High Frequency Asymptotic Methods for Traveltimes and Anisotropy Parameter Estimation in Azimuthally Varying Media

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Traveltimes are conventionally evaluated by solving the zero-order approximation of the Wentzel, Kramers and Brillouin (WKB) expansion of the wave equation. This high frequency approximation is good enough for most imaging applications and provides us with a traveltime equation called the eikonal equation. The eikonal equation is a non linear partial differential equation which can be solved by any of the familiar numerical methods. Among the most popular of these methods is the method of characteristics which yields the ray tracing equations and the finite difference approaches. In the first part of the Master Thesis, we use the ray tracing method to solve the eikonal equation to get P-waves traveltimes for orthorhombic models with arbitrary orientation of symmetry planes. We start with a ray tracing procedure specified in curvilinear coordinate system valid for anisotropy of arbitrary symmetry. The coordinate system is constructed so that the coordinate lines are perpendicular to the symmetry planes of an orthorhombic medium. Advantages of this approach are the conservation of orthorhombic symmetry throughout the model and reduction of the number of parameters specifying the model. We combine this procedure with first-order ray tracing and dynamic ray tracing equations for P waves propagating in
smooth, inhomogeneous, weakly anisotropic media. The first-order ray tracing and
dynamic ray tracing equations are derived from the exact ones by replacing the exact
P-wave eigenvalue of the Christoffel matrix by its first-order approximation. In the
second part of the Master Thesis, we compute traveltimes using the fast marching
method and we develop an approach to estimate the anisotropy parameters. The
idea is to relate them analytically to traveltimes which is challenging in inhomoge-
neous media. Using perturbation theory, we develop traveltime approximations for
transversely isotropic media with horizontal symmetry axis (HTI) as explicit func-
tions of the anellipticity parameter and the symmetry axis azimuth in inhomogeneous
background media. Specifically, our expansion assumes an inhomogeneous elliptically
anisotropic background medium, which may be obtained from well information and
stacking velocity analysis in HTI media. This formulation has advantages on two
fronts: on one hand, it alleviates the computational complexity associated with solv-
ing the HTI eikonal equation, and on the other hand, it provides a mechanism to scan
for the best fitting parameters without the need for repetitive modeling of traveltimes,
because the traveltime coefficients of the expansion are independent of the perturbed
parameters.
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To my lovely family, I dedicate my Master Thesis.
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LIST OF ABBREVIATIONS

HTI       Horizontal Transversely Isotropic
ODE       Ordinary Differential Equation
ORTHO     Orthorhombic
PDE       Partial Differential Equation
TI        Transversely Isotropic
TORTHO    Tilted Orthorhombic
TTI       Tilted Transversely Isotropic
VTI       Vertical Transversely Isotropic
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Chapter 1

Introduction

In exploration geophysics, the role of seismic anisotropy has considerably increased over the past two decades. The advances in parameter estimation, the transition from post-stack to pre-stack depth migration and the wide-azimuthal coverage 3D surveys has motivated this interest. The introduction of a new parametrization for the transversely isotropic (TI) models by Thomsen [28], as well as, the discovery of the P-wave time processing parameter $\eta$ by Alkhalifah and Tsvankin [5] are also some breakthroughs that made anisotropy simpler for seismic exploration. A more detailed historical review on the developments in seismic anisotropy can be found in [13] and [32].

Since anisotropy affects our acquired seismic data in many ways, it is important to construct anisotropic models that well describe the considered type of rocks or sedimentation. Specifically, the transversely isotropic (TI) model is a good assumption for sedimentation and thin layering encountered in the earth subsurface because the layering has a general preferred direction. It has been shown ([21]) that the TI assumption is the most practical type of anisotropy to represent large parts of the subsurface. The vertical symmetry axis (VTI) medium is a special case of TI medium in which the symmetry axis is normal to the typically horizontal acquisition surface. In some cases, the layers may be dipping (e.g., near flanks of salt domes), which leads to an azimuthally anisotropic medium with a tilted TI symmetry axis. In addition
to the VTI media or tilted one, wave propagation in the earth subsurface encounters media with horizontal symmetry axis (HTI) usually attributable to parallel vertical fractures. Luckily, the HTI media can predict many of azimuth variation features including the strength of azimuthal anisotropy ([12]) and its general direction. However, a more realistic representation of the earth subsurface is the orthorhombic symmetry (ORTHO). In fact, the TI model becomes much more restrictive when applied to the description of cracked media that contain fractures. Indeed, only the simplest fractured model with a single system of parallel vertical circular cracks embedded in an isotropic matrix exhibits transverse isotropy with a horizontal symmetry axis media. Deviations from the circular crack shape, misalignment of the crack planes or the presence of anisotropy or thin layering in the matrix lower the symmetry of the medium to orthorhombic. That is why, orthorhombic symmetry is suitable to model sedimentary basins containing parallel vertical cracks ([22]). When the layering is not horizontal and/or the fractures are not vertical, the medium becomes tilted orthorhombic. Currently, many seismic processing methods operate with anisotropic models and there is no doubt that improving the efficiency and accuracy of these tools is necessary.

In this context, the Master Thesis focuses on solving some of the most common and challenging problems in seismic exploration. One of the problems we dealt with is ray tracing which consists in finding the path of seismic energy from a source to a receiver and computing traveltimes along these paths ([7], [34]). The solution to this problem is required in many applications that exploit the high frequency component of seismic records, such as wave tomography, kirchhoff migration and full waveform inversion. The process of tracking the evolution of seismic energy, in addition to enabling traveltime computation, also brings with it the possibility of computing various other wave-related quantities such as amplitude, attenuation, and even the high frequency waveform, which can be compared to observation. We developed ray tracing
equations for the tilted transversely isotropic media (TTI) and the tilted orthorhombic media (TORTHO). Our approach preserves the symmetry (TI or ORTHO) at any point of the model thus it requires the minimum number of anisotropy parameters. In TTI media, three anisotropy parameters and two angles need to be specified, while for the ORTHO symmetry, 6 anisotropy parameters and three angles are required. We apply our procedure to the computation of two-point P-wave rays and corresponding traveltimes and show the efficiency on synthetic examples as well as realistic 3D models.

In the second part of the Master Thesis, we investigated an alternative approach to traveltime computation and anisotropy parameter estimation. It is true that ray tracing is often accurate and efficient, and lends naturally to the prediction of various seismic wave properties. However, it is not robust and may fail to converge to a true two-point path even in mildly heterogeneous media. In addition, it usually provides no guarantee as to whether a path corresponds to a first or later arrival. An alternative way to obtain traveltimes is to solve the non-linear partial differential (PDE) equation called the eikonal equation using finite difference. In addition to providing traveltimes for all grid points, finite difference solutions of the eikonal equation are computationally efficient and highly robust ([33],[16]), which makes them a viable alternative to ray tracing. Disadvantages of the eikonal equation solutions include that in most cases they only contain first arrival traveltime, and their accuracy is generally not as high as ray tracing. A more detailed comparison between both eikonal solution method and ray tracing is given in chapter 2 of the report.

Although the eikonal equation for the isotropic case is easy to solve, the anisotropic eikonal equation offers more challenges. For instance, the numerical solution of the HTI eikonal equation using finite difference requires finding the roots of a quartic polynomial at each computational step, while, for the isotropic case, this requires finding the roots of a quadratic polynomial. The challenging idea we develop in
this research is to solve approximately the HTI eikonal equation at the cost of an isotropic case. This approach not only provides traveltimes in a regular grid but also consists in a very sophisticated anisotropy parameters estimation tool. For an HTI media, through specific parametrization, P-wave traveltimes under the acoustic assumption, become dependent on only three parameters and the symmetry axis azimuth $\phi$. These parameters include the vertical velocity $v_v$, Thomsen parameter $\delta$ and the anellipticity parameter $\eta$. By considering $\eta$ and $\phi$ to be constant and small, we expanded the traveltime solution of the HTI eikonal equation in terms of $\eta$ and $\phi$. However, we allowed the other two parameters $v_v$ and $\delta$ to vary freely. Such process induces solving traveltimes in a simple elliptically anisotropic model having the same order of complexity as the isotropic case in terms of solving the eikonal equation; and solving a series of linear partial differential equations representing the terms of the traveltime expansion. We demonstrate the effectiveness of our scheme with tests on a 3D model and we propose an approach for multi-parameters estimation in TI media.

This report is divided into three parts. In chapter 2, we review the basic anisotropic models and their parametrization. The eikonal equation as well as the ray tracing equations in the isotropic case are derived. We also introduce some concepts used in the development of our algorithms such as the runge kutta integration method, the fast marching method and B-splines interpolation. In chapter 3, we present our development on the P-wave ray tracing and dynamic ray tracing in orthorhombic media of varying symmetry orientation. Finally, chapter 4 is dedicated to our work on anisotropy parameters estimation. Chapters 3 and 4 represent extended texts of our papers submitted respectively to the European Association of Geoscientists and Engineers (EAGE) and Society of Exploration Geophysicists (SEG) conferences.
Chapter 2

Ray Theory and Traveltimes in Anisotropic Media

The aim of this chapter is to introduce the theory and the numerical tools used in the development of this research. In the first section 2.1.1, we start with an overview of the basic anisotropic models such as the transversely isotropic and the orthorhombic models. We also present the notation of these models (section 2.1.2) in terms of the elastic coefficients as well as the anisotropy parameters. Moreover, this chapter is devoted to the solutions of the non linear partial differential equation called the eikonal equation. We will derive this equation in section 2.2 using the high frequency asymptotic approximation and show two distinct methods to solve it. These methods include solving the characteristics of this partial differential equation as we will see in section 2.2.2 or by finite difference using the fast marching method as presented in section 2.3.1. Moreover, some numerical tools such as the runge kutta integration method and B-splines interpolation are introduced in sections 2.3.2 and 2.3.3.
2.1 Brief Overview of Anisotropy

2.1.1 The Basic Anisotropic Models

One definition of anisotropy could be “the dependence of seismic velocity upon angle” [20]. The simplest anisotropy type is the transversely isotropic (TI) medium which has a single axis of rotational symmetry. All seismic signatures in this model depend just on the angle between the propagation direction and the symmetry axis. Moreover, any plane that contains the symmetry axis represents a plane of mirror symmetry while one more symmetry is the plane perpendicular to the symmetry axis. The TI model results from aligned thin layering. Horizontal layering yield a TI medium with a vertical symmetry axis (VTI), see figure 2.1(a). The stiffness matrix (symmetric matrix) of VTI media in Voigt notation is given by:

\[
C_{VTI} = \begin{bmatrix}
  c_{11} & c_{11} - 2c_{66} & c_{13} & 0 & 0 & 0 \\
  c_{11} & c_{11} & c_{13} & 0 & 0 & 0 \\
  c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
  c_{13} & c_{13} & c_{33} & c_{44} & 0 & 0 \\
  0 & 0 & 0 & c_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & c_{66} & 0 \\
\end{bmatrix}.
\] (2.1)

Five independent stiffness coefficients are needed to define a VTI media.

