A Direct Radiative Transfer Equation Solver for Path Loss Calculation of Underwater Optical Wireless Channels

Changping Li, Ki-Hong Park, *Member, IEEE,*

and Mohamed-Slim Alouini, *Fellow, IEEE*

Abstract

In this letter, we propose a fast numerical solution for the steady state radiative transfer equation in order to calculate the path loss due to light absorption and scattering in various type of underwater channels. In the proposed scheme, we apply a direct non-uniform method to discretize the angular space and an upwind type finite difference method to discretize the spatial space. A Gauss-Seidel iterative method is then applied to solve the fully discretized system of linear equations. The accuracy and efficiency of the proposed scheme is validated by Monte Carlo simulations.

Index Terms

Radiative transfer equation, underwater optical path loss, finite difference method, Gauss-Seidel iteration.

I. INTRODUCTION

The underwater environment provides a promising area for the application of optical wireless communications. Channel modeling and characterization are the key steps for efficient, reliable, and robust underwater optical wireless communication (UOWC) system design. Compared with free space optics, the underwater channel is much more complex due to two mechanisms

C. Li, K.-H. Park, and M.-S. Alouini are with the Computer, Electrical, and Mathematical Sciences and Engineering (CEMSE) Division, King Abdullah University of Science and Technology (KAUST), Thuwal, Makkah Province, Saudi Arabia. (E-mail: {changping.li, kihong.park, slim.alouini}@kaust.edu.sa)

corrupting the UOWC channel, namely, absorption and scattering. It is reported in [1] that pure water has 1000 times the attenuation of clear air and that turbid water has more than 100 times the attenuation of the densest fog. Therefore, one of the main targets in UOWC channel modeling is to evaluate the overall path loss which is essential for calculating link budgets and signal-to-noise ratio.

The Beer's law which is essentially the simple exponential attenuation model, is typically applied to calculate the optical path loss owing to its simplicity [2]–[4]. However, it is assumed that all the scattered photons are lost even though in reality some of the scattered photons can still be captured by the receiver after multiple scattering. By doing so, it can severely underestimate the received power, especially in the scattering dominant regime. On the other hand, the radiative transfer equation (RTE) is a more general theoretical model which takes the multiple scattering into account. It clearly describes the behavior of energy conservation for light propagating through an absorbing, scattering and emitting medium (e.g. water). However, the RTE is a complex integro-differential equation of several variables [5] and finding a general analytical solution for this equation is not possible.

In view of this, most of the recent works have proposed Monte Carlo (MC) simulation approaches instead of solving the integro-differential equation directly [6]–[8]. MC simulation is a probabilistic method which finds a solution by sending and tracking large numbers of photons individually through a water body. This method is flexible, easy to program, and accurate. However it has drawbacks: i) it is not suitable for the simulation of a point source and a point detector; ii) it cannot address wave phenomena; and iii) it tends to be very time consuming as millions (or billions) of photons are needed in order to accurately simulate a real-world situation [9]. For instance, if N is the number of photons used in the MC simulation, the rate of convergence is $\propto N^{-1/2}$ in the worst case [10].

In this letter, we develop an efficient numerical solution for the 2-dimensional (2D) steady state RTE to compute the received power of UOWC systems. The previously proposed numerical method, namely, discrete ordinate method (DOM) in [11] cannot handle highly forward peaked volume scattering functions (VSF) which is the case for ocean water [12]. In the proposed scheme, we apply a direct non-uniform angular discretization along an upwind type spatial discretization method. Since the non-uniform discretization of the integral term in RTE is done directly in the angular space, the characterization of the strong forward scattering can be well

captured. To solve the fully discetized large system of linear equations, we adopt the matrix free Gauss-Seidel iterative method. Simulation results show that the proposed scheme computes the received power with an accuracy comparable to MC simulations but with much more reduced computational time.

