Introduction

Forward modeling lies at the heart of all inversion schemes. Its accuracy directly impacts how well the inverted model matches the true one. In addition, computational efficiency of inversion algorithms are usually bottle-necked by the forward modeling tool it resorts to. Thus, development of more accurate and efficient forward modeling algorithms is always needed. Seismologists have long recognized that sedimentary rocks induce anisotropic wave propagation. This anisotropic behavior is attributed to, among other things, the dominant thin layering accumulated in the sedimentation process. Specifically, since the layering has a general preferred direction, the TI assumption is the most practical kind of anisotropy to represent subsurface geology. The tilt is naturally set in the direction orthogonal to the layering. Developing efficient and accurate traveltime formulations for such a model is useful for many applications, including traveltime tomography and the integral-based Kirchhoff imaging.

To help improve the efficiency of TI traveltime calculation, which suffers from the complex dispersion relation relative to the isotropic case, we use perturbation theory to solve the quartic traveltime polynomial resulting from the discretized form of the TI eikonal equation. Specifically, we perturb in terms of the anellipticity parameter η, as this renders an easily solvable quadratic, possibly tilted, elliptic anisotropic equation. Since the perturbation is applied at the grid scale, which inherently assumes a homogenous step, η is allowed to vary freely. We investigate the accuracy and efficiency of the new eikonal solver, and demonstrate these assertions through tests on the VTI Marmousi model, comparing the proposed approach with the exact TI solver. We also compare our approximation with that of Fomel (2004), which illustrates remarkable accuracy and efficiency properties of the proposed formulation.

Traveltime Approximation for TI media

The 2D eikonal equation for a TTI medium, under the acoustic assumption, is given as (Alkhalifah, 1998):

$$v_{nmo}^2(1+2\eta)\left(\cos\theta\frac{\partial\tau}{\partial x}+\sin\theta\frac{\partial\tau}{\partial z}\right)^2+{v_0}^2\left(\cos\theta\frac{\partial\tau}{\partial z}-\sin\theta\frac{\partial\tau}{\partial x}\right)^2\left(1-2\eta v_{nmo}^2\left(\cos\theta\frac{\partial\tau}{\partial x}+\sin\theta\frac{\partial\tau}{\partial z}\right)^2\right)=1. \quad (1)$$

where $\tau(x,z)$ is the traveltime measured from the source to a point with the coordinates (x,z), v_0 and v_{nmo} are the vertical and NMO velocities measured along the symmetry axis, η denotes the anellipticity parameter, and θ represents the symmetry axis angle from vertical.

Numerical solution of equation 1 is computationally cumbersome, as it requires solving a quartic equation at each time evaluation step. Therefore, Stovas and Alkhalifah (2012) proposed the use of perturbation expansion as a trial solution. The proposed expansion was in term of the anellipticity parameter η. However, in perturbing the eikonal equation, all parameters related to the background medium are allowed to vary freely with position whereas the perturbation parameter is considered constant. In order to allow inhomogeneity in the perturbed parameter as well, we discretize eikonal equation 1 first and then perturb it in terms of the anellipticity parameter η. The first order discretization for the traveltime derivatives for an upwind case, as an example, is given by:

$$\frac{\partial\tau}{\partial x}=\frac{\tau_{i,j}-\tau_{i-1,j}}{\Delta x},\qquad \frac{\partial\tau}{\partial z}=\frac{\tau_{i,j}-\tau_{i,j-1}}{\Delta z}, \quad (2)$$

where $\tau_{i,j}$ represents traveltime at a particular grid point (i,j). The traveltimes $\tau_{i-1,j}$ and $\tau_{i,j-1}$ correspond to the neighboring stencil along the x- and z-directions, respectively. Δx and Δz denote the sizes of spatial grids along the x- and z-directions, respectively.