If the layers are dipping this leads to a tilt of the symmetry axis with respect to the earth surface (TTI media). Tilting the symmetry axis all the way to horizontal leads to a horizontal transverse isotropy (HTI) media. The HTI model has two mutually orthogonal vertical planes of symmetry called the symmetry axis plane and the isotropy plane, see figure 2.1(b). In most cases, HTI media are caused by parallel vertical cracks embedded in an isotropy background. If the symmetry...
direction coincides with the x1-axis the stiffness tensor for HTI media has this form:

\[
C_{HTI} = \begin{bmatrix}
    c_{11} & c_{13} & c_{13} & 0 & 0 & 0 \\
    c_{33} & c_{33} - 2c_{44} & 0 & 0 & 0 \\
    c_{33} & 0 & 0 & 0 \\
    c_{44} & 0 & 0 \\
    c_{66} & 0 \\
    c_{66}
\end{bmatrix}.
\]  

(2.2)

A more realistic representation of the earth subsurface is the orthorhombic symmetry (ORTHO). Orthorhombic models are characterized by three mutually perpendicular planes of mirror symmetry, see figure 2.1(c). Orthorhombic symmetry represents well sedimentary basins containing parallel vertical fractures. In a Cartesian coordinate system associated with the symmetry planes, the orthorhombic stiffness matrix is given by:

\[
C^{ORTHO} = \begin{bmatrix}
    c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
    c_{22} & c_{23} & 0 & 0 & 0 \\
    c_{33} & 0 & 0 & 0 \\
    c_{44} & 0 & 0 \\
    c_{55} & 0 \\
    c_{66}
\end{bmatrix}.
\]  

(2.3)

Thus, nine independent elastic coefficients are needed to fully describe orthorhombic symmetry.

### 2.1.2 Parametrization of the Anisotropic Models

Historically, wave propagation was described using the stiffness coefficients. These coefficients are convenient to use for forward modeling such as ray tracing but not for data processing or inversion algorithms. That is why alternative types of parametrization have been introduced (e.g. \[28\], \[17\], \[5\], \[30\]). The idea behind these different
Figure 2.1: Anisotropy models: The VTI model (a) has a vertical axis of rotational symmetry and may be caused by thin horizontal layering, while the HTI model (b) is due to a system of parallel vertical cracks. The orthorhombic model (c) is caused by parallel vertical fractures embedded in a finely layered medium. One of the symmetry planes in the orthorhombic symmetry is horizontal, while the other two planes are parallel and perpendicular to the fractures, after [31].

parametrizations is to combine the elastic coefficients that are most suitable for the description of seismic wavefields.

In this section, we introduce the parametrizations we used in our current study. Thomsen [28] suggested using five elastic coefficients to represent a VTI media. His parameters include the vertical velocities for P-waves and S-waves and three dimensionless parameters denoted $\epsilon$, $\delta$ and $\gamma$. For describing the kinematics of P-waves, only three parameters are relevant (the vertical P-wave velocity, $v_v$, $\epsilon$ and $\delta$, see Appendix [A]). Furthermore, Alkhalifah and Tsvankin [5] introduced a new notation. They showed that, for VTI media, just two parameters are sufficient for performing all time-related processing such as move-out correction or pre-stack or post-stack
migration. Their parameters $V_{nmo}$ and $\eta$ are given as follows:

$$V_{nmo} = \nu_v \sqrt{1 + 2\delta},$$  \hspace{1cm} (2.4)

$$\eta = \frac{\epsilon - \delta}{1 + 2\delta},$$ \hspace{1cm} (2.5)

where $\nu_v$, $\delta$ and $\epsilon$ are Thomsen parameters. Alkhalifah and Tsvankin parameters could also be written directly in terms of the elastic coefficients, see Appendix A. In our current study, both Thomsen and Alkhalifah notation are being used in Chapter 4. Furthermore, Tsvankin [30] introduced a new set of parameters for orthorhombic symmetry. He showed that for P-waves, orthorombic symmetry could be described by the vertical velocity and five dimensionless parameters.

Another approach to parametrize anisotropic models is the weak anisotropy (WA) parameters proposed by Psencik and Gajewski [17]. Their approach is based on the weak anisotropy approximation of the eigenvalues of the Christoffel matrix and leads to a different definitions of anisotropy parameters, see Appendix A. In contrast to Thomsen parameters, the WA parameters are related linearly to the elastic coefficients. Equivalently, to Tsvankin [30] notation, six WA parameters are also needed to describe orthorhombic symmetry. In Chapter 3, Psencik and Gajewski [17] notation is adopted.

### 2.2 High Frequency Asymptotic Approximation

#### 2.2.1 Eikonal Equation in the Isotropic Case

The high frequency asymptotic approximation, on which rely most of the seismic exploration framework, states that the wavelength of the propagating wave is substantially shorter than the seismic heterogeneities that characterize the medium through which they pass.
Below, we derive the eikonal equation for the acoustic isotropic case, using the high frequency asymptotic assumption, then extend it to the anisotropic case. For a seismic P-wave in an isotropic medium, the wave equation can be written as:

\[ \nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial \tau^2} = 0, \]  

(2.6)

where \( \psi \) represents the scalar potential of a P-wave, \( c \) is the P-wavespeed and \( \tau \) is the traveltime. If we assume that the solution of equation 2.6 has the general form

\[ \psi = A \exp(-i\omega \tau), \]  

(2.7)

where \( A = A(x) \) is the wave amplitude, \( \omega \) is the frequency and \( \tau \) is the traveltime, then the laplacian of the scalar potential is:

\[
\nabla^2 \psi = \nabla^2 A \exp(-i\omega \tau) - i\omega \nabla \tau \cdot \nabla A \exp(-i\omega \tau) - i\omega \nabla A \cdot \nabla \tau \exp(-i\omega \tau)
- i\omega A \nabla^2 \tau \exp(-i\omega \tau) - \omega^2 A |\nabla \tau|^2 \exp(-i\omega \tau),
\]

(2.8)

and the second derivative of \( \psi \) with respect to time is:

\[ \frac{\partial^2 \psi}{\partial \tau^2} = -\omega^2 A \exp(-i\omega \tau). \]  

(2.9)

Substitution of the above two expressions into equation 2.6 yields:

\[
\nabla^2 A - \omega^2 A |\nabla \tau|^2 - i \left( 2\omega \nabla A \cdot \nabla \tau + \omega A \nabla^2 \tau \right) = -\frac{A \omega^2}{c^2},
\]

(2.10)

which can be divided into real and imaginary parts. If we take the real part and divide through by \( A\omega^2 \), then we obtain:

\[ \frac{\nabla^2 A}{A \omega^2} - |\nabla \tau|^2 = -\frac{1}{c^2}. \]  

(2.11)
Application of the high frequency approximation ($\omega \to \infty$) yields the eikonal equation for the isotropic case:

$$|\nabla \tau|^2 = \frac{1}{c^2}, \quad (2.12)$$

which can also be written in Cartesian coordinates, as (using Einstein summation convention for repeated subscripts)

$$G(x_i, p_i) = c^2(x_i) p_i p_i = 1, \quad (2.13)$$

where $p_i = \frac{\partial \tau}{\partial x_i}$ and $p_i$ are the components of the slowness vector ($\vec{p} = \nabla \tau$).

In equation 2.13, $G$ denotes the eikonal equation and represents the eigenvalue of the Christoffel matrix ([7]). For anisotropic media, the eikonal equation could be obtained from the equation 2.13 by replacing the wavespeed $c$ by the phase velocity in the considered type of media, which is also derived from the eigenvalues of the Christoffel matrix, ([7]).

### 2.2.2 Rays as Characteristic of the Eikonal Equation

The eikonal equation 2.13 is a non linear partial differential equation of the first order. We can solve this equation using the Fast Marching Method to get the traveltime solution on regular grid points, see Section 2.3.1, or by the method of characteristics.

We can express the eikonal equation 2.13 in the following Hamiltonian form:

$$H(x_i, p_i) = \frac{1}{2} (G(x_i, p_i) - 1) = 0. \quad (2.14)$$

The characteristics of equation 2.14 are 3D space trajectories $x_i = x_i(\sigma)$ (\sigma being some parameter along the trajectory), along which $H(x_i, p_i) = 0$ is satisfied. The characteristic curve is the solution of a system of differential equation of the first order ([6]). The characteristic system of the nonlinear partial differential equation
\[ dx_i \over d\sigma = \frac{1}{2} \frac{\partial G}{\partial p_i} , \quad dp_i \over d\sigma = -\frac{1}{2} \frac{\partial G}{\partial x_i} , \quad d\tau \over d\sigma = \frac{1}{2} p_i \frac{\partial G}{\partial p_i} , \quad (i = 1, 2, 3). \] (2.15)

In 3D, the system consists of seven equations. The six equations for \( x_i(\sigma) \) and \( p_i(\sigma) \) are coupled and must be solved together. The solution of the six equations is \( x_i = x(\sigma) \), the characteristic curve as a 3D trajectory, and \( p_i = p_i(\sigma) \), the components of the slowness vector along the characteristics. The seven equations for traveltime along the characteristics, \( \tau = \tau(\sigma) \) is not coupled with the other six equations, and can be solved independently. However, if we let \( \sigma = \tau \), which we consider in this study, then:

\[ d\tau \over d\tau = \frac{1}{2} p_i \frac{\partial G}{\partial p_i} = G = 1. \] (2.16)

Thus, the ray tracing system reduces to only six equations:

\[ dx_i \over d\tau = \frac{1}{2} \frac{\partial G}{\partial p_i} , \quad dp_i \over d\tau = -\frac{1}{2} \frac{\partial G}{\partial x_i} , \quad (i = 1, 2, 3). \] (2.17)

The initial conditions for the ray tracing system is as follows:

At the source S: \( x_i = x_{i0} , \quad p_i = p_{i0} \), \hspace{1cm} (2.18)

where \( p_{i0} \) satisfies the eikonal equation at S, \( G(x_{i0}, p_{i0}) = 1 \). The components of the slowness vector \( \overrightarrow{p_0} \) (at S) can be expressed in terms of two take off angles denoted here \( \theta_0 \) and \( \phi_0 \), called the ray parameters. Thus, the components of \( \overrightarrow{p_0} \) can be expressed as following:

\[ p_{10} = \frac{n_{10}}{c_0} , \quad p_{20} = \frac{n_{20}}{c_0} , \quad p_{30} = \frac{n_{30}}{c_0} , \] (2.19)

where

\[ n_{10} = \sin \theta_0 \cos \phi_0 , \quad n_{20} = \sin \theta_0 \sin \phi_0 , \quad n_{30} = \cos \theta_0 \] (2.20)
and \(c_0\) is the phase velocity at S in the direction \(\vec{n}_0\).

