Effective Orthorhombic Anisotropic Models for Wavefield Extrapolation

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The thesis of Wilson Ibanez Jacome is approved by the examination committee

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有效正交各向异性模型在波前外推中的应用

Effective Orthorhombic Anisotropic Models for Wavefield Extrapolation

Wilson Ibanez Jacome

波场外推在正交各向异性介质中涉及复杂但现实的模型，以再现波在地球表层的传播现象。与代表更简单对称性的模型，如横向各向同性或各向同性，正交模型需要更广、更详细的表述，从而也涉及到更昂贵的计算过程。声学假设更有效表述正交波方程，同时也提供正交各向异性传播关系的简化表述。然而，这种表述的第六阶性质也是声波方程的特征，它也包括了剪切波的贡献。为了减少波场外推中在这些介质中的计算成本，我开发了有效各向同性不均匀模型，这些模型能够再现正交各向异性波场的第一到达的动态方面。首先，为了计算在垂直正交各向异性介质中的传播时间，我发展了一种稳定、高效和准确的算法，基于快速前进方法。正交各向异性声学传播关系，不同于各向同性或横向各向异性的一个，被表示为一个包含声波第一到达的解的第六阶多项式方程。有效速度模型是通过求解各向同性等效模型的传播时间导数来计算的，该模型是通过解各向同性伊克索尼方程来求解的。倒转的这些有效速度场是源相关的，能够提供正交各向异性介质中波传播的第一到达的等效动态描述。我使用更高效的各向同性操作符来外推这些有效的各向同性波场，结果在波前时间上，尤其是在亲波性上，与更昂贵的各向异性外推器的结果一致。
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LIST OF ABBREVIATIONS

VTI  Vertical Transversely Isotropic.
TI   Transversely Isotropic.
TTI  Tilted Transversely Isotropic.
ORTHO  Orthorhombic.
FFT  Fast Fourier Transform.
NMO  Normal Move Out.
LIST OF SYMBOLS

\( \eta_1 \) Anisotropic parameter \( \eta \) defined in the \([x_1, x_3]\) symmetry plane.

\( \eta_2 \) Anisotropic parameter \( \eta \) defined in the \([x_2, x_3]\) symmetry plane.

\( \delta \) Thomsen’s parameter \( \delta \) defined in the \([x_1, x_2]\) symmetry plane.

\( \gamma \) Factor defined in terms of \( \delta \), such that \( \gamma = \sqrt{1 + 2\delta} \).

\( v_v \) P-wave vertical velocity.

\( v_1 \) NMO P-wave velocity for horizontal reflectors in the \([x_1, x_3]\) plane.
Also represented as \( v_x \) velocity.

\( v_2 \) NMO P-wave velocity for horizontal reflectors in the \([x_2, x_3]\) plane.
Also represented as \( v_y \) velocity.

\( v_{\text{hor}} \) Horizontal velocity component.

\( v_{\text{nmo}} \) NMO P-wave velocity. Notation used for VTI symmetries.

\( v_{\text{eff}} \) Effective velocity.

\( v_{s1}, v_{s2}, v_{s3} \) Cartesian Shear wave velocity components.

\( \tau \) Traveltime variable defined for eikonal solver.

\( c_{ijkl} \) Stiffness tensor components.

\( \delta_{ik} \) Delta Kronekers operator.

\( \rho \) Density.

\( G_{jk} \) Christoffel matrix.

\( U_k \) Displacement vector components of plane waves.

\( n_j \) Unit vector components in the slowness direction.

\( \Gamma(p) \) Christoffel equation for acoustic orthorhombic media.

\( p_x, p_y, p_z \) Cartesian phase vector components.

\( P(x, y, z; t) \) Wavefield in Cartesian system for a specific time \( t \).

\( P_0(x, y, z) \) Initial amplitude of wavefield in Cartesian coordinate system.

\( i \) Imaginary number.

\( \omega \) Angular frequency.

\( \zeta \) Denominator term from orthorhombic eikonal equation.

\( A, B, C, D, E, F, G \) Sequence of coefficients in eikonal equation used to collect all medium properties from orthorhombic symmetry.

\( \tau_{i,j,k} \) Traveltime from finite difference scheme at grid point \( i, j, k \).

\( \sum_{p=0}^{6} \beta_p \) Series of real coefficients in sixth order polynomial equation used to compute P-wave fastest traveltime solution in orthorhombic media.

\( P(\tau) \) Polynomial representation from Bairstow’s method.

\( \epsilon \) Thomsen’s anisotropic parameter.

\( \varepsilon \) Value of tolerance to adjust results inherent to Newton’s iteration method.

\( n_x, n_y, n_z \) Number of samples on each Cartesian axis used for modeling.

\( dx, dy, dz \) Grid spacing on each Cartesian axis used for modeling.
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Chapter 1

Introduction

In exploration geophysics, especially for seismic imaging, it is now a critical aspect to generate reasonable working models of the Earth’s subsurface. New applications in seismic processing sequences based on seismic anisotropy, are now considered a crucial factor for better understanding of wavefield propagation in such more complicated media. The approach presented in this work is naturally related to the field of hydrocarbon exploration, but more importantly, it is connected with a sequence of nonconventional methods that incorporate seismic anisotropy. Most recent improvements in parameter estimation and seismic data processing are integrating the implementation of anisotropic models for a significant extended set of seismic methods [1]. A more detailed historical review on the developments in seismic anisotropy is shown by Helbig and Thomsen [2]. Some important cases regarding the type of anisotropy such as vertical and tilted transverse isotropy are currently considered standard options for most velocity inversion algorithms. These results are subsequently incorporated to important imaging sequences such as prestack depth migration. Since the continuous advancements and developments in data-acquisition technology have spurred the transition from transverse isotropy to lower anisotropic symmetry systems, such as orthorhombic, the current transformation to such more realistic models facilitates the actual implementation of anisotropic parameters in fracture systems detection, and lithology differentiation, among other important reservoir characterization aspects. As a matter of fact, most important geological settings for hydrocarbon exploration (e.g., reservoir formations) are created by large tectonic and irregular stresses that generate systems of vertical fractures in a vertical or tilted thin-layering background (see Figure 1.1(b)). As shown in Figure 1.1(a), the type of systems described by this geological representation can be approximately modeled by orthorhombic symmetries [3,4,5,6]. However, due to the complexity involved in this particular geometry, the corresponding implicit sequences used to model or image seismic data in such complicated media require significant additional computational cost.

Since wavefield extrapolation in anisotropic media is in fact a much more expensive computational process compared to the extrapolations performed for isotropic media, finding an alternative method that reduces the computational cost of such operations
can cause a significantly remarkable impact in this field. The motivational value of this work is conducted to the implementation of an efficient method that reduces the computational cost of wavefield extrapolation in vertical orthorhombic media.

In addition, the computational cost of a wide set of seismic imaging techniques, such as seismic modeling, migration, and full waveform inversion, relies strongly on the methods used for wavefield extrapolation. Since orthorhombic seismic anisotropy implies a more complicated description of seismic wave propagation, orthorhombic wavefield extrapolations are significantly expensive for most standard production processing techniques. In order to reduce the computational cost implied in wavefield extrapolations (initially performed by conventional approaches, such as finite difference methods [7]), spectral methods have started to become a more robust alternative for such demanding computational processes. However, even a lowrank approximation of the corresponding wave extrapolation matrix [8] can still be computationally intensive. A more efficient approach introduced by Alkhalifah [9] for Transversely Isotropic (TI) models, is implemented in this work to facilitate the computation of the kinematic aspect of wavefields propagation in acoustic vertical orthorhombic media.

This study is mainly divided into two parts: First, a fundamental theory based on the fast marching method is used to incorporate the set of equations derived in this work to describe the high frequency asymptotic representation of waves propagating in acoustic vertical orthorhombic media. As a result, an orthorhombic eikonal solver is introduced in this study. Then, a second part of the study is conducted by incorporating the orthorhombic traveltime fields in an efficient approach that involves the construction of effective velocity models used for isotropic inhomogeneous wavefield extrapolations [9].

1.1 Orthorhombic Eikonal Solver

In order to incorporate the orthorhombic eikonal solver into the fast marching method, an orthorhombic dispersion relation for acoustic media is first derived. The approach starts with the description of a Christoffel equation [10] for orthorhombic media under the acoustic assumption [11]. Subsequently, expressing the vertical slowness component as a function of the horizontal part, often referred as the dispersion relation, leads to the actual acoustic orthorhombic dispersion relation. Afterwards, defining the slowness components in terms of the corresponding traveltime-spatial derivatives leads to the first representation of the orthorhombic eikonal equation. Then, for the corresponding numerical implementation, a first order finite difference scheme is applied to the eikonal equation and subsequently reformulated to find the resultant expression that yields a sixth order polynomial equation. Solving this polynomial equation finally leads to a numerical solution that represents the fastest P-wave arrival traveltime of a wavefield propagating in an acoustic vertical orthorhombic media.
Excluding the effects of orthorhombic symmetries (for certain geological settings where orthorhombic symmetries fit for better description), may lead to erroneous traveltime estimations and consequently to an inaccurate understanding of the structural model of the Earth’s subsurface [12]. One of the most important contributions presented in this study is the capability of estimating first arrival traveltimes in orthorhombic media. These can be used in a wide range of seismic imaging techniques, such as seismic tomography [13, 14], and Green function definitions used for Kirchhoff methods [15].

Figure 1.1: Representations of orthorhombic media. General model used to reproduce wave propagation in vertical orthorhombic media (a). Geological formation that can be represented by a tilted orthorhombic symmetry (b).

1.2 Effective Anisotropic Velocity Models for Isotropic Wavefield Extrapolation

The corresponding traveltime fields computed with the orthorhombic eikonal solver are used to explicitly estimate effective isotropic velocity models that encompass all the kinematic effects of the orthorhombic anisotropy. These effective velocity fields are then used to extrapolate an approximate orthorhombic anisotropic wavefield, based on an isotropic operator that implements a lowrank approximation approach [16].

Wavefield extrapolation is considered a computationally very expensive procedure for any 3D depth imaging method [17, 18]. This process in orthorhombic media is well described by the numerical solution of the anisotropic elastic wave equation or the corresponding acoustic approximation [11]. Mixed-domain acoustic wave extrapolators for time marching may be applied using low-rank approximations [16, 19]. The lowrank solution obtained for wavefield extrapolation used in this study is based on the
acoustic approach introduced by Alkhalifah [11], where a sixth order pseudo-acoustic wave equation is used to describe wave propagation in orthorhombic media.

The study that I am presenting is organized as follows: I first describe the fundamental but important concepts of the fast marching method in Chapter 2. Then, in the same chapter, I introduce the corresponding notation implemented in this work to describe wave propagation (kinematic aspect) in orthorhombic media; an orthorhombic dispersion relation is presented. Subsequently, in Chapter 3, an orthorhombic eikonal equation is introduced and used to derive a sixth order polynomial equation, the solutions of which give the fastest P-wave traveltime arrival in orthorhombic media. Lastly, Chapter 4 shows the first direct implementation of the novel study presented in Chapter 3. Effective orthorhombic anisotropic models are generated from the estimated orthorhombic traveltimes and then used for isotropic inhomogeneous wavefield extrapolations. I finally compare the new wavefields with those extracted from the actual orthorhombic wavefield extrapolators.
Chapter 2

Traveltime Computation and Dispersion Relations

One of the major challenges for any anisotropic prestack depth migration is to accurately and efficiently extrapolate wavefields in 3D anisotropic media. Transversely isotropic media (TI) have gradually become a useful assumption of the Earth, representing in one form (tilted) the first order nature of azimuthal anisotropy influence of the Earth [20]. However, a more realistic representation of the Earth’s subsurface that may include the natural thin horizontal layering and parallel vertical cracks may be given by orthorhombic anisotropic models, which assume three mutually orthogonal planes of mirror symmetry [3, 4, 5, 6].

For traveltime computation in any of the previously mentioned media, the different available methods are either based on eikonal or ray tracing equations [21, 22, 23]. Taking into consideration the eikonal equation approach, Popovici [24] proposed a fast-marching finite-difference eikonal solver in Cartesian coordinates, which is very efficient and stable. Based on this approach, the traveltime solution at each grid point in the model is estimated in a simple upwind fashion using the corresponding orthorhombic traveltime solution derived in this work. Therefore, this accurate, fast, and unconditionally stable algorithm [25], can be implemented to compute traveltimes in more realistic anisotropic symmetries such as the ones exhibit for Vertical Transversely Isotropic (VTI) and orthorhombic media. Different methodologies and algorithms are available to compute traveltimes in TI and isotropic symmetries using the eikonal solver approach, in Cartesian, tetragonal or even spherical coordinates [26, 27, 28, 29, 30]. As part of the motivational value of this study, the algorithm presented in this work can be considered the first eikonal solver capable of computing traveltime fields in acoustic vertical orthorhombic media.

2.1 Fast Marching Method

Estimation of seismic traveltime fields is an important process because it leads to a characterization of the kinematic properties of seismic waves, and their corresponding estimations are directly connected with the velocity distribution field presented
in the subsurface. Thus, traveltime estimation methods are considered a very important aspect of any seismic data imaging technique. For instance, different migration and modeling procedures require traveltime calculations that enable the estimation of Green’s functions, containing traveltime values from source points on the surface to a reflected point, back to the surface space [31, 32].

Traveltime can be computed numerically using an upwind finite difference method on a regular grid. First order approximation in the upwind finite difference method provides accurate values of traveltime for seismic data applications [22]. A wide range of different approaches have been presented until now. As an example, Cerveny [33] introduced a very popular technique based on ray tracing methods. It is well known that ray tracing applications also provide ray direction information and ray amplitude [21]. However, since the irregular trajectories are defined through the ray in order to compute traveltimes, this technique requires interpolation methods to determine the traveltime solution at each grid point [34]. Additionally, rays may have traveltime shadow zones during the computational process [35]. Based on these restrictions, ray tracing methods may be considered a less efficient scheme for traveltime estimation.

Different alternative methods have been introduced to compute traveltime on regular grids. Reshef and Kosloff [36] proposed a depth gradient of traveltimes using a finite difference approximation for the eikonal equation, the Hamilton-Jacobi equation which is a first-order, non-linear partial differential equation. Also, implementing a finite difference approach, Vidale [23] approximated the eikonal equation solving for traveltime values using circular or planar wavefront extrapolations. Although, all these methods for regular grids may provide alternatives schemes to compute traveltime of wave propagation, they may fail in complex velocity structures or may be considered computationally very expensive.

