact Sobolev embedding of

$$X \hookrightarrow H \tag{59}$$

it follows that

$$f_{n_j} \rightarrow f \text{ in } L^2(\Omega),$$

$$m_{n_j} \rightarrow m \text{ in } L^2(\Omega),$$

$$s_{n_j} \rightarrow s \text{ in } L^2(\Omega).$$

This enables us to state the following Lemma,

Lemma 3.5 *The semigroup $\{S(t)\}_{t \geq 0}$ for the modified Trojan Y Chromosome model is asymptotically compact in H .*

We can now state the following theorem,

Theorem 3.6 *Consider the modified Trojan Y Chromosome model, (1)-(3). There exists a (H, H) global attractor \mathcal{A} for this system which is compact and invariant in H and attracts all bounded subsets of H in the H metric.*

Proof 3.7 *The proof of bounded absorbing set is provided via (42), (46). This combined with the asymptotic compactness of the semigroup via lemma 3.5 establishes the result.*

4 Finite dimensionality of the global attractor

In this section we show that the dimension of the global attractor for the modified TYC model is finite. We provide upper bounds on its Hausdorff and fractal dimensions in terms of parameters in the model. There is a standard methodology to this end. We consider a volume element in the phase space, and try and derive conditions that will cause it to decay. If \mathcal{A} is the global attractor of the semi-group $\{S(t)\}_{t \geq 0}$ in $L^2(\Omega)$ associated with the modified TYC model, then the trace of the linear operator $A + \delta + F'(S(\tau)u_0)$, where F is the nonlinear map in (1)- (3) can be projected onto an n dimensional subspace formally. Let $q_n = \limsup_{t \rightarrow \infty} q_n(t)$ where

$$\sup_{u_0 \in \mathcal{A}} \sup_{g_i \in H, \|g_i\|=1, 1 \leq i \leq n} \frac{1}{t} \int_0^t \text{Tr}(\Delta U(\tau) - \delta U(\tau) + F'(S(\tau)u_0) \circ Q_n(\tau)) d\tau$$

Here Q_n is the orthogonal projection of the phase space H onto the subspace spanned by $U_1(t), U_2(t), \dots, U_n(t)$, with

$$U_i(t) = L(S(t)u_0)u_i, i = 1, 2, \dots, n.$$

$L(S(t)u_0)$ is the Frechet derivative of the map $S(t)$ at u_0 . We recall the following Lemma from [16],

Lemma 4.1 *If there is an integer n such that $q_n < 0$ then the Hausdorff dimension $d_H(\mathcal{A})$ and the fractal dimension $d_F(\mathcal{A})$ of \mathcal{A} satisfy*

$$d_H(\mathcal{A}) \leq n \text{ and } d_F(\mathcal{A}) \leq 2n$$

For the TYC system, $L(S(t)u_0)U_0 = U(t) = (F(t), M(t), S(t))$, where $u = (f, m, s)$ is a solution to the modified TYC system. Also in our case we will denote $\phi_j = (\phi_j^1, \phi_j^2, \phi_j^3)$ to be an orthonormal basis for the subspace $Q_n(\tau)H$. The first variational equation for this system is explicitly worked out, where f, m, s , are functions of t :

$$\begin{aligned} \frac{\partial F}{\partial t} &= D\Delta F - \delta F + \beta \left((fM + mF) \left(1 - \frac{f+m+s}{K} \right) - fm \frac{F+M+S}{K} \right) \\ \frac{\partial M}{\partial t} &= D\Delta M + -\delta M + \frac{\beta}{2} \left((fM + mF) \left(1 - \frac{f+m+s}{K} \right) - fm \frac{F+M+S}{K} \right) \\ &\quad + \beta \left((fS + sF) \left(1 - \frac{f+m+s}{K} \right) - fm \frac{F+M+S}{K} \right) \\ \frac{\partial S}{\partial t} &= D\Delta S - \delta S + \frac{C}{|s|^q} p \left(\frac{f}{1+f} \right)^{p-1} \frac{1}{(1+f)^2} - \frac{qC}{|s|^{q+1}} \\ F(0) &= F_0, \quad M(0) = M_0, \quad S(0) = S_0, \end{aligned}$$