### 2.2.3 Dynamic Ray Tracing and Geometrical Spreading

Solving the eikonal equation by the method of characteristics not only allows us to get rays trajectories and traveltimes, but also other important quantities along the ray such as geometrical spreading. The concept of geometrical spreading plays an important role in the computation of amplitudes of seismic waves, necessary for wavefield estimation. The geometrical spreading along a ray from S to R is defined as follows:

\[
L(R, S) = |X^{(1)} \times X^{(2)}|^{\frac{1}{2}}. \tag{2.21}
\]

Equation 2.21 holds for the vectors \(X^{(I)}\) and other important vectors \(P^{(I)}\) defined below and determined from solving a system of ordinary differential equations called the dynamic ray tracing system. If we denote the ray parameters (the two take-off angles at the initial source point S, \(\theta_0\) and \(\phi_0\)) by \(\gamma^{(I)}\), the components of the vectors \(X^{(I)}\) and \(P^{(I)}\) can be expressed as shown in \([19]\) and \([18]\):

\[
X^{(I)}_i = \left[ \frac{\partial x_i}{\partial \gamma^{(I)}} \right]_{\tau=\text{const}}, \quad P^{(I)}_i = \left[ \frac{\partial p_i}{\partial \gamma^{(I)}} \right]_{\tau=\text{const}}. \tag{2.22}
\]

The quantities \(X^{(I)}_i\) and \(P^{(I)}_i\) describe variations along the wavefront of the quantities \(x_i\) and \(p_i\) of the slowness vector due to the variations of the parameters \(\gamma^{(I)}\). The values of \(X^{(I)}_i\) and \(P^{(I)}_i\) can be found from solving the dynamic ray tracing system obtained from the differentiation of the ray tracing equations 2.17 with respect to \(\gamma^{(I)}\):

\[
\frac{dX^{(I)}_i}{d\tau} = \frac{1}{2} \left[ \frac{\partial^2 G(x_m, p_m)}{\partial p_i \partial x_j} X^{(I)}_j + \frac{\partial^2 G(x_m, p_m)}{\partial p_i \partial p_j} P^{(I)}_j \right],
\]

\[
\frac{dP^{(I)}_i}{d\tau} = -\frac{1}{2} \left[ \frac{\partial^2 G(x_m, p_m)}{\partial x_i \partial x_j} X^{(I)}_j + \frac{\partial^2 G(x_m, p_m)}{\partial x_i \partial p_j} P^{(I)}_j \right]. \tag{2.23}
\]
We obtain the quantities $X_i^{(J)}$ required for the calculation of the geometrical spreading by specifying proper initial conditions. Psencik and Teles [20] defined these conditions as the following:

$$X_i^{(J)} = 0, \quad P_i^{(J)} = c_0^{-1} \left( Z_{iJ} - p_i^0 \nu_0 Z_{kJ} \right),$$

(2.24)

where

$$Z_{11} = -\sin \phi_0 \cos \theta_0, \quad Z_{21} = \cos \phi_0 \cos \theta_0, \quad Z_{31} = 0,$$

$$Z_{12} = -\cos \phi_0 \sin \theta_0, \quad Z_{22} = -\sin \phi_0 \sin \theta_0, \quad Z_{32} = \cos \theta_0$$

(2.25)

where $c_0$ denotes the phase velocity at the source point, $\nu_0$ denotes the components of the ray velocity vector $\nu_i = dx_i/d\tau$ at the same point. The symbols $\phi_0$ and $\theta_0$ denotes again the two take off angles at the point source.

### 2.3 Numerical Solutions of the Eikonal and Ray Tracing system

#### 2.3.1 Fast Marching Method

The fast marching method has been recognized as one of the most efficient methods of traveltime computation ([25], [26]). At its core, the problem of computing traveltimes is equivalent to tracking an advancing interface. During the 1980s and 1990s, a lot of work was devoted to the issues of tracking evolving interfaces starting with the work of Sethian ([23], [24]), initially leading to the level set methods ([15]), and later to the fast marching method ([25]) which is specifically aimed at the solution of the eikonal equation. The idea behind these techniques is the construction of ”entropy-satisfying” weak solutions, by making use of numerical schemes borrowed from the
technology of hyperbolic conservations laws and aimed at constructing the correct viscosity solution of the eikonal equation [25].

The fast marching method relies on an upwind finite difference scheme which chooses grid points in terms of the direction of the flow of information. Upwind means that if a wave progresses from left to right, then one should use a finite difference scheme which reaches upwind to the left in order to get information to construct the solution downwind to the right. The upwind scheme implies that information propagates one way, that is, from smaller values to larger values.

Considering a first order finite difference scheme with a grid point sequence \( i, j, k \), the eikonal equation (2.12) can be rewritten as

\[
\left( \frac{\tau_{i,j,k} - \tau_{i-1,j,k}}{\Delta x} \right)^2 + \left( \frac{\tau_{i,j,k} - \tau_{i,j-1,k}}{\Delta y} \right)^2 + \left( \frac{\tau_{i,j,k} - \tau_{i,j,k-1}}{\Delta z} \right)^2 = \frac{1}{c_{i,j,k}^2}. \tag{2.26}
\]

Using the idea of upwind scheme, we can rewrite the discretized version (2.26) of the eikonal equation such as:

\[
\text{max}(D^{-x}_{ijk}\tau, D^{+x}_{ijk}\tau, 0)^2 + \text{max}(D^{-y}_{ijk}\tau, D^{+y}_{ijk}\tau, 0)^2 + \text{max}(D^{-z}_{ijk}\tau, D^{+z}_{ijk}\tau, 0)^2 = \frac{1}{c_{ijk}}, \tag{2.27}
\]

where

\[
\begin{align*}
D^{-x}_{i,j,k}\tau &= \tau_{i,j,k} - \tau_{i-1,j,k} / \Delta x, \\
D^{+x}_{i,j,k}\tau &= \tau_{i+1,j,k} - \tau_{i,j,k} / \Delta x, \\
D^{-y}_{i,j,k}\tau &= \tau_{i,j,k} - \tau_{i,j-1,k} / \Delta x, \\
D^{+y}_{i,j,k}\tau &= \tau_{i,j+1,k} - \tau_{i,j,k} / \Delta x, \\
D^{-z}_{i,j,k}\tau &= \tau_{i,j,k} - \tau_{i,j,k-1} / \Delta x, \\
D^{+z}_{i,j,k}\tau &= \tau_{i,j,k+1} - \tau_{i,j,k} / \Delta x.
\end{align*}
\]

The idea of the fast marching method is to sweep the front ahead in an upwind
fashion by considering a set of points in the narrow band around the existing front and to march this narrow band forward, freezing the values of existing points and bringing new ones into the narrow band structure, see figure 2.2. The steps used in the fast marching algorithm can be summarized in the following [26]: Initially, the source point is considered an alive point with a corresponding zero value. Then, the following closest points are computed analytically and tagged as close in the narrow band. Afterward, the rest of the points are tagged as far away with a very large initial value. After generating the initial three domains: accepted, narrow, and far away, then the following procedure is applied iteratively:

- The smallest value in the narrow band is considered as a trial point.
- All points around the trial value that are not alive will be reassigned to the narrow band even if they are far away points.
- Calculate the solution in the new narrow band using the corresponding eikonal traveltime solution.
- Assign the trial solution in the narrow band as an alive point.
- If the narrow band is not empty, start process from the first indicated step.

Figure 2.2: Upwind scheme of the fast marching method, after [26].
The cost of the algorithm is reflected from the binary tree sorting that allows the corresponding ordering from smallest to largest points at each stage of the narrow band. The computational cost of the binary tree sorting is $O(\log N_{NB})$, where $N_{NB}$ is the total number of traveltime values in the narrow band. For $N$ total grid points, the final cost of the process is represented by $O(N \log N_{NB})$.

### 2.3.2 Runge Kutta Integration Method

The ray tracing and dynamic ray tracing equations are first order coupled ordinary differential equations that require robust integration methods. We used the fourth order adaptive runge kutta method to solve these equations. This method is convenient and accurate since its truncation error is on the order of $O(dt^5)$ where $dt$ is the integration step size. Appendix A contains a brief description of the method.

The choice of the integration step in this method is particularly critical since it affects the accuracy of the computation. In the code, we used an adaptive step size which keeps the required accuracy of the computation below some limit that we choose. The idea is the following: at each step $dt$, the computation is performed in two ways, one with the step $dt$ and another with twice step $\frac{dt}{2}$. If the difference between both computations satisfies the accuracy conditions the computation continues. If not, the step is halved, and the computations are repeated. If the accuracy conditions are not satisfied even if the step is halved the step is halved again. After several halvings the required accuracy is usually achieved. Of course, the more complicated the model is the more halvings are required.

### 2.3.3 B-Splines

The integration of ray tracing and dynamic ray tracing equations involve computing first and second order derivatives model parameters, see equations 2.17, 2.23 and appendix B. If these models contain higher-order discontinuities, then their derivatives
may have large values or may not even exist, which necessarily induces numerical issues. Also, if the model parameters are big dataset such as 3D model, then storing and manipulating these data become challenging. Also, one common task in any ray tracing algorithm is interpolation required to evaluate model parameters between grid points. That is why finding an efficient way to handle all these issues is important.

Using cubib B-splines interpolation with smoothing allowed us to overcome the previous mentioned challenges. The cubic B-splines is constructed of piecewise cubic polynomials defined on some interval, which are by construction continuous through to second derivative at the boundaries of intervals.

If we consider a 1D problem and we denote $y_i$ the data point values at the nodes $x_i$, then the smoothing spline $f$ minimizes

$$ p \sum_i \omega_i |y_i - f(x_i)|^2 + (1 - p) \int_{x_1}^{x_n} |D^2 f(x)|^2 dx, \quad (2.28) $$

where $p$ is the smoothness parameter and $D^2$ denotes the second derivative. Notice that the larger the smoothing parameter $p \in [0..1]$ used, the more closely $f$ matches the given data and the less smooth the spline is.

Using these splines functions, we were able to generate smooth version of the input model parameters, ensure the existence of the first and second derivatives required in the computation of the ray tracing and dynamic equations as well as efficient interpolation of model parameters between the grid points.

2.4 Discussion

In this chapter, we discussed the commonly used anisotropy models (orthorhombic, transversely isotropic) and saw two different types of parametrization. Particularly, we emphasized that the orthorhombic symmetry is the most simple but best representative of the earth subsurface comparing to the transversely isotropic models (VTI
Moreover, we derived the eikonal equation using the high frequency asymptotic approximation and we discussed the two common ways of solving this non-linear partial differential equation: directly solving the eikonal equation with the fast marching method or solving its characteristics. Both methods allow one to compute traveltime information required in many applications such as Kirchhoff migration. Solving the eikonal equation by the method of characteristics require solving a set of differential equations of the first order. This approach is often highly accurate and efficient and naturally lends itself to the prediction of various seismic wave properties. However, it becomes non-robust and may fail if the medium is non-smooth, that is why B-splines with interpolation were used in the development of our algorithms. Another limitation in this approach is the need of interpolating traveltimes between the rays especially in shadow zones where the rays cannot penetrate. On the other hand, solving the eikonal equations with the fast marching method enables one to get traveltime field at all grid points even in discontinuous type models and in shadow zones. That is why the method is very stable and robust. Despite these advantages, directly solving the eikonal equation also has some limitations. In fact, the fast marching method only computes first arrival traveltimes, thus features such as wavefront triplications cannot be predicted. Also quantities other than traveltime (such as amplitude) are difficult to compute without first extracting the ray paths. Finally, the accuracy of the method is function of grid spacing. For example, halving the grid spacing in 3D model would increase the cost of computation by 8.
Chapter 3

Ray Tracing and Dynamic Ray Tracing in Tilted Orthorhombic Media

In this chapter, we present our study on ray tracing and dynamic ray tracing in orthorhombic symmetry models. Seismic modeling and imaging with this model have drawn increased attention. Ray tracing in such media has many potential applications among them pre-stack depth migration, traveltime tomography, and even full waveform inversion. We develop an approximate but efficient and sufficiently accurate ray tracing procedure for orthorhombic media whose symmetry planes may vary throughout the medium.

In section 3.1, we develop a ray tracing procedure specified in a curvilinear orthogonal coordinate system valid for anisotropy of arbitrary symmetry. The coordinate system is constructed so that the coordinate lines are perpendicular to the symmetry planes of an orthorhombic medium [14]. Advantages of this approach are the conservation of orthorhombic symmetry throughout the model and reduction of the number of parameters specifying the model. We combine this procedure with first-order ray kinematic and dynamic ray tracing equations for P waves propagating in smooth,
inhomogeneous, weakly anisotropic media ([18], [19]). The first-order ray tracing and
dynamic ray tracing equations are derived from the exact ones by replacing the ex-
act P-wave eigenvalue of the Christoffel matrix by its first-order approximation. In
orthorhombic media, such equations are controlled by 6 weak anisotropy (WA) pa-
rameters, which represent a linearized generalization of the Thomsen [28] parameters.