The approach implemented in this study satisfies the well known hyperbolic conservation laws for the components of the traveltime gradient. The major disadvantage of this method is that it only estimates first arrival traveltimes [37]. Therefore, multivalued traveltime curves cannot be calculated under this scheme. Regardless of this, multiple applications may be based on this traveltime calculation method. For instance, Kirchhoff modeling or migration algorithms implement single-valued traveltime functions [15, 28]. Also, seismic tomography algorithms are based on first arrival traveltime selection (traveltime picking) where efficient algorithms, such as the fast marching method, may lead to a less computationally demanding alternative process [13, 14].

The fast marching method was developed initially by Tsitsiklis [38] and later reformulated by Sethian [25]. The mathematical principle behind this method is based on the fact that the first arrival traveltimes are the viscosity solution (weak solution) of the eikonal equation. The first viscosity solution techniques were created for the concept of hyperbolic laws. The corresponding numerical methods implemented for these
conservation laws apply upwind finite difference schemes that may lead to the viscosity solution of the eikonal equation \[22\].

Considering the nonclassical sense of the numerical solution for the eikonal equation, involves dealing with discontinuous gradients, that are needed to be replaced by the nonclassical or weak solution. Therefore, at the points where the discontinuity is found, an alternative method (nonclassical scheme) needs to be implemented in order to define the required solution. Then, an especial function that contains discontinuous gradients is used to solve the equations on average. This approximation does not affect the computed average, since the extreme anomalous values do not change the final result significantly. However, using this approach implies that a very high range of values may be considered as a solution in the weak or non-classical sense, leading to a non-unique value. Thus, in order to introduce uniqueness to the solution, a viscosity assumption can be applied to the eikonal equation.

Viscosity solutions allow finding unique and weak results that are stable regardless to any perturbation in the initial equation. Implicit numerical errors embedded in the solution of the equation that perturb the corresponding solution do not affect the stability of the result. An extended list of numerical works and theoretical demonstrations verify the assumption that confirms having the first arrival traveltime field as the solution of the eikonal equation. As mentioned previously, the main limitation of this method is that it computes only the first-arrival field. In other words, this first-arrival traveltime solution corresponds to the unique viscosity (thus stable) solution of the eikonal equation.

The process of computing traveltime values at each grid point is solved forward in time where previous estimated values are required for continuous propagation of the traveltime field at the specific grid point. Because of the non-linearity aspect of the equation, it does not define any initial marching direction. Thus, after ordering from smallest arrival time to largest arrival time, the corresponding solution can be found only applying one pass in the region that contains the grid points. Then, the upwind finite difference scheme is used to solve the eikonal equation \[22\]. As mentioned before, one of the major advantages of this approach is that the implied computational procedure is very efficient and easy to program.

While nonlinear systems of equations are normally solved using a large number of iterations, with the implementation of the fast marching methodology, the eikonal equation only needs a unique solution per grid point after the corresponding required sorting from smallest to largest arrival traveltime.

Additionally, the fast marching method can also be extended to a second order of accuracy \[39, 40\], using one-sided finite difference schemes integrated with a switching mechanism that incorporates back the first order accuracy scheme in regions where the solution is not smooth. However, higher finite difference schemes are found to be very
unstable when the order is higher than two. Only the first order approach of this method is implemented in this study. Thus, for the 3D geometry involved in this case, the eikonal equation is represented as

$$\left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial y} \right)^2 + \left( \frac{\partial \tau}{\partial z} \right)^2 = \frac{1}{v^2(x, y, z)},$$  \hspace{1em} (2.1)$$

where $\tau$ represents traveltime for wave propagation and $v(x, y, z)$ corresponds to an isotropic velocity field, which value depends on the Cartesian location $x, y, z$. Now, considering a first order finite difference scheme, with a grid point sequence $i, j, k$, equation (2.1) may be rewritten as

$$\left( \frac{\tau_{i,j,k} - \tau_{i-1,j,k}}{\Delta x} \right)^2 + \left( \frac{\tau_{i,j,k} - \tau_{i,j-1,k}}{\Delta y} \right)^2 + \left( \frac{\tau_{i,j,k} - \tau_{i,j,k-1}}{\Delta z} \right)^2 = \frac{1}{v^2_{i,j,k}},$$  \hspace{1em} (2.2)$$

Considering that the fast marching method uses the upwind discretization of the eikonal equation, equation (2.2) may be expressed as

$$\max (-D^{+x}\tau_{i,j,k}, D^{-x}\tau_{i,j,k}, 0)^2 + \max (-D^{+y}\tau_{i,j,k}, D^{-y}\tau_{i,j,k}, 0)^2 + \max (-D^{+z}\tau_{i,j,k}, D^{-z}\tau_{i,j,k}, 0)^2 = s^2_{i,j,k},$$  \hspace{1em} (2.3)$$

where the term $s^2_{i,j,k}$ represents the slowness of the medium (equivalent to the inverse of $v^2_{i,j,k}$), and the implemented first order differential operators are defined as

$$D^{+x}\tau_{i,j,k} = (\tau_{i+1,j,k} - \tau_{i,j,k})/\Delta x,$$

$$D^{-x}\tau_{i,j,k} = (\tau_{i,j,k} - \tau_{i-1,j,k})/\Delta x,$$

$$D^{+y}\tau_{i,j,k} = (\tau_{i,j+1,k} - \tau_{i,j,k})/\Delta y,$$

$$D^{-y}\tau_{i,j,k} = (\tau_{i,j,k} - \tau_{i,j-1,k})/\Delta y,$$

$$D^{+z}\tau_{i,j,k} = (\tau_{i,j,k+1} - \tau_{i,j,k})/\Delta z,$$

$$D^{-z}\tau_{i,j,k} = (\tau_{i,j,k} - \tau_{i,j,k-1})/\Delta z.$$

In order to compute the traveltime solution at a particular grid point, the fast marching method essentially requires first a sorting process in the narrow band. In general, for the 3D models used in this study, the first six neighboring points around the source form the first narrow band of the process. Next, the fast marching method expands the domain of the narrow band by computing the solution for the neighboring points around the trial solution in a downwind-fashion, and then takes the trial point as an alive (accepted) value. The causality aspect of the algorithm ensures that the trial point considered as accepted (alive) represents the smallest value from the available ones in the narrow band.

The required steps used in the fast marching algorithm can be summarized in the following form: Initially, the source point (location where the process starts) is considered an alive point with a corresponding zero value. Then, the following closest
points are computed analytically and tagged as close in the narrow band. Later, the rest of the points are tagged as far away with a very large initial value. In the code, far away points are ranked with the value of the FLT_MAX macro, which is the maximum number representable in type float. It is at least 1E+37. After generating the initial three domains: accepted, narrow, and far away, then the following procedure is applied iteratively:

- The smallest value in the narrow band is considered as a trial point.
- All points around the trial value that are not alive will be reassigned to the narrow band even if they are far away points.
- Calculate the solution in the new narrow band using the corresponding eikonal travelt ime solution, which in this case corresponds to the travelt ime value estimated with the orthorhombic eikonal solver.
- Assign the trial solution in the narrow band as an alive point.
- If the narrow band is not empty, start process from the first indicated step.

The major cost of the algorithm is reflected from the binary tree sorting that allows the corresponding ordering from smallest to largest points at each stage of the narrow band. The computational cost of the binary tree sorting is given by $O(\log_2 N_b)$, where $N_b$ represents the number of grid points in the close domain, the narrow band. For $N$ total grid points, the final cost of the process is represented by $O(N \log_2 N_b)$. More details of the fast marching algorithm may be found in the document published by Sethian in 1996 [41].

2.2 Notation for Orthorhombic Media

Orthorhombic symmetries are considered to be very important models to describe wave propagation in fractured reservoirs with specific geometries [3, 42, 43, 5]. Although, the implementation of this symmetry may conduct to better descriptions in such media, it may also lead to more complicated and computationally expensive processes. In order to reduce the complexity of the initial equations found for orthorhombic media (in terms of the elastic coefficients [10], which gives a more complicated and less physically-understandable set of equations), Tsvanking [44], Alkhalifah [11], and other authors, established alternative relations to describe wave propagation in orthorhombic media in terms of a series of anisotropic parameters. Most of these derivations are being based on the novel and very well known study named weak elastic anisotropy, published by Leon Thomsen in 1986 [45].

There are different geological aspects that can reproduce the effects of an orthorhombic symmetry. As shown in Figure 1.1(a), the most common setting of this type of
anisotropy in sedimentary basins is given by the combination of parallel vertical cracks
and vertical transverse isotropy as a background medium [5]. Also the presence of a
second crack system (instead of a TI background) may reproduce equivalent effects in
wave propagation [46, 47, 48]. Therefore, orthorhombic symmetry may be considered as
one of the most complicated assumption but the simplest more realistic geometry [49].

Since orthorhombic media are fully described by nine independent stiffness compo-
nants, a more complicated system of equations is required for full description of
wave propagation [44]. VTI case can be completely described by five independent stiff-
ness components [50]. Increasing the symmetry to isotropic models requires only two
independent stiffness components. Orthorhombic or orthotropic symmetries are rep-
resented by three mutually orthogonal planes of mirror symmetry [44]. Therefore, in
this context, high frequency approximations given by Christoffel equations are used
to describe wave propagation in orthorhombic media. On each corresponding plane of
mirror symmetry, the equations used for wavefield propagation description (kinematic
aspect) are equivalent to the equations used in transversely isotropic media. This uni-
ified description provides a suitable relation between the two models, and enables the
application of standard and common procedures to the orthorhombic case, initially
used for TI symmetries [44]. Thus, the stiffness tensor $c_{ijkl}$ for orthorhombic media
can be represented by

$$C_{ortho} = \begin{pmatrix}
  c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
  c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\
  c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & c_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & c_{55} & 0 \\
  0 & 0 & 0 & 0 & 0 & c_{66}
\end{pmatrix}, \quad (2.5)$$

where a two-index notation based on the well known Voigt recipe is used. For $c_{11} = c_{22},$
$c_{13} = c_{23}, c_{44} = c_{55},$ and $c_{12} = c_{11} - 2c_{44}$, equation (2.5) represents the stiffness matrix
for VTI media. With regard to the different phase velocity values $V,$ and displacement
vector components $U_k$ (using plane wave assumption [10]) in any arbitrary anisotropic
media, Christoffel equation satisfies the following relation [10],

$$[G_{ik} - \rho V^2 \delta_{ik}] U_k = 0, \quad (2.6)$$

where $\delta_{ik}$ represents a Kroneker’s operator, $\rho$ is the density, and $G_{ik}$ corresponds to the
Chistoffel matrix, $G_{ik} = c_{ijkl}n_jn_l.$ The values $n_j$ and $n_l$ represent unit vector compo-
nents in the slowness direction. Solving for the eigenvalues ($\rho V^2$) and eigenvectors $U_k$
problem,

$$\det [G_{ik} - \rho V^2 \delta_{ik}] = 0, \quad (2.7)$$

leads to the different phase velocity components and the corresponding anisotropic pa-
rameters. Specific components $G_{ik}$ and directions $n_jn_l$ have to be considered in order to
describe wave propagation in terms of the corresponding velocities and anisotropic parameters, involving the directional dependence aspect of wave propagating in anisotropic media. Complete algebraic sequences used to define phase velocities and anisotropic parameters may be found in *Seismic Waves and Rays in Elastic Media* by M.A. Slawinski [10].

Kinematic signatures of P-waves in pseudo-acoustic anisotropic orthorhombic media depend on only five anisotropic parameters and the vertical velocity as shown by Tsvankin [44]. Instead of rigorously using the notation suggested by Tsvankin [44], I implement a different parametrization given by Alkhalifah [11], where

\[
v_v = \sqrt{\frac{c_{33}}{\rho}},
\]

\[
v_1 = \frac{\sqrt{c_{13}(c_{13} + 2c_{55}) + c_{33}c_{55}}}{\rho(c_{33} - c_{55})},
\]

\[
v_2 = \frac{\sqrt{c_{23}(c_{23} + 2c_{44}) + c_{33}c_{44}}}{\rho(c_{33} - c_{44})},
\]

are the P-wave vertical velocity, and the NMO P-wave velocities for horizontal reflectors defined in the \([x_1, x_3]\) and \([x_2, x_3]\) planes of mirror symmetry respectively (see Figure 1.1(a)). The term \(\rho\) represents the density value. Additionally, the following notation for the anisotropic parameters is also considered,

\[
\eta_1 = \frac{c_{11}(c_{33} - c_{55})}{2c_{13}(c_{13} + 2c_{55}) + 2c_{33}c_{55}} - \frac{1}{2},
\]

\[
\eta_2 = \frac{c_{22}(c_{33} - c_{44})}{2c_{23}(c_{23} + 2c_{44}) + 2c_{33}c_{44}} - \frac{1}{2},
\]

\[
\delta = \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}(c_{11} - c_{66})},
\]

where the first two parameters, \(\eta_1\) and \(\eta_2\) respectively correspond to the anellepticity value in the \([x_1, x_3]\) and \([x_2, x_3]\) symmetry planes, and \(\delta\) represents the anisotropic parameter in the \([x_1, x_2]\) plane, defined with respect to the \(x_1\) coordinate axis. The parameter \(\delta\) is used in the eikonal fast marching algorithm under the following definition

\[
\gamma = \sqrt{1 + 2\delta}\ [11].
\]

Figure 1.1(a) shows the relation between the planes of symmetry and the corresponding anisotropic parameters.

### 2.3 Dispersion Relation

Since seismic reflection data are usually recorded on the Earth’s surface, the horizontal slowness component may be estimated from the corresponding slopes inherent on reflection events collected in the data. Therefore, the equation that characterizes the vertical slowness component in terms of the known (from surface seismic data) horizontal slowness field is usually denoted as a dispersion relation. This equation or the
corresponding vertical component of the slowness field, plays a critical role in seismic imaging \cite{52, 53}. The dispersion relation used in this particular study incorporates the approximations made by Alkhalifah \cite{50}. Since the influence of shear-wave velocities on P-wave propagation is small, Alkhalifah \cite{50} showed that setting the S-wave velocity to zero, does not compromise in general the accuracy for traveltime computation.

