Efficient Modeling and Migration in Anisotropic Media Based on Prestack Exploding Reflector Model and Effective Anisotropy

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ABSTRACT

Efficient Modeling and Migration in Anisotropic Media Based on Prestack Exploding Reflector Model and Effective Anisotropy

Hui Wang

This thesis addresses the efficiency improvement of seismic wave modeling and migration in anisotropic media. This improvement becomes crucial in practice as the process of imaging complex geological structures of the Earth’s subsurface requires modeling and migration as building blocks. The challenge comes from two aspects. First, the underlying governing equations for seismic wave propagation in anisotropic media are far more complicated than that in isotropic media which demand higher computational costs to solve. Second, the usage of whole prestack seismic data still remains a burden considering its storage volume and the existing wave equation solvers.

In this thesis, I develop two approaches to tackle the challenges.

In the first part, I adopt the concept of prestack exploding reflector model to handle the whole prestack data and bridge the data space directly to image space in a single kernel. I formulate the extrapolation operator in a two-way fashion to remove the restriction on directions that waves propagate. I also develop a generic method for phase velocity evaluation within anisotropic media used in this extrapolation kernel. The proposed method provides a tool for generating prestack images without wavefield
cross correlations.

In the second part of this thesis, I approximate the anisotropic models using effective isotropic models. The wave phenomena in these effective models match that in anisotropic models both kinematically and dynamically. I obtain the effective models through equating eikonal equations and transport equations of anisotropic and isotropic models, thereby in the high frequency asymptotic approximation sense. The wavefields extrapolation costs are thus reduced using isotropic wave equation solvers while the anisotropic effects are maintained through this approach.

I benchmark the two proposed methods using synthetic datasets. Tests on anisotropic Marmousi model and anisotropic BP2007 model demonstrate the applicability of my approaches.
I dedicate this thesis to my advisor, professors, colleagues and family, who have helped and supported me during my Master’s research life. I am grateful to KAUST for its financial support during my studies.

I would like to thank my advisor, Tariq Alkhalifah, for his persistent generosity with creative ideas, as well as for providing a collaborative research environment in the Seismic Wave Analysis Group. His continual patience, insightful research advice and constant push and challenge guarantees the completion of the two research projects I have worked on.

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<td>ADCIGs</td>
<td>angle domain common image gathers</td>
</tr>
<tr>
<td>AMO</td>
<td>azimuth moveout</td>
</tr>
<tr>
<td>CFL</td>
<td>Courant-Friedrichs-Lewy</td>
</tr>
<tr>
<td>DFT</td>
<td>discrete Fourier transform</td>
</tr>
<tr>
<td>DMO</td>
<td>dip moveout</td>
</tr>
<tr>
<td>DSR</td>
<td>double square root</td>
</tr>
<tr>
<td>DTI</td>
<td>dip oriented transversely isotropic</td>
</tr>
<tr>
<td>ENO</td>
<td>essentially non-oscillatory</td>
</tr>
<tr>
<td>FFT</td>
<td>fast Fourier transform</td>
</tr>
<tr>
<td>HJE</td>
<td>Hamilton-Jacobi equations</td>
</tr>
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<td>MVA</td>
<td>migration velocity analysis</td>
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<tr>
<td>NMO</td>
<td>normal moveout</td>
</tr>
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<td>PERM</td>
<td>prestack exploding reflector model</td>
</tr>
<tr>
<td>SODCIGs</td>
<td>subsurface offset domain common image gathers</td>
</tr>
<tr>
<td>SVD</td>
<td>singular value decomposition</td>
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<tr>
<td>TI</td>
<td>transversely isotropic</td>
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<td>WENO</td>
<td>weighted essentially non-oscillatory</td>
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Chapter 1

Introduction

This thesis addresses the efficiency improvements of seismic imaging methods in the anisotropic media. Seismic imaging, in its literal sense, is creating images or snapshots of the Earth’s subsurface using seismic data, which is analogous to medical imaging. These images conventionally refer to the Earth’s reflectivity functions of spatial location, from which physical properties of rocks are derived. The process of migrating the recorded seismic data energy to where it came from is named as migration, also known as linearized seismic inversion.

An even more ambitious goal of seismic imaging is to extract the rock property information directly from the seismic data in the context of non-linear seismic inversion. The non-linearity can be intuitively understood through the scattering theory. Scattering theory says that, given an Earth’s subsurface model, the waves travel through the model, get scattered in the model and then travel to receivers being recorded as signals. If we assume the waves interact with model parameters only once (i.e. single scattering or first order Born scattering), the problem becomes linear since the recorded data are linear combinations of model parameters. However if the waves get scattered multiple times, the interaction effects with model parameters are multiplied, then the recorded data non-linearly depends on model parameters.

No matter whether the problem is linear or not, the forward problem is often easier to solve than its inverse counterpart since the mapping from model space to
data space is deterministic while it is not the case in the inverse problem. A typical solution to the inverse problem is by trial and error, that is assuming one physical model, solve forward problem and test if the modeled data match recorded data according to some criteria; if not, adjust the model, solve forward problem until such criteria are satisfied.

Three key aspects in the inversion are worthy of note. First, we should parameterize the model wisely such that the parameters are sensitive to data perturbations as well as physically explainable; the former condition ensures the parameters are easy to invert for while the latter makes sure the inverted parameters make sense and can be interpreted by rock physicists and geologists. Second, forward problems need to be solved several times. Therefore accurate and efficient forward modelings are building blocks of inverse problems. Moreover linear inversion or migration is also building block of non-linear inversions as pointed out in Tarantola (2005) that migration is equivalent to the gradient of misfit function after the first iteration in non-linear seismic inversion. Thirdly, strategies for measuring the fitness of modeled data and updating the model parameters are essential to obtain the inversion results. In this thesis I address the first two aspects, namely considering an Earth’s subsurface model of anisotropic velocity and develop a fast, easy to implement modeling and migration kernel with the anisotropic model.

Motivations

Until now, quantum theory still has no explicit applications in exploration seismology. Instead, the Earth is described as a continuum and the adherent continuum mechanics provides the basics to this area. Explorationists further assume a linear elastic solid Earth model to simplify their world given the fact that material displacement is small compared to seismic wavelength and the material velocity is small compared to the
wave propagation velocity \cite{Slawinski:2010}. Moreover the pressure waves in fluid models are used to approximate the quasi-$P$ elastic waves in solid models \cite{Cerveny:2005}, since the kinematics of acoustic wave well approximate those of elastic wave. This dominant approximation is extremely successful in exploring oil and gas fields under relatively simple geological structures since various well-established migrations techniques only use the kinematics information.

However, with the diminishing of “easy” oil fields, the industry is exploring areas with complex geological features such as salt flanks and faults. Under such situations, we lose the luxury of having correct kinematic information using acoustic assumption especially in the appearance of anisotropy. To image the Earth’s subsurface accurately, the industry has incorporated seismic anisotropy into seismic processing and imaging since the 1980s. Accounting for arbitrarily anisotropy by directly solving the linearized elastodynamics equation is computationally expensive and meaningless in practice since our Earth’s subsurface maintains certain symmetry. For example, with the effect of gravity, the shales and sedimentary layers often exhibit anisotropy in vertical direction while possessing isotropy in the transverse plane. This kind of model is called transversely isotropic (TI) model, and is the simplest, with highest symmetry among all the anisotropic models. These models have been the most commonly used in seismic imaging. A number of simplified quasi-$P$ wave equations are derived based on this model to account for anisotropy which only deal with $P$ wave seismic data \cite{Alkhalifah:2000, Fletcher et al.:2009, Grechka et al.:2004, Zhan et al.:2012}. However these equations computationally cost more than modeling isotropic $P$ wave equation, especially in tridimensional cases. One of my objectives therefore is to reduce the computational cost through an effective anisotropy idea, which I call effective anisotropy through traveltime and amplitude matching.

There is another category of seismic inverse problems which needs accurate, efficient and neat migration algorithms which is called migration velocity analysis MVA.
This technique takes advantage of prestack seismic data and utilizes the fact that with a correct migration velocity, the prestack migrated images should be consistent. This is quite analogous to traveltime tomography while its misfit criteria is measured in the migrated image space. Angle domain common image gathers (ADCIGs) and subsurface offset domain common image gathers (SODCIGs) are two popular image spaces to do MVA. Poynting vectors (Yoon and Marfurt, 2006) and extended imaging conditions (Sava and Fomel, 2006; Sava and Vasconcelos, 2011) are two methods of extracting these gathers using wave equation extrapolation. The latter is preferable since the former requires the knowledge of time and spatial derivatives of wavefields and the division of wavefields pose another numerical issue. However extended imaging conditions necessitates the cross-correlation of wavefields which is computationally expensive. My second objective of this thesis is to construct a modeling and migration kernel in anisotropic media without wavefields cross-correlation to generate prestack images that are ready for MVA. I refer to my method as prestack exploding reflector modeling and migration which already has its name in the geophysical literature.

**Contributions**

Several investigations similar to my work appear in geophysical literature.

The first description about prestack exploding reflector model (PERM) appears in Biondi (2002), where the author formulates the double square root (DSR) equation to time evolution wave equation and solve that in a standard reverse time migration manner. Guerra (2010) adopted the PERM using one-way wave equation solver to do MVA. Alkhalifah et al. (2013a) stabilize the PERM formula by filtering out the singularity and extrapolating the wavefield using spectral method. However all of them are done in isotropic media.

In this thesis, I follow Alkhalifah et al. (2013a), extending it to anisotropic media.
The major original contributions of this work are as follows:

- Anisotropic wavefields extrapolators along time axis using low rank decomposition in a spectral sense.

- Systematic ways to evaluate the phase angle and phase velocity used in the extrapolators.

The effective anisotropy idea originally comes from Alkhalifah et al. (2013b), where the authors approximate the anisotropic models using source dependent effective isotropic velocity model. The equivalence lies in the kinematic match of anisotropic wavefields and isotropic wavefields in a high frequency asymptotic sense. I follow this method and extend it to dynamical match of the wavefields. The major original contributions of this work are as follows:

- Both kinematic and dynamical match of wavefields through equating the eikonal and transport equations.

- One possibility of evaluating effective density without solving transport equations.

**Thesis Overview**

This thesis is organized as follows:

**Fundamentals of Seismic Anisotropy**

In Chapter 2, fundamentals of seismic anisotropy are introduced. I introduce the phase velocity formulas for TI media.
Prestack Exploding Reflector Modeling and Migration

In Chapter 3, workflows of modeling and migration with PERM in TI media are introduced. I briefly review the spectral methods and one possibility of reducing the cost by low rank decomposing the extrapolation operator. I describe the systemic ways of evaluating phase angles and phase velocities for PERM formula. I also show synthetic examples to validate the proposed method.