In section 3.3.1 we show the accuracy of our procedure by tests on synthetic
models, for which exact results are available. Then in section 3.3.2 we present an
application of the procedure to the computation of two-point P-wave rays and corre-
sponding traveltimes in a generalization of the BP model [27]. Discussion of the main
features and limitations of our approach is given in section 3.4. Note that Einstein
summation convention for repeated subscripts is used in this chapter.

3.1 Ray Tracing in Curvilinear Coordinates

3.1.1 Curvilinear Coordinate System and Transformation Ma-
trix

Let us introduce an orthogonal curvilinear coordinate system, (ξ₁, ξ₂, ξ₃) with its
origin (ξ₁, ξ₂, ξ₃) = (0, 0, 0) fixed relative to the origin of the Cartesian coordinate
system (x₁, x₂, x₃). We define the coordinate lines of the curvilinear system so that
they are perpendicular to the symmetry planes of an orthorhombic medium at any
point of the medium. The transformation matrix H from curvilinear to Cartesian
coordinates in terms of Euler angles φ, θ and ν reads:

$$
H = \begin{pmatrix}
\cos\phi \cos\theta \cos\nu - \sin\phi \sin\nu & \cos\phi \cos\theta \sin\nu + \sin\phi \cos\nu & \cos\phi \sin\theta \\
-\sin\phi \cos\theta \cos\nu - \cos\phi \sin\nu & -\sin\phi \cos\theta \sin\nu + \cos\phi \cos\nu & -\sin\phi \sin\theta \\
-\sin\theta \cos\nu & -\sin\theta \sin\nu & \cos\theta
\end{pmatrix}
$$

(3.1)
Angle $\phi$ controls the rotation around $x_3$ axis of the Cartesian coordinates. It transforms axes $x_1$ and $x_2$ to $x'_1$ and $x'_2$, and is positive if the rotation is anticlockwise. Angle $\theta$ controls the rotation around $x'_2$ axis. It transforms axis $x'_1$ to $x''_1$ and axis $x_3$ to $x'_3$, and is positive if the rotation is anticlockwise. Angle $\nu$ controls the rotation around the $x'_3$ axis.

### 3.1.2 Ray Tracing Equations

As we saw in section 2.2.2, the ray tracing system is governed by a system of differential equations for position vector $\mathbf{x}$ and slowness vector $\mathbf{p}$. In Cartesian coordinates, we can write these equations in this form:

$$
\frac{d}{d\tau} \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{H} & 0 \\ -\mathbf{K} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v}^\xi \\ \rho^\xi \end{pmatrix}. 
$$

(3.2)

In equation (3.2), $\mathbf{H}$ is the transformation matrix, $\mathbf{I}$ is a $3 \times 3$ identity matrix and $\mathbf{K}$ is a $3 \times 3$ matrix with elements:

$$
K_{mk} = \frac{\partial H_{ik}}{\partial x_m} p_i. 
$$

(3.3)

The vectors $\mathbf{v}^\xi$ and $\mathbf{\rho}^\xi$ have components:

$$
v^\xi_i = \frac{1}{2} \frac{\partial G^\xi}{\partial p_i^\xi}, \quad \rho^\xi_i = -\frac{1}{2} \frac{\partial G^\xi}{\partial x_i}.
$$

(3.4)

Symbol $G^\xi$ denotes the first-order P-wave eigenvalue of the Christoffel matrix, expressed in curvilinear coordinates $\xi$. For the orthorhombic case, $G^\xi$ reads

$$
G^\xi = \alpha^2 \left( p^\xi_k p^\xi_k + 2(\xi_x (p_1^\xi)^2 + \epsilon_y (p_2^\xi)^2) + \epsilon_z (p_3^\xi)^2 \right) + 2(p^\xi_k p^\xi_k)^{-1} \left[ \eta_x (p_2^\xi)^2 (p_3^\xi)^2 + \eta_y (p_1^\xi)^2 (p_3^\xi)^2 + \eta_z (p_1^\xi)^2 (p_2^\xi)^2 \right],
$$

(3.5)
where the weak-anisotropy (WA) parameters $\epsilon_x$, $\epsilon_y$, $\epsilon_z$, $\eta_x$, $\eta_y$ and $\eta_z$ are defined in appendix A. The quantity $\alpha$ in equation 3.5 is a constant reference velocity used in the definition of WA parameters. Equation 3.5 and the ray tracing equations are independent of $\alpha$. Thus $\alpha$ can be chosen arbitrarily. In our experiments, we use $\alpha$ that makes the WA parameters as small as possible.

Note that the ray tracing system 3.2 reduces exactly to the system 2.17 shown in section 2.2.2 if the medium is not tilted. In a none tilted case, the rotation matrix $H$ reduces to an identity matrix and the $K$ matrix to a null matrix. Here, the left hand side of the ray tracing equations $(\upsilon^\xi \rho^\xi)^T$ is consistent locally with the orientation of the symmetry planes. Thus, we see that our approach consists in applying a rotational operator which rotates back $(\upsilon^\xi \rho^\xi)^T$ to the Cartesian coordinates and ensures the conservation of the symmetry at every point of the model. The derivation of the ray tracing equations in a coordinate system consistent with the orientation of the symmetry planes is given in appendix B.

### 3.1.3 Second-Order Traveltime Correction

The ray tracing equations 3.2 provides first-order traveltime $\tau$ since the eigenvalue $G(x_m, p_m)$ of the Christoffel matrix is approximate. We improve the accuracy of the traveltime by computing a second order traveltime correction. In fact, [10] and [7] show that an approximate traveltime $\tau$ along a ray $\Omega$, specified by an approximate eigenvalue $G(x_m, p_m)$ can be improved by applying this correction:

$$\Delta \tau = -\frac{1}{2} \int_{\Omega} \Delta G(x_m, p_m) d\tau.$$ (3.6)

The symbol $\Delta G$ denotes the difference between the exact eigenvalue and its approximation. For the evaluation of $\Delta G$, we need an estimation of the exact $G$. Therefore, we borrow an expression from [11] in which an approximation of the second order
approximation of the eigenvalue is given as follows:

\[ G(x_m, p_m) = 1 + [c(x_m, n_m)]^{-2} \frac{B_{13}^2 + B_{23}^2}{V_P^2 - V_S^2}, \]  

where the symbols \( B_{13} \) and \( B_{23} \) denote elements of the Christoffel matrix projected into a local coordinate system connected with the ray, see appendix \[B\]. \( c \) is the approximate phase velocity for the orthorhombic case, \( V_P \) and \( V_S \) are P- and S-wave velocities. In practice, we choose \( V_P^2 = (p_k p_k)^{-1} \), where \( p_k \) are the components of the slowness vector. For the S-wave velocity \( V_S \) we choose is as \( V_S^2 = V_P^2 / 3 \). Therefore, the following traveltime correction is applied:

\[ \Delta \tau = -\frac{1}{2} \int_\Omega [c(x_m, n_m)]^{-2} \frac{B_{13}^2 + B_{23}^2}{V_P^2 - V_S^2} d\tau. \]  

### 3.2 Two Point Ray Tracing and Dynamic Ray Tracing

#### 3.2.1 Two Point Ray Tracing Method

Two point ray tracing consists in tracking the path taken by seismic energy between two points, generally a source and a receiver. The difficulty associated with locating a two-point path arises from the non-linearity relationship between the velocity (or the model parameters) and the path geometry. If the medium is homogeneous, then the raypaths are simply straight lines. However, if the medium is heterogeneous then the rays are curved and their paths become difficult to predict.

In our work, we consider a single wave (P-waves) in a smooth medium, so we use a simple two-point ray tracing procedure, in which, using the results of dynamic ray tracing, we convert deviations of a ray from the prescribed receiver position into the corrections of shooting angles at the source. The trajectory of a ray and its final
position are controlled by the ray parameters which represent the shooting angles $\gamma_1$, $\gamma_2$ and the traveltime $\tau$.

We denote $\Delta x_i$ the error between the ending ray position and a receiver in Cartesian coordinates. By expanding $\Delta x_i$ in terms of the independent variables $\gamma_1$, $\gamma_2$ and $\tau$, we obtain the following expression

$$\Delta x_i = \frac{\partial x_i}{\partial \gamma_1} \Delta \gamma_1 + \frac{\partial x_i}{\partial \gamma_2} \Delta \gamma_2 + \frac{\partial x_i}{\partial \tau} \Delta \tau,$$  \hspace{1cm} (3.9)

in which $\Delta \gamma_1$ and $\Delta \gamma_2$ are the angles update used in our two point ray tracing procedure. The quantities $\frac{\partial x_i}{\partial \gamma_1}$ and $\frac{\partial x_i}{\partial \gamma_2}$ in equation (3.9) are obtained from integrating the dynamic ray tracing system (see next section 3.2.2) for the ray coordinates $\gamma_1$ and $\gamma_2$ while $\frac{\partial x_i}{\partial \tau}$ is obtained from integrating the ray tracing system 3.2.

### 3.2.2 Dynamic Ray Tracing

For two-point ray tracing and other useful applications, we need dynamic ray tracing. The dynamic ray tracing system for our specification reads [14]:

$$\frac{d}{d\tau} \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} H & 0 \\ -K & I \end{pmatrix} \begin{pmatrix} R^T & T \\ S & -R -S \end{pmatrix} \begin{pmatrix} I & 0 \\ K^T & H^T \end{pmatrix} \begin{pmatrix} X \\ P \end{pmatrix} + \begin{pmatrix} V^T & 0 \\ -U & -V \end{pmatrix} \begin{pmatrix} X \\ P \end{pmatrix}. \hspace{1cm} (3.10)$$

Here $X$ and $P$ are $3 \times 1$ matrices with elements:

$$X_i = \frac{\partial x_i}{\partial \gamma}, \hspace{0.5cm} P_i = \frac{\partial p_i}{\partial \gamma}. \hspace{1cm} (3.11)$$

The quantities $X_i$ and $P_i$ describe variations along the wavefront of the coordinates $x_i$ and of the components $p_i$ of the slowness vector due to the variation of the ray coordinate $\gamma$. We use two ray coordinates, $\gamma_1$ and $\gamma_2$, which represent two shooting angles at the source. The matrices $R$, $S$, $T$, $U$ and $V$ in equation 3.10 are $3 \times 3$
matrices with elements:

\[
R_{ij} = \frac{1}{2} \partial^2 G^\xi_{ij}, \quad S_{ij} = \frac{1}{2} \partial^2 G^\xi_{ij}, \quad T_{ij} = \frac{1}{2} \partial^2 G^\xi_{ij},
\]

\[
U_{ij} = \frac{1}{2} \partial^2 G^\xi_{ik} \partial^2 p^\xi_{kj}, \quad V_{ij} = \frac{1}{2} \partial^2 G^\xi_{ik} \partial^2 p^\xi_{kj}.
\]

Symbol \( G^\xi \) denotes again the first order approximate P-wave eigenvalue of the Christoffel matrix, see equation 3.5.

### 3.3 Numerical Example

#### 3.3.1 Synthetic Data Examples

Solving the two point ray tracing problem allows us to compute raypaths between a source and a set of receivers as well as the traveltimes along these paths. To show the accuracy of our method, we present in this section some examples on synthetic data. The examples consist in a vertical seismic profiling (VSP) configuration, in which the source and the borehole are situated in a vertical plane \((x_1, x_3)\). The borehole is parallel with the \(x_3\)-axis and the source is located on the surface \((x_3 = 0\text{km})\) at a one kilometer distance from the intersection of the borehole with the surface. The spacing of receivers in the borehole is 40m. The output traveltimes from our codes are compared with an exact standard ray tracing code called ANRAY ([9]).