### 2.3.1 Dispersion Relation in Orthorhombic Media

Based on the pseudo-acoustic approximation shown in Alkhalifah \cite{11}, after setting the S-wave velocity field components \(v_{s1}, v_{s2}, \) and \(v_{s3}\) to zero, Christoffel equation reduces to

\[
\Gamma(p) = \begin{pmatrix}
(p_x^2 v_1^2 (1 + 2 \eta_1) - 1 & \gamma p_x p_y v_1^2 (1 + 2 \eta_1) & p_x p_z v_1 v_v \\
p_x p_z v_1 v_v & (1 + 2 \eta_2) p_y^2 v_2^2 (1 + 2 \eta_2) - 1 & p_y p_z v_2 v_v \\
p_y p_z v_2 v_v & p_x p_z v_1 v_v & (1 + 2 \eta_1) p_x^2 v_1^2 - 1
\end{pmatrix},
\]

where the values of \(p_x, p_y, \) and \(p_z\) represent the Cartesian components of the phase vector \(p\). As previously shown in equation (2.7), taking the determinant of \(\Gamma(p)\), setting the resultant linear equation (in terms of \(p_z^2\)) to zero, and solving for the squared vertical component of the phase vector \(p_z^2\), gives the dispersion relation for orthorhombic media,

\[
p_z^2 = \frac{1 - (1 + 2 \eta_2) p_y^2 v_2^2}{\xi(p_x, p_y)} (1 + 2 \eta_1) p_x^2 v_1^2 (1 + 2 \eta_1) \gamma^2 v_1^2 - (1 + 2 \eta_2) v_2^2 \] \xi(p_x, p_y),
\]

where

\[
\xi(p_x, p_y) = v_v^2 \left(1 - 2 \eta_2 p_y^2 v_2^2 - p_x^2 v_1^2 \left[2 \eta_1 + \gamma^2 p_y^2 v_2^2 + 4 \eta_1 (1 + \eta_1) \gamma^2 p_y^2 v_1^2 - 2 (1 + 2 \eta_1) \gamma p_y^2 v_2 + (1 - 4 \eta_1 \eta_2) p_y^2 v_2^2 \right]\right).
\]

Equation (2.9) represents one of the most important relations implemented in this particular study. Since the corresponding orthorhombic eikonal equation is found from equation (2.9), after defining the phase vector components in terms of the spatial derivatives of traveltimes, it consequently leads to a sixth order polynomial equation that is then used to numerically estimate the fastest P-wave traveltime arrival in orthorhombic media.

### 2.3.2 VTI Case from Orthorhombic Dispersion Relation

VTI media models may be interpreted as a simplified symmetry (with respect to orthorhombic media) where no distinction between the vertical planes of mirror symmetry
can be made. In this case, an increase of symmetry is implied after adopting the azimuthal isotropic aspect with respect to the P-wave NMO velocities $v_1$ and $v_2$, and the anellepticity values $\eta_1$ and $\eta_2$. Since the horizontal planes of mirror symmetry are considered to be isotropic for the VTI case [20], the Thomsen’s parameter $\delta = 0$ and therefore $\gamma = 1$. Thus, based on previous simplifications, where $v_{nmo} = v_1 = v_2$, $\eta_1 = \eta_2 = \eta$, and $\gamma = 1$, a VTI dispersion relation may be found from equation (2.9).

From an alternative point of view, considering a different interpretation of the simplification mentioned earlier, setting one of the phase vector components, $p_y = 0$ or $p_x = 0$ in equation (2.9), equivalently yields

$$p_z^2 = \frac{1}{v_y^2} \left(1 - \frac{v_x^2 p_x^2}{1 - 2\eta_1 v_x^2 p_x^2}\right),$$

(2.11)

which represents the VTI acoustic dispersion relation. Note that in this case, $p_z$ involves a two dimensional plane domain that inherently represents the azimuthal symmetric aspect implied in 3D VTI media.

### 2.3.3 Isotropic Case from Orthorhombic Dispersion Relation

The simplest case may be found when all common medium properties are represented by the same magnitude, in this case, the velocity field. Therefore, this simplification indicates that no distinction between different velocity fields can be made. With respect to the anisotropic parameters, $\eta_1$ and $\eta_2$, this implies that under the definition $\eta = \left(v_{hor}^2/v_{nmo}^2 - 1\right)/2$, explained with more details by Alkhalifah [50], since $v_{hor} = v_{nmo}$, then, all $\eta$ parameters are considered to be zero. Additionally, since Thomsen’s parameters are required to be zero in isotropic media, then $\delta = 0$ and therefore $\gamma = \sqrt{1 + 2\delta} = 1$. All these simplifications imply that $v_v = v_1 = v_2 = v_{hor}$, $\eta_1 = \eta_2 = 0$, and $\gamma = 1$, which lead to an increase of symmetry for equation (2.9) in terms of the phase vector components. It respectively provides the isotropic dispersion relation given by

$$p_z^2 = \frac{1}{v^2} - p_x^2 - p_y^2,$$

(2.12)

where $v$ represents the unique velocity field for isotropic media.

### 2.4 Discussion

The theoretical background presented in this chapter illustrates the fundamental starting concepts used to develop what is considered the most relevant contribution of this study; an orthorhombic eikonal solver. Initially, a fast marching method is presented in general terms. In order to develop a fast marching implementation for orthorhombic media, the foremost starting point is to build the orthorhombic eikonal solver from
the available fast marching algorithms, used to compute traveltime in isotropic media (available in Madagascar\(^1\)). The actual critical aspect of the implementation is to solve a sixth order polynomial equation (presented in next chapter) and incorporate the corresponding solution (P-wave fastest arrival) into the sequence of steps mentioned earlier for the fast marching method. More detailed information about the developed algorithm is shown in Appendix A.

The sequence of different simplifications shown in previous subsections are used to compare and comprehend the geometrical aspects related to each one of the presented symmetries; from the most complicated model given by the orthorhombic formulation, to the simplest case, where no directional distinction between velocity factors are made (isotropic symmetry). The different traveltime solutions obtained and shown in the next section (for each symmetry), are used to more clearly identify the significant distinction between isotropic, VTI, and orthorhombic solutions. In fact, a crucial aspect during the traveltime computational process relates the inherent directional dependence of the fast marching method with the corresponding geometry defined by each media (isotropic, VTI, or, orthorhombic). Therefore, depending on the direction in which the traveltime is computed, in other words, the direction in which the wave propagates, then, even for the orthorhombic model, the solution could be obtained by only solving the VTI-related equations. This is one of the possible options that can be obtained when the wave travels through one of the vertical planes of mirror symmetry. Since these planes are solely described by VTI-type of equations, then, an orthorhombic problem is simplified to a VTI traveltime computation. A more complete analysis of the common geometrical aspects between orthorhombic and VTI media is shown by Tsvankin [44].

\(^{1}\)Madagascar is an open-source software package for multidimensional data analysis and reproducible computational experiments. More information at [http://www.ahay.org/wiki/Main_Page](http://www.ahay.org/wiki/Main_Page).
Chapter 3

Orthorhombic Eikonal Solver

The eikonal solver presented in this work provides the kinematic response of wavefields propagating in orthorhombic media. First arrival traveltimes are computed by solving a sixth order polynomial equation, that is derived from the orthorhombic eikonal equation presented in this chapter. The corresponding traveltime computational process requires a set of input fields that indicate the general directional dependence of the traveltime \( \tau(x) \) computed at each grid point. A set of three different velocities; vertical velocity \( v_v \), NMO P-wave velocity \( v_1 \), and NMO P-wave velocity \( v_2 \), are used in this symmetry as well as two anellepticity parameters \( \eta_1 \) and \( \eta_2 \), along with an additional factor \( \gamma \), which is defined in the horizontal plane domain [11].

3.1 Orthorhombic Eikonal Equation

In general, the eikonal equation relates the magnitude value of the slowness (inverse of velocity) of a wavefield with the properties of the medium through which it propagates. In the mathematical context,

\[
[\nabla \tau(x)]^2 = \frac{1}{v^2(x, \frac{\nabla \tau}{|\nabla \tau|})}, \tag{3.1}
\]

where \( x \) defines location in the Cartesian system and the relation \( \frac{\nabla \tau}{|\nabla \tau|} \) represents the directional dependence of the corresponding velocity field in anisotropic media. In general terms, for any anisotropic media, the velocity field depends not only on the location but the direction in which the wave propagates. Thus, for orthorhombic symmetries, equation (3.1) defines the spatial variation of the traveltime in a medium that is not uniquely characterized by only one scalar value (velocity field at certain location) but a sequence of scalar fields that may change the resultant kinematic aspect of the wave, depending on the direction in which it propagates. One approach that can be used in order to understand the origin of the isotropic eikonal equation (3.1), could be based on the assumption of plane wave propagation [54]. For a plane wave function \( P(x, y, z; t) \), with an initial amplitude \( P_0(x, y, z) \) and time \( \tau(x, y, z) \) variation in space, the wavefield is represented by

\[
P(x, y, z; t) = P_0(x, y, z) \exp \{-i\omega [t - \tau(x, y, z)]\}. \tag{3.2}
\]
Assuming that the plane wave solution (3.2) satisfies the isotropic scalar wave equation
\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = \frac{1}{v^2(x, y, z)} \frac{\partial^2 P}{\partial t^2}.
\] (3.3)

Thus, evaluating equation (3.3) with the input function \(P(x, y, z; t)\) (as an initial approximation, no amplitude variation with respect to space is considered in this case) leads to the expression
\[
\left(\frac{\partial \tau}{\partial x}\right)^2 + \left(\frac{\partial \tau}{\partial y}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2 = \frac{1}{v^2(x, y, z)},
\] (3.4)
which corresponds to the isotropic eikonal equation. This result is considered a ray-theoretical or high frequency asymptotic approximation to the scalar wave equation shown in (3.3). Solving the scalar wave equation yields the wavefield solution \(P(x, y, z; t)\), that fully describes the amplitude and traveltime effects produced by the medium. However, the solution given by the eikonal equation only represents the traveltime \(\tau(x, y, z)\) at location \((x, y, z)\) with a velocity value of \(v(x, y, z)\). Now, for an arbitrary inhomogeneous function \(v(x, y, z)\) and an amplitude field \(P_0(z, y, z)\) changing in the space domain, the traveltime \(\tau(x, y, z)\) shown in equation (3.2), does not correspond to the solution of the eikonal equation shown in equation (3.4) (an additional factor may be found when computing space derivatives of the field \(P(x, y, z; t)\)). Therefore, for the plane wave function shown in (3.2) that defines spatial amplitude variations, the eikonal equation becomes an acceptable approximation valid for a high frequency domain. This high frequency assumption can be interpreted in terms of the relative wavelength variation value with respect to the changes of medium properties given by the velocity gradient. High frequency ranges refer to small wavelengths. Thus, to satisfy the limit-frequency condition, the velocity gradient changes must be significantly less than the wave frequency. Therefore, velocity models should not be represented by large velocity gradients when used to compute traveltime fields through the eikonal equation.

Now, instead of defining a wave solution for the pseudo-acoustic orthorhombic wave equation to obtain an orthorhombic eikonal solution, an alternative much simpler approach can be implemented by only solving the determinant of the Christoffel equation \(\Gamma(p)\) shown in equation (2.8), and then use the corresponding resultant expression to define the orthorhombic dispersion relation shown in equation (2.9). Thus, rewriting equation (2.9) in terms of the corresponding time derivatives, yields the orthorhombic eikonal equation,
\[
\left( \frac{\partial \tau}{\partial z} \right)^2 = \frac{1 - (1 + 2\eta_2) \left( \frac{\partial \tau}{\partial y} \right)^2 v_2^2}{\zeta \left( \frac{\partial \tau}{\partial z}, \frac{\partial \tau}{\partial y} \right)} \tag{3.5}
\]

\[
- (1 + 2\eta_1) \left( \frac{\partial \tau}{\partial x} \right)^2 v_1^2 \left( 1 + \left( \frac{\partial \tau}{\partial y} \right)^2 \left( (1 + 2\eta_1) \gamma^2 v_1^2 - (1 + 2\eta_2) v_2^2 \right) \right)
\]

where

\[
\zeta \left( \frac{\partial \tau}{\partial x}, \frac{\partial \tau}{\partial y} \right) = v_2 \left( 1 - 2\eta_2 \left( \frac{\partial \tau}{\partial y} \right)^2 v_2^2 - \left( \frac{\partial \tau}{\partial x} \right)^2 v_1^2 \left( 2\eta_1 + \gamma^2 \left( \frac{\partial \tau}{\partial y} \right)^2 v_1^2 \right) \right)
\]

+ 4\eta_1 (1 + \eta_1) \gamma^2 \left( \frac{\partial \tau}{\partial y} \right)^2 v_2^2 - 2(1 + 2\eta_1) \gamma \left( \frac{\partial \tau}{\partial y} \right)^2 v_1 v_2
\]

+ (1 - 4\eta_1 \eta_2) \left( \frac{\partial \tau}{\partial y} \right)^2 v_2^2 \right) \right). \tag{3.6}
\]

As mentioned earlier, for simpler symmetries, such as VTI or isotropic, different parameters become equivalent to others or in some cases they are just considered to be zero values.

Now, to facilitate the implementation of a finite difference scheme in equation (3.5), we may factorize all the common coefficients in terms of the corresponding time derivative components. By doing this, a simplified form of the eikonal equation (3.5) may be found, bringing together all medium properties in a new sequence of coefficients such that

\[
A \left( \frac{\partial \tau}{\partial x} \right)^2 + B \left( \frac{\partial \tau}{\partial y} \right)^2 + C \left( \frac{\partial \tau}{\partial z} \right)^2
\]

+ \[ D \left( \frac{\partial \tau}{\partial x} \right)^2 \left( \frac{\partial \tau}{\partial y} \right)^2 + E \left( \frac{\partial \tau}{\partial x} \right)^2 \left( \frac{\partial \tau}{\partial z} \right)^2
\]

+ \[ F \left( \frac{\partial \tau}{\partial y} \right)^2 \left( \frac{\partial \tau}{\partial z} \right)^2 + G \left( \frac{\partial \tau}{\partial x} \right)^2 \left( \frac{\partial \tau}{\partial y} \right)^2 \left( \frac{\partial \tau}{\partial z} \right)^2 \right) = 1. \tag{3.7}
\]

Ordering all the derivative terms in equation (3.5), as shown in equation (3.7), may allow to separate a set of coefficients from the series of space derivatives components. By doing so, the values of the coefficients found in equation (3.5) after the corresponding
sorting applied in equation (3.7) are represented by

\[ A = v_1^2(1 + 2\eta_1), \]
\[ B = v_2^2(1 + 2\eta_2), \]
\[ C = v_0^2, \]
\[ D = (1 + 4\eta_1 + 4\eta_1^2)\gamma_1^2 v_1^4 - (1 + 2\eta_1 + 2\eta_2 + 4\eta_1\eta_2)v_1^2 v_2^2, \]
\[ E = -2\eta_1 v_1^2 v_0^2, \]
\[ F = -2\eta_2 v_2^2 v_0^2, \]
\[ G = 2(1 + 2\eta_1)\gamma_1 v_1^3 v_0 v_2 + (4\eta_1\eta_2 - 1)v_1^2 v_2^2 v_0^2 - (1 + 4\eta_1 + 4\eta_1^2)\gamma_2 v_1^4 v_0^2. \]

This sequence of coefficients may be reduced for simpler media, such as VTI and isotropic. For the VTI case, \( v_1 = v_2, \eta_1 = \eta_2, \) and \( \gamma = 1, \) therefore, \( A = B, E = F, \)
\( D = G = 0, \) and equation (3.7) becomes the VTI dispersion relation as shown in equation (2.11). For isotropic media, \( v_0 = v_1 = v_2, \eta_1 = \eta_2 = 0, \) and \( \gamma = 1, \) which case gives, \( D = E = F = G = 0, \) therefore, this simplification leads to the isotropic dispersion relation shown in equation (2.12).