Substituting the expressions 2 into equation 1, we get the discretized TI eikonal equation:

$$v_{nmo}^2(1+2\eta)\left(\cos\theta\left(\frac{\tau_{i,j}-\tau_{i-1,j}}{\Delta x}\right)+\sin\theta\left(\frac{\tau_{i,j}-\tau_{i,j-1}}{\Delta z}\right)\right)^2+{v_0}^2\left(\cos\theta\left(\frac{\tau_{i,j}-\tau_{i,j-1}}{\Delta z}\right)-\sin\theta\left(\frac{\tau_{i,j}-\tau_{i-1,j}}{\Delta x}\right)\right)^2$$
$$\times\left(1-2\eta v_{nmo}^2\left(\cos\theta\left(\frac{\tau_{i,j}-\tau_{i-1,j}}{\Delta x}\right)+\sin\theta\left(\frac{\tau_{i,j}-\tau_{i,j-1}}{\Delta z}\right)\right)^2\right)=1. \quad (3)$$

Note that in equation 3 and what follows, v_0, v_{nmo} and θ denote the vertical velocity, NMO velocity and tilt, respectively, at grid point (i, j).

Let us consider the trial traveltime solution for equation 3,

$$\tau_{i,j} \approx \tau^0_{i,j} + \tau^1_{i,j}\,\eta_{i,j} + \tau^2_{i,j}\,(\eta_{i,j})^2, \tag{4}$$

where $\tau^0_{i,j}$, $\tau^1_{i,j}$, and $\tau^2_{i,j}$ are the coefficients of expansion with dimensions of traveltime. Note that $\tau^0_{i,j}$ is the solution of the discretized elliptical anisotropic eikonal equation. For practical purposes, we consider here only three terms of the expansion.

Next, we substitute the trial solution 4 into the discretized eikonal equation 3. Expanding the resulting expression and comparing the coefficients of $\eta_{i,j}$ from the left hand side to those on the right hand side yields the expressions for traveltime coefficients $\tau^0_{i,j}$, $\tau^1_{i,j}$, and $\tau^2_{i,j}$. Solving the resulting quadratic equation for $\tau^0_{i,j}$ gives:

$$\tau^0_{i,j} = \frac{\left(b_0\tau_{i-1,j}+c_0\tau_{i,j-1}\right)\cos^2\theta+a_0\left(\tau_{i-1,j}+\tau_{i,j-1}\right)+\left(d_0\tau_{i,j-1}+e_0\tau_{i-1,j}\right)\sin^2\theta+\sqrt{\Delta}}{(b_0+c_0)\cos^2\theta+2a_0+(d_0+e_0)\sin^2\theta}, \tag{5}$$

where

$$a_0 = \left(v^2_{nmo}-v^2_0\right)\Delta x\,\Delta z\sin\theta\cos\theta,\ b_0=(v_{nmo}\Delta z)^2,\ c_0=(v_0\Delta x)^2,\ d_0=(v_{nmo}\Delta x)^2,\ e_0=(v_0\Delta z)^2,$$

$$\Delta = \Delta x^2\,\Delta z^2\left(-v_0^2v^2_{nmo}\left(\tau_{i-1,j}-\tau_{i,j-1}\right)^2\cos^4\theta+2a_0+\left(b_0+c_0-2v^2_{nmo}v^2_0\left(\tau_{i-1,j}-\tau_{i,j-1}\right)^2\sin^2\theta\right)\cos^2\theta\right.$$
$$\left.+\left(d_0+e_0-v^2_{nmo}v^2_0\left(\tau_{i-1,j}-\tau_{i,j-1}\right)^2\sin^2\theta\right)\sin^2\theta\right).$$

Knowing $\tau^0_{i,j}$, we can solve the resulting linear equation in $\tau^1_{i,j}$ to obtain:

$$\tau^1_{i,j} = \frac{v^2_{nmo}(b_1\cos\theta+a_1\sin\theta)^2\left(v^2_0(a_1\cos\theta-b_1\sin\theta)^2-\Delta x^2\,\Delta z^2\right)}{\Delta x^2\,\Delta z^2\left(\left(v^2_{nmo}\Delta z\,b_1+v^2_0\Delta x\,a_1\right)\cos^2\theta+c_1\cos\theta\sin\theta+\left(v^2_{nmo}\Delta x\,a_1+v^2_0\Delta z\,b_1\right)\sin^2\theta\right)}, \tag{6}$$

where

$$a_1=\Delta x\left(\tau^0_{i,j}-\tau_{i,j-1}\right),\quad b_1=\Delta z\left(\tau^0_{i,j}-\tau_{i-1,j}\right),\quad c_1=\left(v^2_{nmo}-v^2_0\right)\Delta x\,\Delta z\left(2\tau^0_{i,j}-\tau_{i-1,j}-\tau_{i,j-1}\right).$$