We consider a 3D orthorhombic model specified by its stiffness matrices at the top and at the bottom of the model then a linear interpolation is used between the two levels. The density normalized stiffness matrices of this model with elements
measured in \((kms^{-1})^2\), are at the top, \(z = 0km\):

\[
C^{TOP} = \begin{pmatrix}
9.00 & 3.60 & 2.25 & 0 & 0 & 0 \\
9.84 & 2.40 & 0 & 0 & 0 \\
5.94 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1.60 & 0 & 0 & 0 & 0 \\
2.18 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (3.13)
\]

and at the bottom at \(z = 3km\):

\[
C^{BOT} = \begin{pmatrix}
19.80 & 7.92 & 4.95 & 0 & 0 & 0 \\
21.65 & 5.28 & 0 & 0 & 0 \\
13.07 & 0 & 0 & 0 & 0 \\
4.40 & 0 & 0 & 0 & 0 \\
3.52 & 0 & 0 & 0 & 0 \\
4.80 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (3.14)
\]

Figure 3.1 shows a set of results from our code where figure 3.1(a) represents rays projected into the vertical plane containing the source and the borehole while the projection of rays into the horizontal plane is shown in figure 3.1(b). In the same set of plots, figure 3.1(c) represents the traveltime at each receiver location while figure 3.1(d) shows the relative traveltime error for the case the second order traveltime correction is used (* symbol) and the case it is not used (+ symbol). Note that the use of the second order traveltime correction reduces considerably the relative error.

The example shown in figure 3.1 corresponds to a none tilted symmetry planes. To assess the effectiveness of the method on more complex examples, we consider six cases of orthorhombic symmetry with arbitrary symmetry planes orientation. The idea behind these examples is to apply different rotations to the symmetry planes and to check the accuracy of the traveltimes at the receivers locations of the VSP profile. Table 3.1 summarizes the values \(\theta\) and \(\phi\) of the symmetry planes rotations, where subscripts \(upper\) and \(lower\) denote respectively the value of the angle at the
(a) Rays projected into the vertical plane containing the source and the borehole.

(b) Transversal rays.

(c) Traveltimes for each receiver position in seconds.

(d) Relative traveltime error without using second order correction (+) and with the second order traveltime correction (*).

Figure 3.1: Projections of ray diagram into the vertical plane containing the source and the borehole (a), into the horizontal plane (b), the corresponding traveltime curve (c) and the relative traveltime error (d).

upper and lower surfaces between which linear interpolation of the angles is applied. For all these examples, we keep the angle $\nu$ constant and equals to zero. We use the same stiffness matrices 3.13 3.14 respectively at the upper and bottom surfaces, as well as the same VSP configuration.

The resulting relative traveltime errors for the six examples are shown in figure 3.2. Notice that for all the examples, except the example ORTHO-2, the relative travelt ime error without the use of the second order traveltime correction does not
Table 3.1: Values of $\theta$ and $\phi$ corresponding to the symmetry planes angle rotations for six examples of orthorhombic models

<table>
<thead>
<tr>
<th>Angles in degree</th>
<th>$\phi_{\text{upper}}$</th>
<th>$\phi_{\text{lower}}$</th>
<th>$\theta_{\text{upper}}$</th>
<th>$\theta_{\text{lower}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ORTHO-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>ORTHO-2</td>
<td>45</td>
<td>0</td>
<td>45</td>
<td>0</td>
</tr>
<tr>
<td>ORTHO-3</td>
<td>0</td>
<td>0</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>ORTHO-4</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>ORTHO-5</td>
<td>45</td>
<td>45</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>ORTHO-6</td>
<td>45</td>
<td>45</td>
<td>45</td>
<td>45</td>
</tr>
</tbody>
</table>

exceed 1.5%. Application of the second order traveltime correction (3.8) substantially decreases the error. Except at several shallow receivers, the second order corrected traveltime error does not exceed 0.2%. Note that only example ORTHO-2 has a relative error that exceeds 2% even when the second order correction is used. The reason behind that is the exact code (ANRAY) we use is inaccurate in interpolating the stiffness matrices if the rotation angles of the symmetry planes are not constant. Moreover, only in the example ORTHO-2, a linear interpolation of angles is being used compared to the 5 other examples where only constant angles are tested, which explains why we obtain accurate traveltimes only for these 5 examples.

3.3.2 Realistic Data Example

We further apply our procedure for ray tracing and traveltime computations to a 3D model of a realistic orthorhombic medium. The model is a generalization of a part of the 2D BP transversely isotropic model with varying axis of symmetry (27). We extended this model to a 3D orthorhombic one by assuming that the structural features of the 2D model do not vary in the direction perpendicular to the plane of the 2D model, and by appropriate choices of additional WA parameters. The parameters of the BP model are $v_u$, $\epsilon$, $\delta$ (28) and a tilt angle $\theta$ specified in a regular grid with 0.05 km spacing in the vertical and the horizontal directions. In each grid point, we used the values of $v_u$, $\epsilon$ and $\delta$ and converted them into WA parameters $\epsilon_x$, $\epsilon_y$, $\epsilon_z$, 

Figure 3.2: Relative traveltime error without using second order traveltime correction (+) and with the second order traveltime correction (*) for six orthorhombic models with arbitrary symmetry planes orientation, see table 3.1.
δ_x, δ_y and δ_z. The values and derivatives of WA parameters at an arbitrary point of
the model were determined using the cubic spline interpolation with smoothing. The
velocity α was chosen, α = 3.5 km/s. As to angles φ, θ and ν, we let only angle θ to
vary as in the BP model and we kept angles φ and ν constant. We show results for
three values of φ, specifically 15°, 30° and 45° and we keep ν = 0°.

The distribution of the WA parameters ε_x, ε_y, ε_z, δ_x, δ_y and δ_z as well as of the
angles θ, φ and ν in the plane (x_1, x_3) are shown in figure 3.3.

We can see in figures 3.3(a) to 3.3(f) that models’ parameters decrease at the
top of the salt dome, which leads to a shadow zone effect. We also notice in figures
3.3(g), 3.3(h) and 3.3(i) that θ varies from −45° to +45°, while the angles φ and ν
are constants.

We use a vertical seismic profiling (VSP) configuration in which the source and
the borehole are located in a vertical plane (x_1, x_3). The borehole is parallel with
the x_3-axis and the source is located on the surface (x_3 = 0km) at a 9 km distance
from the intersection of the borehole with the surface. The spacing of receivers in the
borehole is 0.2km. Figure 3.4 shows rays projected into the vertical plane containing
the source and the borehole, while the projection of rays into the horizontal plane is
shown in Figure 3.5. Both figures 3.4 and 3.5 correspond to the case φ = 15°. We can
see that rays deviate from the vertical plane (x_1, x_3) due to the deviations of planes
of symmetry from vertical and horizontal planes. Figure 3.6 illustrates the effect of
varying angle φ on the traveltimes. In this figure, one can see that the traveltime
decreases as the angle φ increases.

3.4 Discussion

In this chapter, we derived an approach for ray tracing and dynamic ray tracing
in inhomogeneous, weakly orthorhombic media with varying orientation of planes
Figure 3.3: Distribution of WA: $\epsilon_x$, $\epsilon_y$, $\epsilon_z$, $\delta_x$, $\delta_y$, $\delta_z$ and the rotation angles $\theta$, $\phi$ and $\nu$ of the symmetry planes.

Figure 3.4: Projection of rays into the vertical plane containing the source and the borehole for the case $\phi = 15^\circ$ with the distribution of $\delta_z$ in the background.
of symmetry. Our approach guarantees the conservation of considered anisotropy throughout the model and reduces the number of parameters necessary for the specification of the model. In the case of P-wave propagating in an orthorhombic media, only six WA parameters and three euler angles need to be specified.

We demonstrated our approach on synthetic data and realistic data examples. We used the stiffness matrix representation for the synthetic examples since the exact code uses this representation, while for the realistic case WA parameters were used. We can move from one set of parameterization to the other easily since the WA parameters are linear functions of the stiffness coefficients.

The synthetic examples showed that the relative traveltime error for complicated symmetry planes rotation does not exceed 0.2% when the second order traveltime correction is applied. This confirms the reliability and stability of our scheme. Further
ther, we applied this approach to a realistic model. This model is complicated since it contains a salt dome with a low velocity zone making a region where the rays cannot penetrate. Because the ray tracing and dynamic ray tracing equations involve applying the model first and second derivatives, cubic splines with smoothing was applied for polynomial representation of the models which ensures the existence of these derivatives.

We implemented a simple two point ray tracing in which using the result of dynamic ray tracing, we converted deviations of a ray from the prescribed receiver position into the corrections of shooting angles at the source. The limitation of this approach is that it ignores multipathing. In other words, the existence of possible multiple rays that go from one source to one receiver is ignored. Thus, there is no guarantee that the traveltime we measure corresponds to the most energetic one (late arrival) or to the less energetic (the first arrival). However, the main advantage of ray tracing compared to other methods such as solving the eikonal equation directly is the possibility of computing several other information such as amplitudes and geometrical spreading.
Chapter 4

Multi-Parameters Scanning in HTI Media

This chapter is dedicated to our work on traveltimes computation by solving the eikonal equation using the finite difference methods, as well as to the presentation of the anisotropy parameter estimation tool we developed. One way to estimate anisotropy parameters is to relate them analytically to traveltime, which is challenging in inhomogeneous media ([2],[3]). Using perturbation theory, we develop traveltime approximations for transversely isotropic media with horizontal symmetry axis (HTI) as explicit functions of the anellipticity parameter $\eta$ and the symmetry axis azimuth $\phi$ in inhomogeneous background media [4]. Specifically, our expansion assumes an inhomogeneous elliptically anisotropic background medium (section 4.1.1), which may be obtained from well information and stacking velocity analysis in HTI media.

As explained in section 4.1.2, this formulation has advantages on two fronts: on one hand, it alleviates the computational complexity associated with solving the HTI eikonal equation, and on the other hand, it provides a mechanism to scan for the best fitting parameters $\eta$ and $\phi$ without the need for repetitive modeling of traveltimes. Then in section 4.1.3, we show how we enhance the accuracy of our expansion by the use of Shanks transform. We show the effectiveness of our scheme with tests on a 3D model in section 4.2.1 and we propose an approach for multi-parameters scanning in
4.1 HTI Eikonal Equation and Traveltime Computation

4.1.1 Elliptical Anisotropic Model

Traveltimes are generally evaluated by solving a nonlinear partial differential equation referred to as the eikonal equation. In anisotropic media, through specific parametrization of HTI media, P-wave traveltimes in 3D, under the acoustic assumption, dependent on three parameters and the symmetry axis azimuth. These parameters include the vertical velocity \( v_v \), the normal moveout velocity \( v_{nmo} = v_v \sqrt{1 + 2\delta} \), and the anellipticity parameter \( \eta = \frac{\epsilon - \delta}{1 + 2\delta} \) (where \( \delta \) and \( \epsilon \) are Thomsen parameters \[28\]). Numerically, solving the HTI eikonal equation using finite difference is generally hard, especially because such a process requires finding the root of a quartic equation at each computational step. However, traveltime computation for a simple elliptically anisotropic model is far more efficient requiring solving a quadratic equation at each computational step and thus has the same complexity as the isotropic model in terms of solving the eikonal equation.