Now, in order to approximate the derivatives of the first order nonlinear partial differential equation shown in expression (3.5) or equivalently in equation (3.7), a finite difference method is used, based on a first order scheme. Thus,

\[ A \left( \frac{\tau_{i,j,k} - \tau_{i-1,j,k}}{\Delta x} \right)^2 + B \left( \frac{\tau_{i,j,k} - \tau_{i,j-1,k}}{\Delta y} \right)^2 + C \left( \frac{\tau_{i,j,k} - \tau_{i,j,k-1}}{\Delta z} \right)^2 \]
\[ + D \left( \frac{\tau_{i,j,k} - \tau_{i,j-1,k}}{\Delta x} \right)^2 \left( \frac{\tau_{i,j,k} - \tau_{i,j-1,k}}{\Delta y} \right)^2 + E \left( \frac{\tau_{i,j,k} - \tau_{i-1,j,k}}{\Delta x} \right)^2 \left( \frac{\tau_{i,j,k} - \tau_{i,j,k-1}}{\Delta z} \right)^2 \]
\[ + F \left( \frac{\tau_{i,j,k} - \tau_{i-1,j,k}}{\Delta y} \right)^2 \left( \frac{\tau_{i,j,k} - \tau_{i,j,k-1}}{\Delta z} \right)^2 + G \left( \frac{\tau_{i,j,k} - \tau_{i,j-1,k}}{\Delta x} \right)^2 \left( \frac{\tau_{i,j,k} - \tau_{i,j-1,k}}{\Delta y} \right)^2 \left( \frac{\tau_{i,j,k} - \tau_{i,j,k-1}}{\Delta z} \right)^2 = 1. \]

Expanding the corresponding quadratic terms and collecting all the common traveltime solutions \( \tau_{i,j,k} \) with respect to its corresponding exponential value, equation (3.9) may be rewritten in a polynomial form as
\[
\begin{align*}
C \tau_{i,j,k-1}^2 &+ \frac{D \tau_{i,j-1,k}^2}{\Delta x^2} + \frac{F \tau_{i,j,k-1}^2}{\Delta y^2} + A \tau_{i-1,j,k}^2 + \frac{E \tau_{i,j,k-1}^2}{\Delta z^2} \\
 &+ \frac{G \tau_{i,j-1,k}^2}{\Delta x^2 \Delta y^2} + \frac{\tau_{ijk}}{\Delta z^2} - 2 A \tau_{i-1,j,k} - 2 D \tau_{i,j-1,k} - 2 B \tau_{i,j,k} \\
&= \frac{2 \tau_{i,j-1,k} \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} - \frac{2 \tau_{i,j,k} \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} - \frac{2 \tau_{i,j,k} \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} + \frac{2 \tau_{i,j,k} \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} - \frac{2 \tau_{i,j,k} \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} \\
&+ G \tau_{i,j-1,k} \tau_{i,j,k-1} + G \tau_{i,j-1,k} \tau_{i,j,k-1} + G \tau_{i,j-1,k} \tau_{i,j,k-1} + G \tau_{i,j-1,k} \tau_{i,j,k-1} + G \tau_{i,j-1,k} \tau_{i,j,k-1} + G \tau_{i,j-1,k} \tau_{i,j,k-1}.
\end{align*}
\]

This equation represents the first order finite difference scheme of the orthorhombic dispersion relation shown in equation (3.5). The approach shown in equation (3.10) facilitates the separation of medium properties and the finite difference contribution factors, all used to find the required solutions \( \tau_{ijk} \). Taking advantage of the sorting scheme presented in equation (3.10), it is possible to define a sixth order polynomial equation which solutions contain the orthorhombic P-wave first arrival traveltime required for this study. Similar approaches are conducted for TTI media in other previous studies [55,56,57]. Now, equation (3.10) may be rewritten in a polynomial form as

\[
\beta_0 \tau_{ijk}^6 + \beta_5 \tau_{ijk}^5 + \beta_4 \tau_{ijk}^4 + \beta_3 \tau_{ijk}^3 + \beta_2 \tau_{ijk}^2 + \beta_1 \tau_{ijk} + \beta_0 = 0, \quad (3.11)
\]
where the sequence of coefficients $\sum_{\beta = 0}^{6} \beta_\beta$ are represented by

\[
\beta_0 = \frac{C \tau_{i,j,k-1}}{\Delta x^2} + \frac{B \tau_{i,j,k-1}}{\Delta y^2} + \frac{F \tau_{i,j,k-1}}{\Delta y^2 \Delta z^2} + \frac{A \tau_{i-1,j,k}}{\Delta x^2} + \frac{E \tau_{i,j,k-1}}{\Delta x^2 \Delta z^2} - 1,
\]

\[
\beta_1 = -\frac{2C \tau_{i,j,k-1}}{\Delta x^2} - \frac{2B \tau_{i,j,k-1}}{\Delta y^2} - \frac{2F \tau_{i,j,k-1}}{\Delta y^2 \Delta z^2} - \frac{2A \tau_{i-1,j,k}}{\Delta x^2} - \frac{2E \tau_{i,j,k-1}}{\Delta x^2 \Delta z^2},
\]

\[
\beta_2 = \frac{A}{\Delta x^2} + \frac{B}{\Delta y^2} + \frac{C}{\Delta y^2} + \frac{E \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} + \frac{4F \tau_{i,j,k-1}}{\Delta y^2 \Delta z^2} + \frac{4A \tau_{i-1,j,k}}{\Delta x^2 \Delta y^2} + \frac{4E \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} - \frac{4C \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} - \frac{4B \tau_{i,j,k-1}}{\Delta y^2 \Delta z^2} - \frac{4A \tau_{i-1,j,k}}{\Delta x^2 \Delta y^2} - \frac{4E \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2},
\]

\[
\beta_3 = -\frac{2E \tau_{i,j,k-1}}{\Delta x^2 \Delta z^2} - \frac{2F \tau_{i,j,k-1}}{\Delta y^2 \Delta z^2} - \frac{2A \tau_{i-1,j,k}}{\Delta x^2 \Delta y^2} - \frac{2E \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} - \frac{2F \tau_{i,j,k-1}}{\Delta y^2 \Delta z^2} - \frac{2A \tau_{i-1,j,k}}{\Delta x^2 \Delta y^2} - \frac{2E \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2} - \frac{2F \tau_{i,j,k-1}}{\Delta y^2 \Delta z^2} - \frac{2A \tau_{i-1,j,k}}{\Delta x^2 \Delta y^2} - \frac{2E \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2}.
\]

\[
\beta_4 = -\frac{2G \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2 \Delta z^2} - \frac{2G \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2 \Delta z^2} - \frac{2G \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2 \Delta z^2} - \frac{2G \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2 \Delta z^2} - \frac{2G \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2 \Delta z^2} - \frac{2G \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2 \Delta z^2} - \frac{2G \tau_{i,j,k-1}}{\Delta x^2 \Delta y^2 \Delta z^2}.
\]

The set of coefficients $\beta_i$, shown in equation (3.12), represents the combined contribution of physical properties related to the orthorhombic symmetry, as well as initial traveltime conditions and grid spacing values. All these parameters are used in the finite difference algorithm for the fast marching method, to compute traveltimes by solving for the polynomial roots in equation (3.11). As shown before, the sequence of
parameters, $A$, $B$, $C$, $D$, $E$, $F$, and $G$, represents the description of medium properties that are defined in an orthorhombic symmetry. Note that the first three parameters $A$, $B$, and $C$, are function of a second-degree velocity field which is isolatedly connected to the isotropic solution case. A fourth-degree contribution is given by the additional values of $D$, $E$, and $F$, representing a VTI solution where NMO P-wave velocity $v_{nmo}$ and $\eta$ are considered azimuthally symmetric. A last sixth-degree contribution is defined by the parameter $G$, where the azimuth-dependence for the total velocity field can be defined as the major contribution of this symmetry, used for orthorhombic media. As shown in the next section, this proportionality is associated with the degree of symmetry of the medium. Isotropic symmetries incorporate intrinsically a function of quadratic velocity. VTI and TTI cases involve functions of velocities in a fourth-degree fashion.

3.1.1 Bairstow’s Method

In order to find the corresponding traveltime solutions for the orthorhombic symmetry, it is required to solve for the roots of the polynomial equation

$$
\beta_6 \tau_{ijk}^6 + \beta_5 \tau_{ijk}^5 + \beta_4 \tau_{ijk}^4 + \beta_3 \tau_{ijk}^3 + \beta_2 \tau_{ijk}^2 + \beta_1 \tau_{ijk} + \beta_0 = 0,
$$

which is represented here in a canonical form. In order to calculate the kinematic response of wavefields propagating in orthorhombic media, a root-finding algorithm based on a numerical method is implemented to compute the required traveltime solutions $\tau_{i,j,k}$. Continuous iterations lead to a series of numbers that may converge to an optimum solution or limit that represents the corresponding polynomial root, giving the P-wave first arrival traveltime solution. On the other hand, a sequence of analytical equations, such as the well known quadratic formula, can be used to find the roots of polynomials with a degree higher than two. However, according to Abel-Ruffini theorem [58], polynomial equations of degree-five or higher than four cannot be solved by a general algebraic technique. Therefore, a numerical method is required in this case, in order to compute the corresponding traveltime solutions.

The numerical process implemented in this study is the Bairstow’s method [59, 60]. It is based on the well known Newton’s method. It provides a procedure to decompose a polynomial with real coefficients, into a sequence of second quadratic factors. Finding these second order quadratic factors from the original sixth order polynomial allows to determine the corresponding roots (sometimes as complex conjugate pairs), by only solving a quadratic formula. Bairstow’s approach uses Newton’s method to adapt the coefficients $u$ and $v$ in the equation $\tau^2 + u\tau + v$, until the roots of the quadratic polynomial are also the roots of the initial polynomial. After finding optimum adjustment for $u$ and $v$, the first two roots are found. Then, the polynomial being solved is divided by the corresponding quadratic equation to eliminate the respective roots. This process is
iteratively applied until all the roots are calculated. Therefore, for a given polynomial, such as the one presented in equation (3.13),
\[ P(\tau) = \sum_{p=0}^{n} \beta_p \tau^p = \beta_6 \tau_6^6 + \beta_5 \tau_5^5 + \beta_4 \tau_4^4 + \beta_3 \tau_3^3 + \beta_2 \tau_2^2 + \beta_1 \tau_1^1 + \beta_0 = 0. \]

Applying long division to \( P(\tau) \) by the quadratic equation \( \tau^2 + u\tau + v \), leads to the quotient \( Q(\tau) = \sum_{p=-2}^{n-2} b_p \tau^p \) with a first remainder \( c\tau + d \). Thus,
\[ P(\tau) = (\tau^2 + u\tau + v) \left( \sum_{p=0}^{n-2} b_p \tau^p \right) + (c\tau + d). \]

Then, a second long division is applied to the quotient \( Q(\tau) \) by the same quadratic equation \( \tau^2 + u\tau + v \), giving as a result a new quotient \( R(\tau) = \sum_{p=-4}^{n-4} f_p \tau^p \) and remainder \( g\tau + h \). Therefore,
\[ Q(\tau) = (\tau^2 + u\tau + v) \left( \sum_{p=0}^{n-4} f_p \tau^p \right) (g\tau + h). \]

The adjustment-iteration process is stopped when the quadratic equation \( \tau^2 + u\tau + v \) evenly divides the polynomial \( P(\tau) \), therefore \( c(u, v) = d(u, v) < \varepsilon \). The parameter \( \varepsilon \) is considered to be a very small quantity for the iterative estimation used in the algorithm.

The presented set of variables \( c, d, h, g, b_p, f_p \) are all function of the coefficients \( u \) and \( v \). Their values can be estimated through the following recursive relations,
\[
\begin{align*}
    b_n &= b_{n-1} = 0, \\
    c &= \beta_1 - ub_0 - vb_1, \\
    f_n &= f_{n-1} = 0, \\
    g &= b_1 - uf_0 - vf_1.
\end{align*}
\]

The sequence of values for \( u \) and \( v \) for which the polynomial is evenly divided are given by the Newton’s iteration method, where
\[
\begin{pmatrix}
    u \\
    v
\end{pmatrix}
= \begin{pmatrix}
    \frac{\partial c}{\partial u} & \frac{\partial c}{\partial v} \\
    \frac{\partial d}{\partial u} & \frac{\partial d}{\partial v}
\end{pmatrix}^{-1}
\begin{pmatrix}
    u \\
    v
\end{pmatrix}
= \begin{pmatrix}
    -h & g \\
    -gv & gu - h
\end{pmatrix}
\begin{pmatrix}
    c \\
    d
\end{pmatrix}.
\]

Depending on the type of symmetry (isotropic, VTI or orthorhombic), a set of coefficients in the series of terms \( \beta_p \) are reduced to zero. For instance, as mentioned before, isotropic symmetries imply \( v_p = v_1 = v_2 \), \( \eta_1 = \eta_2 = 0 \), and \( \gamma = 1 \), where \( A = B = C \) and \( D = E = F = G = 0 \). Thus, from the given definition of the sequence \( \sum_{p=0}^{6} \beta_p \) shown in equation (3.12), \( \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0 \). This simplification reduces the initial polynomial equation to the simplest case in the algorithm, where the numerical roots are not required to be found. It only requires solving for the
quadratic equation $\beta_2 \tau^2 + \beta_1 \tau + \beta_0 = 0$, where the solutions represent the first arrival traveltime for the outgoing and incoming P-wave.

For VTI symmetries, $v_1 = v_2$, $\eta_1 = \eta_2$, and $\gamma = 1$. This type of anisotropy reflects a complete azimuthal symmetry in terms of medium properties with respect to the vertical axis. Therefore, for the series of coefficients shown in equation (3.8), $D = G = 0$, $A = B$, and $E = F$, therefore, $\beta_5 = \beta_6 = 0$. These results lead to a fourth order polynomial equation represented by $\beta_4 \tau^4 + \beta_3 \tau^3 + \beta_2 \tau^2 + \beta_1 \tau + \beta_0 = 0$. Unlike the isotropic symmetry, traveltime computation in VTI media requires solving for a fourth order polynomial equation [61]. This particular case can be solved through the implementation of an algebraic technique for solving general quartic equations, such as Ferrari’s method [62]. However, in order to follow the same scheme presented in this study, it can be also solved by using the numerical method implemented in this work.