II. PROPOSED RTE SOLVER

We modify the general time dependent 3-dimensional (3D) RTE given in [5] to the steady state 2D RTE as:

$$\vec{n} \cdot \nabla L(\vec{r}, \vec{n}) = -cL(\vec{r}, \vec{n}) + \int_{2\pi} \beta(\vec{n}, \vec{n}')L(\vec{r}, \vec{n}')d\vec{n}' + E(\vec{r}, \vec{n}), \qquad (1)$$

where we note that all the quantities in (1) are wavelength dependent. In this letter, we omit the wavelength for brevity by assuming that single wavelength is used (blue or green). In (1), $c = a+b$ (m^{-1}) is the attenuation coefficient which is the summation of the absorption coefficient a and scattering coefficient b, ∇ is the divergence operator. $L(\vec{r}, \vec{n})$ is the optical radiance at position $\vec{r}$ propagating towards direction $\vec{n}$, with unit of $Wm^{-2}sr^{-1}$, $E(\vec{r}, \vec{n})$ is the source radiance, and $\beta(\vec{n}, \vec{n}')$ is the VSF which is related to the scattering phase function $\tilde{\beta}(\vec{n}, \vec{n}')$ as

$$\beta(\vec{n}, \vec{n}') = b\tilde{\beta}(\vec{n}, \vec{n}'). \qquad (2)$$

The phase function describes the angular distribution of the scattered photons. In this letter, we apply the 2D Henyey-Greenstein (H-G) phase function

$$\tilde{\beta}(\vec{n}, \vec{n}') = \frac{1 - g^2}{2\pi(1 + g^2 - 2g\vec{n} \cdot \vec{n}')}. \qquad (3)$$

Defining θ as the scattering angle between $\vec{n}$ and $\vec{n}'$, i.e., $\vec{n} \cdot \vec{n}' = \cos\theta$, the phase function can be rewritten as

$$\tilde{\beta}(\theta) = \frac{1 - g^2}{2\pi(1 + g^2 - 2g\cos\theta)}, \qquad (4)$$

where g is the asymmetry parameter (ranging from 0 to 1) which decides the scattering type. For instance, when isotropic scattering dominates, $g = 0$, while g is close to 1 in presence of peaked scattering. As an example, ocean waters are strong forward scattering media with g values in the range of 0.8 to 0.95 [9], which is potentially beneficial for the communication link.
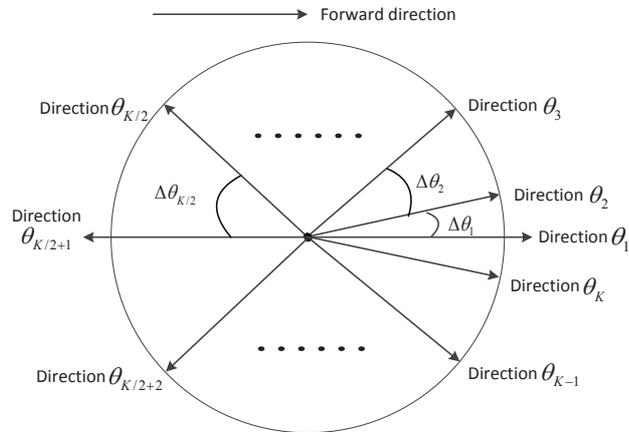
Fig. 1. Direct non-uniform angular discretization.

(a) 2D mesh structure

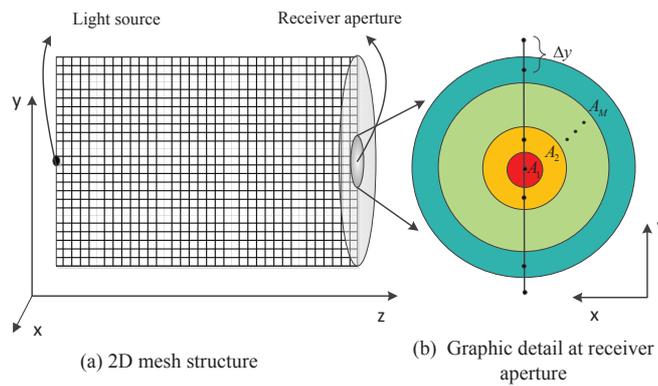(b) Graphic detail at receiver aperture

Fig. 2. The overall system configuration.

To solve the integro-differential RTE numerically, we firstly discretize both angular and spatial variables and then solve the fully discretized large system of linear equations by a Gauss-Seidel iteration method.