The eikonal equation for the isotropic case as we have seen in chapter \[2\] has the following form:

\[
v^2(x, y, z)|\nabla \tau|^2 = 1 \tag{4.1}
\]

where \( \tau \) is the traveltime measured from the source to any point with coordinates \( (x, y, z) \) and \( v \) is the velocity at that point. For elliptically anisotropic media, the eikonal equation has this form:

\[
v^2_v (x, y, z) (1 + 2\delta) \left( \left( \frac{\partial \tau}{\partial y} \right)^2 + \left( \frac{\partial \tau}{\partial z} \right)^2 \right) + v^2_v (x, y, z) \left( \frac{\partial \tau}{\partial x} \right)^2 = 1, \tag{4.2}
\]
The elliptical eikonal equation 4.2 is similar to equation 4.1 except for different coefficients in front of the traveltime derivatives. The different coefficients result in an elliptical wavefront in homogeneous media. Note that elliptically anisotropic medium is uncommon in nature but since it has the same order of complexity as isotropic media in terms of solving the eikonal equation, we use this model as the background medium for perturbing traveltime for the more practical HTI model.

4.1.2 The HTI Eikonal Equation for Arbitrary Symmetry-Axis Azimuth

The eikonal equation for HTI media under the acoustic approximation, can be extracted from the VTI version [1] by a rotation of the axis of symmetry, has the form:

\[
\nu_v^2 (1 + 2\delta) (1 + 2\eta) \left( \left( \frac{\partial \tau}{\partial y} \right)^2 + \left( \frac{\partial \tau}{\partial z} \right)^2 \right) + \nu_v^2 \left( \frac{\partial \tau}{\partial x} \right)^2 \times \\
\left( 1 - 2\eta \nu_v^2 (1 + 2\delta) \left( \left( \frac{\partial \tau}{\partial y} \right)^2 + \left( \frac{\partial \tau}{\partial z} \right)^2 \right) \right) = 1, \quad (4.3)
\]

where \( \tau (x, y, z) \) is the traveltime measured from the source to a point with coordinates \((x, y, z)\). For an arbitrary symmetry axis azimuth in the x-y plane, the traveltime derivatives in equation 4.3 are taken with respect to the azimuthal angle \( \phi \). Thus, we use the following rotation operator:

\[
\begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Rotating the traveltime derivatives of equation 4.3 with the operator 4.4 yields the following eikonal equation:

\[
v'_u^2 (1 + 2\delta) (1 + 2\eta) \left( \left( -\sin \phi \frac{\partial \tau}{\partial x} + \cos \phi \frac{\partial \tau}{\partial y} \right)^2 + \left( \frac{\partial \tau}{\partial z} \right)^2 \right) + v'_u^2 \left( \cos \phi \frac{\partial \tau}{\partial x} + \sin \phi \frac{\partial \tau}{\partial y} \right)^2 \times \\
\left( 1 - 2\eta v'_u^2 (1 + 2\delta) \left( \left( -\sin \phi \frac{\partial \tau}{\partial x} + \cos \phi \frac{\partial \tau}{\partial y} \right)^2 + \left( \frac{\partial \tau}{\partial z} \right)^2 \right) \right) = 1.
\]  
(4.5)

The numerical solution of equation 4.5 requires solving a quartic equation at each time step of the finite difference implementation or it can be solved approximately by solving a series of simpler equations using perturbation theory. By considering \(\eta\) and \(\phi\) to be constant and small, we expand the traveltime solution as a series expansion in \(\eta\) and \(\phi\). The constant \(\eta\) and \(\phi\) assumption assumes a factorized medium useful for model-development applications. However, all other velocities, including, \(v_u\) and \(v_{nmo}\) (or \(\delta\)), are allowed to vary freely. Therefore, we substitute the following trial solution

\[
\tau (x, y, z) \approx \tau_0 (x, y, z) + \tau_\eta (x, y, z) \eta + \tau_\phi (x, y, z) \sin \phi + \\
\tau_{\eta^2} (x, y, z) \eta^2 + \tau_{\eta\phi} (x, y, z) \eta \sin \phi + \tau_{\phi^2} (x, y, z) \sin^2 \phi,
\]  
(4.6)

where \(\tau_0, \tau_\eta, \tau_\phi, \tau_{\eta^2}, \tau_{\eta\phi}\) and \(\tau_{\phi^2}\) are coefficients of the expansion with units of traveltime, into the eikonal equation 4.5. As a result, and as shown in appendix \(C\), \(\tau_0\) satisfies the eikonal equation for elliptical anisotropy, which is easy solvable, whereas the coefficients of the expansion satisfy linear partial differential equations having as shown in appendix \(C\) the following general form:

\[
\frac{\partial \tau_0}{\partial x} \frac{\partial \tau_i}{\partial x} + (1 + 2\delta) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_i}{\partial y} + \frac{\partial \tau_0}{\partial z} \frac{\partial \tau_i}{\partial z} \right) = f_i (x, y, z),
\]  
(4.7)

with \(i = \eta, \phi, \eta^2, \eta\phi, \phi^2\). The function \(f_i (x, y, z)\) becomes more complicated for \(i\)
corresponding to the second-order term and it depends on the terms for the first order and background medium solutions. Therefore, the linear partial differential equations must be solved in succession starting with \( i = \eta \) and \( i = \phi \).

### 4.1.3 Shanks Transform

Once we compute all \( \tau_i \) coefficients, we use them as suggested by Alkhalifah [4] to estimate the traveltime using Shanks transform [8]. We first define the following parameters:

\[
A_0 = \tau_0 + \tau_\phi \sin \phi + \tau_{\phi_2} \sin^2 \phi, \\
A_1 = A_0 + (\tau_\eta + \tau_{\eta\phi} \sin \phi)\eta, \\
A_2 = A_1 + \eta_{\eta_2} \eta^2.
\]

The first sequence of Shanks transform uses \( A_0, A_1 \) and \( A_2 \) and has this form:

\[
\tau(x, y, z) \approx \frac{A_0 A_2 - A_1^2}{A_0 - 2A_1 + A_2} \\
= \tau_0 (x, y, z) + \tau_\phi (x, y, z) \sin \phi + \tau_{\phi_2} (x, y, z) \sin^2 \phi \\
+ \frac{\eta (\tau_\eta (x, y, z) + \tau_{\eta\phi} (x, y, z) \sin \phi)^2}{\tau_\eta (x, y, z) + \tau_{\eta\phi} (x, y, z) \sin \phi - \eta \tau_{\eta_2} (x, y, z)}. \tag{4.9}
\]

To scan for \( \eta \) and \( \phi \), the coefficients \( \tau_i \) need to be evaluated only once and can be used with equation [4.9] to search for the best traveltime fit to the traveltime extracted from the data.
4.2 Anisotropy Parameters Estimation

4.2.1 Scanning for an effective $\eta$ and $\phi$

In this section, we test the accuracy of the Shanks transform formulation and show its efficiency in estimating effective $\eta$ and $\phi$ values in complex media. Specifically, we test our approach on a 3D model containing a salt structure. This model is interesting, since most of the azimuthal anisotropy exists near salt diapirs, where radial faulting caused by salt emplacement may be observed. Figure 4.1 shows the model parameters: the vertical velocity (a), the tilt $\theta$ of the axis symmetry (b), the anelipticity parameter $\eta$ (c) and the azimuthal angle $\phi$ (d). Firstly, notice the sharp velocity discontinuity especially around the salt body which jumps from 1500 m/s at the edge of the salt structure to 4000 m/s inside the salt body. Secondly, as the tilt angle $\theta$ is measured from the vertical direction, notice that $\theta$ varies from 85° to 95° which means that the symmetry axis does not completely remain in the horizontal plane. The azimuthal angle varies considerably from 10° to 90°. For the two experiments we present, we consider the Thomsen parameter $\delta$ to be homogeneous and equals 0.1.

To assess the accuracy of equation 4.9 and its scanning capabilities, we present two examples. The idea behind these experiments is first to choose some source locations inside the model likewise the exploding reflector assumption. Then, we solve for the HTI eikonal equation with the model parameters using a fast marching type eikonal solver. Afterwards, we assume complete ignorance of $\eta$ and $\phi$ model and we scan for the effective $\eta$ and $\phi$ that best fit the traveltime, using the formulation given in equation 4.9. Specifically, we compute the Root-Mean-Square-Error (RMSE) between the exact (observed) and approximated (predicted) traveltimes. Thus the minimum
Figure 4.1: The 3D salt model parameters, (a) the vertical velocity $v_y$, (b) the tilt angle $\theta$, (c) $\eta$ and (d) the azimuthal angle $\phi$.

RMSE, given as follows:

$$\min_{\phi, \eta} \text{RMSE}(\eta, \phi) = \min_{\phi, \eta} \sqrt{\frac{\sum_{i,j} (\widehat{\tau}_{ij}(\eta, \phi) - \tau_{ij})^2}{n_1 \times n_2}},$$  \hspace{1cm} (4.10)$$

provides the criteria to choose the best effective $\eta$ and $\phi$ values. In equation 4.10, the summation is made over the number of receivers $(n_1 \times n_2)$ which are considered to be on the surface $(z = 0)$, with $\widehat{\tau}_{ij}$ and $\tau_{ij}$, respectively as the predicted and observed traveltimes.
Table 4.1: Scanning for constant η and φ

<table>
<thead>
<tr>
<th>Model parameters</th>
<th>Estimated parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>η</td>
<td>φ°</td>
</tr>
<tr>
<td>0.1</td>
<td>5</td>
</tr>
<tr>
<td>0.1</td>
<td>10</td>
</tr>
<tr>
<td>0.1</td>
<td>15</td>
</tr>
<tr>
<td>0.1</td>
<td>20</td>
</tr>
<tr>
<td>0.1</td>
<td>30</td>
</tr>
<tr>
<td>0.05</td>
<td>5</td>
</tr>
<tr>
<td>0.1</td>
<td>5</td>
</tr>
<tr>
<td>0.15</td>
<td>5</td>
</tr>
<tr>
<td>0.2</td>
<td>5</td>
</tr>
<tr>
<td>0.25</td>
<td>5</td>
</tr>
</tbody>
</table>

4.2.2 Homogeneous η and φ models

In the first experiment, we will not use the η and φ models shown in figure 4.1(c) and 4.1(d), but we consider homogeneous η and φ models and let νυ and θ vary as they are in Figure 4.1(a) and 4.1(b). We compute the exact traveltimes using a fast marching eikonal solver. Then, we invert for η and φ using the perturbation formulations and we apply our scanning scheme to check if we can retrieve the constant η and φ values. We tested this experiment for different homogeneous η and φ models and we show the results in Table 4.1. For all cases, we placed only one source at the bottom of the model located at (X = 7km, Y = 7km, Z = 4km).

The first half of table 4.1 in blue corresponds to experiments with η = 0.1 and different φ angles. We notice that the smaller the actual φ values, the more accurate estimate of the angle we obtain. This is evident since our expansion is around φ = 0. We also notice that η cannot be retrieved exactly. The tilt of the symmetry axis is one reason that prevents us obtaining exact values for the model parameters. However, considering the varying tilt, the results are reasonable. The second half of table 4.1 in red, shows examples with φ = 10° and different η cases. Here, as well, the estimated η and φ are close to the model ones.
We check the accuracy of the traveltime expansion with the effective \( \eta \) and \( \phi \) by computing the absolute relative error. Figure 4.2 shows such error for two cases taken from table 4.1. We can see that the maximum error is around 5%.