The last and most relevant case is found for the actual orthorhombic symmetry. Since all the terms in equation (3.8) are independent and different than zero, solving for a traveltime solution requires in this case finding the roots of a sixth order polynomial. Since the eikonal equation represents the high frequency asymptotic approximation of the wave equation, the orthorhombic acoustic wave equation has also a sixth term implicit in it [11]. The following general sequence was used to find the sixth and fifth order roots of the polynomial equation shown in (3.13).

```plaintext
if(a6!=0 && a5!=0){
    c = 1; d = 1; u0 = a5/a6; v0 = a4/a6;
    b6 = 0; b5 = 0; f6 = 0; f5 = 0;
    while (fabs(c)>0.0001 && fabs(d)>0.0001){
        .
        .
        .
        << Bairstow’s Method >>
        .
    }
    else{
        b4=a4; b3=a3; b2=a2; b1=a1; b0=a0;
    }
if ( a6 != 0 && a5 != 0){
    tf5 = (-u0+csqrt(u0*u0-4*v0))/(2.0);
    tf6 = (-u0-csqrt(u0*u0-4*v0))/(2.0);
} else{
    tf5=0.; tf6=0.;
}
```
Variables $a_6$ and $a_5$ represent the coefficients $\beta_6$ and $\beta_5$ in the algorithm respectively. Once the magnitudes of the parameters $c$ and $d$ have been adjusted to zero (very small value referred as $\varepsilon$), the first two solutions of the sixth order polynomial equation are computed analytically in $t_{f6}$ and $t_{f5}$. An equivalent procedure is then applied to obtain the remained roots $t_{f4}$, $t_{f3}$, $t_{f2}$, and $t_{f1}$. Once the set of solutions are found, only one traveltime value is chosen to represent the first arrival traveltime solution of the corresponding outgoing P-wave field. More details of the algorithm are shown in Appendix A.

3.2 Numerical Results

The sequence of numerical results presented in this section shows the kinematic effects of an orthorhombic symmetry in which a wavefield propagates. This set of examples does not only emphasize the results on the difference between traveltime computed, under VTI and isotropic symmetries, but more importantly, it highlights the accuracy of the orthorhombic traveltimes compared with the full wave solution given by the orthorhombic wavefield extrapolations [19].

Figure 3.1 shows the Ricker wavelet source function used for the corresponding series of wavefield extrapolations. In order to identify more clearly the actual overlay between first arrival traveltime solutions and the wavefields, instead of overlaying the first arrival traveltime solution and the corresponding fastest arrival for the wavefield, the specific traveltime contours have been shifted (100 ms in this case, based on the time shift given by the source wavelet shown in Figure 3.1) to overlay the maximum positive amplitude of the wavefield. This convention is usually implemented when traveltimes solutions are produced to be overlaid with the corresponding wavefield results as shown by Xiaolei Song [19].

Some of the figures presented in this section are used to compare the VTI and the isotropic solutions (given by the algorithms available in Madagascar) with the respective solutions (VTI and isotropic) obtained with the orthorhombic eikonal solver introduced in this work. Velocity models, anisotropic parameter models, traveltime fields, traveltime contours, and traveltime solutions overlaying wavefields, are the main sequences of plots presented in this section. All models are generated with a specific number of samples, $n_x = n_y = n_z = 200$, and grid spacing, $dx = dy = dz = 0.025$ km. Source location is defined at $s_x = 2.5$ km, $s_y = 2.5$ km, $s_z = 2.5$ km.

3.2.1 Isotropic Traveltime Fields

A first test for the orthorhombic eikonal solver can be performed by comparing its isotropic solution (isotropic solution from orthorhombic eikonal solver) with the actual isotropic eikonal solution given by the isotropic eikonal solver available in Madagascar.
Figure 3.1: Source Ricker function of 20 Hz used for wavefield extrapolation.

Figure 3.2(a) shows the isotropic traveltime field solution obtained from the orthorhombic eikonal solver, where \( v_v = v_1 = v_2 = 1.8 \text{ km/s}, \eta_1 = \eta_2 = 0, \text{ and } \gamma = 1. \) Figure 3.2(b) shows the corresponding traveltime contours of the same traveltime field. Figure 3.2(c) represents the computed difference between the corresponding isotropic traveltime solutions given by the orthorhombic eikonal solver and the actual isotropic eikonal solver available in Madagascar. In order to compare the traveltime solution with the actual wave equation solution, a lowrank algorithm is used to generate wavefields for the corresponding symmetries [16]. Figure 3.2(d) shows a wavefield estimated for an isotropic medium with a velocity field of magnitude \( v = 1.8 \text{ km/s}. \) The corresponding eikonal solution is overlaying the wavefield at the specific time. As mentioned earlier, for better representation of the results, the first arrival solution given by the eikonal solver is overlaying the maximum wavefield amplitude value instead of the initial very weak first-arrival shown for the wavefield.

### 3.2.2 VTI Traveltime Fields

Only three parameters (acoustic case) are required in this symmetry to calculate traveltimes [50]. Two velocity fields; vertical velocity \( v_v, \) NMO P-wave velocity \( v_{nmo}, \) and an azimuthally symmetric anellepticity value \( \eta. \) Therefore, considering the VTI case from the orthorhombic eikonal solver, \( v_v = v_1 = v_2, v_{nmo} = v_1 = v_2, \eta = \eta_1 = \eta_2 \text{ and } \gamma = 1. \) Figure 3.3(a) shows the traveltime field estimated in a VTI medium with \( v_v = 2.0 \text{ km/s}, v_{nmo} = 2.2 \text{ km/s} \) and \( \eta = 0.1. \) Figure 3.3(b) shows the corresponding VTI eikonal solution overlaying the VTI wavefield generated with the Ricker source wavelet shown in Figure 3.1.

On the other hand, an interesting difference is found between the VTI solution given by the orthorhombic eikonal solver and the VTI solution obtained with the actual VTI eikonal solver available in Madagascar. An apparent divergence can be identified when high values of \( \eta \) are used in the VTI eikonal solver. As shown in Figure 3.4(a), travel-
time solutions along the diagonal component (Depth-Inline or Depth-Crossline domain) seems to diverge from the wavefield solution when the magnitude of $\eta$ increases. Figure 3.4(b) represents the VTI solution using the orthorhombic eikonal solver presented in this work. The VTI solution given by the orthorhombic eikonal solver seems to represent more accurately the kinematic aspect of the wavefield. Figures 3.4(c) and 3.4(d) show how the difference between the two VTI solutions increases when the value of $\eta$ is incremented. In order to be consistent with the given results, as required by the VTI eikonal solver, $v_{nmo}$ velocity fields were transformed into horizontal velocity models such that $v_{hor} = v_{nmo}\sqrt{1 + 2\eta}$ \cite{50}. The difference between the two solutions is the result of assuming weak anisotropy in the VTI eikonal solver.

### 3.2.3 Orthorhombic Traveltine Fields

The series of examples shown in this section represents the main and most important contribution made for this research. Since no other eikonal solver for orthorhombic media is available in an open-source software package such as Madagascar, the results of the algorithm introduced in this study can be considered a novel contribution to the field.

As mentioned previously, six different input values or fields are required in this case to compute the first arrival traveltine response of P-waves propagating in the acoustic orthorhombic media. Figure 3.5 shows the eikonal traveltine solution and the corresponding wavefield response for an orthorhombic media where $v_p = 2.0$ km/s, $v_1 = 2.2$ km/s, $v_2 = 2.6$ km/s, $\eta_1 = 0.1$, $\eta_2 = 0.25$, and $\gamma = 1.2$. More specifically, Figure 3.5(a) represents the corresponding orthorhombic traveltine field computed with the orthorhombic eikonal solver introduced in this work. Figure 3.5(b) shows some traveltine contours from the estimated orthorhombic traveltine field represented in Figure 3.5(a). Figure 3.5(c) shows the corresponding orthorhombic eikonal solution overlaying the isotropic (computed with $v = 2.0$ km/s) and VTI ($v_p = 2.0$ km/s, $v_{nmo} = 2.2$ km/s, and $\eta = 0.1$) solutions. Last plot shown in Figure 3.5(d) shows how the orthorhombic eikonal solution accurately overlays the orthorhombic wavefield at the corresponding time. More information of the orthorhombic wavefield extrapolation methodology used in this work may be found in \textit{Modeling of pseudo-acoustic P-waves in orthorhombic media with lowrank approximation} by Song and Alkhalifah \cite{19}. The following two examples shown in Figures 3.6 and 3.7 represent an equivalent sequence, in which different values for $v_p$, $v_1$, $v_2$, $\eta_1$, $\eta_2$, $\gamma$, are implemented.

The first three examples represent traveltine computation in homogeneous orthorhombic media. Another important aspect to assess in the orthorhombic eikonal solver is the implementation of inhomogeneous models for the corresponding velocity and anisotropic parameter fields. As a first example, a series of inhomogeneous models were generated with the following 3D function prototype $\varsigma = a + bx_1 + c(x_2 - 1)(x_2 - 1) +$
\[ d(x_3 - 1)(x_3 - 1), \] where \( x_1, x_2, \) and \( x_3 \) represent the respective Cartesian components, and \( a, b, c, \) and \( d, \) are scalar coefficients differently chosen for each model. Figure 3.8 shows the sequence of models \( v_v, v_1, v_2, \eta_1, \eta_2, \) and \( \gamma, \) generated with the function \( \zeta. \) Figure 3.9 shows an equivalent sequence of results for the orthorhombic traveltime estimations, as shown in Figures 3.5, 3.6, and 3.7 but in this particular case, the results are based on the inhomogeneous input models shown in Figure 3.8.

Figures 3.10 and 3.11 represent an additional example of orthorhombic first arrival traveltime estimation in inhomogeneous models. The models were generate with a function-type of the form \( \varsigma = a + b \sqrt{x_1 x_2 x_3}. \)

Figure 3.12 shows a sequence of models in which a high velocity and anisotropic parameter contrast is located at 2 km depth. This model allows to demonstrate the versatility of the algorithm when high-contrast velocity changes are included. The corresponding traveltime field computed in this model is shown in Figure 3.13. Additionally, as a second example that considers reflections, Figure 3.14 shows a series of input models where two main high contrasts are located. The corresponding first arrival traveltime fields are shown in Figure 3.15.

### 3.3 Discussion

Most geological anisotropic settings are represented by a continuous velocity increment from the vertical domain (perpendicular to the surface) towards the horizontal component [63]. Therefore, for most cases, the magnitude of the P-wave horizontal velocity field \( v_{\text{hor}}, \) is higher than the NMO P-wave velocities \( v_{\text{nmo}}. \) Moreover, under the same geological model (major characteristic of polar anisotropic systems), the magnitudes of these NMO velocities are usually higher than the vertical velocity fields. This pattern of change on the velocity fields can establish the physical reason why the \( \eta \) values are most likely to be represented by positive quantities [64, 65]. This is supported by the definition of the anellepticity value \( \eta \) given by

\[
\eta = \frac{\epsilon - \delta}{1 + 2\delta} = \frac{1}{2} \left( \frac{v_{\text{hor}}^2}{v_{\text{nmo}}^2} - 1 \right),
\]

where \( \epsilon \) and \( \delta \) correspond to the well known Thomsen’s parameters [45]. As shown in all previous examples, most velocity fields are represented by this particular tendency, where horizontal velocities are higher than NMO velocities, as well as, NMO velocities are higher than vertical velocities. This characteristic pattern is being found in most geological common settings [64, 65].

Now, with regard to the models presented in this section, the first example shown in Figure 3.5 represents a medium in which an angular increase in velocity (from vertical to horizontal component) governs the kinematic aspect of the wavefield. Depth-inline planes contain the contribution from \( v_2 \) and \( \eta_2 \) fields. On the other hand, since \( v_1 \) and
\( \eta_1 \) are represented by a lower magnitude, then, a lower contribution of the anisotropic aspect of the medium is observed in the depth-crossline domain (less curvature with respect to the horizontal component domain). Figure 3.5(c) shows different kinematic responses given by the isotropic and VTI cases. As a general rule for this plot and the subsequent ones (plots overlaying orthorhombic, isotropic and VTI solutions), isotropic solutions are always computed with the vertical velocity field \( v_v \), used for the corresponding orthorhombic solution. Also, for the given examples, VTI solutions are always computed with the same vertical velocity \( v_v \) used for the orthorhombic case, assigning the first NMO velocity \( v_1 \) (from orthorhombic case) as the unique NMO velocity in VTI media, and allocating the first \( \eta_1 \) (from orthorhombic case) to the unique anellepticity parameter \( \eta \) implemented for the corresponding VTI solution. Thus, based on this condition, since the vertical velocity components are equivalent for the three cases, all vertical first arrival traveltimes should match exactly at the same location (same fastest arrival), as shown in Figures 3.5, 3.6, 3.7, 3.9, 3.11, 3.13, 3.15. Also, since the VTI solution takes as input the values \( v_1 \) and \( \eta_1 \), contained in the depth-crossline domain, the corresponding VTI solution should overlay exactly the orthorhombic first arrival traveltime. This can be observed in the same sequence of figures mentioned previously.

Figure 3.6 shows the traveltime effects after using very high contrasts between the anellepticity values defined for each plane of mirror symmetry (large difference between \( \eta_1 \) and \( \eta_2 \)). Figure 3.6(c) shows that for a very small value of \( \eta_1 \), orthorhombic and VTI solutions are closer to the obtained isotropic solution. This apparent small difference in traveltime has a maximum value on the horizontal component axis. It is represented by the difference in velocity between the given vertical component \( v_v \), and the inherent horizontal velocity field represented by \( v_{\text{hor}} = v_{\text{nmo}} \sqrt{1 + 2\eta} \). Since \( v_v = 1.8 \text{ km/s} \) and \( v_{\text{hor}} = 1.9\sqrt{1 + 2(0.03)} = 1.96 \text{ km/s} \), the difference between \( v_v \) and \( v_{\text{hor}} \) should represent the delay of the isotropic traveltime field. As shown in Figure 3.7 once the magnitude of \( \eta_1 \) increases from \( \eta_1 = 0.03 \) in Figure 3.6 to \( \eta_1 = 0.33 \) in Figure 3.7, a more remarkable difference is found between the isotropic and the orthorhombic or VTI solutions, defined in the depth-crossline domain. In general terms, the same concepts and analysis may be applied to the inhomogeneous, two-layers, and three-layers examples shown in Figures 3.9, 3.11, 3.13 and 3.15 respectively.
Isotropic traveltime field solution computed with orthorhombic eikonal solver.

Traveltime contours taken from isotropic traveltime field shown in Figure 3.2(a).

Traveltime difference between isotropic eikonal solution of orthorhombic eikonal solver. Note that the two eikonal solvers give exactly the same solution.

Isotropic traveltime solution (solid yellow curve) overlaying isotropic wavefield at 0.99 s.

