Introduction

Diffracted waves carry valuable information regarding the subsurface geometry and velocity. They are particularly useful for anisotropic media as they inherently possess a wide range of dips necessary to resolve the angular dependence of velocity. Inspite of that, diffractions had long been regarded as noise in seismic processing and migration. During the last decade, there has been a steady increase of interest in diffracted waves. The attempts for diffraction imaging, however, have mainly focused on isotropic media. A transversely isotropic (TI) model with tilted symmetry axis (TTI) is regarded as one of the most effective approximations to the Earth subsurface, especially for imaging purposes (Zhou et al., 2006). Therefore, diffraction imaging based on the TTI approximation will be more accurate, in addition to the potential that diffractions provide in estimating velocity variation with angle.

We develop a scheme to compute traveltimes for the case of a point diffractor using perturbation theory. Specifically, we expand the source and receiver traveltimes with regards to a fixed anellipticity parameter η. This results in more accurate forward modeling scheme for diffraction data than the simplified isotropic model of the Earth. The accuracy of such formulation is further enhanced by using Shanks transform (an approach to predict the behavior of a series from the first few terms) to obtain better high-order representation. A scheme for scanning over η is suggested and later tested. We demonstrate the usefulness of our formulation on a homogeneous TTI model and the BP TTI model (Billette and Brandsberg-Dahl, 2005).

The TTI Eikonal Equation

The 2D eikonal equation for a TTI media, under the acoustic assumption, is given as (Alkhalifah, 1998):

$$v_{nmo}^2(1+2\eta)\left(\cos\theta\frac{\partial\tau}{\partial x}+\sin\theta\frac{\partial\tau}{\partial z}\right)^2+{v_0}^2\left(\cos\theta\frac{\partial\tau}{\partial z}-\sin\theta\frac{\partial\tau}{\partial x}\right)^2\left(1-2\eta v_{nmo}^2\left(\cos\theta\frac{\partial\tau}{\partial x}+\sin\theta\frac{\partial\tau}{\partial z}\right)^2\right)=1. \quad (1)$$

where $\tau(x,z)$ is the traveltime measured from the source to a point with the coordinates (x,z), v_0 and v_{nmo} are the vertical and NMO velocities measured along the symmetry axis, η denotes the anellipticity parameter, and θ represents the tilt direction. Here, we present the 2D case for simplicity. A full 3D version of the TTI eikonal equation can be found in Appendix D of Alkhalifah (2011).

Numerical solution of equation 1 requires solving a quartic equation at each time step of the finite difference implementation. Alternatively, Alkhalifah (2011) proposed the use of perturbation theory by approximating equation 1 with a series of simpler linear equations. Here, we use a tilted elliptical isotropic (TEI) medium as a background model and expand in terms of the parameter η (Stovas and Alkhalifah, 2012).

The 2D eikonal equation in TEI media (setting $\eta=0$ in equation 1) takes the form:

$$v_{nmo}^2\left(\cos\theta\frac{\partial\tau}{\partial x}+\sin\theta\frac{\partial\tau}{\partial z}\right)^2+{v_0}^2\left(\cos\theta\frac{\partial\tau}{\partial z}-\sin\theta\frac{\partial\tau}{\partial x}\right)^2=1. \quad (2)$$

The proposed trial solution is:

$$\tau(x,z)\approx\tau_0(x,z)+\tau_1(x,z)\eta+\tau_2(x,z)\eta^2, \quad (3)$$

where τ_0, τ_1 and τ_2 are coefficients of the expansion with dimension of traveltime. For practical purposes, we consider only three terms of the expansion. Thus, τ_0 satisfies the TEI eikonal equation 2, whereas τ_1 and τ_2 satisfy linear first-order PDEs having the following form:

$$\left((v_{nmo}^2\cos^2\theta+v_0^2\sin^2\theta)\frac{\partial\tau_0}{\partial x}+\sin\theta\cos\theta\,(v_{nmo}^2-v_0^2)\frac{\partial\tau_0}{\partial z}\right)\frac{\partial\tau_i}{\partial x}+\left(\sin\theta\cos\theta\,(v_{nmo}^2-v_0^2)\frac{\partial\tau_0}{\partial x}+(v_{nmo}^2\sin^2\theta+v_0^2\cos^2\theta)\frac{\partial\tau_0}{\partial z}\right)\frac{\partial\tau_i}{\partial z}=f_i(x,z), \quad (4)$$

where $i=1,2$. The right hand side functions $f_i(x,z)$ depend on terms that can be evaluated sequentially starting with $i=1$. The exact expressions for the right hand side functions $f_1(x,z)$ and $f_2(x,z)$ are given in Waheed et al. (2012).