![Figure 4.2](image)

Figure 4.2: Maps of the absolute relative error in traveltimes as a function of \( X \) and \( Y \) for the case the model parameters are \( (\eta = 0.1, \phi = 15^\circ) \) and the estimated parameters are \( (\eta = 0.12, \phi = 17^\circ) \) (a) and for the case the model parameters are \( (\eta = 0.15, \phi = 10^\circ) \) and the estimated parameters are \( (\eta = 0.18, \phi = 11^\circ) \) (b).

### 4.2.3 Inhomogeneous \( \eta \) and \( \phi \) Models

In the second experiment, we compute the traveltimes with the inhomogeneous model parameters as shown in figure 4.1. Then, we assume complete ignorance of \( \eta \) and \( \phi \) and we scan for the effective \( \eta \) and \( \phi \) that gives the minimum RMSE. In this experiment we use 4 sources located at different locations as shown in table 4.2. The number of receivers (placed at \( z = 0 \)) used to compute the RMSE correspond to a 4\( km \) lateral aperture away from the source location. Figure 4.3 shows maps of the RMSE as a function of \( \eta \) and \( \phi \) for the 4 sources. The effective values found are given in table 4.2. We notice that the 4 sources do not give us exactly the same effective values. This is expected since the exact traveltime is computed with inhomogeneous \( \eta \) and \( \phi \) models, which will affect the results of scanning using different source locations.
Table 4.2: Effective $\eta$ and $\phi$ values from the minimum RMSE for 4 different source locations.

<table>
<thead>
<tr>
<th>Source location in Km</th>
<th>Effective $\eta$</th>
<th>Effective $\phi^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X=3 Y=3 Z=4</td>
<td>0.11</td>
<td>12</td>
</tr>
<tr>
<td>X=5 Y=5 Z=4</td>
<td>0.1</td>
<td>15</td>
</tr>
<tr>
<td>X=7 Y=7 Z=4</td>
<td>0.1</td>
<td>16</td>
</tr>
<tr>
<td>X=9 Y=9 Z=4</td>
<td>0.11</td>
<td>19</td>
</tr>
</tbody>
</table>

Figure 4.3: Maps of the RMSE as a function of $\eta$ and $\phi$ for 4 different source locations.

4.3 Discussion

By expanding the traveltime solutions of the HTI eikonal equation in a power series in the anelipticity parameter $\eta$ and the azimuthal angle $\phi$, we developed an efficient tool to estimate these parameters in a generally inhomogeneous background medium. Shanks transform allowed us to better represent the expansion using fewer terms, and thus fewer equations to solve. We showed the effectiveness of our approach.
on a realistic 3D model containing a salt structure. Although the multi-parameters expansion we present is in terms of \( \eta \) and \( \phi \), the approach could be easily extended to any combination of anisotropy parameters, specifically to scan for an effective \( \eta \) and \( \delta \), which we plan to do as future work. In fact, if prior information of the azimuth angle is available, we could do velocity analysis in the isotropic plane to determine an inhomogeneous velocity field. We could then scan for effective \( \eta \) and \( \delta \).

Limitation of our approach is that it becomes not accurate for large azimuthal angle since the expansion is with respect to \( \phi = 0 \). However, as Alkhalifah [4] proposed, we can use a similar expansion around \( \phi = 90^\circ \) to extend the accuracy limit to angles near \( \phi = 90^\circ \).

Moreover, by approximating the tilt angle by the structure of the model, we can incorporate the tilt effect on the background model by the following coordinate transformation applied to the eikonal equation 4.3:

\[
\begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}.
\]

Because this coordinate transform is linear and the perturbation PDEs are linear, as well, then the resulting equations will remain linear. Also the background medium is elliptically anisotropic with a tilted axis of symmetry, which is easily solvable and has the same complexity as the isotropic eikonal equation.

The challenging part in such an approach is to estimate accurate tilt angles. In all cases, the choice of the parameters to scan for depends on the prior information we have. Its accuracy will determine our ability to estimate the rest of the parameters.
Chapter 5

Conclusion and Future Work

5.1 Concluding remarks

In this Master Thesis, we applied the method of characteristics for the computation of P-wave rays and traveltimes in orthorhombic models with arbitrary orientation of symmetry planes. By using a curvilinear coordinate system constructed such that its basis elements are consistent with the orientation of the symmetry planes, we make use of the minimum number of model parameters. Synthetic model examples with exact traveltimes have been presented in this report and validated the accuracy of our approach. Although the method relies on the weak anisotropy approximation, the relative traveltime errors did not exceed 0.2 % for complicated symmetry plane rotation. Furthermore, we showed the results of P-wave rays and traveltimes on a realistic 3D model containing a salt structure. The method of characteristics is accurate. It not only provides traveltime information but also other important quantities such as the geometrical spreading and amplitudes useful for wavefield construction. However, this method works only for smooth models.

Moreover, we used the fast marching method to solve the eikonal equation and to estimate the anisotropy parameter $\eta$ and the azimuthal angle $\phi$. Solving the eikonal equation for the transversely isotropic media is difficult since it requires finding the roots of a quartic polynomial at each computational step of the finite difference
method. Using perturbation theory and considering $\eta$ and $\phi$ to be small, we expanded the traveltime solution of the horizontal transversely isotropic (HTI) eikonal equation in terms of these parameters. This process results in a scheme solving for the simple elliptical eikonal equation, thus making the computation equivalent to the cost of solving the isotropic eikonal equation, which requires only solving the roots of a quadratic polynomial. The coefficients of the traveltime expansion also require solving some linear partial differential equations. We showed some examples on a 3D model containing a salt body in this report. The approach consists of solving first the exact HTI eikonal equation by the fast marching method and getting traveltime information. By comparing the approximate traveltime and the observed traveltime (exact) on the surface of the model, we were able to estimate the anisotropy parameters.

Both methods have their advantages and disadvantages. On one hand, we have seen that the main limitation of the characteristics method is its weakness to handle nonsmooth models and also its computational cost. However, it provides more accurate traveltime as well as important information along the ray such as geometrical spreading. On the other hand, solving the eikonal equation by finite difference with the fast marching method has the main feature of providing traveltimes at all grid points even for nonsmooth models. The main limitation of this method is that it only computes first arrival traveltimes which are the less energetic.

5.2 Future Work

The approach developed for the ray tracing and dynamic ray tracing is valid for P-waves propagating in smooth inhomogeneous models without structural interfaces. In the future, we can use paraxial methods to estimate traveltimes in the vicinity of the computed rays and make use of interpolation methods to compute traveltimes at all grid points. Furthermore, our approach could be applied in seismic tomography
to compute traveltime fields and update the corresponding velocities and anisotropy parameters. In this case, the computed travel times fields can be used for the updating process required for the optimization method in seismic tomography. Another direction is to extend the method to models with interfaces. This will require the computation of the transmission and reflection angles as well as to take into account the ray code in the two point ray tracing method. Moreover, we can consider the propagation of S waves since they hold valuable information such as the presence of liquid in the subsurface.

The idea of multi-parameters search we developed could also be extended to any combination of anisotropy parameters. Specifically, our next step is to implement the procedure that estimates Thomsen’s $\delta$ parameter and the anellipticity parameter $\eta$. In fact, if prior information of the azimuth angle $\phi$ is available, we could do velocity analysis in the isotropic plane to determine an inhomogeneous velocity field. We could then scan for effective $\eta$ and $\delta$. Moreover, we are considering introducing the tilt angle $\theta$ in the background model which provides more accurate traveltimes and parameter estimation if an information of the tilt angle is available.
REFERENCES


APPENDICES

A  Anisotropy Parameters in Terms of the Elastic Moduli

A.1  Thomsen Parameters

The P-wave Thomsen [28] parameters for VTI media are the following:

\[ v_v = \sqrt\frac{c_{33}}{\rho}, \quad \epsilon = \frac{c_{11} - c_{33}}{2c_{33}}, \quad \delta = \frac{(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})} \]  

(A.1)

where \( c_{ij} \) are the elastic moduli in Voight notation.

A.2  Weak Anisotropy Parameters

The expressions of weak anisotropy parameters (WA) suggested by Psencik and Gajewski [17] are related linearly to the elastic coefficients. If we denote by \( \alpha \) the velocity of a reference isotropic medium, the 6 P-wave weak anisotropy parameters necessary to describe an orthorhombic symmetry are the following:

\[ \epsilon_x = \frac{A_{11}-\alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22}-\alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33}-\alpha^2}{2\alpha^2}, \]

\[ \delta_x = \frac{A_{13}+2A_{55}-\alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23}+2A_{44}-\alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12}+2A_{66}-\alpha^2}{\alpha^2}. \]  

(A.2)
In the above expressions, $A_{ij}$ are the density normalized elastic moduli in the Voight notation. The following variables are also used:

$$\eta_x = \delta_y - \epsilon_y - \epsilon_z, \quad \eta_y = \delta_x - \epsilon_x - \epsilon_z, \quad \eta_z = \delta_z - \epsilon_x - \epsilon_y. \quad (A.3)$$

**A.3 Fourth Order Runge Kutta Method**

Runge kutta is used for the integration of the ray tracing and dynamic ray tracing equations. If we consider the following initial value problem:

$$\dot{y} = f(t, y), \quad y(t_0) = y_0 \quad (A.4)$$

where $y$ is the unknown function which depends on the time $t$. $\dot{y}$ is the rate at which $y$ changes which is a function of $t$ and of $y$ itself. At the initial time $t_0$, the corresponding $y$-value is $y_0$ The solution to this problem at at time $t_{i+1} = t_i + h$, where $h$ is the step-size, is as follows:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h,$$

$$k_1 = f(t_i, y_i),$$

$$k_2 = f(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h),$$

$$k_3 = f(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h),$$

$$k_4 = f(t_i + h, y_i + k_3h). \quad (A.5)$$
B  
RT and DRT Equations for the 
Orthorhombic Symmetry

The ray tracing equations in Cartesian coordinates read:

\[
\frac{dx_i}{dt} = \frac{1}{2} \frac{\partial G}{\partial p_i}, \quad \frac{dp_i}{dx_i} = -\frac{1}{2} \frac{\partial G}{\partial x_i}, \quad (i=1,2,3) 
\]  

(B.1)

where G is the eigenvalue of the Christoffel matrix and has this expression in orthorhombic media:

\[
G(x_m, p_m) = \alpha^2 \left\{ p_k p_k + 2(\epsilon_x p_1^2 + \epsilon_y p_2^2 + \epsilon_z p_3^2) 
+ 2(p_k p_k)^{-1} \left[ \eta_x p_2^2 p_3 + \eta_y p_1^2 p_3 + \eta_z p_1^2 p_2^2 \right] \right\}, 
\]  

(B.2)

where \( \epsilon_x, \epsilon_y, \epsilon_z, \eta_x, \eta_y \) and \( \eta_z \) are the weak anisotropy parameters, \( \alpha \) is a reference velocity used in the definition of the parameters and \( p_k \) are the components of the slowness vector. Below the ray tracing equations valid in a coordinate system attached to the orientation of the symmetry planes of an orthorhombic symmetry are derived:

\[
\frac{dx_1}{d\tau} = \alpha^2 \left\{ p_1 + 2\epsilon_x p_1 + 2c^4 \left[ A p_{23} p_1 - D p_1 p_2 p_3 \right] \right\}, 
\]

\[
\frac{dx_2}{d\tau} = \alpha^2 \left\{ p_2 + 2\epsilon_y p_2 + 2c^4 \left[ E p_{13} p_2 - H p_1 p_2 p_3 \right] \right\}, 
\]

\[
\frac{dx_3}{d\tau} = \alpha^2 \left\{ p_3 + 2\epsilon_z p_3 + 2c^4 \left[ P p_{12} p_3 - S p_1 p_2 p_3 \right] \right\}, 
\]

\[
\frac{dp_i}{d\tau} = -\alpha^2 \left\{ T_{i,i} + c^2 \left[ P_i p_3^2 + S_i p_1 p_2 \right] \right\}, 
\]  