Effective Anisotropy

In Chapter 4, workflows of obtaining effective isotropic models are presented. I review the strategies of solving eikonal and transport equations. I provide one possibility of effective density evaluation without solving the transport equations. I give numerical examples to justify this effective anisotropy technique.
Chapter 2

Basics of Anisotropic Seismic Wave Propagation

Seismic wave propagation are propagation of deformations through the interactions of grains within the Earth’s materials. Seismologists describe the Earth’s materials as continua by looking at the structure of the Earth on macroscopic scale. However, the wave propagation phenomena can be quite complicated depending on which continuum models we use to approximate the real Earth. The dominant assumed model among exploration seismologists is the linearized elastic solid model. Seismic wave propagation through this kind of media can be anisotropic and this phenomena is attracting more interests recently both from academia and from industry. A continuum is anisotropic with respect to certain property if that property behaves differently when observed from different directions. For exploration seismologists, the velocity is the most important parameter or rock property to be determined. Since this thesis is focused on the media with anisotropic velocities based on some assumed models, in this chapter I introduce the basics of anisotropic elastic theory.
2.1 Linear Elasticity

As the Earth’s materials are continua, the kinematics of continuum mechanics should apply to our Earth model whatever further constitutive assumptions we make. The balance of linear momentum gives the elastodynamics equation written in Lagrangian description

\[ \rho_0 \ddot{\varphi} = \nabla^X \cdot (\hat{P}(\nabla \varphi)) + \rho_0 \mathbf{b}_m, \]  

(2.1)

where I adopt the conventions that \( X = (X_1, X_2, X_3)^T \) denote the material coordinates in reference configuration \( B_0 \), \( x = (x_1, x_2, x_3)^T \) denote the spatial coordinates in the present configuration \( B_t \). \( \varphi(X) : B \rightarrow B_t \) is the deformation mapping from reference configuration to present configuration. \( \nabla \varphi \) denote the deformation gradient. \( \rho_0(X) \) denote the mass density in the reference configuration. \( P(X,t) \) is the first Piola-Kirchhoff stress tensor and \( \mathbf{b}_m(X,t) \) is the body force in material description. \( \hat{P}(\nabla \varphi) \) is the first Piola-Kirchhoff stress tensor written in terms of deformation gradient, i.e. \( \hat{P}(\nabla \varphi) = \hat{P}(\nabla \varphi) \).

If we assume the Earth is undergoing an infinitesimal deformation, then the first Piola-Kirchhoff stress tensor is equivalent to Cauchy stress tensor and the Lagrangian description is equivalent to Eulerian description, i.e.

\[
\begin{cases}
P(X,t) \approx \sigma(x) \\
X \approx x
\end{cases},
\]

(2.2)

where \( \sigma(x) \) is the Cauchy stress tensor. We also notice that

\[ \varphi = u + X, \]

(2.3)
where \( \mathbf{u} \) is displacement of the body. Then we have

\[
\ddot{\phi} = \ddot{\mathbf{u}},
\]

(2.4)
since \( \mathbf{X} \) is not a function of time. We then substitute equations (2.2) and (2.4) into (2.1) to arrive at the linearized elastodynamics equation which is

\[
\rho \ddot{\mathbf{u}} = \nabla^x \cdot \mathbf{\sigma}(x) + \rho \mathbf{b},
\]

(2.5)
where I dropped the subscripts of \( \rho \) and \( \mathbf{b} \) to indicate they do not change under infinitesimal deformation.

To close the system, we need the constitutive assumption which is generalized Hooke’s law written in index notation

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl},
\]

(2.6)
where \( \mathbf{C} = C_{ijkl} \) is a fourth order tensor known as stiffness tensor or elastic moduli and \( \varepsilon \) is the strain tensor defined as

\[
\varepsilon = \text{sym}(\nabla \mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).
\]

(2.7)
Further notice the symmetry property of the Cauchy stress tensor, the strain tensor, and the stiffness tensor, equation (2.6) can be written as

\[
\sigma_{ij} = C_{ijkl} u_{k,l}.
\]

(2.8)
Substitute equation (2.8) into (2.5), we then have a system of second order PDEs in terms of displacements describing the linear elasticity which can be written in index
notation as

\[ \rho \ddot{u}_i = (C_{ijkl} u_{k,l})_{,j} + \rho b_i, \]  

(2.9)

\subsection*{2.2 Phase Velocity}

From the governing equation (2.9) of elastic solid Earth, we see the rock property information are all embedded into the stiffness tensor \( C \). Anisotropy is applying certain constraints on the stiffness tensor and assign certain physical explanation to the constraints. Although the stiffness tensor is symmetric and has 21 degrees of freedom in general, it can be further simplified by its intrinsic symmetry, i.e. it can be rotation invariant about certain axes (symmetry axes). The classification depending on the number of symmetry axes gives rise to different types of anisotropic media models. Among them, triclinic media is the most general without any intrinsic symmetry, i.e. with 21 independent parameters in stiffness tensor; media with orthorhombic symmetry and hexagonal symmetry (transverse isotropy) are gaining interests in industry recently. Isotropic media are the simplest cases which are used by explorationists for decades.

To see how the stiffness tensor affects the propagation of elastic waves, we do harmonic analysis to the equation (2.9). In the case of homogeneous media without external body force, equation (2.9) takes the form of

\[ \rho \ddot{u}_i = C_{ijkl} u_{k,lj}, \]  

(2.10)

dropping the divergence term of stiffness tensor \( C_{ijkl,j} \). A time-harmonic (plane wave) trial solution of equation (2.10) has the form of

\[ u_i = U_i e^{i(k_j x_j - \omega t)} = U_i e^{i\omega(n_j x_j - V t)} = U_i e^{i\omega(n_j x_j / V - t)}, \]  

(2.11)
where \( U_i \) is the \( i \)th component of polarization vector \( U \), \( \omega \) is the angular frequency, \( k \) is wavenumber vector, \( p \) is the slowness vector, \( n \) is the unit vector along phase slowness direction and \( V \) is the magnitude of phase velocity. Substitute (2.11) into (2.10), we arrive at

\[
C_{ijkl} n_j n_l u_k = \rho V^2 u_i. \tag{2.12}
\]

Denote \( \Gamma_{ik} = C_{ijkl} n_j n_l \) as the Kelvin-Christoffel matrix, we arrive at an eigenvalue problem written in matrix form

\[
\Gamma u = \rho V^2 u, \tag{2.13}
\]

with \( \rho V^2 \) and \( u \) as its eigensystem. Clearly the Kelvin-Christoffel matrix symmetric, further it is positive definite (Musgrave, 1970). Therefore \( \Gamma \) has three real and positive eigenvalues. These eigenvalues can be found by solving the characteristic equation

\[
\det(\Gamma_{ik} - \rho V^2 \delta_{ik}) = 0. \tag{2.14}
\]

Notice the Kelvin-Christoffel matrix depends on the wave propagation direction \( n \) (phase velocity direction), therefore its eigenvalues and the wave propagation velocity \( V \) in general should depend on the direction that wave propagates. This is the mathematical explanation on seismic anisotropy. Therefore we can use the relationship between of velocity or slowness and the propagation direction to describe the anisotropic signature of certain wave types, and the relations are usually denoted as slowness surfaces.

Once we solve the equation (2.14) for eigenvalues \( \rho V^2 \) given certain wave propagate direction \( n \), we are able to solve for their corresponding eigenvectors \( u \) which has the same direction as polarization vector \( U \). The polarization direction is a
function of wave propagation direction yet is necessarily parallel or perpendicular to wave propagation direction although it is the case in isotropic media where we denote $P$-wave and $S$-wave depending on whether their polarization direction is perpendicular or is parallel to the wave propagation direction. However if the characteristic equation (2.14) has three distinct roots, we are then able to choose three orthogonal eigenvectors and we can in particular choose them to be along or perpendicular to the wave propagation direction. That is why we have the name quasi-$P$ and quasi-$S$ wave modes in general elastic media rather than $P$-wave and $S$-wave in isotropic case.

However the three eigenvalues are not guaranteed to be distinct, they depend on the structure of stiffness tensor and the wave propagation direction. I now give the specific example of stiffness tensor and determine the phase velocity expression. I focus on the transversely isotropic (TI) media which has a hexagonal intrinsic symmetry. In particular VTI is discussed and extension to tilted TI should apply through a change of coordinates.

First I adopt the convention that stiffness tensor is converted to stiffness matrix through the Voigt notation (Thomsen 1986), i.e.

$$
i j \text{ or } k l : 11 \ 22 \ 33 \ 32 = 23 \ 31 = 13 \ 12 = 21$$

$$\downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ .$$

(2.15)
With this notation, the stiffness matrix of VTI media can be written as

\[
c = \begin{pmatrix}
  c_{11} & c_{11} - 2c_{66} & c_{13} & 0 & 0 & 0 \\
  c_{11} - 2c_{66} & c_{11} & c_{13} & 0 & 0 & 0 \\
  c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & c_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & c_{44} & 0 \\
  0 & 0 & 0 & 0 & 0 & c_{66}
\end{pmatrix},
\]

(2.16)

substitute to the *Kelvin-Christoffel* matrix, we have the components \(^{(2.17)}\) \(^{(Tsvankin \, 2001)}\)

\[
\Gamma_{11} = c_{11}n_1^2 + c_{66}n_2^2 + c_{44}n_3^2,
\]

\[
\Gamma_{22} = c_{66}n_1^2 + c_{11}n_2^2 + c_{44}n_3^2,
\]

\[
\Gamma_{33} = c_{44}(n_1^2 + n_2^2) + c_{33}n_3^2,
\]

\[
\Gamma_{12} = (c_{11} - c_{66})n_1n_2,
\]

\[
\Gamma_{13} = (c_{13} + c_{44})n_1n_3,
\]

\[
\Gamma_{23} = (c_{13} + c_{44})n_2n_3.
\]

(2.17)

Express the unit vector of the wave propagation direction in terms of phase angles which are measured away from the symmetry axis \(\theta\) and substitute \(^{(2.17)}\) into \(^{(2.14)}\), we can solve for the three eigenvalues

\[
2\rho V_k^2(\theta) = (c_{11} + c_{44})\sin^2 \theta + (c_{33} + c_{44})\cos^2 \theta
\]

\[
+ \left\{ [(c_{11} - c_{44})\sin^2 \theta - (c_{33} - c_{44})\cos^2 \theta]^2 + 4(c_{13} + c_{44})^2\sin^2 \theta \cos^2 \theta \right\},
\]

(2.18)

\[
2\rho V_V^2(\theta) = (c_{11} + c_{44})\sin^2 \theta + (c_{33} + c_{44})\cos^2 \theta
\]

\[
- \left\{ [(c_{11} - c_{44})\sin^2 \theta - (c_{33} - c_{44})\cos^2 \theta]^2 + 4(c_{13} + c_{44})^2\sin^2 \theta \cos^2 \theta \right\},
\]

(2.19)

\[
2\rho V_H^2(\theta) = c_{66}\sin^2 \theta + c_{44}\cos^2 \theta,
\]

(2.20)
which corresponds to quasi-$P$, quasi-$SV$ and quasi-$SH$ waves, respectively. In this thesis, I only discuss the quasi-$P$ wave mode. Since the individual component of stiffness matrix lacks physical meaning, Thomsen (1986) introduces the equivalent set of parameters: vertical velocities $V_{P0}$ and $V_{S0}$ of $P$- and $S$-waves, and three dimensionless parameters $\varepsilon$, $\delta$ and $\gamma$ defined by

\[
\begin{align*}
V_{P0} &= \sqrt{\frac{c_{33}}{\rho}}, \\
V_{S0} &= \sqrt{\frac{c_{44}}{\rho}}, \\
\varepsilon &= \frac{c_{11} - c_{33}}{2c_{33}}, \\
\delta &= \frac{(c_{13} - c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}, \\
\gamma &= \frac{c_{66} - c_{44}}{2c_{44}}.
\end{align*}
\tag{2.21}
\]

Using this parameterization, the quasi-$P$ wave phase velocity is formulated as (Tsvankin, 2001)

\[
\frac{V^2(\theta)}{V_{P0}^2} = 1 + \varepsilon \sin^2 \theta - \frac{f}{2} + \frac{f}{2} \sqrt{1 + \frac{4\sin^2 \theta}{f}(2\delta \cos^2 \theta - \varepsilon \cos 2\theta) + \frac{4\varepsilon^2 \sin^4 \theta}{f^2}},
\tag{2.22}
\]

where $f \equiv 1 - \frac{V_{S0}^2}{V_{P0}^2} \equiv 1 - \frac{c_{44}}{c_{33}}$. This phase velocity formula is exact and often is the starting point of many derived pseudo acoustic wave equations. The most widely used one, is the acoustic approximation made by Alkhalifah (2000), which set the vertical $S$-wave velocity to zero, i.e. $V_{S0} = 0$. This is the case I discuss in the following chapters.
Chapter 3

Prestack Exploding Reflector Modeling and Migration in TI Media

3.1 Prestack Exploding Reflector Model

For many years, seismic imaging techniques were only applied in the poststack domain with the help of \textit{exploding reflector model} (ERM) \cite{Claerbout1985,Loewenthal1976}, partially due to the limited computational capability and the prestack seismic data are voluminous. ERM is a powerful analogy to zero-offset acquired realistic data by simply assuming a theoretical experiment in which waves explode at reflectors with strength proportional to reflection coefficients, and propagate with a halved velocity to the Earth’s surface where they are observed. However ERM fails to predict prismatic waves and multiples \cite{Claerbout1985} since ERM by assumption necessitates the downgoing wavepath from real sources to reflector coincides with the upgoing wavepath from reflector to receivers. This assumption satisfies in areas of simple geology which is often the case in early years’ seismic exploration. Despite the assumption, ERM is only limited to zero-offset acquisition geometry while we
in reality do not even have zero-offset data. Poststack data is obtained and used to approximate the zero-offset data through processing and stacking prestack data. Routinely used processing techniques include normal moveout (NMO), dip moveout (DMO) [Hale 1984] and the more general azimuth moveout (AMO) [Biondi et al. 1998]. These correction and stacking techniques all transform prestack data to post-stack data but fail in the presence of strong lateral velocity variation.