These right hand side functions become more complicated as we increase the number of terms in the Taylor's series expansion. Therefore, we consider expansion upto second order terms. However, we can increase the accuracy of expansion in equation 3 by using Shanks transform. Once τ_0, τ_1 and τ_2 have been evaluated, traveltimes can be calculated using the first-sequence of Shanks transform given as (Bender and Orszag, 1978).

$$\tau(x,z) \approx \tau_0(x,z) + \frac{\eta\,\tau_1^2(x,z)}{\tau_1(x,z)-\eta\,\tau_2(x,z)}. \tag{5}$$

Diffraction Traveltimes in TTI Homogeneous Medium

In this section we present equations for diffraction traveltimes in a TTI homogeneous medium and assess their accuracy.

Traveltime formulation discussed in equation 3 and the Shanks representation of it in equation 5 describe a one-way wave. For diffracted wave, we need to add two such components: one from source to diffractor and the other from diffractor to receiver. The coefficients of traveltime expansion in equation 3 are then given as (Golikov and Stovas, 2012):

$$\tau_0 = \tau_{0s}(x_s, x_d, z_d) + \tau_{0r}(x_r, x_d, z_d), \tag{6}$$

where x_s, x_r denote the source and receiver locations on the surface, respectively, while the diffractor is placed at (x_d, z_d) in the subsurface. τ_{0s} and τ_{0r} are given as:

$$\tau_{0s}(x_s,x_d,z_d) = \sqrt{\frac{s_1^2}{v_{nmo}^2} + \frac{s_2^2}{v_0^2}}, \qquad \tau_{0r}(x_r,x_d,z_d) = \sqrt{\frac{r_1^2}{v_{nmo}^2} + \frac{r_2^2}{v_0^2}}, \tag{7}$$

where,

$$s_1 = (x_s - x_d)\cos\theta + z_d\sin\theta, \quad s_2 = -(x_s - x_d)\sin\theta + z_d\cos\theta, \tag{8}$$

$$r_1 = (x_d - x_r)\cos\theta + z_d\sin\theta, \quad r_2 = -(x_d - x_r)\sin\theta + z_d\cos\theta. \tag{9}$$

The coefficient τ_1 is given as:

$$\tau_1 = -\frac{\tau_{0s}\,v_0^4\,s_1^4}{(v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^2} - \frac{\tau_{0r}\,v_0^4\,r_1^4}{(v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^2}, \tag{10}$$

and τ_2 is given as:

$$\tau_2 = \frac{3\,\tau_{0s}\,v_0^6\,s_1^6\,(v_0^2 s_1^2 + 4 v_{nmo}^2 s_2^2)}{2\,(v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^4} + \frac{3\,\tau_{0r}\,v_0^6\,r_1^6\,(v_0^2 r_1^2 + 4 v_{nmo}^2 r_2^2)}{2\,(v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^4}, \tag{11}$$

where $\tau_{0s}, \tau_{0r}; s_1, s_2$; and r_1, r_2 are defined by equations 7, 8, and 9, respectively.

Substituting equations 6, 10, and 11 into the trial solution given by equation 3, results in the following eikonal solution approximation:

$$\tau = \tau_{0s}\left(1 - \frac{v_0^4 s_1^4\,\eta}{(v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^2} + \frac{3 v_0^6 s_1^6 (v_0^2 s_1^2 + 4 v_{nmo}^2 s_2^2)\,\eta^2}{2\,(v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^4}\right) + \tau_{0r}\left(1 - \frac{v_0^4 r_1^4\,\eta}{(v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^2} + \frac{3 v_0^6 r_1^6 (v_0^2 r_1^2 + 4 v_{nmo}^2 r_2^2)\,\eta^2}{2\,(v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^4}\right). \tag{12}$$