Figure 3.2: Isotropic traveltime solution computed with orthorhombic eikonal solver with a unique velocity field defined by the magnitude $v = 1.8$ km/s.
Figure 3.3: Traveltime from a VTI medium with $v_v = 2.0 \, \text{km/s}$, $v_{nmo} = 2.2 \, \text{km/s}$ and $\eta = 0.1$. Note that for a VTI solution using the orthorhombic eikonal solver, it is required to equalize the fields $v_1 = v_2$, $\eta_1 = \eta_2$, and $\gamma = 1$. This reduction inherently represents the azimuthal symmetric aspect of these type of media, represented by VTI symmetries.
(a) Traveltime VTI solution (yellow curve) overlaying VTI wavefield at 0.87 s. VTI traveltime represented with the yellow curve is calculated with VTI eikonal solver available in Madagascar, where \( v_v = 2.0 \) km/s, \( v_{nmo} = 2.2 \) km/s, and \( \eta = 0.4 \).

(b) Traveltime VTI solution (yellow curve) overlaying VTI wavefield at 0.87 s. VTI traveltime represented with the yellow curve is calculated with the orthorhombic eikonal solver presented in this study, where \( v_v = 2.0 \) km/s, \( v_{nmo} = v_1 = v_2 = 2.2 \) km/s, \( \eta_1 = \eta_2 = 0.4 \), and \( \gamma = 1 \).

(c) Traveltime difference between VTI eikonal solver and VTI eikonal solution from orthorhombic eikonal solver. In this example, \( v_v = 2.0 \) km/s, in eikonal solver. In this example, \( v_v = 2.0 \) km/s, \( v_{nmo} = 2.2 \) km/s and \( \eta = 0.05 \).

(d) Traveltime difference between VTI eikonal solver and VTI eikonal solution from orthorhombic eikonal solver. In this example, \( v_v = 2.0 \) km/s, \( v_{nmo} = 2.2 \) km/s and \( \eta = 0.3 \).

Figure 3.4: Traveltime from VTI medium with \( v_v = 2.0 \) km/s, \( v_{nmo} = 2.2 \) km/s and \( \eta = 0.4 \) computed with VTI eikonal solver (a) and orthorhombic eikonal solver (b). Traveltime difference between VTI solutions from the orthorhombic eikonal solver and the actual VTI eikonal solution (VTI eikonal solver algorithm available in Madagascar) with the same velocity fields \( v_v = 2.0 \) km/s, \( v_{nmo} = 2.2 \) km/s and different anellipticity values, \( \eta = 0.05 \) (c) and \( \eta = 0.3 \) (d). Note the progressive increase of the difference between the two solutions for higher \( \eta \) values.
(a) Orthorhombic traveltime field where $v_0 = 2.0$ km/s, $v_1 = 2.2$ km/s, $v_2 = 2.6$ km/s, $\eta_1 = 0.1$, and $\eta_2 = 0.25$.

(b) Orthorhombic traveltime contours taken from traveltime field shown in Figure 3.5(a).

(c) Orthorhombic traveltime solution overlaying isotropic and VTI solutions. For isotropic case, vertical velocity from orthorhombic model is taken as the unique velocity field $v = 2.0$ km/s. For VTI case, vertical velocity is set to be the same vertical velocity from orthorhombic model, $v_0 = 2.0$ km/s. The NMO and $\eta$ parameter are set to be the first two values $v_1$ and $\eta_1$ from the orthorhombic model.

(d) Orthorhombic traveltime overlaying orthorhombic wavefield at 0.71 seconds. A very accurate match can be observed between the traveltime found with orthorhombic eikonal solver presented in this study and the corresponding orthorhombic wavefield extrapolation. The first arrival traveltime represented by the solid yellow curve overlays the maximum amplitude of the wavefield.

Figure 3.5: Traveltime in orthorhombic media with $v_0 = 2.0$ km/s, $v_1 = 2.2$ km/s, $v_2 = 2.6$ km/s, $\eta_1 = 0.1$, $\eta_2 = 0.25$, and $\gamma = 1.2$. In Figure [3.5(c)], the solid yellow curve represents the orthorhombic solution, the dashed red curve corresponds to the VTI solution and the dotted purple line represents the isotropic solution. The following examples are also represented under the same sequence of curves and colors.
(a) Orthorhombic traveltime field where \( v_v = 1.8 \) km/s, \( v_1 = 1.9 \) km/s, \( v_2 = 2.0 \) km/s, \( \eta_1 = 0.03 \), \( \eta_2 = 0.41 \), and \( \gamma = 1.1 \).

(b) Orthorhombic traveltime contours computed from traveltime field shown in Figure 3.6(a).

(c) Orthorhombic traveltime solution overlaying isotropic and VTI solutions. For isotropic case, \( v_v = 1.8 \) km/s. For VTI case, \( v_v = 1.8 \) km/s, \( v_{nmo} = 1.9 \) km/s, and \( \eta = 0.03 \). An equivalent explanation regarding the order and magnitude of the chosen values for the VTI and isotropic cases can be found in Figure 3.5(c).

(d) Orthorhombic traveltime (yellow curve) overlaying orthorhombic wavefield at 0.80 seconds. An accurate matching may be observed between the traveltime solution and the orthorhombic wavefield.

Figure 3.6: Traveltime in orthorhombic media with \( v_v = 1.8 \) km/s, \( v_1 = 1.9 \) km/s, \( v_2 = 2.0 \) km/s, \( \eta_1 = 0.03 \), \( \eta_2 = 0.41 \), and \( \gamma = 1.1 \). This example shows a significant major difference between the given solution on the vertical planes of mirror symmetry. Since the horizontal velocity depends on \( \eta \) such that \( v_{hor} = v_{nmo}\sqrt{1+2\eta} \), having a high contrast between the two anellepticitics \( \eta_1 \) and \( \eta_2 \) produces a more remarkable difference on the corresponding first arrival traveltime solutions.
(a) Orthorhombic traveltime field $v_v = 1.8$ km/s, $v_1 = 1.9$ km/s, $v_2 = 2.0$ km/s, $\eta_1 = 0.33$, $\eta_2 = 0.41$, and $\gamma = 1.1$.

(b) Orthorhombic traveltime contours taken from traveltime field shown in Figure 3.7(a).

(c) Orthorhombic traveltime solution overlaying isotropic and VTI solutions. For isotropic case, $v_v = 1.8$ km/s. For VTI case, $v_v = 1.8$ km/s, $v_{nmo} = 1.9$, and $\eta = 0.33$.

(d) Orthorhombic traveltime (yellow curve) overlaying orthorhombic wavefield at 0.80 seconds.

Figure 3.7: Traveltime in orthorhombic media with $v_v = 1.8$ km/s, $v_1 = 1.9$ km/s, $v_2 = 2.0$ km/s, $\eta_1 = 0.33$, $\eta_2 = 0.41$, and $\gamma = 1.1$. Different than the example shown in Figure 3.6, this model represents a compensation for the anellepticity value $\eta_1$. Thus, increasing the anellepticity from $\eta_1 = 0.03$ to $\eta_1 = 0.33$ reflects an increase of the horizontal velocity component, as mentioned in Figure 3.6.
Figure 3.8: Inhomogeneous velocity and anisotropic parameter models used in the orthorhombic eikonal solver to compute traveltime solutions shown in Figure 3.9. Vertical velocity \( v_v \) (a), NMO velocities \( v_1 \) (b), \( v_2 \) (c), anellepticity parameters \( \eta_1 \) (d), \( \eta_2 \) (e), and factor \( \gamma \) (f). This sequence of models attempts to represent a geological setting with a parabolic velocity anomaly with a symmetry axis located at 1 km on the inline and crossline domain.
(a) Orthorhombic traveltime field computed with the set of models shown in Figure 3.8.

(b) Orthorhombic traveltime contours taken from traveltime field shown in Figure 3.9(a).

(c) Orthorhombic traveltime solution overlaying isotropic and VTI solutions. Isotropic solution is computed with $v_v$ shown in Figure 3.8(a). For VTI case, $v_v$, $v_{omo}$ and $\eta$ are shown in Figures 3.8(a), 3.8(b) and 3.8(d).

(d) Orthorhombic traveltime (yellow curve) overlaying orthorhombic wavefield at 0.74 seconds. An accurate match is also found for traveltime for inhomogeneous orthorhombic media.

Figure 3.9: Traveltime in orthorhombic media computed with the inhomogeneous velocity and anisotropic parameter models shown in Figure 3.8.
Figure 3.10: Inhomogeneous velocity and anisotropic parameter models used in the orthorhombic eikonal solver to compute traveltime solutions shown in Figure 3.11. Vertical velocity $v_v$ (a), NMO velocities $v_1$ (b), $v_2$ (c), anellepticity parameters $\eta_1$ (d), $\eta_2$ (e), and factor $\gamma$ (f). In this example, a nonlinear increase is used to represent a progressive high change on the values of the models. The kinematic response inherent in the first arrival traveltime solution is shown in Figure 3.11.
Orthorhombic traveltime field computed from orthorhombic model shown in Figure 3.10.

Orthorhombic traveltime contours taken from traveltime field shown in Figure 3.11(a).

Orthorhombic traveltime solution overlaying isotropic and VTI solutions. Isotropic solution is computed with $v_n$ shown in Figure 3.10(a). For VTI case, $v_e$, $v_{nma}$ and $\eta$ are shown in Figures 3.10(a), 3.10(b) and 3.10(d).

Orthorhombic traveltime (yellow curve) overlaying orthorhombic wavefield at 0.80 seconds.

Figure 3.11: Traveltime in orthorhombic media computed with the inhomogeneous velocity and anisotropic parameter models shown in Figure 3.10. Based on previous results, the orthorhombic eikonal solver presented in this study shows an accurate behavior compared to the wavefield solution given by the lowrank wavefield extrapolation process. The introduced eikonal solver demonstrates its versatility when inhomogeneous models are used as shown in this and previous example. The next two models represent the response of the algorithm with fields describing high velocity contrasts used to generate reflections.
Figure 3.12: Velocity and anisotropic parameter models with high contrast values located at 2 km depth, used in the orthorhombic eikonal solver to compute the traveltime solutions shown in Figure 3.13. Vertical velocity $v_v$ (a), NMO velocities $v_1$ (b), $v_2$ (c), anellepticity parameters $\eta_1$ (d), $\eta_2$ (e), and factor $\gamma$ (f). Smooth gradients are also included in all models.
(a) Orthorhombic traveltime field computed from orthorhombic model shown in Figure 3.12. A clear indication of the high velocity contrast is represented by the curvature changes shown in this traveltime field.

(b) Orthorhombic traveltime contours taken from the traveltime field shown in Figure 3.13(a). A clearer representation of the change of curvature for traveltime values is shown in this plot at 2 km depth.

(c) Orthorhombic traveltime contours overlaying isotropic and VTI solutions. Isotropic solution is computed with $v_w$ shown in Figure 3.12(a). For VTI case, $v_w$, $v_{nmo}$ and $\eta$ are shown in Figures 3.12(a), 3.12(b) and 3.12(d).

(d) Orthorhombic traveltime overlaying orthorhombic wavefield at 0.7 seconds. Since only first arrival events are computed during the traveltime calculation, only wavefield components representing first arrival perturbations can be matched with the corresponding eikonal traveltime solution indicated with the solid yellow curve.

Figure 3.13: Traveltime in orthorhombic media computed with the inhomogeneous velocity and anisotropic parameter models shown in Figure 3.12. Information regarding reflected or late events are now part of the wavefield extrapolation.
(a) Vertical velocity $v_v$.
(b) NMO velocity $v_1$.
(c) NMO velocity $v_2$.
(d) Anellepticity parameter $\eta_1$.
(e) Anellepticity parameter $\eta_2$.
(f) Factor $\gamma$.

Figure 3.14: Velocity and anisotropic parameter models used in the orthorhombic eikonal solver to compute traveltime solutions shown in Figure 3.15. Two high velocity and $\eta - \gamma$ contrasts are included in this example. Vertical velocity $v_v$ (a), NMO velocities $v_1$ (b) and $v_2$ (c), anellepticity parameters $\eta_1$ (d), $\eta_2$ (e), and factor $\gamma$ (f).
(a) Orthorhombic traveltime field computed from Figure 3.14. Location of high velocity contrasts can be identified in the traveltime field due to changes in curvature.

(b) Orthorhombic traveltime contours taken from traveltime field shown in Figure 3.15(a). This plot clearly indicates the location of high velocity changes at 2 and 3 km depth. Note the particular form of the traveltime curvature at 3 km depth, that indicates the presence of head waves in the traveltime computation.

(c) Orthorhombic traveltime solution overlaying isotropic and VTI solutions. Isotropic solution is computed with $v_v$ shown in Figure 3.14(a). For VTI case, $v_v$, $v_nmo$ and $\eta$ are shown in Figures 3.14(a), 3.14(b) and 3.14(d).

(d) Orthorhombic traveltime overlaying orthorhombic wavefield at 0.75 seconds. As shown in Figure 3.13(d), matching between traveltime solutions and wavefields cannot be applied for reflected or late arrivals.

Figure 3.15: Traveltime in orthorhombic media computed with the inhomogeneous velocity and anisotropic parameter models shown in Figure 3.14. A second reflector was included in this example to model the kinematic effect of head waves in the estimated traveltime field as shown in Figure 3.15(b). Possible additional effects for the inversion of effective velocity models are discussed in the next section.
Chapter 4

Effective Orthorhombic Anisotropic Models for Wavefield Extrapolation

Effective velocity models are defined based on the fact that the total kinematic contribution of a set of properties may be represented by a unique model, in this case, a unique velocity field. This resultant model should quantify the total effects of all sources of changes with respect to the kinematic aspect of the wavefield. In order to generate such effective models, only one physical property that is able to collect all the kinematic effects of the medium, can be used to recreate the equivalent kinematic aspect of the wavefield propagation. In this particular study, such physical property is considered to be the corresponding first arrival traveltine. Since traveltine fields represent the main source of characterization for wavefield propagation, they are implemented to reproduce the total kinematic aspect of the medium in which the wave propagates. Reproducing all the kinematic effects in only one model may allow to reduce the complexity of certain procedures such as anisotropic wavefield extrapolation to a much simpler and less expensive isotropic but inhomogeneous computation. Therefore, the main topic presented in this chapter is based on the estimation of effective orthorhombic anisotropic models for isotropic wavefield extrapolation. This effective velocity model approach was originally introduced by Alkhalifah for a TI model [9].