A. Angular Discretization

As the VSF of ocean water is highly peaked in the forward direction, we take advantage of this inherent quality to accelerate the calculation by a direct non-uniform angular discretization. In the 2D case, the angular variable ranges in $[0, 2\pi)$. As shown in Fig. 1, we discretize the angular space into K directions unequally with the angle interval $\Delta\theta_k$. We note that the angular discretization in $(0, \pi)$ and $(\pi, 2\pi)$ are symmetric in regard to the forward direction. The value

of the H-G phase function in (4) depends on the scattering angle θ between two direction $\vec{n}$ and $\vec{n}'$. For the highly peaked H-G phase function, it possesses larger value at smaller θ, which means that the value at the small θ is dominating. Therefore, we refine the discretization in dominant region and coarsen it otherwise as shown in Fig. 1. The proposed scheme can capture the characteristic of scattering in water more effectively and maintain good accuracy with much reduced number of discrete angular directions K.

As shown in Fig. 2(a), we assume that light beam is transmitted along the Z axis. We solve the 2D steady state RTE on YOZ plane. The angular direction $\vec{n}$ can be expressed in cartesian coordinates with $\vec{e}_y \cdot \vec{n} = \sin\theta$ and $\vec{e}_z \cdot \vec{n} = \cos\theta$, where $\vec{e}_y$ and $\vec{e}_z$ are the unit vectors along the Y and Z axis. After replacing the integral term by the summation of discrete form of phase function, $w_{k,k'}$, the RTE on a particular discrete direction k can be given as

$$
\sin\theta_k \frac{\partial L_k(\vec{r})}{\partial y} + \cos\theta_k \frac{\partial L_k(\vec{r})}{\partial z}
$$
$$
= -cL_k(\vec{r}) + b\sum_{k'=1}^{K} w_{k,k'} L_{k'}(\vec{r}) + E_k(\vec{r}).
$$
(5)

We apply the Simpson's rule [13] which is a method for numerical integration to compute $w_{k,k'}$. Firstly, we compute $w_{1,k'}$ as

$$
w_{1,k'} = \begin{cases}
\frac{\Delta\theta_1}{3}(2\tilde{\beta}(\theta_1) + \tilde{\beta}(\theta_2)), & k' = 1 \\
\frac{(\Delta\theta_{k'} + \Delta\theta_{k'-1})}{12}(\tilde{\beta}(\theta_{k'-1}) + 4\tilde{\beta}(\theta_{k'}) + \tilde{\beta}(\theta_{k'+1})), & k' = 2, \ldots, \frac{K}{2} \\
\frac{\Delta\theta_{K/2}}{3}(2\tilde{\beta}(\theta_{K/2+1}) + \tilde{\beta}(\theta_{K/2})), & k' = \frac{K}{2} + 1 \\
w_{1,K+2-k'}, & k' = \frac{K}{2} + 2, \ldots, K,
\end{cases}
$$
(6)

where $\Delta\theta_{k'}$ is the angle interval between discrete angular direction k' and $k'+1$ and $\theta_{k'}$ is the corresponding angle of the angular direction k'. Note that the scattering phase function is a probability density function (PDF) which means that the integration of it is one. To preserve this statistical property, we scale the weight as $w_{1,k'} \Big/ \sum_{k'=1}^{K} w_{1,k'}$ so that the summation of each row of the weight matrix remains one. Since the phase function only depends on the angle between two directions, we can derive all $w_{k,k'}$ simply from $w_{1,k'}$ as follows

$$
w_{k,k'} = w_{1,|k'-k|+1}.
$$
(7)

B. Spatial Discretization

As shown in Fig. 2(a), the highly collimated light source is placed in the middle of the left boundary and the receiver is aligned on the opposite side XOY plane. We assume vacuum boundary condition which means that the incoming radiance on the boundary of the interested area of water body is zero.

A rectangular spatial mesh is applied with I grid points on the Y coordinate and J grid points on the Z coordinate. The step sizes along Y axis and Z axis are Δy and Δz, respectively. The smaller the step size is, the more accurate results can be obtained. However, there is trade-off between accuracy and computational time. Let $L_{k,i,j}$ represent the radiance at the grid point (i, j) towards angular direction k. We replace the spatial derivatives in (5) with the upwind type finite difference. Each spatial derivative has two distinct difference formulas according to the sign of $\sin \theta_k$ and $\cos \theta_k$. Letting $\eta_k = \sin \theta_k$ and $\xi_k = \cos \theta_k$, we have

$$\frac{\partial L_{k,i,j}}{\partial y} \approx \begin{cases} \frac{L_{k,i,j} - L_{k,i-1,j}}{\Delta y}, & \eta_k > 0 \\ \frac{L_{k,i+1,j} - L_{k,i,j}}{\Delta y}, & \eta_k < 0. \end{cases} \tag{8}$$

$$\frac{\partial L_{k,i,j}}{\partial z} \approx \begin{cases} \frac{L_{k,i,j} - L_{k,i,j-1}}{\Delta z}, & \xi_k > 0 \\ \frac{L_{k,i,j+1} - L_{k,i,j}}{\Delta z}, & \xi_k < 0. \end{cases} \tag{9}$$