(B.3)

where the expressions for \( A, D, E, H, T, P \) and \( S \) are given in appendix B.6.
The dynamic ray tracing equations read:

\[
\frac{dX_i^{(l)}}{d\tau} = \frac{1}{2} \left[ \frac{\partial^2 G(x_m, p_m)}{\partial p_i \partial x_j} X_j^{(l)} + \frac{\partial^2 G(x_m, p_m)}{\partial x_i \partial p_j} P_j^{(l)} \right],
\]

\[
\frac{dP_i^{(l)}}{d\tau} = -\frac{1}{2} \left[ \frac{\partial^2 G(x_m, p_m)}{\partial x_i \partial x_j} X_j^{(l)} + \frac{\partial^2 G(x_m, p_m)}{\partial x_i \partial p_j} P_j^{(l)} \right],
\]

where the quantities \( \frac{\partial^2 G(x_m, p_m)}{\partial p_i \partial x_j} \), \( \frac{\partial^2 G(x_m, p_m)}{\partial p_i \partial p_j} \) and \( \frac{\partial^2 G(x_m, p_m)}{\partial x_i \partial x_j} \) are the following:

\[
\frac{\partial^2 G}{\partial p_1^2} = 2\alpha^2 \left\{ 1 + 2\epsilon_x + 2\epsilon_y \left[ (Ap_{23} - Dp_2p_3)(p_{23} - 3p_1^2) \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial p_1 \partial p_2} = 4\alpha^2 \left\{ c^4 \left[ 2\epsilon_x p_1 p_2 p_{23} - \eta_x p_1 \eta_y p_2 p_{23} \right] - c^6 \left[ 2(Ap_{23} - p_1^2)p_1 p_2 + D(p_{13} - 3p_2^2)p_1 p_3 \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial p_1 \partial p_3} = 4\alpha^2 \left\{ c^4 \left[ 2\epsilon_x p_1 p_3 p_{23} - \eta_x p_1 \eta_y p_3 p_{23} \right] - c^6 \left[ 2(Ap_{23} - p_1^2)p_1 p_3 + D(p_{12} - 3p_3^2)p_1 p_2 \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial p_2^2} = 2\alpha^2 \left\{ 1 + 2\epsilon_y + 2\epsilon_y \left[ (Ep_{13} - H p_1 p_3)(p_{13} - 3p_2^2) \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial p_2 \partial p_3} = 4\alpha^2 \left\{ c^4 \left[ 2\epsilon_x p_3 p_2 p_{13} - \eta_x p_1 \eta_y p_3 p_{23} \right] - c^6 \left[ 2(Ep_{13} - p_2^2)p_2 p_3 + H(p_{12} - 3p_3^2)p_1 p_2 \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial p_3^2} = 2\alpha^2 \left\{ 1 + 2\epsilon_z + 2\epsilon_y \left[ (P p_{12} - S p_1 p_2)(p_{12} - 3p_3^2) \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial x_i \partial p_1} = 4\alpha^2 \left\{ \epsilon_{x,i} p_1 + c^4 \left[ A_{i,23}p_1 - D_{i,1}p_1 p_2 p_3 \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial x_i \partial p_2} = 4\alpha^2 \left\{ \epsilon_{y,i} p_2 + c^4 \left[ E_{i,13}p_2 - H_{i,1}p_1 p_2 p_3 \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial x_i \partial p_3} = 4\alpha^2 \left\{ \epsilon_{z,i} p_3 + c^4 \left[ P_{i,12}p_3 - S_{i,1}p_1 p_2 p_3 \right] \right\},
\]

\[
\frac{\partial^2 G}{\partial x_i \partial x_j} = 2\alpha^2 \left\{ T_{ij} + c^2 \left[ P_{i,j}p_3^2 + S_{i,j}p_1 p_2 \right] \right\}.\]

The following notation is being used:

\[
c = (p_k p_k)^{1/2}, \quad p_{13} = p_1^2 + p_3^2, \quad p_{23} = p_2^2 + p_3^2, \quad p_{12} = p_1^2 + p_2^2,
\]

\[
A = \eta_x p_3^2 + \eta_x p_2^2, \quad D = \eta_x p_2 p_3, \quad E = \eta_x p_3^2 + \eta_y p_1^2, \quad H = \eta_y p_1 p_3,
\]

\[
P = \eta_y p_1^2 + \eta_z p_2^2, \quad S = \eta_z p_1 p_2, \quad T = \epsilon_x p_1^2 + \epsilon_y p_2^2 + \epsilon_z p_3^2.\]
The expressions of $B_{13}$ and $B_{23}$ needed in the second order traveltime correction are:

\[
B_{13} = \alpha^2 D^{-1} \left\{ n_3^3 (\eta_y n_1^2 + \eta_x n_2^2) + n_3 \left[ n_1^2 n_2^2 (2\eta_z - \eta_x - \eta_y) - n_1^4 \eta_y - n_2^4 \eta_x \right.ight.
\]
\[
+ n_1^2 (\epsilon_x - \epsilon_z) + n_2^2 (\epsilon_y - \epsilon_z) \left. \right]\},
\]
\[
B_{23} = \alpha^2 D^{-1} \left\{ n_3^2 n_1 n_2 (\eta_x - \eta_y) + n_3^3 n_2 \eta_z - n_1 n_2^3 \eta_z + n_1 n_2 (\epsilon_y - \epsilon_x) \right\},
\]

(B.7)

where

\[
D = (n_1^2 + n_2^2)^{1/2}.
\]

(B.8)

The symbols $n_i$ are the components of the unit vector normal to the wavefront.
C Higher Order Expansions

We expand the traveltimes solution of equation 4.5 in terms of the independent parameters \( \eta \) and \( \phi \):

\[
\tau(x, y, z) \approx \tau_0(x, y, z) + \tau_\eta(x, y, z) \eta + \tau_\phi(x, y, z) \sin \phi + \tau_{\eta^2}(x, y, z) \eta^2 + \tau_{\eta \phi}(x, y, z) \eta \sin \phi + \tau_{\phi^2}(x, y, z) \sin^2 \phi.
\] (C.1)

Inserting the trial solution, equation C.1 into equation 4.5 yields a long formula but by setting both \( \sin \phi = 0 \) and \( \eta = 0 \), we get the zeroth-order term given by:

\[
\nu_v^2(x, y, z) (1 + 2\delta) \left( \left( \frac{\partial \tau_0}{\partial y} \right)^2 + \left( \frac{\partial \tau_0}{\partial z} \right)^2 \right) + \nu_v^2(x, y, z) \left( \frac{\partial \tau_0}{\partial x} \right)^2 = 1, \] (C.2)

which is simply the elliptical eikonal equation. By equating the coefficients of the powers of the independent parameters \( \sin \phi \) and \( \eta \), in succession starting with first powers of the two parameters, we end up first with the coefficients of first power in \( \sin \phi \) and zeroth power in \( \eta \):

\[
\frac{\partial \tau_0}{\partial x} \frac{\partial \tau_\phi}{\partial x} + (1 + 2\delta) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_\phi}{\partial y} + \frac{\partial \tau_0}{\partial z} \frac{\partial \tau_\phi}{\partial z} \right) = 2\delta \frac{\partial \tau_0}{\partial x} \frac{\partial \tau_0}{\partial y}, \] (C.3)

which is a first order linear partial differential equation in \( \tau_\phi \). The coefficients of zero power in \( \sin \phi \) and the first power in \( \eta \), simplified by using equation C.2, is given by:

\[
\frac{\partial \tau_0}{\partial x} \frac{\partial \tau_\eta}{\partial x} + (1 + 2\delta) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_\eta}{\partial y} + \frac{\partial \tau_0}{\partial z} \frac{\partial \tau_\eta}{\partial z} \right) = -\nu_v^2 (1 + 2\delta)^2 \left( \left( \frac{\partial \tau_0}{\partial z} \right)^2 + \left( \frac{\partial \tau_0}{\partial y} \right)^2 \right)^2 \] (C.4)
The coefficients of the square term in \(\sin \phi\) with some manipulation, result in the following relation:

\[
\frac{\partial \tau_0}{\partial x} \frac{\partial \tau_0}{\partial x} + (1 + 2\delta) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial y} + \frac{\partial \tau_0}{\partial z} \frac{\partial \tau_0}{\partial z} \right) = -\frac{1}{2} \left( 1 + 2\delta \right) \left( \left( \frac{\partial \tau_0}{\partial z} \right)^2 + \left( \frac{\partial \tau_0}{\partial y} \right)^2 \right) + \left( \frac{\partial \tau_0}{\partial x} \right)^2 + 2\left( \frac{\partial \tau_0}{\partial x} \right)^2 + 2\delta \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial x} + \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial y} \right) + \delta \left( \left( \frac{\partial \tau_0}{\partial y} \right)^2 - \left( \frac{\partial \tau_0}{\partial x} \right)^2 \right),
\]

(C.5)

which is again a first-order linear PDE in \(\tau_0\) with a more complicated source function given by the right hand side.

The coefficients of the square terms in \(\eta\), with also some manipulation, result also in a linear first-order PDE with the following relation:

\[
\frac{\partial \tau_0}{\partial x} \frac{\partial \tau_0}{\partial x} + (1 + 2\delta) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial y} + \frac{\partial \tau_0}{\partial z} \frac{\partial \tau_0}{\partial z} \right) = -\frac{1}{2} \left( 1 + 2\delta \right) \left( \left( \frac{\partial \tau_0}{\partial z} \right)^2 + \left( \frac{\partial \tau_0}{\partial y} \right)^2 \right) + \left( \frac{\partial \tau_0}{\partial x} \right)^2 + 2(1 + 2\delta) \left( \left( \frac{\partial \tau_0}{\partial x} \right)^2 + \left( \frac{\partial \tau_0}{\partial y} \right)^2 \right) + 2(1 + 2\delta) \left( \left( \frac{\partial \tau_0}{\partial y} \right)^2 - \left( \frac{\partial \tau_0}{\partial x} \right)^2 \right),
\]

(C.6)

Finally, the coefficients of the first-power terms in both \(\sin \phi\) and \(\eta\) with some manipulation result into the following first order PDE:

\[
\frac{\partial \tau_0}{\partial x} \frac{\partial \tau_0}{\partial x} + (1 + 2\delta) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial y} + \frac{\partial \tau_0}{\partial z} \frac{\partial \tau_0}{\partial z} \right) = -\left( 1 + 2\delta \right) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial y} + \frac{\partial \tau_0}{\partial z} \frac{\partial \tau_0}{\partial z} \right) + \frac{\partial \tau_0}{\partial x} \frac{\partial \tau_0}{\partial x} + 2(1 + 2\delta) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial z} + \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial y} \right) + 2(1 + 2\delta) \left( \frac{\partial \tau_0}{\partial y} \frac{\partial \tau_0}{\partial y} - \frac{\partial \tau_0}{\partial x} \frac{\partial \tau_0}{\partial x} \right),
\]

(C.7)
D  Conferences Submitted

• Masmoudi N. and Psencik I, 2014, “Approximate P-wave ray tracing and dynamic ray tracing in weakly orthorhombic media of varying symmetry orientation”, Submitted to 76th Annual EAGE meeting, accepted.