Finally, solving the linear equation in $\tau^2_{i,j}$, and plugging the already computed traveltimes $\tau^0_{i,j}$ and $\tau^1_{i,j}$, yields:

$$\begin{aligned}\tau^2_{i,j} = \frac{-\tau^1_{i,j}}{D}&\left(-4\Delta x^2\,\Delta z^2\,a_2b_2c_2v^2_{nmo}v^2_0\cos^4\theta-4\Delta x\,\Delta z\left(-\Delta z^2\,a^2_2d_2+\Delta x^2\,b^2_2e_2\right)v^2_{nmo}v^2_0\cos^3\theta\sin\theta\right.\\&+2\Delta x\,\Delta z\cos\theta\sin\theta\left(\Delta x^2\,\Delta z^2\left(4\tau^0_{i,j}v^2_{nmo}-2\left(\tau_{i-1,j}+\tau_{i,j-1}\right)v^2_{nmo}+\tau^1_{i,j}\left(v^2_{nmo}-v^2_0\right)\right)+2\left(-\Delta z^2\,a^2_2d_2+\Delta x^2b^2_2e_2\right)v^2_{nmo}v^2_0\sin^2\theta\right)\\&+\cos^2\theta\left(\Delta x^2\,\Delta z^2\left(\Delta z^2\left(4\tau^0_{i,j}+\tau^1_{i,j}-4\tau_{i-1,j}\right)v^2_{nmo}+\Delta x^2\tau^1_{i,j}v^2_0\right)-8\left(\Delta z^4\,a^3_2+\Delta x^4\,b^3_2-2\Delta x^2\,\Delta z^2\,a_2b_2c_2\right)v^2_{nmo}v^2_0\sin^2\theta\right)\\&\left.+\Delta x^2\,\Delta z^2\sin^2\theta\left(4\Delta x^2\,\tau^0_{i,j}v^2_{nmo}+\Delta x^2\,\tau^1_{i,j}v^2_{nmo}-4\Delta x^2\,\tau_{i,j-1}v^2_{nmo}+\Delta z^2\,\tau^1_{i,j}v^2_0-4a_2b_2c_2v^2_{nmo}v^2_0\sin^2\theta\right)\right),\end{aligned} \tag{7}$$

where

$$D = 2\Delta x^2\,\Delta z^2\left(\left(\Delta z^2\,a_2v^2_{nmo}+\Delta x^2\,b_2v^2_0\right)\cos^2\theta+\Delta x\,\Delta z c_2\left(v^2_{nmo}-v^2_0\right)\cos\theta\sin\theta+\left(\Delta x^2\,b_2v^2_{nmo}+\Delta z^2\,a_2v^2_0\right)\sin^2\theta\right),$$

$$a_2=\tau^0_{i,j}-\tau_{i-1,j},\ b_2=\tau^0_{i,j}-\tau_{i,j-1},\ c_2=2\tau^0_{i,j}-\tau_{i-1,j}-\tau_{i,j-1},\ d_2=4\tau^0_{i,j}-\tau_{i-1,j}-3\tau_{i,j-1},\ e_2=4\tau^0_{i,j}-3\tau_{i-1,j}-\tau_{i,j-1}.$$

For the case of TI media with vertical symmetry axis (VTI), the traveltime coefficients given by equations 5, 6 and 7 reduces to:

$$\tau^0_{i,j} = \frac{\Delta x^2\,\tau_{i,j-1}v^2_0+\Delta z^2\,\tau_{i-1,j}v^2_{nmo}+\sqrt{\Delta x^2\Delta z^2\left(v^2_0\left(\Delta x^2-v^2_{nmo}(\tau_{i-1,j}-\tau^2_{i,j-1}\right)+\Delta z^2v^2_{nmo}\right)}}{\Delta x^2v^2_0+\Delta z^2v^2_{nmo}}, \tag{8}$$

$$\tau^1_{i,j} = \frac{v^2_{nmo}(\tau^0_{i,j}-\tau_{i-1,j})^2\left(v^2_0(\tau^0_{i,j}-\tau_{i,j-1})^2-\Delta z^2\right)}{\Delta x^2v^2_0(\tau^0_{i,j}-\tau_{i,j-1})+\Delta z^2v^2_{nmo}(\tau^0_{i,j}-\tau_{i-1,j})}, \tag{9}$$