For $\eta_k > 0$ and $\xi_k > 0$, substituting (8) and (9) into (5), we get the fully discretized form of RTE as

$$\eta_k \frac{L_{k,i,j} - L_{k,i-1,j}}{\Delta y} + \xi_k \frac{L_{k,i,j} - L_{k,i,j-1}}{\Delta z} + cL_{k,i,j}$$
$$= b \sum_{k'=1}^{K} w_{k,k'} L_{k',i,j} + E_{k,i,j}. \tag{10}$$

For the other three cases, the procedure is just straightforward and we will not go into detail hereafter.

C. Gauss-Seidel Iterative Method

We apply the Gauss-Seidel iteration to solve the fully discretized system of linear equations. This iteration accelerates the convergence by guaranteeing that all the values are updated by newly

calculated results without delay. After some basic manipulation on (10), we get the following iterative formula

$$L_{k,i,j}^{l+1} =$$

$$\frac{(\frac{\eta_k}{\Delta y})L_{k,i-1,j}^{l+1} + (\frac{\xi_k}{\Delta z})L_{k,i,j-1}^{l+1} + b \sum_{k'=1}^{K} w_{k,k'}L_{k',i,j}^{l} + E_{k,i,j}}{\frac{\eta_k}{\Delta y} + \frac{\xi_k}{\Delta z} + c}, \tag{11}$$

where l is the iterative index. The iteration process is repeated until the relative error norm $\frac{\left\| L_{k,i,j}^{l+1} - L_{k,i,j}^{l} \right\|}{\left\| L_{k,i,j}^{l+1} \right\|}$ at the grid point (i,j) is smaller than a predetermined termination value σ.

III. COMPUTATION OF THE RECEIVED POWER

As shown in Fig. 2, the receiver is fixed on the plane perpendicular to Z axis (e.g., XOY plane). Since the radiance calculation in section II is processed on 2D rectangular mesh on YOZ plane, we finally get the radiance at the grid points along Y axis at the receiver. In this section, we show the scheme to compute the received power from the radiance calculated by the 2D steady state RTE. As shown in Fig. 2(b), $A_1, A_2, \ldots, A_M$ are the areas of the regions with the grid points located in its middle and marked with different colours respectively, which can be calculate as

$$A_m = \begin{cases} \pi\left(\frac{\Delta y}{2}\right)^2, & m = 1 \\ \pi\left(\frac{\Delta y}{2} + (m-1)\Delta y\right)^2 - \pi\left(\frac{\Delta y}{2} + (m-2)\Delta y\right)^2, & m = 2, 3, \ldots, M, \end{cases} \tag{12}$$

where M is the number of regions within the receiver aperture. Let R be the radius of the receiver aperture, we have $M = R/\Delta y$. We assume that the scattering is symmetric in the azimuthal direction, which means that the radiance for any azimuthal angle on the same circle is the same. As such the radiance at each grid point along Y axis can be regarded as the average radiance in the corresponding region where it located. Therefore, the received power can be obtained by

$$P_r = \sum_{m=1}^{M} A_m \left(\sum_{k=1}^{K'} L_{m+\left(\frac{I-1}{2}\right),J,k} \right), \tag{13}$$

where K' is the number of discrete directions within the receiver field of view (FOV).

TABLE I

PARAMETERS OF THREE WATER TYPES

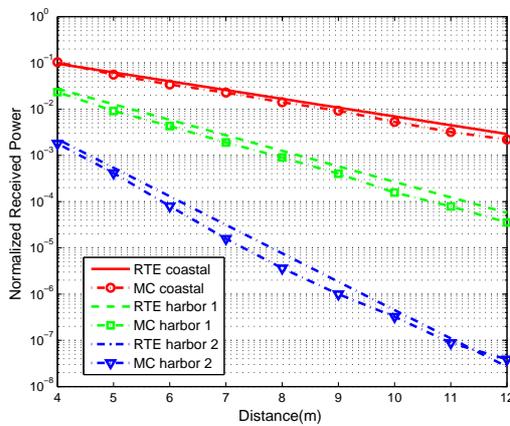| Water type	$c(m^{-1})$	Albedo
Coastal	0.568	0.60
Harbor 1	1.1	0.85
Harbor 2	2.19	0.85

Fig. 3. Normalized received power versus transmit distance for receiver aperture = 0.1m and receiver FOV = 180°.