We now estimate

$$\begin{aligned}
& Tr(\Delta U(\tau) - \delta U(\tau) + F'(S(\tau)u_0) \circ Q_n(\tau)) \\
&= \sum_{j=1}^n \left(\langle \Delta \phi_j(\tau), \phi_j(\tau) \rangle + \langle F'(S(\tau)u_0) \phi_j(\tau), \phi_j(\tau) \rangle - \delta \langle \phi_j(\tau), \phi_j(\tau) \rangle \right) \\
&\leq \sum_{j=1}^n 3D |\nabla \phi_j(\tau)|^2 + 3\delta |\phi_j(\tau)|^2 + J_1 + J_2 + J_3
\end{aligned}$$

Here, with f , m , s , and r as functions of τ ,

$$\begin{aligned}
J_1 &= \sum_{j=1}^n \int_{\Omega} ((f\phi_j^2(\tau)\phi_j^1(\tau) + m|\phi_j^1|^2) \left(1 - \frac{f+m+s}{K}\right) \\
&\quad - fm(|\phi_j^1|^2 + \phi_j^1\phi_j^2 + \phi_j^1\phi_j^3) d\mathbf{x} \\
&\leq \sum_{j=1}^n \int_{\Omega} ((f\phi_j^2(\tau)\phi_j^1(\tau) + m|\phi_j^1|^2) - fm(|\phi_j^1|^2 + \phi_j^1\phi_j^2 + \phi_j^1\phi_j^3) d\mathbf{x} \\
&\leq (K^2 + 2K) \sum_{j=1}^n (|\phi_j|_2^2), \\
J_2 &= \sum_{j=1}^n \int_{\Omega} ((f|\phi_j^2(\tau)|^2 + m\phi_j^1\phi_j^2) \left(1 - \frac{f+m+s}{K}\right) \\
&\quad - fm(\phi_j^1\phi_j^2 + |\phi_j^2|^2 + \phi_j^2\phi_j^3) d\mathbf{x} \\
&\quad + \sum_{j=1}^n \int_{\Omega} ((f|\phi_j^1(\tau)\phi_j^3(\tau) + m\phi_j^3\phi_j^2) \left(1 - \frac{f+m+s}{K}\right) \\
&\quad - fm(\phi_j^1\phi_j^2 + |\phi_j^2|^2 + \phi_j^2\phi_j^3) d\mathbf{x} \\
&\leq 3 \sum_{j=1}^n \int_{\Omega} ((f\phi_j^2(\tau)\phi_j^1(\tau) + m|\phi_j^1|^2) + K^2(|\phi_j^1|^2 + \phi_j^1\phi_j^2 + \phi_j^1\phi_j^3) d\mathbf{x} \\
&\quad + K^2(|\phi_j^1|^2 + \phi_j^1\phi_j^2 + \phi_j^1\phi_j^3) d\mathbf{x} \\
&\quad + 3 \sum_{j=1}^n \int_{\Omega} ((f|\phi_j^1(\tau)\phi_j^3(\tau) + m\phi_j^3\phi_j^2) + K^2(|\phi_j^1|^2 + \phi_j^1\phi_j^2 + \phi_j^1\phi_j^3) d\mathbf{x} \\
&\leq 9(K + K^2) \sum_{j=1}^n |\phi_j|_2^2, \\
J_3 &= \sum_{j=1}^n \int_{\Omega} \left(\frac{C}{|s|^q} p \left(\frac{f}{1+f} \right)^{p-1} \frac{1}{(1+f)^2} - \frac{qC}{|s|^{q+1}} \right) \phi_j^2 \\
&\leq Cq \sum_{j=1}^n |\phi_j|_2^2
\end{aligned}$$

Thus we obtain the estimate,

$$\begin{aligned}
 & Tr(\Delta U(\tau) - \delta U(\tau) + F'(S(\tau)u_0) \circ Q_n(\tau)) \\
 \leq & -\sum_{j=1}^n 3D|\nabla\phi_j(\tau)|_2^2 - 3\delta|\phi_j(\tau)|_2^2 + (9(K + K^2) + Cq) \sum_{j=1}^n |\phi_j|_2^2 \\
 = & -3D \sum_{j=1}^n |\nabla\phi_j(\tau)|_2^2 + (-3\delta + 9(K + K^2) + Cq) \sum_{j=1}^n |\phi_j(\tau)|_2^2 \\
 \leq & -3D \sum_{j=1}^n |\nabla\phi_j(\tau)|_2^2 + (9(K + K^2) + Cq - 3\delta)n
 \end{aligned}$$