$$\tau^2_{i,j} = -\frac{\tau^1_{i,j}\left(v_0^2\left(\Delta x^2\tau^1_{i,j}-4v^2_{nmo}(\tau^0_{i,j}-\tau_{i-1,j})(\tau^0_{i,j}-\tau_{i,j-1})(2\tau^0_{i,j}-\tau_{i-1,j}-\tau_{i,j-1})\right)+\Delta z^2 v^2_{nmo}(4\tau^0_{i,j}+\tau^1_{i,j}-4\tau_{i-1,j})\right)}{2\left(\Delta x^2 v_0^2(\tau^0_{i,j}-\tau_{i,j-1})+\Delta z^2 v^2_{nmo}(\tau^0_{i,j}-\tau_{i-1,j})\right)}, \tag{10}$$

respectively.

We can increase the accuracy of expansion in equation 4 by using Shanks transform. Once $\tau^0_{i,j}$, $\tau^1_{i,j}$ and $\tau^2_{i,j}$ are evaluated, traveltimes can be calculated using the first-sequence of Shanks transform given by (Bender and Orszag, 1978):

$$\tau_{i,j} \approx \tau^0_{i,j} + \frac{\eta_{i,j}(\tau^1_{i,j})^2}{\tau^1_{i,j} - \eta_{i,j}\tau^2_{i,j}}. \tag{11}$$

Numerical Tests

In this section, we study the computational efficiency and accuracy properties of our proposed scheme. For comparison, we use the exact solution of equation 1 obtained by solving a quartic polynomial at each time step. In addition, we also present a comparison of our approximation with that of Fomel (2004).

First, we test our approximation on the VTI Marmousi model (Alkhalifah, 1997). The source is located at x=5000 m on the surface. Figure 1 plots absolute traveltime errors (in seconds) obtained by considering up to the zeroth, first, second and third order terms in the traveltime expansion. Figure 1(a) presents the errors caused by ignoring anellipticity (elliptical anisotropy). Notice that the effect of anellipticity parameter η corresponds to the fronts propagating mostly orthogonal to the symmetry axis where the maximum error is approximately 97 ms. As we include the first order term in the expansion in Figure 1(b), the maximum error is reduced to about 15ms. As we include additional terms in the traveltime expansion, the errors are significantly reduced (note the change of scale of colorbars). We present results up to third-order terms, for which the maximum error has been reduced to approximately 5.5 ms.

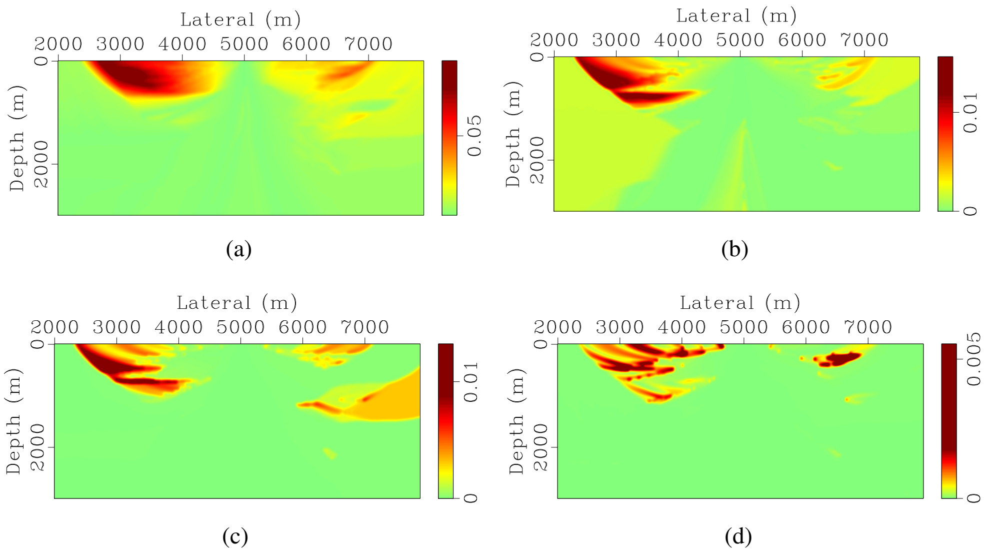
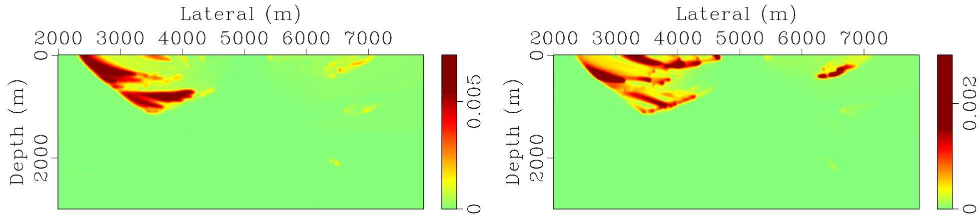