IV. SIMULATION RESULTS

The parameters of three water types investigated in the simulation are listed in Table I. All of them are the typical values of coastal and harbor waters. The single scattering albedo is defined as the ratio of b/c. The step size $\Delta y = 0.01m$, $\Delta z = 0.05m$, number of angular directions $K = 16$, and iterative termination value $\sigma = 10^{-4}$ are used in the proposed RTE solver. The asymmetry parameter $g = 0.924$ is applied for the H-G phase function, which is a good approximation for the practical situations [7] . We set the angle intervals as $\Delta\theta_k = k\Delta\theta_1$, for $k \in \{1, 2, \ldots, K/2\}$. Although it is not optimal, it is still quiet efficient in comparison with the simple uniform angular discretization. At the receiver, the receiver aperture=$0.1m$ and FOV =180° are used in the simulation. MC simulation provided in [14] is applied for comparison.

The normalized received power at different transmit distance for different water types is shown in Fig. 3. We can see that the results from the proposed RTE solver and MC simulation agree
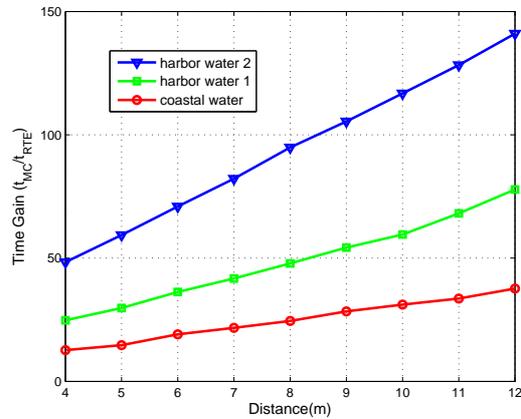
Fig. 4. Time gain versus transmit distance for receiver aperture = 0.1m and receiver FOV = 180°.

with each other. Fig. 4 shows the efficiency of the proposed numerical RTE method in terms of ratio of simulation running time between MC and RTE. The proposed scheme is much faster than MC simulation and the former can provide results in a few seconds. As shown in Fig. 4, the time gain becomes more prominent as the turbidity of the water increases. This is because the proposed RTE solver is not sensitive to the c value. Changes in c barely affects the simulation time since we do not change the step size. However, the MC simulation is highly affected by c value since the photons encounter more scattering with higher c.

V. CONCLUSION

In this work, an efficient numerical scheme was proposed to solve the 2D steady state RTE. As a result, we can calculate the path loss of UOWC channel at the receiver by the radiance obtained from the RTE slover. Simulation results show that the proposed scheme can compute the received power with much faster speed than that of MC, while keeping a comparable accuracy.