To better image areas of complex geology, we might revert to use the whole prestack data. Then we might ask the question whether there exists a corresponding hypothetical model to mimic our prestack data just as the ERM does to poststack data. Prestack exploding reflector model (PERM) is such a generalization of ERM which aims to extend ERM to apply to data seismic data at nonzero offset.

To understand the concept of PERM, we parameterize the model in an extended coordinate system. Let us denote the original source and receiver positions as having the coordinates $x_s$ and $x_r$. The then extended model is parameterized as $\xi_x = (x_x, x_s, x_r)$, where $x$ is our original coordinate of an image point which can either be $x_x = (x, z)$ in 2D, or $x_x = (x, y, z)$ in 3D. Figure 3.1 shows the extended models with an additional axis $(s, r)$ compared to original models. Each slice is a model corresponds to one source-receiver configuration. We easily notice that under this extended coordinate system, the source coordinate is defined as $\xi_s = (x_s, x_r)$ which is exactly the receiver coordinate $\xi_r = (x_s, x_r)$. This means each slice in Figure 3.1 is a model with source and receiver sharing the same position in the extended coordinate system. Recall the definition of zero-offset acquisition geometry is source and receiver sharing the same position. We therefore can call each slice of Figure 3.1 a ‘zero-offset’ acquisition geometry in the extended coordinate system sense.

We can therefore apply ERM to each slice of Figure 3.1 and package and title these models ‘prestack exploding reflector model’ as a whole. PERM can be thought as follows. Given a prestack image which is a conventional image parameterized with
an additional offset or angle axis, reflectors of this prestack image suddenly explode and generate waves propagating up to the Earth’s surface where they are observed by a hypothetical string of receivers. Apart from ERM, the waves penetrate into the additional axis, while waves in ERM only propagate in zero-offset slice. Such PERM modeled data is regarded as an approximate of prestack data.

Before delving into exploring the usage of PERM concept, I first consider how to utilize ERM. To simplify the illustration, I consider the constant density acoustic wave equation (the governing equation does not affect the utility of ERM). The well-known acoustic wave equation has the form of

$$\frac{1}{v^2(x)} \frac{\partial^2 P}{\partial t^2} - \nabla^2 P = f(x, t),$$

where $v(x)$ is media velocity, $f(x)$ is the forcing term, and $P$ is the scalar pressure field. A modeling process is to solve for pressure filed $P(x, t)$ given media velocity $v(x)$ and source function $f(x, t)$ while its adjoint process is migration. Notice we have to do modeling or migration for each shot separately even in the case of zero-offset configuration, which is quite time consuming. However with ERM concept, we only
need one modeling to generate the whole zero-offset data or one migration to image
the data. ERM concept can be used either with depth-extrapolation or with time-
extrapolation. The migration process that extrapolates wavefields in depth is known
as downward continuation (Claerbout [1985]) while extrapolating in time is known as
reverse time migration (Baysal et al., 1983; Whitmore et al., 1983).

With ERM, the modeling mode by depth-extrapolation is simply solving the one-
way wave equation corresponding to upgoing waves:

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial z} + i \sqrt{\frac{\omega^2}{v^2} - |k|^2} \right) P(x, \omega) = r(x) \\
P(x, y, z = z_{\text{max}}, \omega) = 0
\end{array} \right. ,
\end{align*}
\]

where \(r(x)\) is the reflectivity function acts as source function in the equation. The
wavefields are recursively extrapolated along depth axis from bottom to top of the
model. The zero-offset data is obtained by taking an inverse Fourier transform of
the wavefields after the extrapolation process reach the depth of receivers. In the
migration mode of depth-extrapolation, the problem becomes solving the one-way
wave equation corresponding to the downgoing waves:

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial z} - i \sqrt{\frac{\omega^2}{v^2} - |k|^2} \right) P(x, \omega) = 0 \\
P(x, y, z = z_0, \omega) = d(x, \omega)
\end{array} \right. ,
\end{align*}
\]

where \(d(x, \omega)\) is the zero-offset data acting as boundary values. The wavefields are
recursively extrapolated along depth axis from top to bottom of the model. The
image is obtained by summing over all frequencies to extract the zero-time slice of
the wavefields at each depth-extrapolation step.

The time-extrapolation approach is solving the two-way wave equation (3.1). The
modeling mode is to evolve the time axis from time zero to maximum recording time
with source functions at every reflector position. The zero-offset data is a surface
slice of the wavefields at each time step. The migration mode takes the time reversal mirror of zero-offset data and injects it to the wave equation as a time dependent boundary condition. The image is obtained by evolving the time axis to zero time which is the imaging condition.

Notice with ERM, using either depth-extrapolation or time-extrapolation, no cross-correlation of wavefields is involved in the imaging condition and only one kernel is enough to model or migrate the whole zero-offset data. However the media velocity should be halved as well.

Therefore, analogous to ERM, it would be beneficial to seek a method that connects the prestack data to the prestack image in one single kernel while the imaging condition does not involve cross-correlation of wavefields. In the geophysical literature, double square root formulation (DSR) or survey sinking \cite{Claerbout1985, DeHoop2003, Popovici1996} provides such a natural platform. The DSR formula can be derived from the geometrical representation or kinematics of wavefields.

To derive the general anisotropic DSR formula, we can start from the general anisotropic eikonal equation given by

$$|\nabla_x T|^2 = s^2(\theta),$$

(3.4)

where $T$ is the traveltime, $s(\theta) = 1/v(\theta)$ is the magnitude of phase slowness vector and $\theta$ is the phase angle. In the 2D case (extension to 3D is straightforward), we have

$$\frac{\partial T}{\partial z} = \pm \sqrt{s^2(\theta) - \left(\frac{\partial T}{\partial x}\right)^2}.$$ 

(3.5)

If we parameterize the traveltime from source to image point then to receiver in the prestack coordinates, i.e. $T = T(x, s, r)$ and we perturb the source and receiver position by a small amount along depth direction $z$, i.e. $\delta z_s$ and $\delta z_r$, the induced
variation of traveltime then reads:

\[ \delta T = \partial T \frac{\partial z_s}{\partial z} \delta z_s + \partial T \frac{\partial z_r}{\partial z} \delta z_r. \] (3.6)

Now the DSR assumption comes in. DSR presumes that source and receiver are redatumed by the same amount of depth shift \( \delta z = \delta z_s = \delta z_r \). This is equivalent to putting the seismic survey at the subsurface which is \( \delta z \) deep from the original survey surface. In other words, the survey sinks from surface to subsurface, hence the name *survey sinking*. By this assumption, equation (3.6) becomes

\[ \delta T = \left( \partial T \frac{\partial z_s}{\partial z} + \partial T \frac{\partial z_r}{\partial z} \right) \delta z \] (3.7)

Substitute equation (3.5) (assume plus sign for increasing depth) into equation (3.7) then written in differential form, we have

\[ \frac{\partial T}{\partial z} = \sqrt{s^2_s(\theta) - \left( \frac{\partial T}{\partial x_s} \right)^2} + \sqrt{s^2_r(\theta) - \left( \frac{\partial T}{\partial x_r} \right)^2}. \] (3.8)

To recover the dispersion relation representation of the DSR formula, we first denote the wavefield as \( U \), then by chain rule we have

\[
\frac{\partial U}{\partial z} = \frac{\partial U}{\partial T} \frac{\partial T}{\partial z} = \left( \sqrt{s^2_s(\theta) - \left( \frac{\partial T}{\partial x_s} \right)^2} + \sqrt{s^2_r(\theta) - \left( \frac{\partial T}{\partial x_r} \right)^2} \right) \frac{\partial U}{\partial T}.
\] (3.9)

With plane wave representation of the wavefield, we have the equivalence of Fourier duals: \( \partial / \partial z \) with \( ik_z \) and \( \partial / \partial T \) with \( -i\omega \). Along with the relation \( \partial T / \partial x = 1/v(x, \theta) = k_x/\omega \), equation (3.9) becomes

\[ k_z = \sqrt{\omega^2 s^2_s(\theta) - k^2_s} + \sqrt{\omega^2 s^2_r(\theta) - k^2_r}, \] (3.10)

which is the dispersion relation representation of DSR formula. With this dispersion
relation, the prestack data can be migrated to prestack image by evolving the following equation along depth:

\[ \left\{ \begin{array}{l}
\left( \frac{\partial}{\partial z} + ik_z \right) P(x, s, r) = 0 \\
k_z = \sqrt{\omega^2 s_s^2(\theta) - k_s^2 + \sqrt{\omega^2 s_r^2(\theta) - k_r^2}}, \\
P(x, y, z = z_0, s, r) = d(s, r)
\end{array} \right. \]

which is known as \textit{downward continuation} with DSR in source-receiver domain. It can also be recast into midpoint-offset formulation using the relations

\[ \left\{ \begin{array}{l}
x_m = \frac{1}{2}(x_s + x_r) \\
x_h = \frac{1}{2}(x_r - x_s) \\
k_m = k_s + k_r \\
k_h = k_r - k_s
\end{array} \right. \]

where \( x_m \) and \( x_h \) are midpoint and half-offset position, \( k_m \) and \( k_h \) are their corresponding wavenumbers.