4.1 Effective Velocity Models

Since traveltine estimations inherently represent the kinematic essence of the velocity field in which the wave propagates, it is possible to invert for effective velocity models using the well known isotropic dispersion relation, where

\[
\left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial y} \right)^2 + \left( \frac{\partial \tau}{\partial z} \right)^2 = \frac{1}{v^2},
\]  

(4.1)

If the traveltine \( \tau \) is estimated from an anisotropic modeling sequence, such that a series of initial velocity and anisotropic parameter models were involved during the computation, then, the velocity field inverted in equation (4.1) should represent the
exact kinematic effects of the combination given by the corresponding velocity and anisotropic parameter fields. Therefore, the main purpose of this approach is to implement the orthorhombic first arrival traveltime solutions shown in previous chapter to invert for an effective velocity model $v_{eff}$, that may reproduce the equivalent kinematic effects of the complete sequence of models required by the actual orthorhombic anisotropic media. This new velocity field $v_{eff}$ depends on the spatial location $x, y, z$, but most importantly, it is also source-location dependent. Since all the total kinematics characteristics are inserted in the effective velocity, this approach tends to embed the anisotropic influence in the resultant inhomogeneity representation. Therefore,

$$v_{eff}(x, y, z, s_x, s_y, s_z) = \frac{1}{\sqrt{\left(\frac{\partial \tau_{ort}}{\partial x}\right)^2 + \left(\frac{\partial \tau_{ort}}{\partial y}\right)^2 + \left(\frac{\partial \tau_{ort}}{\partial z}\right)^2}}, \quad (4.2)$$

where $s_x, s_y, s_z$, indicate the coordinates of the source location. Once the orthorhombic traveltimes $\tau$ are estimated, a simple sequence of first order partial derivatives computation is required in order to invert for the effective velocity model shown in equation (4.2). We now closely examine the approach used to estimate the first order spatial derivatives shown in equation (4.2). As a first example of the inversion process, we start from using a homogeneous velocity field, compute the corresponding first arrival traveltimes, and then invert back the original velocity model using equation (4.2). Figure 4.1 shows a sequence of plots where the mentioned inversion process is applied. A second example of the inversion is shown in Figure 4.2 where an inhomogeneous initial velocity model is implemented for the process. Results shown in Figures 4.1(d) and 4.2(d) indicate that the inversion process allows to recover the initial velocity with a high degree of accuracy. Since a singular value is being replaced at the source location for the $v_{eff}$ model, some small errors are found around this region. This singular value appears due to the implementation of a central finite difference scheme used to compute the spatial derivatives shown in equation (4.2). Thus, when computing the finite difference terms at the source location, a non-physical high value needs to be replaced in order to invert back the original velocity field (initial velocity field for isotropic examples). More information regarding the corresponding implementation used to invert $v_{eff}$ is shown in Appendix B.

A last example is shown in Figure 4.3 where an inhomogeneous velocity field with two reflectors is used as an input model for the inversion of $v_{eff}$. Since only first arrival traveltime values are estimated by the eikonal solver, this characteristic or property allows to identify the presence of head waves located at 3 km depth in the model, as shown in Figure 4.3(b). As represented in Figure 4.3(c), the presence of head waves in the traveltime field produces higher velocity values during the inversion of $v_{eff}$. The result of this particular irregularity is shown in Figure 4.3(d).
4.2 Wavefield Extrapolation

The scalar wavefield extrapolation method implemented in this study, uses the exact dispersion relation for acoustic orthorhombic media to model acoustic wavefields [66, 19]. A lowrank approximation approach is used in this case to operate the inherent space-wavenumber mixed domain operator (see Appendix C). As shown by Song and Alkhalifah [66, 19], the acoustic orthorhombic wavefield extrapolator used in this work does not suffer from quasi-P and quasi-SV waves in the wavefield, is accurate, practically free of dispersion and does not require any constraints on Thomsen’s parameters for stability.

Figure 4.5(c) shows an effective velocity field estimated from the set of inhomogeneous \( v, v_1, v_2, \eta_1, \eta_2, \) and \( \gamma \) models shown in Figure 4.4 with a source located at \( x = 2.5 \) km, \( y = 2.5 \) km, and \( z = 2.5 \) km. Figures 4.6(a), 4.6(c), and 4.6(e) show different views of time snapshots of the orthorhombic wavefield modeled with the corresponding effective velocity, shown in Figure 4.5(c). An isotropic wavefield extrapolation based on the lowrank approximation approach [16] is applied to obtain the results shown in Figures 4.6(a), 4.6(c), and 4.6(e). In terms of the kinematic aspects of wavefield propagation, equivalent results are found between the wavefield computed with the isotropic effective velocity approach and the orthorhombic wavefield extrapolation [19], shown in Figures 4.6(b), 4.6(d), and 4.6(f). Note that the wavefield extrapolated with the effective velocity experiences a loss in amplitude around the region of highest velocity variation. Despite the difference in amplitude, first arrival traveltimes, as expected, are found to be equivalent. The set of plots shown in Figure 4.7 represents the difference in amplitude found from the effective and orthorhombic wavefield extrapolation approaches. More importantly, and as shown by Alkkhalifah for the TI case [9], fastest P-wave arrivals computed with the effective wavefield extrapolation method match very accurately the first arrival events produced with the actual orthorhombic wavefield extrapolation. The solid yellow curve superimposed on all the wavefield snapshots shown in Figure 4.6 represents the eikonal traveltime solution at the equivalent time. This traveltime solution is estimated using the orthorhombic eikonal solver proposed in this study. These effective velocities reduce to isotropic velocity fields when anisotropy is zero. Their variation depends on the strength of anisotropy, so the wavefield produced, is a correction of the isotropic full wavefield to hopefully include the anisotropic correct traveltime for at least the first arrival.

A second example is also shown in the set of plots represented in Figures 4.8, 4.9, 4.10, 4.11. This time, a sequence of velocity and anisotropic parameter models indicates the presence of two reflectors, at 2 and 3 km depth respectively. Different important aspects should be taken into consideration for this particular case. As shown in Figure 4.3, the rapid change of velocity at 3 km depth, produces a smooth transition of the traveltime curves, as shown in Figure 4.9(b) [54]. As mentioned earlier, the change in
curvature notably affects the inversion of the effective velocity around the corresponding region. For that reason, a particular pattern is generated in the effective velocity model represented with higher velocity values. However, in terms of the kinematic aspect of wavefield propagation, this anomaly does not affect the arrival traveltime of the corresponding wavefield. This can be seen in Figure 4.11 where the first arrival events given by the effective wavefield extrapolation and the actual orthorhombic wavefield extrapolation are found to be equivalent.

Seismic wave extrapolation using lowrank symbol approximation is the method implemented in this study to generate isotropic and orthorhombic anisotropic wavefields. A general mathematical derivation of the method, description of the lowrank approximation algorithm, and some numerical examples are shown by Fomel and Song [16, 19].

4.3 Discussion

Despite only the first arrival matching with the isotropic wavefield extrapolator, as Alkhalifah showed for the TI case [9], this method can significantly reduce the computational cost of wavefield extrapolation in orthorhombic media. Therefore, as shown for the examples presented in Figure 4.4 and Figure 4.8, the implemented methodology provides accurate and stable results with a much simpler and less expensive technique, used to generate wavefields in acoustic vertical orthorhombic media. Implementation of this alternative method allows to run orthorhombic wavefield extrapolations with a much lesser computational cost. Once the effective velocity models shown in Figures 4.5(c) and 4.9(c) are constructed for the particular source located at \( x = 2.5 \) km, \( y = 2.5 \) km, \( z = 2.5 \) km, they are used to solve the isotropic wave equation. Solving the wave equation in these cases involves wavefield extrapolation in inhomogeneous isotropic media, which implies a much lower cost for the computational process.

The computational approach implemented through the lowrank approximation, used for the wavefields shown in Figures 4.6 and 4.10 involves selecting a reduced set of representative wavenumbers and spatial locations from the full wave extrapolation matrix. Regular mixed-domain operators [67] used for wavefield extrapolation can cost \( \mathcal{O}(N_x^2) \), where \( N_x \) corresponds to the total size of the three dimensional grid. Even for considerable small models, this approach is significantly expensive (Regular mixed-domain operators). On the other hand, the lowrank approximation approach, used in this study, reduces the cost to \( \mathcal{O}(NN_x \log N_x) \), where \( N \) is a small value that physically represents a selected set of representative spatial locations [16]. If this approximation is implemented, the complete process does not require the access of the full matrix, it only accesses a selected group of columns and rows. More detailed information regarding the lowrank approximation is shown in Appendix C.

For a time step of \( \Delta t = 0.001 \) s, the lowrank approximation for the two orthorhombic models shown in Figures 4.4 and 4.8 is \( N = M = 5 \), with a relative error less than
This leads to a total cost of $O((5 + 1)N_x \log N_x)$. The extra term "1" included here, is required by the cost of the additional spatial Fourier transform implied in each iteration. On the other hand, the approximation rank decomposition for the respective wavefield extrapolations, applied to the effective velocity fields shown in Figures 4.5(c) and 4.9(c) is $N = M = 2$, with a relative error less than $1.5^{-5}$. Therefore, the total cost in this case corresponds to $O((2 + 1)N_x \log N_x)$. Based on these results, to relate the theoretical costs between the two approaches (from the two examples), the corresponding wavefield extrapolation for acoustic vertical orthorhombic media is found to be approximately 2 times more expensive than the wavefield extrapolations computed with the effective velocity approach. This value only represents the very approximate (theoretical) relation between the costs of the two approaches. A better estimation of the cost-relation is obtained by comparing the actual computational running-time taken for each process. With respect to the two-reflectors example, shown in Figure 4.8, the effective velocity wavefield extrapolation takes 16 minutes using one-CPU core (Intel(R) Xeon(R) CPU-X5550 of 2.67GHz). On the other hand, the actual orthorhombic case takes 42 minutes (under the same CPU characteristics). Therefore, the actual real relation between the costs is represented by a reduction of 2.65 (times less expensive), when using the effective velocity approach.

Higher relations are found for orthorhombic models where at least one of the velocities or anisotropic parameters are represented by a wider range of values. For example, changing the velocity models shown in Figure 4.8 using a different range of values, where, $v_v$ varies between 0.2 km/s and 1.87 km/s, $v_1$ varies between 0.3 km/s and 1.97 km/s, and $v_2$ varies between 0.4 km/s and 2.64 km/s, yields a low rank decomposition of $N = 11$ for the orthorhombic wavefield extrapolator, and $N = 2$ for the effective velocity extrapolation approach. This leads to a theoretical cost relation of $12/3 = 4$. However, with regard to the real computational running-time, the effective approach takes 17 minutes, whereas the orthorhombic wavefield extrapolation requires 115 minutes of processing. This gives a factor of 6.76 higher cost for the full orthorhombic wavefield extrapolation. This example shows that superior cost-relations can be found, when velocity models with a remarkable wide range of values are used.

It is also important to understand that the reduction of cost achieved by the lowrank approximation approach depends also on the time step size $\Delta t$. Higher values of $\Delta t$ do not stop the algorithm but increment the corresponding rank of the approximation and therefore, the number of required Fourier transforms. As an example, increasing $\Delta t$ from 0.001 s to 0.005 s, for the model shown in Figure 4.8 yields $N = 4$, initially represented by $N = 2$ from $\Delta t = 0.001$ s, for the effective velocity case (isotropic inhomogeneous extrapolation).

Last, with regard to the reflected or late events obtained in the effective wavefield extrapolation, note that accurate overlays are only given by the first arrival component.
of the corresponding two wavefields, from effective and orthorhombic models. Since only a fitting process based on first arrival traveltime solution is applied, it exclusively equals first arrival components. Thus, as shown in Figures 4.10(a)-4.10(b), 4.10(c)-4.10(d) and 4.10(e)-4.10(f), reflected or late arrivals in the corresponding wavefields, from effective and the actual orthorhombic approach, do not lead to an accurate overlaying for most cases. Based on the particular example shown in Figure 4.10, late arrivals are faster for the effective orthorhombic model. Since only first arrival travel-time fields are implemented for the inversion process, we should not expect equivalent results for accurate overlays of late arrivals. The sequence of traces shown in Figure 4.11 provides a much clearer representation of the delays (in space domain), found between the corresponding reflected or late events.
(a) Initial homogeneous velocity field with \( v = 2 \) km/s used to compute traveltimes shown in Figure 4.1(b).

(b) First arrival traveltimes calculated from velocity field shown in Figure 4.1(a).

(c) Inverted homogeneous velocity field computed with equation (4.2).

(d) Velocity difference between initial velocity 4.1(a) and inverted effective velocity 4.1(c).

Figure 4.1: Computation of effective velocity \( v_{eff} \) from a homogeneous initial velocity \( v_i \). Note that the magnitude of the difference between initial and inverted models may be considered negligible. A very important aspect to ensure the reproducibility of orthorhombic wavefields using the effective model method presented in this study, is to guarantee that the inversion of \( v_{eff} \) involves minimum or negligible errors, as shown in Figure 4.1(d).
Figure 4.2: Computation of effective velocity $v_{\text{eff}}$ shown in Figure 4.2(c) from an inhomogeneous initial velocity $v_i$ shown in Figure 4.2(a). Minimum errors are found near the source location where a high-value singularity is located and replaced from the velocity field $v_{\text{eff}}$ shown in Figure 4.2(c).
(a) Initial inhomogeneous velocity model with two reflectors used to compute traveltime contours shown in Figure 4.3(b).
(b) Contour plot of first arrival traveltime solution computed from velocity model shown in Figure 4.3(a). Due to the velocity difference at 3 km depth, a smooth effect of change may be observed in the curvature of the traveltime contour at that location.
(c) Inverted effective velocity model computed from traveltime field shown in Figure 4.3(b). Note that a different higher range of values near the second reflector is inverted due to the presence of head waves in the traveltime field. It is shown in further results that these high frequency changes do not affect the kinematic aspect of the first arrival events in the effective orthorhombic wavefield extrapolations.
(d) Difference between initial and effective velocity models shown in Figures 4.3(a) and 4.3(c). The highlighted pattern shown in this figure represents the contribution of inverting head waves in the corresponding effective velocity shown in Figure 4.3(c). As shown here, computation of derivatives at the reflector location (where rapid changes in traveltime are located) also lead to residual values around these regions (horizontal planes).

Figure 4.3: Computation of effective velocity $v_{eff}$ shown in Figure 4.3(c) from an inhomogeneous initial velocity $v_i$ shown in Figure 4.3(a) represented by two main high velocity contrasts located at 2 and 3 km depth.
Figure 4.4: Velocity and anisotropic parameter models used in the orthorhombic eikonal solver to compute traveltime solutions shown in Figure 4.5. After computing the corresponding traveltime field, an inversion process using equation (4.2) is applied to compute the corresponding effective velocity field. In this figure, vertical velocity $v_v$ (a), NMO velocities $v_1$ (b), $v_2$ (c), anellepticity parameters $\eta_1$ (d), $\eta_2$ (e), and $\gamma$ (f).
(a) Orthorhombic traveltime field computed from the set of input models shown in Figure 4.4.

(b) Traveltime contours of traveltime field shown in Figure 4.5(a).