The accuracy of this expansion is further enhanced by the use of Shanks transform. Substituting expressions for τ_0, τ_1 and τ_2 into equation 5, we get:

$$\tau \approx (\tau_{0s} + \tau_{0r})\left(\frac{1 + (\Phi_1 + \Phi_2)\eta}{1 + \Phi_2\eta}\right), \tag{13}$$

where,

$$\begin{aligned}
\Phi_1 &= -\frac{\tau_{0s}\,v_0^4 s_1^4 (v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^2 + \tau_{0r}\,v_0^4 r_1^4 (v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^2}{(\tau_{0s} + \tau_{0r})\left((v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^2 + (v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^2\right)},\\
\Phi_2 &= \frac{3\,v_0^6}{2(v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^2\,(v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^2}\left(\frac{1}{\tau_{0s}\,v_0^4 s_1^4 (v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^2 + \tau_{0r}\,v_0^4 r_1^4 (v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^2}\right)\\
&\quad \times \left(\tau_{0s}\,s_1^6 (v_0^2 s_1^2 + 4 v_{nmo}^2 s_2^2)(v_0^2 r_1^2 + v_{nmo}^2 r_2^2)^4 + \tau_{0r}\,r_1^6 (v_0^2 r_1^2 + 4 v_{nmo}^2 r_2^2)(v_0^2 s_1^2 + v_{nmo}^2 s_2^2)^4\right).
\end{aligned} \tag{14}$$

To validate the accuracy of equation 13, we consider a TTI homogeneous model with $v_0 = 2\,\text{km/s}$, anisotropic parameters $\delta = 0.1$, $\eta = 0.1$ and varying symmetry direction. The diffractor is located beneath the source at a depth of 1 km while receivers extend upto an offset of 5 km. Figure 1 presents relative error in traveltimes computed using zeroth, first and second order approximations and the Shanks transform representation for the considered model with varying tilt values. In all cases, Shanks transform yields highly accurate traveltimes, thereby demonstrating the effectiveness of the proposed formulation given by equation 13.

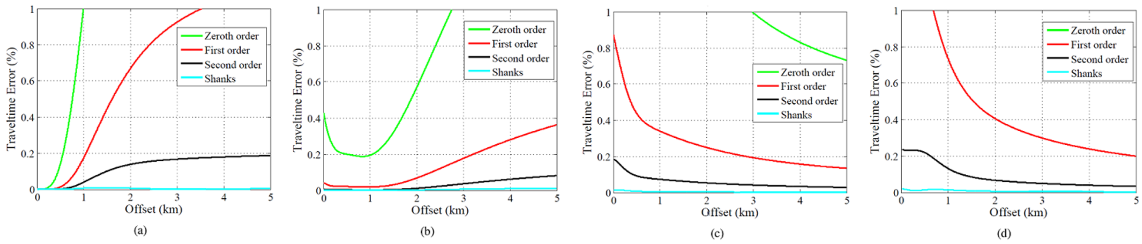

Figure 1 *Relative error in traveltimes versus offset for zeroth, first and second order expansions in η and the Shanks transform representation of it for a TTI homogeneous model having $v_0 = 2\,km/s$, $\delta = 0.1$, $\eta = 0.1$ with: (a) $\theta = 0°$, (b) $\theta = 30°$, (c) $\theta = 60°$ and (d) $\theta = 90°$. The diffractor is located at a depth of 1 km beneath the source while receivers extend upto an offset of 5 km. The error curve for zeroth order in (d) is outside the plotted error range.*

Diffraction Traveltimes in TTI Inhomogeneous Medium

Let τ_s and τ_r represent expansion in terms of η for traveltimes from source to diffractor and from diffractor to receiver, respectively, then we can write using equation 3:

$$\tau_s \approx \tau_{0s} + \tau_{1s}\,\eta + \tau_{2s}\,\eta^2, \qquad \tau_r \approx \tau_{0r} + \tau_{1r}\,\eta + \tau_{2r}\,\eta^2, \tag{15}$$

where τ_{0s}, τ_{1s} and τ_{2s} are coefficients of expansion for source to diffractor wave, while τ_{0r}, τ_{1r} and τ_{2r} denote the expansion coefficients for the wave going from diffractor to receiver. Again, using Shanks transform can lead to higher accuracy. By using the transform given by equation 5, we get the following traveltime representation for diffracted wave:

$$\tau \approx (\tau_{0s} + \tau_{0r}) + \frac{\eta\left(\tau_{1s}^2 + \tau_{1r}^2\right)}{(\tau_{1s} + \tau_{1r}) - \eta\,(\tau_{2s} + \tau_{2r})}. \tag{16}$$

Since the traveltime coefficients are computed using the background tilted elliptical isotropic inhomogeneous model, equation 16 allows us to search for the best η that could fit the diffraction curve.