Figure 1 *Accuracy tests for the traveltime solution in VTI Marmousi model, considering: (a) the zero-order, (b) first order, (c) second order, and (d) third order approximations of the discretized TI eikonal. The source is located at $x = 5000$ m on the surface.*

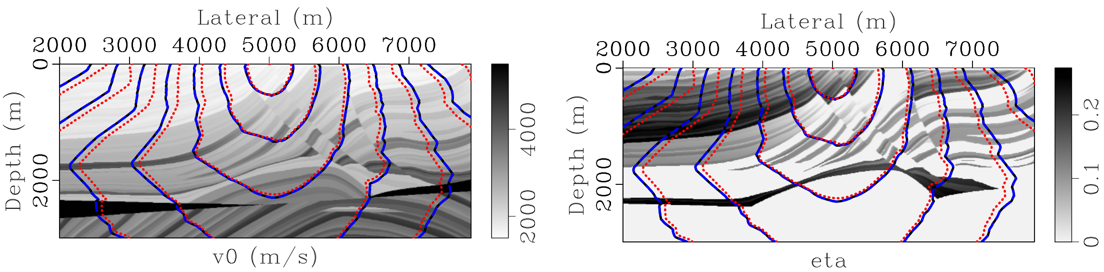
The accuracy obtained in Figure 1 can be further enhanced by using Shanks transform. The first sequence of Shanks transform requires the first three terms of the expansion, whereas the second Shanks sequence requires four terms of the expansion. Figure 2(a) and 2(b) shows error plots obtained using the first and second sequence of Shanks transform, respectively. Notice that with the second Shanks sequence, the maximum error is reduced to approximately 2.5 ms. This is remarkable accuracy given the presence of large η values and complex variations in the model.



(a) (b)

Figure 2 *Accuracy tests for the traveltime solution in VTI Marmousi model, considering (a) the first sequence, and (b) second sequence of the Shanks transform. The source is located at* $x = 5000$ *m on the surface.*

Next, we compare the accuracy obtained using the first sequence of Shanks transform with the approximation of Fomel (2004). Both approximations use fast marching algorithm and have almost the same computational cost ($\sim$ 16% of the exact TI solver). Figure 3 shows plots of traveltime contours for the exact eikonal solution (solid black), our approximation (dashed blue) and the approximation of Fomel (2004) (dotted red). Notice that the approximation by Fomel (2004) is far less accurate, especially in regions of large η. However, our proposed approximation yields near perfect traveltimes even for a highly complex model such as the Marmousi with large η values.



(a) (b)

Figure 3 *Traveltime contours for VTI Marmousi model using the exact VTI solution (solid black), first sequence of Shanks transform (dashed blue) and the approximation by Fomel (2004) (dotted red) mapped on : (a) velocity model and (b)* η *model. The source is located at* $x = 5000$ *m on the surface.*

Conclusions

A highly efficient approximation for traveltime computation in TI media has been developed. The suggested approximation relies on the application of perturbation theory to the discretized TI eikonal equation. In addition to achieving near perfect accuracy, the proposed formulation is highly efficient with regards to computational cost. Numerical stability is ensured by using the fast marching scheme. In addition to the VTI Marmousi model, numerical tests on the BP TTI model (Billette and Brandsberg-Dahl, 2005) will be presented at the 75$^{\text{th}}$ EAGE conference & exhibition.

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