Notice DSR formulation does not involve cross-correlation of wavefields in the imaging condition. It maintains the one-way extrapolation feature. Its corresponding depth-extrapolation imaging technique is called \textit{survey sinking}. However we can also imagine the possibility of time-extrapolating DSR formula and assign a physical meaning to this process. PERM fortunately provides such physical meaning, since time-extrapolating DSR satisfies the two requirements: first, source and receiver share the same position in extended coordinates; second, one exploding reflector modeling or migration in this extended coordinates directly relates prestack wavefields to prestack images without cross-correlations.
3.2 Recursive Integral Time Extrapolator of DSR

PERM migration, or reverse time migration of DSR is analogous to reverse time migration with ERM in the zero-offset case. In fact, zero-offset reverse time migration with ERM is a degenerate case of PERM. To see this point, we recall the DSR dispersion relation (3.10) and the conditions for zero-offset, i.e. \( k_s = k_r = \frac{1}{2}k_x \), \( s_s = s_r \). The dispersion relation in zero-offset case then reads

\[
k_z = 2\sqrt{\omega^2 s_s^2 - \frac{k_x^2}{4}} = \sqrt{\frac{\omega^2}{(v/2)^2} - k_x^2}.
\] (3.13)

To achieve time extrapolation, we represent the frequency \( \omega \) in terms of wavenumbers \( k_x \) and \( k_z \), which is

\[
\omega^2 = \left(\frac{v}{2}\right)^2(k_x^2 + k_z^2).
\] (3.14)

Inverse Fourier transform to temporal-spatial domain, we can recover an acoustic wave equation with a halved velocity which is

\[
\frac{\partial^2 U}{\partial t^2} - \left(\frac{v}{2}\right)^2 \nabla^2 U = 0.
\] (3.15)

Solve that second-order two-way wave equation using time reversal mirror gives the reverse time migration with ERM in zero-offset case. Notice using DSR formulation, the velocity is automatically halved which is the requirement for ERM.

Therefore reverse time migration with DSR can be done using the same trick. We start from the DSR dispersion relation (3.10), represent frequency \( \omega \) in terms of wavenumbers \( k_s \) and \( k_r \) (using midpoint-offset representation), and recover the associated PDE in temporal-spatial domain by inverse Fourier transform. Solving that PDE in a time reversal mirror manner gives the reverse time migration of DSR which is described in [Biondi (2002)](#). The author however declares the approach
unstable considering the inherent singularity of horizontally propagating waves.

Here, I adopt the spectral approach to do time marching which follows the idea of Alkhalifah et al. (2013c). This provides the possibility to isolate the singularity and to extrapolate the wavefields in time using DSR formulation.

Solving acoustic wave equation (3.1) using finite difference has been explored for decades and the pseudo-spectral methods that evaluate the Laplacian in wavenumber domain has also been adopted to alleviate the numerical grid dispersion. However the stability issue of finite difference in time controlled by the Courant-Friedrichs-Lewy (CFL) condition often requires the time stepping to be small. To remove the stability constraint for large time stepping, spectral or pseudo-analytical in time methods are proposed (Chu, 2011; Du et al., 2013; Etgen et al., 1989; Fomel et al., 2013; Tal-Ezer, 1986; Tal-Ezer et al., 1987; Zhang and Zhang, 2009). These methods which are based on mixed spatial-wavenumber domain extrapolators in integral form, approximate the analytical time marching rather than discrete finite difference scheme and therefore avoid the stability issue. Du et al. (2013) refer to these methods as recursive integral time extrapolation methods and I adopt this notion here.

To see how these work, we take the spatial Fourier transform of homogeneous acoustic wave equation (3.1) gives the ODE

$$\frac{\partial^2 U}{\partial t^2} = -v(x)^2 |k|^2 U,$$

(3.16)

where $|k|$ is the absolute value of wavenumber vector. In constant velocity case, equation (3.16) becomes constant coefficient thus has the analytical solution

$$U(k, t_0 + \Delta t) = e^{\pm iv|k|\Delta t} U(k, t_0).$$

(3.17)

An inverse Fourier transform to spatial domain gives the solution to wave equation
which is

\[ U(x, t_0 + \Delta t) = \int e^{\pm i|k| \Delta t} U(k, t_0) e^{-i k \cdot x} d^3k, \quad (3.18) \]

where the spatial forward Fourier transform is defined as

\[ U(k, t_0) = \int U(x, t_0) e^{i k \cdot x} d^3x. \quad (3.19) \]

When the velocity varies in space, equation (3.18) becomes an approximation. The approximation is good given that the non-stationary phase shift operator \( e^{\pm i|k| \Delta t} \) is small. Physically, it results from the local plane decomposition and Huygens principle. Suppose we are given the wavefield at certain time step \( U(x, t_0) \), we can consider each point as a potential point source as the Huygens principle states. We then decompose the source into plane wave components and propagate each plane wave component outward for time \( \Delta t \). During this time period \( \Delta t \), the media velocity is assumed to be constant, therefore a phase shift \( e^{\pm i|k| \Delta t} \) is applied to each component. Then a wavefield reconstruction is called by summing over all plane wave components to obtained the total wavefield at the next time step \( U(x, t_0 + \Delta t) \).

Although the stability issue of time extrapolation is compensated by sacrificing some accuracy, the computational cost of direct implementation of equation (3.18) is restrictive, since the inverse Fourier transform is the discrete Fourier transform (DFT) of \( \mathcal{O}(N^2) \) time complexity rather than the fast Fourier transform (FFT) of \( \mathcal{O}(N \log N) \) time complexity.

Notice the similarity between this time extrapolation and the well-established mixed-domain depth extrapolation. To speed up the time extrapolation, we can adopt the heuristic methods used in mixed-domain downward continuation methods, for example ‘phase-shift plus interpolation’, ‘split-step’, and ‘fourier finite difference’. These methods are extensively reviewed in Du et al. (2013). In thesis I adopt an
operator splitting method called lowrank decomposition (Fomel et al., 2013) which takes advantage of the sparsity of the extrapolation matrix. The method is fully explained in Fomel et al. (2013). I briefly introduce it here.

First, in the heterogeneous velocity case, we discretize equation (3.18) as

\[
U(x, t_0 + \Delta t) \approx F^{-1} \left\{ \sum_{i=0}^{N} \sum_{j=0}^{N} e^{\pm iv_i |k_j| \Delta t} F\{U(x, t_0)\} \right\},
\]

(3.20)

where \(F\{\cdot\}\) and \(F^{-1}\{\cdot\}\) are forward and inverse Fourier transforms with respect to spatial coordinates. Then the phase shift extrapolator or the propagator is a square matrix and can be denoted as

\[
W(v, k, \Delta t) = (W)_{ij} = e^{\pm iv_i |k_j| \Delta t}
\]

(3.21)

Fomel et al. (2013) observed this propagator has a lowrank feature provided \(\Delta t\) is sufficiently small. That says the propagator and its Hermitian have large null spaces. Such a sparsity feature gives the possibility of representing its column space and row space using relatively small number of column vectors and row vectors. A natural choice of selecting these vectors would be singular value decomposition (SVD). However as the dimensions of this propagator matrix can be huge for 3D problems (typically with \(10^9\) rows and \(10^9\) columns), SVD is prohibitive. Instead, a randomized algorithm based sparse matrix decomposition by selecting certain columns and rows of the original propagator matrix reduces the computational flops to a cost linear in
the number of columns or rows of the matrix. Symbolically the decomposition takes the form of

\[ W(v, k, \Delta t) = U \Lambda V, \]  

(3.22)

where \( W \) is the propagator with dimension \( N \times N \), \( U \) is the matrix of selected columns with dimension \( N \times m \), \( V \) is the matrix of selected rows with dimension \( n \times N \), \( \Lambda \) is a full matrix of relatively small size \( m \times n \), which is titled as the approximate numerical rank of \( W \). Here \( m \) and \( n \) are numbers much smaller than \( N \). Notice the obvious differences of this decomposition with SVD are that columns of \( U \) are a subset of columns of \( W \) rather than the eigenvectors of \( WW^* \), rows of \( V \) are a subset of rows of \( W \) rather than the eigenvectors of \( W^*W \), and \( \Lambda \) is a small full matrix rather than a diagonal matrix storing eigenvalues in SVD case. More illustratively, the decomposition can be represented as

\[
W = \begin{pmatrix}
\vdots \\
k \\
\vdots
\end{pmatrix} \approx \begin{pmatrix}
\vdots \\
U_{N \times m} \Lambda_{m \times n} \\
\vdots
\end{pmatrix} \begin{pmatrix}
\vdots \\
V_{n \times N}
\end{pmatrix}.
\]

With this decomposition of the time extrapolator \( W(v(\mathbf{x}), k, \Delta t) \), we can recursively evolve the wavefields in time to achieve modeling or migration. However the aforementioned extrapolator is for isotropic acoustic equation. I now show how the DSR associated PDE can be solved through this method. Recall the DSR dispersion relation (3.10), it is suitable for general anisotropic media provided the phase velocity/s-lowness \( s(\theta) \) is given. To obtain time marching equation, we square the square roots
and rearrange the equation (3.10) in terms of velocities to arrive at

\[
(v^2_\text{r}(\theta) \omega^2)^2 + 2v^2_\text{s}(\theta)v^2_\text{r}(\theta)(v^2_\text{r}(\theta)(k^2_\text{s} - k^2_\text{r}) - v^2_\text{s}(\theta)k^2_\text{r}) \omega^2 \\
+ v^4_\text{s}(\theta)v^4_\text{r}(\theta)(k^2_\text{s} + k^2_\text{r} + k^2_\text{z})^2 - 4k^2_\text{s}k^2_\text{r}] = 0,
\]

(3.23)

where \( v^2_\text{r}(\theta) = v^2_\text{s}(\theta) - v^2_\text{r}(\theta), v^2_\text{s}(\theta) = v^2_\text{r}(\theta) + v^2_\text{r}(\theta) \). Solve equation (3.23) for \( \omega^2 \), we get

\[
\omega^2 = \frac{v^2_\text{r}(\theta)v^2_\text{s}(\theta)[(k^2_\text{s} + k^2_\text{r} + k^2_\text{z})^2 - 4k^2_\text{s}k^2_\text{r}]}{v^2_\text{r}(\theta)k^2_\text{r} - v^2_\text{s}(\theta)k^2_\text{s} - v^4_\text{s}(\theta)k^2_\text{r} \pm 2D},
\]

(3.24)

where \( D = \sqrt{k^2_\text{s}[v^2_\text{s}(\theta)v^2_\text{r}(\theta)(k^2_\text{s} + k^2_\text{r} + k^2_\text{z}) - v^4_\text{s}(\theta)k^2_\text{r} - v^4_\text{s}(\theta)k^2_\text{r}]}. \) The sign before \( 2D \) in denominator should be chosen to be plus because squaring the DSR formula introduces an artificial solution to \( \omega^2 \). Inverse Fourier transform back to time domain gives the variable coefficient ODE

\[
\frac{\partial^2 U(s, r, t)}{\partial t^2} = -\omega^2(x, k, \theta)U(s, r, t).
\]

(3.25)

Equation (3.25) has the same form of (3.16) and therefore the solution is given approximately by

\[
U(s, r, t + \Delta t) \approx \mathcal{F}^{-1}\left\{ \sum_{i=0}^{N} \sum_{j=0}^{N} e^{\pm i\omega(x_i, k_j, \theta)\Delta t} \mathcal{F}\{U(s, r, t)\} \right\}.
\]

(3.26)

Further apply the lowrank decomposition to the new extrapolator \( W(x, k, \Delta t) = e^{\pm i\omega(x, k, \theta)\Delta t} \) using the method described before, we can achieve the recursive integral time extrapolation of DSR formula. However there are two issues with the phase shift extrapolator. First, equation (3.24) has a singularity of negative number inside square root of \( D \) in the denominator. Second, the DSR formula of general anisotropic media requires the knowledge of phase velocity which is a function of phase angle. However this phase angle is typically unknown as the wavefields extrapolate. The
second issue is addressed in the next section with the help of a specially chosen kind of anisotropic model, dip oriented transversely isotropic (DTI) model.