Numerical Tests

In this section, we test the accuracy of the Shanks transform expansion for diffraction traveltimes in complex media. Specifically, we test the scheme on the BP TTI model (Billette and Brandsberg-Dahl, 2005).

First, we consider a diffractor located at (30km,9km) in the BP TTI model (see the background model in Figure 3 for the geometry of the BP model) and a source located at the surface with coordinates (32km,0km). Receivers are spread all over the surface. We solve the TI eikonal equation 1 for diffraction curve at the surface using η values given by the model (shown in the background of Figure 3b). The obtained curve is shown in Figure 2a (solid black curve). We then assume complete ignorance to the η model and scan for effective η value that best fits this diffraction curve using the formulation given by equation 16. Figure 2a also plots diffraction curves associated with η ranging from 0 to 0.05 in steps of 0.01. In Figure 2b, we show relative error associated with these effective η values. Repeating the procedure for several source positions on the surface and solving an error-minimization problem in the L_2-norm, we get an optimal effective η value for the chosen diffractor position.

Using this obtained effective η, we then plot traveltime contours in Figure 3 from this diffractor position, comparing our perturbation formulation (dashed curves) with exact traveltime (solid curves) obtained by solving equation 1. We obtain remarkable accuracy, even though we assumed complete ignorance to the η model for the approximation (dashed curves) and used an effective η obtained after scanning.

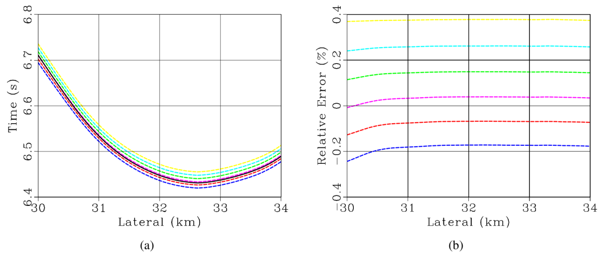

Figure 2 *Scanning for effective* η *value in the BP model. (a) Diffraction curves observed at the surface and (b) Relative error for these curves obtained for a range of* η *values from 0 to 0.05 in steps of 0.01 (yellow dashed curve at the top corresponds to* $\eta = 0$ *while blue dashed curve at the bottom corresponds to* $\eta = 0.05$*, moving sequentially). Black solid curve in (a) represents the exact diffraction curve.*

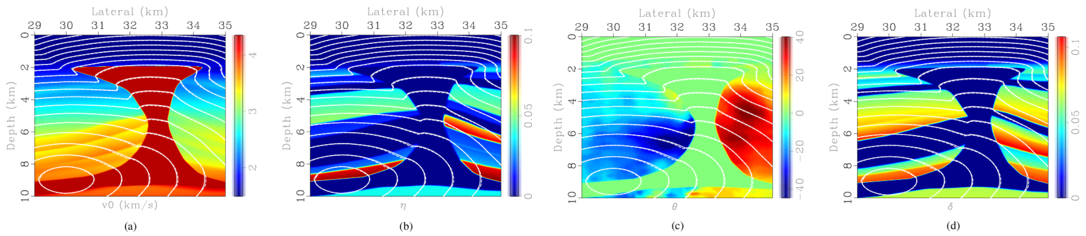

Figure 3 *Traveltime contours for the BP model using Shanks transform expansion (dashed) and the exact TTI eikonal solution (solid) mapped on: (a) velocity model, (b)* η *model, (c)* θ *model and (d)* δ *model. The exact solution uses* η *values given by the model in (b) at each grid point while Shanks transform expansion uses an effective* η *value of 0.03.*

Conclusions

Accurate diffraction imaging requires efficient forward modeling schemes that can model the complexities of real Earth. Diffraction traveltime formulation based on perturbation theory alleviates the huge computational burden associated with solving the exact TTI eikonal equation. Shanks transform enhances the accuracy of the expansion-based approximation. An added advantage of this formulation lies in our ability to use it to scan for the best η that fits the diffraction curve without the need to compute traveltimes again, even in complex media, like the BP TTI model. These assertions are supported through tests on a homogeneous TTI model and the BP TTI model. These tests demonstrate the usefulness of the proposed scheme, in anisotropic media, for computationally efficient diffraction traveltime modeling and the potential for estimating the anisotropy parameter, η in complex media.

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