APPENDIX A
MATLAB CODE OF THE RTE SOLVER

```
clear all
clc
c=2.19;
```

```matlab
albedo = 0.85;
b = c * albedo;
a=c-b;
length_z =12;
start=4;
step_z=0.05;
length_y=0.2;
aperture=0.1;
step_y=0.01;
I=(length_y/step_y)+1
J=(length_z/step_z)+1
M=(aperture/step_y)+1
K=16;
g=0.924;
r(1)=step_y/2;
s=zeros(1,(M-1)/2+1);
s(1,1)=pi*r(1)^2;
for n=2:(M-1)/2+1
    r(n)=r(n-1)+step_y;
    s(n)=pi*r(n)^2-s(n-1);
end
deta_theta1=2/(K/2*(K/2+1))*pi;
av_deta_theta=2*pi/K;
radiance=zeros(I,J,K);
q=zeros(I,J,K);
q(((I-1)/2)+1,1,1)=1/s(1,1)/deta_theta1;
[w, theta]=weight(K,g)
tic
max_iter=300;
for l=1:max_iter
    l
```

```
for k=1:K
    denominator1(k)=theta(k,2)/step_y+theta(k,1)/step_z+c;
    denominator2(k)=theta(k,2)/step_y-theta(k,1)/step_z+c;
    denominator3(k)=-theta(k,2)/step_y-theta(k,1)/step_z+c;
    denominator4(k)=-theta(k,2)/step_y+theta(k,1)/step_z+c;
    if theta(k,1)>0 && theta(k,2)>0
        for i=2:I
            for j=2:J
                for n=1:K
                    sum1(n)=radiance(i,j,n)* w(k,n);
                end
                numerator(i,j,k)=sum(sum1)*b
                +(radiance(i-1,j,k)
                *theta(k,2)/step_y)+(radiance(i,j-1,k)
                *theta(k,1)/step_z)+q(i-1,j-1,k);
                radiance_temp(i,j,k)
                =numerator(i,j,k)/denominator1(k);
            end
        end
    elseif theta(k,1)<0 && theta(k,2)>0
        for i=2:I
            for j=1:J-1
                for n=1:K
                    sum1(n)=radiance(i,j,n)* w(k,n);
                end
                numerator(i,j,k)=sum(sum1)*b
                +(radiance(i-1,j,k)
                *theta(k,2)/step_y)-(radiance(i,j+1,k)
                *theta(k,1)/step_z)+q(i-1,j,k);
                radiance_temp(i,j,k)
                =numerator(i,j,k)/denominator2(k);
```

```
            end
        end
    elseif theta(k,1)<0 && theta(k,2)<0
        for i=1:I-1
            for j=1:J-1
                for n=1:K
                    sum1(n)=radiance(i,j,n)* w(k,n);
                end
                numerator(i,j,k)=sum(sum1)*b
                -(radiance(i+1,j,k)
                *theta(k,2)/step_y)-(radiance(i,j+1,k)
                *theta(k,1)/step_z)+q(i,j,k);
                radiance_temp(i,j,k)
                =numerator(i,j,k)/denominator3(k);
            end
        end
    else
        for i=1:I-1
            for j=2:J
                for n=1:K
                    sum1(n)=radiance(i,j,n)* w(k,n);
                end
                numerator(i,j,k)=sum(sum1)*b
                -(radiance(i+1,j,k)
                *theta(k,2)/step_y)+(radiance(i,j-1,k)
                *theta(k,1)/step_z)+q(i,j-1,k);
                radiance_temp(i,j,k)
                =numerator(i,j,k)/denominator4(k);
            end
        end
    end
```

```
    end
    radiance=radiance_temp;
end
radiance;
toc
min=toc/60
intensity=zeros(I,J);
for k=1:K
    intensity=intensity+radiance(:,:,k);
end
intensity=intensity*av_deta_theta;
s;
power=s*intensity((I-1)/2+1:(M-1)/2+(I-1)/2+1,:);
for ii=1:1:J-start/step_z
    powertrc(ii)=power(ii+start/step_z);
end
powertrc;
z=start:step_z:length_z;
semilogy(z,powertrc)
hold on
power2=exp(-c*z);
semilogy(z,power2)
hold on
power3=exp(-a*z);
semilogy(z,power3)

function [w theta]=weight(N_angle,g)
w=zeros(N_angle,N_angle);
m=(N_angle+2)/2;


dx=2/(m*(m-1))*pi;
```

```
x(1)=0;
for i=2:m
    x(i)=x(i-1)+(i-1)*dx;
end
f=fun(x,g);


w(1,1)=2*1/6*(2*f(1)+f(2))*dx;
w(1,m)=2*1/6*(f(m-1)+2*f(m))*(x(m)-x(m-1));


for i=2:m-1
    w(1,i)=1/6*(f(i-1)+2*f(i))*(x(i)-x(i-1))+1/6*(2*f(i)
    +f(i+1))*(x(i+1)-x(i));
    w(1,2*m-i)=w(1,i);
end
w(1,:)=w(1,:)/sum(w(1,:));


for i=2:N_angle
    for j=1:N_angle
            nj=mod(j+1-i,N_angle);
       % nj=abs(j+1-i);
        if nj==0
             nj=N_angle;
        end
        w(i,j)=w(1,nj);
    end
end


theta=zeros(N_angle,3);
for i=1:N_angle
    theta(i,3)=2*pi*(i-1)/N_angle;
    theta(i,1)=cos(theta(i,3));
```

```
    theta(i,2)=sin(theta(i,3));
end


function f=fun(x,g)
% Henyey-Greenstein phase function here.
f=1/(2*pi)*(1-g^2)./(1+g^2-2*g*cos(x));
```

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