To resolve the first issue of anisotropic DSR recursive time extrapolation, I adopt the strategy from Wu and Alkhalifah (2013), where they approximate \( D \) in equation (3.24) to be

\[
D \approx \frac{k_s^2 v^2}{2}.
\] (3.27)

Therefore the phase operator (3.24) takes the form of

\[
\omega^2 = \frac{v_s^2(\theta)v_s^2(\theta)[(k_s^2 + k_r^2 + k_z^2)^2 - 4k_s^2k_m^2]}{2v_x^2(\theta)k_z^2 - v_x^2(\theta)(k_s^2 - k_r^2)},
\] (3.28)

which can also be written in midpoint-offset wavenumber coordinates using the relation (3.12)

\[
\omega^2 = \frac{v_r^2(\theta)v_z^2(\theta)(k_r^2 + k_m^2 + k_z^2)}{2v_x^2(\theta)k_z^2 + v_x^2k_h^2k_m^2}.
\] (3.29)

Then given this phase operator, we do lowrank decomposition to the time extrapolator using the following formula

\[
W(x, k, \theta, \Delta t) = e^{-i\omega(x,k,\theta)\Delta t} \approx \sum_{j=1}^{m} \sum_{i=1}^{n} W(x(\theta), k_j) a_{ji} W(x_i(\theta), k).
\] (3.30)

After decomposition, the inverse Fourier transform in equation (3.20) becomes FFT rather than DFT as shown in

\[
U(s, r, t + \Delta t) \approx \sum_{j=1}^{m} W(x(\theta), k_j) \left( \sum_{i=1}^{n} a_{ji} \left( \mathcal{F}^{-1} \{ W(x_i(\theta), k) \mathcal{F}\{U(s, r, t)\} \} \right) \right).
\] (3.31)

The only thing left is the evaluation of phase operator \( \omega^2 \) in anisotropic case where
the velocity depends on phase angle.

### 3.3 Phase Velocity Evaluation

In anisotropic media, the phase operator of DSR formula \((3.28)\) requires the knowledge of phase velocities at source and receiver position, i.e. \(v_s(\theta)\) and \(v_r(\theta)\). To develop the strategy for evaluation of phase velocity, I specify the model to be acoustic approximate TTI [Alkhalifah 1998] which is widely accepted in industry.

According to [Alkhalifah 1998], the phase velocity in TTI media takes the form of

\[
v^2(\theta) = \frac{1}{2} (v_{\text{nmo}}^2 (2 \eta + 1) \sin^2 \theta + v_T^2 \cos^2 \theta) + \frac{1}{4} \sqrt{a \sin^4 \theta + b \sin^2 2\theta + c \cos^4 \theta},
\]

(3.32)

where \(a = 4v_{\text{nmo}}^4 (2 \eta + 1)^2\), \(b = 2v_{\text{nmo}}^2 v_T^2 (1 - 2\eta)\), \(c = 4v_T^4\), \(v_{\text{nmo}}\) is NMO velocity, \(v_T\) is P-wave velocity along symmetry axis, and \(\eta\) is the anelliptical parameter relating the NMO velocity to the velocity normal to the tilted symmetry axis. As the formula shows, phase velocity is parameterized by the phase angle rather than the group angle. In fact, there’s no closed-form expression of phase velocity in terms of group angle. However when we extrapolate wavefields, the phase angle, as opposed to group angle, is typically unknown. This leads to the difficulty of determination of phase velocity. One natural solution is that we use the group angle information to get phase angle, then solve for phase velocity using equation (3.32). However this native procedure is infeasible because the group angle is a function of phase angle and phase velocity which has the form of

\[
\tan \phi(\theta) = (\tan \theta + \frac{1}{v(\theta)} \frac{dv(\theta)}{d\theta})(1 - \frac{\tan \theta \frac{dv(\theta)}{d\theta}}{v(\theta)}),
\]

(3.33)

where \(\phi\) is group angle, \(\theta\) is phase angle, and \(v(\theta)\) is phase velocity. Clearly, equa-
tions (3.32) and (3.33) are coupled and we have to solve for phase angle and phase velocity simultaneously. Remember, we need to invert for these two variables at every grid point and repeat this process at each time step. It is therefore too expensive to solve this inversion without an analytical formula. Fortunately, using the concept of DTI (Alkhalifah and Sava, 2010) reduces the complexity of TTI, and makes the phase angle and phase velocity available.

\[ \text{Figure 3.2: Illustration of DTI model. Red arrowed line is source ray-parameter vector (P}_s). \text{ Blue arrowed line is receiver ray-parameter vector (P}_r). \text{ Magenta arrowed line is } P_m = P_s + P_r. \text{ Green arrowed line is } P_h = P_r - P_s. \text{ Yellow line represents reflector. Dashed line is parallel to } P_m \text{ which is normal to reflector and represents the symmetry axis in TI media. Black lines from source S to reflector then to receiver R represents the general ray path. The angle } \theta \text{ is the phase angle. The angle } 2\alpha \text{ is the scattering angle.} \]

In a DTI medium, the tilted symmetry axis is constrained to be normal to the reflector dip, hence the name DTI. Figure 3.2 shows a cartoon of DTI media depicting waves propagating to a reflector then get reflected. The yellow line denotes a reflector. The anisotropic symmetry axis is shown as the dashed black line which is tilted and normal to the reflector. \( P_s \) and \( P_r \) are source and receiver ray-parameter vectors.
at the reflection point. Obviously, the angle between $\mathbf{P}_s$ and the reflector normal is the phase angle of source side wavefield $\theta_s$, the angle between $\mathbf{P}_r$ and the reflector normal is the phase angle of the receiver side wavefield $\theta_r$. They are equal according to Snell’s law. Moreover, the angle between $P_s$ and $P_r$ is known as the scattering angle (Sava and Fomel, 2005). These angles are related to source and receiver ray parameter vectors or wavenumber vectors through a local plane wave decomposition. These relations are intensively used in extracting angle gathers either from Kirchhoff type migration or wave equation type migration. The relations are derived in Sava and Fomel (2005) and are given by

$$
\begin{align*}
|\mathbf{P}_h|^2 &= |\mathbf{P}_s|^2 + |\mathbf{P}_r|^2 - 2|\mathbf{P}_s||\mathbf{P}_r|\cos 2\alpha \\
|\mathbf{P}_m|^2 &= |\mathbf{P}_s|^2 + |\mathbf{P}_r|^2 + 2|\mathbf{P}_s||\mathbf{P}_r|\cos 2\alpha \\
\mathbf{P}_h \cdot \mathbf{P}_m &= |\mathbf{P}_r|^2 - |\mathbf{P}_s|^2
\end{align*}
$$

(3.34)

A simplification of (3.34) comes from the equality of $|\mathbf{P}_r| = |\mathbf{P}_s|$ which is always true in isotropic cases. Further applying Fourier central slice theorem, and written in terms of midpoint-offset wavenumbers, equation (3.34) looks like

$$
\begin{align*}
|\mathbf{k}_h|^2 &= (2\omega S \sin \alpha)^2 \\
|\mathbf{k}_m|^2 &= (2\omega S \cos \phi)^2, \\
\mathbf{k}_h \cdot \mathbf{k}_m &= 0
\end{align*}
$$

(3.35)

where $\mathbf{k}_m$ and $\mathbf{k}_h$ are midpoint and offset wavenumbers, $S$ is the slowness at the specular reflection point, and $\phi$ is half the scattering angle. However in our DTI media, we can still derive equation (3.35) from equation (3.34) since we have assumed the source and receiver side phase angles are equal and is half the scattering angle, i.e. $\theta = \alpha$. This gives the equality $|\mathbf{P}_s(\theta)| = |\mathbf{P}_r(\theta)| = S(\theta)$ at the specular reflection
point. We therefore have the DTI version of equation (3.35) which is

\[
\begin{align*}
|k_h|^2 &= (2\omega S(\theta) \sin \theta)^2 \\
|k_m|^2 &= (2\omega S(\theta) \cos \theta)^2. \\
k_h \cdot k_m &= 0
\end{align*}
\] (3.36)

In general, both \(k_h\) and \(k_m\) have three components for 3D and two components for 2D. We can eliminate \(k_{hz}\) then solve for \(k_{mx}\) to do one-way wavefield extrapolation. However in our time extrapolation case, we eliminate \(\omega\) and \(S(\theta)\) to have

\[
\begin{align*}
\tan \theta &= \frac{|k_h|}{|k_m|}, \\
0 &= k_h \cdot k_m
\end{align*}
\] (3.37)

and write explicitly as

\[
\begin{align*}
\tan \theta &= \frac{k_{hx}^2 + k_{hz}^2}{k_{mx}^2 + k_{mz}^2}, \\
k_{mx} k_{hx} + k_{mz} k_{hz} &= 0
\end{align*}
\] (3.38)

We eliminate \(k_{hz}\) to obtain

\[
\tan \theta = \frac{k_{hx}}{k_{mx}}. \tag{3.39}
\]

Equation (3.39) simply says, given a pair of horizontal offset wavenumber and vertical midpoint wavenumber, we can compute the phase angle without much effort. This is exactly what we want in our DSR recursive time extrapolation since we are given the wavenumbers to compute the phase angle then the phase velocity at source and receiver position. The phase velocity is then computed using the formula (3.32). However this DTI has made an intrinsic assumption that the spatial position we used for computing phase velocity is the specular reflection point which is unknown in wavefield extrapolation. We should instead use the source and receiver spatial
positions to compute the phase velocity at source and receiver position. This gives us a clue that we should compute the phase angle at source and receiver position rather than the specular reflection point. This leads to extension of phase velocity evaluation for DTI to VTI and TTI media.

In DTI case, we achieve the phase angle evaluation by looking at the vicinity of the specular reflection point, do local plane wave decomposition and impose the direction constrain of tilted symmetry axis. Now let us consider standing at the source point, we image a wave coming in, hit the source point, and get reflected back. Such waves can be from secondary sources according to Huygens’s principle. In this situation, the source can be regarded as specular reflection point. Then we apply the same trick of phase angle evaluation which is developed in DTI case. However the equation (3.39) should instead take the form of

\[
\tan \gamma_s = \frac{k_s}{k_{sz}}.
\] (3.40)

The receiver side phase angle can be evaluated by the same trick and takes the form of

\[
\tan \gamma_r = \frac{k_r}{k_{rz}}.
\] (3.41)

Notice Equations (3.40) and (3.41) has no DTI assumption, and suitable for general TI media like VTI or TTI. However these two phase angles are measured from vertical rather than the symmetry axis. In VTI case when the symmetry axis is always vertical, we directly use the evaluated phase angle to compute the phase velocity, i.e. \(\gamma_s = \theta_s\) and \(\gamma_r = \theta_r\). In TTI case, however we should incorporate the symmetry axis tilt angle information \(\phi\). By observing that the phase velocity in formula (3.32) is even and periodic with respect to phase angles \(\theta\), we can use geometrical relations to
find the equivalent phase angles in TTI case which are

\[
\begin{align*}
\theta_s &= \gamma_s + \phi_s \\
\theta_r &= \gamma_r + \phi_r
\end{align*}
\]  \hspace{1cm} (3.42)

So far we have all the information to do DSR recursive time extrapolation which I refer to as PERM modeling and migration. The phase angles are computed using formulas (3.42) for TTI, (3.40) and (3.41) for VTI, and (3.39) for DTI media. With the phase angles at hand, we are able to compute the phase velocity using formula (3.32) and the phase shift operator using formula (3.28) and (3.30) then recursive time evolve the wavefields using formula (3.31).

In next section, I use two synthetic numerical examples to demonstrate the feasibility of my proposed DSR recursive time extrapolation in DTI, VTI and TTI media.