Now via the generalized Sobolev-Lieb-Thirring inequalities [16] we obtain

$$\sum_{j=1}^n |\nabla\phi_j(\tau)|_2^2 \geq K_1 \frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}}$$

Here K_1 depends only on the shape and dimension of Ω . Thus we obtain

$$Tr(\Delta U(\tau) - \delta U(\tau) + F'(S(\tau)u_0) \circ Q_n(\tau)) \leq -3DK_1 \frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (9(K + K^2) + Cq - 3\delta)n,$$

for $\tau > 0$, $u_0 \in A$. We now obtain

$$\begin{aligned}
 q_n(t) &= \sup_{u_0 \in A} \sup_{g_i \in H, \|g_i\|=1, 1 \leq i \leq n} \frac{1}{t} \int_0^t Tr(\Delta U(\tau) - \delta U(\tau) + F'(S(\tau)u_0) \circ Q_n(\tau)) d\tau \\
 &\leq -3DK_1 \frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (9(K + K^2) + Cq - 3\delta)n, \quad \forall t > 0.
 \end{aligned}$$

This yields

$$q_n = \limsup_{t \rightarrow \infty} \leq -3DK_1 \frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (9(K + K^2) + Cq - 3\delta)n < 0$$

If the integer n satisfies

$$n - 1 < \left(\frac{9(K + K^2) + Cq - 3\delta}{3DK_1} \right)^{\frac{3}{2}} |\Omega| < n$$

We can now use lemma 4.1 to obtain the following result

Theorem 4.2 *Consider the modified Trojan Y Chromosome model, (1)-(3). The global attractor \mathcal{A} of the system is of finite dimension. Furthermore explicit upper bounds for its Hausdorff and fractal dimensions are as follows*

$$d_H(\mathcal{A}) \leq \left(\frac{9(K + K^2) + Cq - 3\delta}{3DK_1} \right)^{\frac{3}{2}} |\Omega| + 1$$

$$d_F(\mathcal{A}) \leq 2 \left(\frac{9(K + K^2) + Cq - 3\delta}{3DK_1} \right)^{\frac{3}{2}} |\Omega| + 2$$

5 Conclusion

In this work we have shown rigorously the existence of global attractor for the modified TYC system. Note that a necessary condition for the existence of a global attractor is the presence of a bounded absorbing sets in the phase space, whose existence imply that indeed the population of invasive species will be confined to bounded regions after long time. However, Showing finite time extinction is much harder due to the nonlinear source term considered, and although much more practical, might altogether not be mathematically possible. Thus rigorously showing finite time extinction, with a decaying or weakly decaying source, remains a current challenge. We further show that the global attractor is finite dimensional. Upper bounds on the dimension of the global attractor gives us information on the degrees of freedom of the modified TYC system. This is always desirable from the point of view of numerical computation.

The modified TYC model assumes that the rate of introduction μ of Trojan individuals s is decaying in s . It is conceivable to think of μ as a periodic function in time, e.g. Trojan fish introduced every week or every month to the domain. It is an extremely interesting question to consider a stochastic μ , as a future direction of research. Such as perhaps a white noise perturbation.

The modified TYC strategy has its own challenges. The model assumes that all s fish added to the system are reproductively competent adults. This may not always be the case, and there are curious issues on the longevity of the trojan individuals. It would be interesting to consider weaker or nonlinear death rates, for the s , in this case. This would also force the system to be only weakly dissipative. Also hormone-induced sex reversal of fish is most easily accomplished by feeding juveniles with feed containing hormone. This, method works well for Nile tilapia, but was found to be inefficient for use with grass carps [21]. Efficient sex reversal of grass carp requires a more labor-intensive procedure involving the use of hormone-containing silastic implants. However, sex reversal of fish using silastic implants may not be amenable to high levels

of production. Since the eradication strategy require a significant investment in technology to construct the Trojan fish, it is conceivable that adding just a “supermale” to a population, as we do in the modified model, is more practical than adding a feminised “supermale”, as was done earlier.

We believe these results will provide an effective strategy for eradication/containment of invasive aquatic species. Our ultimate goal is to help biologists carry out this strategy in a realistic scenario, in order to combat invasive species in an eco friendly way, aiding ailing fishing industries, and reduce government and industry expenditures.

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