3.4 Numerical Examples

The first example consists of applying an isotropic, DTI and VTI PERM migration to the Marmousi VTI model. The model is shown in Figure 3.3 covering the lateral area from 2900 m to 7700 m. The horizontal and depth spacing are equal to 12.5 m. The data used for migration are generated by a VTI pseudo-spectral modeling program. The acquisition geometry is a split-spread setup that includes 385 shots starting from 2900 m with an interval of 12.5 m, each shot has 401 receivers with the largest offset equal to 2500 m and the offset interval is 25 m. Early arrival data are muted at far-offset before migration to mitigate low-wavenumber artifacts at shallow part of the image. Only gathers inside the model are used for migration. The time step used for migration is 1 ms. The recording time length is 3.5 s.

The PERM migrated images are shown in Figures 3.4, 3.5 and 3.6. Figures 3.4a, 3.5a, and 3.6a show the subsurface offset image gathers generated by PERM migra-
tion. The energies are quite well focused around zero-offset since I use the correct migration velocity and anisotropic parameters. Figures 3.4b, 3.5b, and 3.6b show the zero-offset slices of migrated images after Laplacian filtering. All three images show structural features of the model. However, the DTI and VTI migration despite the general weakness of the anisotropy influence here show more focused images and both manage to image the faults better. The amplitudes of the strong reflectivity and deeper parts of the model including the reservoir reflections are enhanced in the DTI and VTI images. We can also note higher resolution in the VTI and DTI images partially due to the better focusing. All three images have some artifacts near the bottom since spectral extrapolation uses fast Fourier transform which assumes periodic models. This makes the shallow low wavenumber artifacts wrap around to the bottom.

The second example includes applying an isotropic, DTI and TTI PERM migration to the BP2007 TTI model. The model is shown in Figure 3.7 covering the lateral area from 42 km to 57 km. I replicate the standard benchmark data using reciprocity in midpoint-offset domain before migration. The data are muted at the far-offset before migration and only gathers inside the model are used. The horizontal and depth spacing are 25 m. The shot and receiver intervals are 50 m. The time step used for migration is 1 ms. The total recording time is 9.2 s.

The PERM migrated images are shown in Figures 3.8, 3.9 and 3.10. Figures 3.8a, 3.9a, and 3.10a show the subsurface offset image gathers generated by PERM migration. Again, the energies are quite well focused because true migration parameters are used. Figures 3.8b, 3.9b, and 3.10b show zero-offset slices of the image gathers. Notice the anticlines are more focused (horizontally squeezed) in TTI migrated image, especially around positions at depth 6 km. Moreover deeper part structures are more clearly imaged with TTI PERM results.
3.5 Conclusions

In this chapter, I introduced the concept of prestack exploding reflector model (PERM) which I refer to as the time extrapolation of DSR formula. The recursive time extrapolation of DSR is implemented using a lowrank operator decomposition. This method is then applied to DTI media which allows for evaluation of phase angle and phase velocity for DSR formula. This phase angle and phase velocity evaluation are extended from DTI media to the more general VTI and TTI media. As a result, PERM can be used to extrapolate prestack wavefields in anisotropic media and do imaging without a cross-correlation step. Two numerical examples applied to anisotropic Marmousi model and BP2007 model demonstrated the accuracy and utility of the approach.
Figure 3.3: VTI Marmousi model.
(a) Isotropic PERM migrated subsurface offset common image gathers.

(b) Zero offset slice of isotropic PERM migrated images after Laplacian filtering.

Figure 3.4: Isotropic PERM migrated images of VTI Marmousi Model.
(a) DTI PERM migrated subsurface offset common image gathers.

(b) Zero offset slice of DTI PERM migrated images after Laplacian filtering.

**Figure 3.5:** DTI PERM migrated images of VTI Marmousi Model.
(a) VTI PERM migrated subsurface offset common image gathers.

(b) Zero offset slice of VTI PERM migrated images after Laplacian filtering.

Figure 3.6: VTI PERM migrated images of VTI Marmousi Model.
Figure 3.7: BP2007 TTI model.
(a) Isotropic PERM migrated subsurface offset common image gathers.

(b) Zero offset slice of isotropic PERM migrated images.

*Figure 3.8:* Isotropic PERM migrated images of BP2007 TTI Model.
(a) DTI PERM migrated subsurface offset common image gathers.

(b) Zero offset slice of DTI PERM migrated images.

**Figure 3.9:** DTI PERM migrated images of BP2007 TTI Model.
(a) TTI PERM migrated subsurface offset common image gathers.

(b) Zero offset slice of TTI PERM migrated images.

Figure 3.10: TTI PERM migrated images of BP2007 TTI Model.
Chapter 4

Effective Anisotropy

In this chapter, I introduce an effective anisotropy idea to reduce the computational cost of extrapolating anisotropic wavefields which are extensively used in the seismic inverse problems. The original idea comes from Alkhalifah et al. (2013b) where the authors embed anisotropy in the inhomogeneity of effective velocity models. I follow their procedure and extend it to effective density in order to obtain a dynamically matching of approximate wavefields with the original wavefields.

4.1 Problem Formulation

Introducing anisotropy to seismic wave propagation reveals more realistic physics of our Earth’s subsurface as compared to the isotropic assumption. However wavefield modeling in anisotropic media still suffers from the computational burdens, in particular, with complex anisotropy such as transversely isotropic (TI) and orthorhombic anisotropy. We therefore wish to incorporate the effect of anisotropy while retaining the same cost of modeling isotropic wavefield, or at least on the same order of computational cost. By recognizing that differentiating the anisotropy from heterogeneity of the Earth’s subsurface depends on the scale we are looking at and the wavelength we are using, we can partially conclude that anisotropy and heterogeneity are equivalent to some degree. Therefore, we project the effect of anisotropy onto
heterogeneity of model parameters. The possible bridge of this procedure would be traveltime and amplitude matching which are two characteristics of wavefield often used by seismologists. Traveltimes and amplitudes are of wavefields are high frequency asymptotic representations of wavefields which are governed by the eikonal equations and transport equations.

To approach the traveltime and amplitude matching, we first take a look at these governing equations. For simplicity, I focus on acoustic vertically transversely isotropic (VTI) media [Alkhalifah, 2000]. Extensions to other anisotropic models are possible and can be treated in a similar manner. In this section I give a brief derivation of VTI eikonal equations and transport equations.

**VTI eikonal and transport equation**

The full VTI P-wave phase velocity written in Thomsen parameters [Tsvankin, 1996] has the form of

\[
\frac{V^2(\theta)}{V^2_{P0}} = 1 + \varepsilon \sin^2 \theta - \frac{f}{2} + \frac{f}{2} \sqrt{1 + 4 \sin^2 \theta f (2\delta \cos^2 \theta - \varepsilon \cos 2\theta) + \frac{4\varepsilon^2 \sin^4 \theta}{f^2}},
\]

(4.1)

where \( f \equiv 1 - \frac{V_{S0}^2}{V_{P0}^2} = 1 - \frac{c_{44}}{c_{33}} \) is the only term containing the S-wave vertical velocity. Alkhalifah (1998) proposed an acoustic approximation by setting S-wave vertical velocity \( V_{S0} \) to zero and reformulating the phase velocity using parameters \( V_{nmo} \) and \( \eta \) which are defined as

\[
V^2_{nmo} = V^2_{P0}(1 + 2\delta)
\]

\[
\eta = \frac{\varepsilon - \delta}{1 + 2\delta}
\]

(4.2)
Further substitute the relations

\[
\left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial z} \right)^2 = \frac{1}{V^2(\theta)}
\]

\[
\frac{\sin \theta}{V_p^2} = \frac{\partial \tau}{\partial x},
\]

\[
\frac{\cos \theta}{V_p^2} = \frac{\partial \tau}{\partial z},
\]

back to phase velocity formula \((4.1)\), we can recover the 2D VTI eikonal equation after some algebra manipulation which is

\[
V_{nmo}^2(1 + 2\eta)\left( \frac{\partial \tau}{\partial x} \right)^2 + V_{P_0}^2\left( \frac{\partial \tau}{\partial z} \right)^2 \left( 1 - 2\eta V_{nmo}^2 \left( \frac{\partial \tau}{\partial x} \right)^2 \right) = 1. \quad (4.4)
\]

Equation \((4.4)\) degenerates to the isotropic eikonal equation

\[
|\nabla \tau|^2 = 1/V^2, \quad (4.5)
\]

if we set \(\eta = 0\), \(V_{nmo} = V_{P_0} = V\), where \(V\) is the isotropic velocity. Using the dispersion relation \(k = \omega p = \omega \nabla \tau\) and taking inverse Fourier transform in wavenumbers for equation \((4.4)\) yields the 2D acoustic VTI wave equation

\[
\omega^2 V_{P_0}^2 \frac{\partial^2 U}{\partial z^2} + 2\eta V_{nmo}^2 V_{P_0}^2 \frac{\partial^4 U}{\partial x^2 \partial z^2} + \omega^2(1 + 2\eta)V_{P_0}^2 \frac{\partial^2 U}{\partial x^2} + \omega^4 U = 0. \quad (4.6)
\]

The above frequency domain wave equation \((4.6)\) has a trial solution in the high frequency asymptotic sense given by the Debye series \(\text{Bleistein et al., 2001}\)

\[
U(x, \omega) \sim e^{i\omega \tau(x)} \sum_{n=0}^{\infty} \frac{A_n(x)}{(i\omega)^n}. \quad (4.7)
\]

Substitute a zeroth-order term of the series into equation \((4.6)\) and collect coefficients of \(\omega^3\) then equate them to zero, we will arrive at the constant density VTI transport
equation

\[
\left( 1 + 2\eta \right) V_{nmo}^2 - 2\eta V_{nmo}^2 V_P^2 \left( \frac{\partial \tau}{\partial z} \right)^2 \times \left( 2 \frac{\partial A}{\partial x} \frac{\partial \tau}{\partial x} + \frac{\partial^2 A}{\partial x^2} \right) + V_P^2 \left( 2 \frac{\partial A}{\partial z} \frac{\partial \tau}{\partial z} + A \frac{\partial^2 \tau}{\partial z^2} \right) - 2\eta V_{nmo}^2 V_P^2 \left( A \frac{\partial^2 \tau}{\partial z^2} + 2 \frac{\partial A}{\partial z} \frac{\partial \tau}{\partial z} \right) \times \left( \frac{\partial \tau}{\partial x} \right)^2 - 8\eta V_{nmo}^2 V_P^2 A \frac{\partial \tau}{\partial z} \times \left( \frac{\partial \tau}{\partial x} \frac{\partial^2 \tau}{\partial x \partial z} \right) = 0.
\]

(4.8)

Setting \( \eta = 0 \), \( V_{nmo} = V_P = V \), equation (4.8) degenerates to constant density isotropic transport equation:

\[
\nabla \cdot (A^2 \nabla \tau) = 0.
\]

(4.9)

A variable density version of the isotropic transport equation can be similarly derived and is given by

\[
\nabla \cdot \left( \frac{A^2}{\rho} \nabla \tau \right) = 0.
\]

(4.10)

With all these anisotropic and isotropic eikonal equations and transport equations at hand, the effective anisotropic simply involves solving these equations. In the next section, I review some fast methods to solve these equations.

### 4.2 Fast Methods for Static Hamilton-Jacobi Equations

Eikonal equations (4.4) and (4.5) are first-order nonlinear equations in traveltime \( \tau \). Mathematically they belong to the class of Hamilton-Jacobi equations (HJE) which can be found in many different applications. More specifically, the eikonal equations are static Hamilton-Jacobi equations. To see this point, we check the general form of
Hamilton-Jacobi equation which is

\[ \mathcal{H} + \frac{\partial S}{\partial t} = 0, \]  

(4.11)

where \( \mathcal{H} = \mathcal{H}(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}) \) is the classical Hamiltonian function, \( S = S(\mathbf{q}, t) \) is the Hamilton’s principle function, also called action in mechanics, and \( \mathbf{q} \equiv (q_1, q_2, \cdots, q_{N-1}, q_N) \) is the generalized coordinates which defines the configuration of the system. By analogy, if we let the Hamilton’s principal function \( S \) be the traveltime \( \tau \) and the generalized coordinates \( \mathbf{q} \) be coordinates of the Earth’s subsurface system \( \mathbf{x} \) under Eulerian description, the Hamiltonian can be represented as

\[ \mathcal{H} = \frac{1}{2} (|\nabla \tau|^2 - V^2(\mathbf{x})) \]  

(4.12)

for isotropic media. However we do not have the time dependence in traveltime function since, as discussed in Chapter 1, we have approximate our Earth’s subsurface undergoing infinitesimal deformation which means present configuration always stays close to reference configuration and thus no time dependence of our system. Therefore we drop the time dependence in equation (4.11) to have static HJE

\[ \mathcal{H}(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}) = 0, \]  

(4.13)

which is the general form of our eikonal equations. To see why our traveltime function can be the Hamilton’s principal function \( S \), we check the total time derivative of \( S \) which is

\[ \frac{dS}{dt} = \frac{\partial S}{\partial \mathbf{q}} \mathbf{\dot{q}} + \frac{\partial S}{\partial t}. \]  

(4.14)
Let $S$ be the *type-2* generating function for a canonical transformation of Hamiltonian

\[ \mathcal{H} = \mathcal{H}(q, p, t), \quad (4.15) \]

where $p$ is the conjugate momenta, which together with $q$ defines the phase space in physics, or cotangent bundle in mathematics. Then by definition the conjugate momenta is

\[ p = \frac{\partial S}{\partial q}. \quad (4.16) \]

Now after substituting equations (4.16) and (4.11) back to equation (4.14), we arrive at

\[ \frac{dS}{dt} = p \cdot q - \mathcal{H}, \quad (4.17) \]

with inverse Legendre transform and integrate over time, we see

\[ S = \int L \, dt, \quad (4.18) \]

where $L$ is the classical Lagrangian. Clearly, $S$ can be the *action* and we know our traveltime $\tau$ can be one kind of *action* through the Fermat’s principle which simply is a special case of stationary action in mechanics. Therefore we see that our traveltime function $\tau$ can be the Hamilton’s principle function $S$ in equation (4.13). Thus we can solve our traveltime governing equations (eikonal equations) through static Hamilton-Jacobi formulations. Therefore the general anisotropic eikonal equations we need to solve, takes the form of

\[
\begin{cases}
\mathcal{H}(x, \nabla \tau) = 0, & x \in \Omega \\
\tau(x) = \tau_0(x), & x \in \Gamma \subset \Omega
\end{cases}
\quad (4.19)
\]
Here $\Omega$ is the domain of interest, and $\Gamma$ is the prescribed curve or surface where the boundary values of $\tau$ are assigned.

Such first order nonlinear PDE may be solved through method of characteristics in phase space (so-called ray tracing method in seismic area). It is obvious that the solution of (4.19) is not globally $C^1$ in physical space for generic HJE since we can observe the caustics of rays in physical space. Thus weak solutions need to be defined. The weak solution would not satisfy equation (4.19) for every point. Instead it only satisfies the equation at points where the solution has continuous first derivatives. To single out a reasonable weak solution, the equation necessitates certain requirements at the discontinuities of the derivatives of the solution. These requirements are characterized using some inequalities which are quite similar to entropy conditions for hyperbolic conservation laws and the resultant solution is physically called “viscosity solution”. The viscosity solution is introduced by Crandall and Lions (1983) and corresponds to the first arrival traveltimes in seismic area.

There are mainly two categories of numerical static HJE solvers in literature. One is through reformulating the static HJE into “time” dependent HJE using level-set ideas (Osher, 1993), i.e. extending the problem one dimension higher. System evolution along certain axis is the essence of this method. Paraxial formulations (Qian and Symes, 2001) and fixed-point iteration (Zhang et al., 2006) are variants of this “time” dependent formulations. The convergence of these methods are typically slow due to the finite speed of propagation and the CFL condition restrictions.

Another category of methods directly discretize the static HJE into a system of non-linear equations and solve the system efficiently. Such numerical methods often are referred to as fast methods for HJE. Among them Dijkstra-like programming approaches, e.g. fast marching method (Sethian, 1996), ordered upwind method (Sethian and Vladimirsky, 2003) and iterative approaches, e.g. fast sweeping method (Zhao, 2005) are the best known.
I now describe the fast marching method and fast sweeping method which are used in my development of effective anisotropy approach.

**Fast Marching Method**

In fast marching method, grid points are categorized into three classes, *accepted*, *considered*, and *far*. At the initialization step, the known boundary values are assigned, marked as *accepted* and never get changed throughout the whole process. The neighbors of these boundary grids are marked as *considered*, their values are computed by solving an upwind finite difference discretization of the HJE. All other grid points are marked as *far* and their values are initialized as large values. The *considered* grid points consists of a binary tree (heap) that is maintained as min-priority queue. After initialization, we start a loop until all grid points are marked as *accepted*:

- Extract the minimum value grid point from the *considered* heap, call it *trial*, mark it as *accepted* and delete it from the heap.
- Recompute the values of *trial*’s neighbors and update them only if recomputed values are smaller than previous values.
- Mark all the neighbors of *trial* as *considered* if they are *far*. Push these neighbors into the heap.
- Maintain the min-priority feature of the heap.

The stopping criterion can also be based on whether the heap is empty or not. The total cost of this algorithm is $O(N \log N)$, where $N$ is the total grid points, since maintaining the min-priority queue needs $O(\log N)$ while there are $N$ rounds to maintain such heap.
Another essential ingredient of fast marching method is the local solver for every grid point after finite difference discretization. The difference scheme should be upwind so that the causality along the characteristics is followed. A simple discretization in 2D have the form of

$$\hat{H}^G(\tau_x^-, \tau_x^+, \tau_z^-, \tau_z^+) = H(\tau_x = \max(\tau_x^-, -\tau_x^+, 0), \tau_z = \max(\tau_z^-, -\tau_z^+, 0)),$$

(4.20)

where $\tau_x^\pm$ and $\tau_z^\pm$ are finite difference approximations to their first order differentials. This discretization, if in first order accuracy simply means that the neighbor grid with smallest traveltime should be chosen to construct the finite difference scheme. Clearly, this finite difference scheme can go beyond first order accuracy by adding more grid points in the finite difference scheme. However, in fast marching implementation this requires that the stencils involved in finite difference scheme do not fall in the category of far, and further neighboring stencils should have smaller traveltime values than closer neighboring stencils so that the causality requirement satisfies. Moreover the singularity in the boundary values will also pollute the accuracy of solution. Therefore a construction of higher order accurate fast marching is non-trivial and Sethian (1999) reports approximate second order accurate solution. Another critical defect of fast marching method is that it can not handle generic static Hamilton-Jacobi equations on Cartesian grids. Some anisotropic eikonal equations like TTI eikonal equations will not be solved by fast marching method on Cartesian grids since they violate the stricter Osher’s fast marching criterion (Tsai et al., 2003)

$$\begin{align*}
\begin{cases}
p \frac{\partial H}{\partial p} \geq 0, \\
q \frac{\partial H}{\partial q} \geq 0.
\end{cases}
\end{align*}
$$

(4.21)

Notice $\mathbf{q} = \partial_q H$ is the ray direction, therefore the criterion simply means the angle between the phase velocity direction and the group velocity direction should be acute.
**Fast Sweeping Method**

Unlike fast marching, the fast sweeping method adopts a non-linear Gauss-Seidel iteration with alternating sweeping directions. The Gauss-Seidel philosophy comes from the fact that once a grid value is updated it is immediately used for calculating its neighboring values. The convergence of this iterative process is guaranteed with alternating sweeping directions which is aimed at covering the characteristics from different ordering of grids. A typical sweeping orderings in 2D look like

\[
(1) \quad i = 1 : I, \quad j = 1 : J \\
(2) \quad i = I : 1, \quad j = 1 : J \\
(3) \quad i = I : 1, \quad j = J : 1 \\
(4) \quad i = 1 : I, \quad j = J : 1
\]  

The general algorithm has the following steps:

1. Initialization: assign exact values to boundary and large values to other points inside computation domain.

2. Iterations: for each sweeping ordering, compute grid value according to one local solver using Gauss-Seidel iterations.

3. Convergence check under some norm measure \( \ell^p \) according to

\[
||\tau^{\text{new}} - \tau^{\text{old}}||_{\ell^p} \leq \varepsilon.
\]  \hspace{1cm} (4.23)

Fast sweeping method has plenty of choices for the local solver construction depending on which type of numerical Hamiltonian we use. Godunov type, Lax-Friedrichs type, local Lax-Friedrichs type are among the best known. One thing to note is that fast sweeping method is easier to extend to higher-order than fast marching method with the help of essentially non-oscillatory (ENO) \cite{Osher1991} or weighted essentially non-oscillatory (WENO) \cite{Jiang2000, Jiang1996} to approximate the derivatives. These methods avoid the oscillatory feature of solutions.
near discontinuities that are often seen in central difference approximation. The essence of these methods are adaptively adding stencils based on some criteria to achieve higher order difference approximations. ENO reconstruction adds one stencil at a time while WENO reconstruction adds a convex combinations of several stencils at a time, therefore ENO is not an efficient method.

In my developing of effective anisotropy idea, a higher order traveltime solver is needed since transport equations involves second order derivatives of traveltime.

### 4.3 Effective Velocity and Density Model

With the solvers for eikonal equations and transport equations at hand, I now describe the workflows to build effective velocity and density models thus to achieve the effective anisotropy idea.

The kinematic parts of wavefields are described by traveltime. If we are able to equate the traveltime of two wavefields, we can say they are kinematically matched. The traveltimes are governed by eikonal equations in the high frequency asymptotic sense. Therefore, one natural way to match the traveltimes would be to solve the anisotropic eikonal equations first to get an anisotropic traveltime, then substitute the traveltime back to isotropic eikonal equations to solve for the isotropic model parameters. This procedure simply involves two eikonal solvers, one for anisotropic and the other for isotropic. In VTI case, the workflow is summarized as follows:

1. Solve the anisotropic VTI eikonal equation \(4.4\) for traveltime \(\tau(x)\).

2. Solve the isotropic eikonal equation \(4.5\) for effective velocity \(V_{\text{eff}}(x)\) knowing traveltime \(\tau(x)\).

This procedure can be easily extended to other anisotropic models given that the corresponding eikonal equations can be solved efficiently. We also note that we should
use viscosity solution of eikonal equations since computing the effective velocity step needs taking gradients of traveltimes.

Similarly, we can apply the same trick to compute the effective density with the following steps:

1. Solve the anisotropic transport equation (4.8) for anisotropic amplitude field $A(x)$ knowing traveltime field $\tau(x)$, which is obtained from the anisotropic eikonal solver.

2. Solve the variable density isotropic transport equation (4.10) for effective density $\rho_{\text{eff}}(x)$ knowing the amplitude $A(x)$ and traveltime $\tau(x)$.

However this approach requires a highly accurate anisotropic eikonal solution to ensure an acceptable accurate amplitude. Furthermore, the initialization of boundary values for density could be an issue if either fast marching or fast sweeping methods are used to solve equation (4.10). I therefore evaluate the effective density directly without ever computing the amplitude.

By noticing the isotropic transport equations can be written in conservation forms like (4.9) and (4.10), we wish to have similar conservation form of VTI transport equation (4.8). After tedious but straightforward algebra manipulations on equation (4.8), we can arrive at

$$F(\eta, V_{\text{nmo}}, V_{P0}) + \nabla \cdot A^2 \begin{pmatrix} (1 + 2\eta)V_{\text{nmo}}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}(\frac{\partial \tau}{\partial z})^2 \frac{\partial \tau}{\partial x} \\ (1 + 2\eta)V_{\text{nmo}}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}(\frac{\partial \tau}{\partial z})^2 \frac{\partial \tau}{\partial y} \\ V_{P0}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}(\frac{\partial \tau}{\partial x})^2 + (\frac{\partial \tau}{\partial y})^2 \frac{\partial \tau}{\partial z} \end{pmatrix} = 0, \quad (4.24)$$

where $F(\eta, V_{\text{nmo}}, V_{P0})$ contains the weighted divergence term of anisotropic parameters $\eta, V_{\text{nmo}}$, and $V_{P0}$. If we ignore this term, we can arrive at the conservation form
VTI transport equation

$$\nabla \cdot A^2 \left( \begin{array}{c} 
(1 + 2\eta)V_{nmo}^2 - 2\eta V_{nmo}^2 V_P^2 \left( \frac{\partial \tau}{\partial z} \right)^2 \frac{\partial \tau}{\partial x} \\
(1 + 2\eta)V_{nmo}^2 - 2\eta V_{nmo}^2 V_P^2 \left( \frac{\partial \tau}{\partial z} \right)^2 \frac{\partial \tau}{\partial y} \\
V_P^2 - 2\eta V_{nmo}^2 V_P^2 \left( \left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial y} \right)^2 \right) \frac{\partial \tau}{\partial z} 
\end{array} \right) = 0. \quad (4.25)$$

The justification of ignoring $F(\eta, V_{nmo}, V_P)$ in (4.24) is that we assume the parameters to be slowly varying in the high frequency asymptotic sense. Then we compare the variable density isotropic transport equation (4.10) and the conservation form VTI transport equation (4.25), the amplitude and traveltime should be same and all equal to the corresponding anisotropic transport and eikonal solutions. We then directly equate the flux functions inside the divergence, i.e.

$$\nabla \frac{\tau}{\rho} = \left( \begin{array}{c} 
(1 + 2\eta)V_{nmo}^2 - 2\eta V_{nmo}^2 V_P^2 \left( \frac{\partial \tau}{\partial z} \right)^2 \frac{\partial \tau}{\partial x} \\
(1 + 2\eta)V_{nmo}^2 - 2\eta V_{nmo}^2 V_P^2 \left( \frac{\partial \tau}{\partial z} \right)^2 \frac{\partial \tau}{\partial y} \\
V_P^2 - 2\eta V_{nmo}^2 V_P^2 \left( \left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial y} \right)^2 \right) \frac{\partial \tau}{\partial z} 
\end{array} \right). \quad (4.26)$$

This is by no means the only solution to matching the two transport equations (4.10) and (4.25). However it can serve as one possible solution if we want the system to be solved. Notice the units of density in equation (4.26) may not be correct, we first multiply both sides in equation (4.10) by the effective velocity $V_{\text{eff}}^2$, apply high frequency asymptotic approximation to get

$$\nabla \cdot \left( \frac{A^2}{\rho} V_{\text{eff}}^2 \nabla \tau \right) = 0, \quad (4.27)$$

equate the flux function with that in equation (4.25) again, and ten drop the gradient
of traveltime, we will then have

\[
V_{\text{eff}}^2 \begin{pmatrix}
1/\rho_x \\
1/\rho_y \\
1/\rho_z \\
\end{pmatrix} = \begin{pmatrix}
(1 + 2\eta)V_{\text{nmo}}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z}\right)^2 \\
(1 + 2\eta)V_{\text{nmo}}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial y}\right)^2 \\
V_{P0}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z}\right)^2 \\
\end{pmatrix}. \tag{4.28}
\]

I now propose to treat the effective density as an anisotropic parameter, and match equation (4.28) term by term to get an effective density field \( \rho(x, y, z) = (\rho_x, \rho_y, \rho_z) \).

Using this artificial anisotropic density will not increase computational cost of isotropic wavefield extrapolation too much since the variable density term \( \frac{1}{\rho} \nabla \rho \cdot \nabla U \) in the wave equation is evaluated axis by axis. Through this approach, we avoid evaluating the amplitude while obtaining an effective density to match the dynamic part of the original anisotropic wavefield with that of effective isotropic wavefield.

With the effective isotropic velocity and effective anisotropic density at hand, we easily feed them into the variable density acoustic wave equation

\[
\frac{1}{v^2} \frac{\partial^2 U}{\partial t^2} - \nabla^2 U + \begin{pmatrix}
\partial_x \rho/\rho_x \\
\partial_y \rho/\rho_y \\
\partial_z \rho/\rho_z \\
\end{pmatrix} \cdot \nabla U = 0, \tag{4.29}
\]

where the density is no longer a scalar field, but a vector.

However other solutions for equation (4.26) are possible, for example, we can solve for an isotropic scalar density field through optimization applied to the system under some carefully chosen norm. We could also solve through variational formulations by integrating the two transport equations (4.10) and (4.25).

**Numerical Examples**

I now demonstrate the use of our effective isotropic velocity and effective anisotropic density approach with a VTI Marmousi example.
Figures 4.1a, 4.1b and 4.1c show the vertical $P$-wave velocity ($V_z$), NMO velocity $V_{\text{nmo}}$ and $\eta$ distribution of VTI Marmousi model. The source location is placed at lateral position 4000 m and depth position 500 m. I choose this shot position because anisotropy is relatively stronger in this region. I then use a third-order WENO scheme Godunov type Hamiltonian fast sweeping method to compute the traveltime. In my implementation, the first-order scheme is executed for several iterations to ensure a good traveltime initialization then I switch to the WENO scheme. Boundaries are numerically filled with large values as well as large variations so that the Godunov Hamiltonian can automatically decide the upwind direction. The convergence criterion is implemented in the $\ell_1$ norm. The traveltime field is then shown in Figure 4.2a. A wavefield snapshot form anisotropic wave equation modeling at $t = 0.39$ s overlaid with first arrival traveltime is shown in Figure 4.2b which is a zoom-in of the red box area shown in 4.2a.

An effective isotropic velocity field is obtained by solving equation (4.5) and shown in Figure 4.3a. From the figure, we see that the largest difference between this effective isotropic velocity and the original vertical velocity is located in the upper-left part of the model where the source is located (an amplitude singularity) with large traveltime gradient.

Using the effective isotropic velocity, we then compute the anisotropic density field using equation (4.28). The resulting horizontal and vertical components of the effective density field are shown in Figures 4.3b and 4.3c. This effective density field is also source dependent and has obvious source signature influence at and around the source.

I then do forward modeling with the effective models and compare the wavefields with those using the original anisotropic model. Figures 4.4a and 4.4c show the effective modeled wavefield snapshots, without and using effective density, respectively. Both of them show a good match with the anisotropic eikonal solver which confirms
the kinematic match of effective modeled wavefield with the original anisotropic wavefield. To check the dynamic match, we slice the wavefields at depth position of 500 m, and show the slices in Figures 4.4b and 4.4d. The amplitudes are normalized. It is clear from the figures that adding an effective density correction gives better matching in amplitudes between the effective modeled wavefield and the original anisotropic modeled wavefield, especially near the first arrival areas where the green arrows point to. This, therefore justifies our effective density approach.

4.4 Conclusions

In this chapter, I demonstrated the idea of effective anisotropy. It embeds anisotropy into the inhomogeneity of the velocity and density models in the high frequency asymptotic sense to reduce the cost of extrapolating anisotropic wavefields. This is done by matching both the kinematic ad dynamic parts of the approximated wavefields to original ones, specifically matching the anisotropic eikonal and transport equations to the isotropic eikonal and transport equations. I also provide one possibility of evaluating the effective density efficiently without solving the transport equation. The numerical example demonstrates the feasibility of the approach.
Figure 4.1: VTI Marmousi model.
Figure 4.2: VTI eikonal solution from fast sweeping solver and VTI anisotropic wave equation solution.
(a) Effective isotropic velocity field.

(b) Horizontal component of effective density field.

(c) Vertical component of effective density field.

*Figure 4.3:* Effective velocity and density models.
Isotropic modeled wavefield snapshots at $t = 0.39$ s overlaid with anisotropic eikonal solution only using effective velocity only.

Slices of wavefield snapshots at $t = 0.39$ s from Figure 4.4a.

Isotropic modeled wavefield snapshots at $t = 0.39$ s overlaid with anisotropic eikonal solution only using effective velocity and effective density.

Slices of wavefield snapshots at $t = 0.39$ s from Figure 4.4c.

**Figure 4.4:** Isotropic wave equation modeled wavefield snapshots using effective models. Blue lines in 4.4b and 4.4d are reference slices from original anisotropic wavefield snapshot. Red lines in 4.4b and 4.4d are slices from effective modeled wavefield snapshots without and using effective density, respectively.
Chapter 5

Conclusions and Future Work

The main result of this thesis is a collection of two projects, modeling and migration using prestack exploding reflector model in TI media, and effective anisotropy through traveltime and amplitude matching, both of which are aimed at improving the efficiency of seismic wave modeling and migration in anisotropic media.

The PERM modeling and migration is addressed in chapter 3. It gains the efficiency improvement by directly relating the whole prestack data to the prestack images in one single modeling or migration kernel. There is no cross-correlation of wavefields when applying the imaging condition as the conventional reverse time migration does. The theoretical basis comes from the DSR formula. However it differs from the depth extrapolation of DSR. The procedure I use is a recursive Fourier integral time extrapolation also known as spectral extrapolation, in particular the cost of propagator is reduced through a lowrank decomposition approach. I evaluate the phase angle and phase velocity of DSR formula through the concept of DTI media, the result phase velocity is simple to implement and has a closed-form. The phase angle and velocity evaluation is also extended to more general TI media with a physical explanation similar to DTI media. Numerical examples from VTI Marmousi model and BP2007 TTI model demonstrates the feasibility of my approach and I see some improvements on the results using TTI PERM migration than DTI PERM migration.

The effective anisotropy through traveltime and amplitude matching which is
The second part of this thesis improves the efficiency of modeling by approximating the original anisotropic wavefield with an effective isotropically modeled wavefield. The theoretical basis of this approach is the high frequency asymptotic signature of the wavefields. I use the eikonal equations to embed the kinematics information of anisotropic model into effective velocity model. I use transport equations to approximate the dynamics information using effective density model. These equations are solved through Hamilton-Jacobi formulations using fast sweeping and fast marching methods. I found fast sweeping methods more suitable for higher order traveltime calculation. After equating eikonal and transport equations, the resultant effective velocity models and density models are then fed into an isotropic wave equation solver to approximate the original anisotropic wavefield at a reduced cost. A numerical example from anisotropic Marmousi model demonstrates the capability of my approach.

**Future Work**

There are still some remaining parts of the projects to be investigated in the future:

1. In PERM modeling and migration, horizontal propagated waves are not fully resolved and certain assumptions have been made on the phase operator to make it stable. One of the future work would be to use more accurate phase operator to resolve these waves.

2. In my implementation of PERM modeling and migration, no absorbing boundary conditions are used. The boundary is intrinsically periodic due the fast Fourier transform. Another future work would be to incorporate the boundary conditions.

3. In the effective anisotropy approach, I directly equate the two transport equations by approximating them in conservation form. This direct equating is
inaccurate. Future work would focus on an efficient higher-order accurate anisotropic transport equation solver so as to avoid the direct equating.
REFERENCES


6 Conference Papers Submitted