(c) Effective orthorhombic velocity model obtained from traveltime field shown in Figure 4.5(a) and computed using equation (4.2). An important aspect of this inversion is to understand that the effective velocity depends on the source location. Therefore, a set of sources located at different positions generate most likely a series of different effective velocity fields.

(d) Source Ricker wavelet of 20 Hz used for effective and orthorhombic wavefield extrapolations shown in Figure 4.6. As mentioned previously, due to the progressive decay and therefore weak amplitude of the sides of the wavelet, first arrival traveltimes are overlaid on the maximum amplitude value of the wavefield. Considering the Ricker wavelet example, a traveltime solution is located at 0.1s.

Figure 4.5: Traveltime plots (a) and (b), computed with orthorhombic model shown in Figure 4.4. Effective velocity field (c) calculated from orthorhombic traveltime solution shown in Figure 4.5(a). Source Ricker wavelet used for orthorhombic and effective wavefield extrapolations shown in Figure 4.6.
Figure 4.6: Wavefield snapshots at \( t = 0.8 \) s (a), (c), (e) from effective isotropic wavefield extrapolation and (b), (d), (f), from orthorhombic wavefield extrapolation. Solid yellow curves represent orthorhombic traveltime solution.
(a) Traces representing overlapping of orthorhombic and effective wavefields at in-line 2.5 km from Figures 4.6(a) and 4.6(b) respectively.

(b) Traces representing overlapping of orthorhombic and effective wavefields at crossline 2.5 km from Figures 4.6(a) and 4.6(b) respectively.

(c) Traces representing overlapping of orthorhombic and effective wavefields at depth 2.5 km from Figures 4.6(c) and 4.6(d) respectively.

(d) Traces representing overlapping of orthorhombic and effective wavefields at in-line 2.5 km from Figures 4.6(c) and 4.6(d) respectively.

(e) Traces representing overlapping of orthorhombic and effective wavefields at depth 2.5 km from Figures 4.6(e) and 4.6(f) respectively.

(f) Traces representing overlapping of orthorhombic and effective wavefields at crossline 2.5 km from Figures 4.6(e) and 4.6(f) respectively.

Figure 4.7: Traces from wavefield snapshots shown in Figure 4.6. Dotted and solid curves represent the orthorhombic and effective wavefield solutions respectively. Despite the amplitude differences, the accurate match between first arrival traveltimes is the main objective of the implemented method.
(a) Vertical velocity $v_v$.  

(b) NMO velocity $v_1$.  

(c) NMO velocity $v_2$.  

(d) Anellepticity parameter $\eta_1$.  

(e) Anellepticity parameter $\eta_2$.  

(f) Factor $\gamma$.  

Figure 4.8: Second example for wavefield extrapolation comparison. Two high velocity and anisotropic parameter contrasts are included in this case. Traveltime field estimated from this orthorhombic anisotropic model is shown in Figure 4.9(a). Smooth changes representing overall inhomogeneity are also included in all the corresponding velocity and anisotropic parameter models. In this figure, vertical velocity $v_v$ (a), NMO velocities $v_1$ (b) and $v_2$ (c), anellepticity parameters $\eta_1$ (d), $\eta_2$ (e), and factor $\gamma$ (f).
(a) Orthorhombic traveltine field computed from the set of input models shown in Figure 4.8. Note shown in Figure 4.9(a). As mentioned earlier the location of the high velocity contrast represented in the traveltine field as relatively rapid at 3 km depth, the presence of head waves can be easily identified in the traveltine contours.

(b) Traveltime contours of traveltine field shown in Figure 4.9(a). As mentioned earlier in Figure 4.3, due to the high velocity contrast at 3 km depth, the presence of head waves can be easily identified in the traveltine contours.

(c) Effective orthorhombic velocity model obtained from traveltine field shown in Figure 4.9(a). Velocity values inverted for head wave contributions can be easily detected above the second reflector at 3 km depth.

(d) Source Ricker wavelet of 20 Hz used for effective and orthorhombic wavefield extrapolations shown in Figure 4.10.

Figure 4.9: Traveltime plots (a) and (b), computed from the orthorhombic model integrated by the sequence of fields shown in Figure 4.8. Effective velocity field (c) calculated from the orthorhombic traveltine solution shown in Figure 4.9(a). Source Ricker wavelet (d) used for orthorhombic and effective wavefield extrapolations shown in Figure 4.10. As shown in Figure 4.9(c), computing spatial derivatives from traveltine fields that include headwaves, may reproduce additional effects on the inverted effective velocity model.
Figure 4.10: Wavefield snapshots at $t = 0.8$ s (a), (c), (e) from effective isotropic wavefield extrapolation and from orthorhombic wavefield extrapolation (b), (d), (f). Solid yellow curves represent orthorhombic traveltime solution.
(a) Traces representing overlapping of orthorhombic and effective wavefields at in-line 2.5 km from Figures 4.10(a) and 4.10(b) respectively.
(b) Traces representing overlapping of orthorhombic and effective wavefields at crossline 2.5 km from Figures 4.10(a) and 4.10(b) respectively.
(c) Traces representing overlapping of orthorhombic and effective wavefields at depth 2.5 km from Figures 4.10(c) and 4.10(d) respectively.
(d) Traces representing overlapping of orthorhombic and effective wavefields at crossline 2.5 km from Figures 4.10(c) and 4.10(d) respectively.
(e) Traces representing overlapping of orthorhombic and effective wavefields at in-line 2.5 km from Figures 4.10(e) and 4.10(f) respectively.
(f) Traces representing overlapping of orthorhombic and effective wavefields at line 2.5 km from Figures 4.10(e) and 4.10(f) respectively.

Figure 4.11: Traces from wavefield snapshots shown in Figure 4.10. Dotted and solid curves represent the orthorhombic and effective wavefield solutions respectively. Since only first arrival traveltimes are used to reproduce the kinematic aspect of the orthorhombic model, reflected or late events do not match their corresponding arrivals.
Chapter 5

Conclusions

I have presented a novel algorithm for first arrival traveltime estimations in acoustic vertical orthorhombic media. The numerical examples demonstrate that the presented algorithm based on the fast marching method is stable and accurate for calculating first arrival traveltimes. The high-frequency asymptotic solutions implicit in the orthorhombic eikonal solver fit adequately the wavefronts extrapolated from the orthorhombic lowrank solution. Different models including inhomogeneous fields with high velocity contrasts proved the versatility of the presented algorithm.

High velocity changes allowed to demonstrate that head waves, which correspond to first-arrival events, are also included in the solution given by the orthorhombic eikonal solver.

I have shown that the estimated traveltime values overlay very accurately the corresponding orthorhombic wavefields, even for models where seismic anisotropy is considered to be strong.

In addition, the effective isotropic wavefield extrapolation approach is kinematically accurate when compared to results obtained from the orthorhombic wavefield extrapolation. The kinematic aspect of the corresponding wavefields, calculated from the effective and the actual orthorhombic approach, are found to be equivalent. However, amplitude values mostly do not match, especially in regions where large velocity gradients are located in the effective velocity model. Furthermore, since the effective wavefields are computed from the first arrival traveltime solutions, they do not accurately reproduce reflected or late events in the data. Despite the presented dynamic variations between the respective solutions, the method implemented in this study serves as a platform for evaluating approximate anisotropic wavefields using efficient isotropic extrapolators. For all the models considered for this study, the actual full orthorhombic wavefield extrapolations is found to be 2 to 3 times more expensive than the effective approach. Higher differences can be found when using models with wider range of velocities or anisotropic parameter values.

Additionally, since a straightforward 3D TTI implementation of wavefield extrapolators may involve approximately six times the amount of computation as a VTI solution [68], more significant computational-cost reductions are expected for the tilted
ortho-rhombic case. This implementation is indicated in the future directions of this work.

Also, for models where strong anisotropy is presented, the VTI eikonal solution computed with the orthorhombic eikonal solver is found to be more accurate than the VTI eikonal solver available in Madagascar. A different approach is implemented in the VTI eikonal solver to compute the roots of the corresponding fourth order polynomial equation.

Lastly, with the kinematic wavefield attributes obtained from the efficient effective velocity method implemented in this study, an entire integrated seismic reflection imaging work-flow can be established to generate kinematically accurate seismic images of orthorhombic media with a significant lower computational cost.

5.1 Advantages and Limitations

The method presented in this study clearly demonstrates the capability to reproduce equivalent kinematic effects with a much more efficient approach. The most critical aspect to consider is this case is the reduction of the lowrank approximation. Since the effective velocity models require a lower number of representative elements from the wave extrapolation matrix, it therefore implies a lower number of operations for every time step during the extrapolation. The efficiency and ease of this approach reinforce the possible implementation of this technique in most seismic imaging applications. Acceptable kinematically accurate results are found to support the advantages of the proposed method. However, since the dynamic aspect of wavefield propagation is not considered at any stage of the process, then no amplitude-aspects can be taken into account for additional seismic data characterization. Thus, as a major limitation, true-amplitude analysis should not be considered under the application of this approach.

A second restriction of the presented method is the limited reproducibility of only one type of seismic events. Since only first arrival traveltimes are naturally estimated from the eikonal equation, as shown in this study, it is not possible to reproduce reflected or late events using such a simple and efficient method. Further studies are required to extend the application of this approach to a second level where late or reflected events can be isolated from effective velocity models and then used to reproduce the kinematic response given by the actual anisotropic models.

5.2 Future Directions

A more practical and realistic implementation of this approach can be obtained if the effects of a tilted geometry are also included in the orthorhombic symmetry. This will allow to describe wave propagation in more realistic media where not only system of cracks are embedded into a thin layering background, but also a tilted symmetry axis.
Since the fast marching method will not reproduce the correct solution for the tilted orthorhombic case—which would probably require the application of the fast sweeping method—additional studies are needed for the implementation of this particular approach. Considering that wavefield extrapolations are much more computationally expensive in tilted geometries [68], building a tilted orthorhombic eikonal solver will lead to a more significant and superior reduction of the computational cost.

The orthorhombic eikonal solver introduced in this study, could be also implemented in seismic tomography to compute traveltine fields and update the corresponding velocities and anisotropic parameter models. In this case, the computed first arrival traveltimes fields can be used for the updating process required for the optimization method implicit in seismic tomography. Additional sources of anisotropy or velocity information, such as well data, may be required in order to reduce the ambiguity effects found between the velocities and the anisotropic parameter models.

The wavefields computed with the effective velocity approach and the actual orthorhombic models are kinematically equivalent only for the first arrival events. Incorporating a method where secondary sources (reflector locations) are replaced by primary sources of energy, may lead to a better match between the reflected or late events. A second approach could be also introduced based on a sequence of different effective velocity models. This sequence of velocity models could represent the complete kinematic response of the actual orthorhombic media. For example, a first effective velocity model is estimated to reproduce only first arrival events, a second model is generated to reproduce second arrivals, a third model is generated to reproduce third order arrivals, and so forth. In this case, every reflector point in the model becomes a potential source of energy. Then, due to the linear aspect of the wave equation, a final wavefield may be obtained after superimpose all the individual contributions. However, even when this approach may provide a more complete description of wave propagation in orthorhombic media (and in general for anisotropic models), it will probably become computationally more expensive than the application of the actual full orthorhombic wavefield extrapolation. Thus, further studies are needed to develop alternative solutions that may incorporate reflected or late arrivals under the implementation of the effective velocity approach.
REFERENCES


APPENDICES

A Code Development.
Orthorhombic Eikonal Solver

The codes reproduced and developed to build the orthorhombic eikonal solver are mainly written in programming languages C and Python. The description presented in this section intents to cover the most general but relevant aspects highlighted in the eikonal solver application. Only the parts of the code that contain the most physically understandable definitions are included in this section.

The fast marching orthorhombic eikonal algorithm is mainly integrated by three source codes: MeikonalORTHO.c, fastmarchORTHO.c, and neighborsORTHO.c. The MeikonalORTHO.c code is built based on a sequence of conditionals statements that allow to perform the initial evaluation of the fields available for the corresponding traveltime computation. Therefore, MeikonalORTHO.c represents the check-in and check-out stage that permits to define the fundamental input fields in a way that vel = sf_input("in"); refers to the vertical velocity \( v_v \), \( \text{vnmo} = \text{sf_input(file)}; \), and \( \text{votra} = \text{sf_input(file)}; \) represent the NMO \( v_1 \) and NMO \( v_2 \) velocities respectively, \( \text{ETA1} = \text{sf_input(file)}; \), and \( \text{ETA2} = \text{sf_input(file)}; \) correspond to the anellepticity values \( \eta_1 \) and \( \eta_2 \) respectively. Last, the \( \gamma \) factor is defined under the definition \( \text{thomsengamma} = \text{sf_input(file)}; \). If no value is declared for \( v_1, v_2, \eta_1, \eta_2, \gamma \), a set of null spaces is defined as follows

\[
\begin{align*}
\text{if (NULL != (file = sf_getstring("ETA1")))} & \{ \\
\text{ETA1} = \text{sf_input(file)}; \\
\text{free(file)};
\} \text{ else }
\text{ETA1} = \text{NULL};
\end{align*}
\]
if (NULL != (file = sf_getstring("ETA2"))) {
    ETA2 = sf_input(file);
    free(file);
} else {
    ETA2 = NULL;
}

if (NULL != (file = sf_getstring("vnmo"))) {
    vnmo = sf_input(file);
    free(file);
} else {
    vnmo = NULL;
}

if (NULL != (file = sf_getstring("votra"))) {
    votra = sf_input(file);
    free(file);
} else {
    votra = NULL;
}

if (NULL != (file = sf_getstring("thomsengamma"))) {
    thomsengamma = sf_input(file);
    free(file);
} else {
    thomsengamma = NULL;
}

<< other definitions >>

For the case of no input fields for all the parameters, except the vertical velocity \( v_v \), the algorithm leads to the isotropic solution. However, to compute traveltimes or validate the domain time = sf_output("out"), one velocity field (which is considered to be \( v_v \) in the orthorhombic case) has to be at least defined. Therefore,

<< other definitions >>

if (SF_FLOAT != sf_gettype(vel))
If no input is available in this case, then the code breaks. If no input fields are available for \( v_1 \) or \( v_2 \), then, the NMO velocity field becomes equivalent to the initial velocity, where

\[
\eta_1 = \eta_2 = 0.
\]

If no input fields are available for \( \eta_1 \) or \( \eta_2 \), then, the corresponding values become zero,

\[
\eta_1 = \eta_2 = 0.
\]
} else { /* assume elliptic anisotropy first plane */
    for(i = 0; i < n123; i++)
        fETA1[i] = 0.;
}

if(NULL != ETA2) {
    sf_floatread(fETA2,n123,ETA2);
    sf_fileclose(ETA2);
} else { /* assume elliptic anisotropy second plane */
    for(i = 0; i < n123; i++)
        fETA2[i] = 0.;
}

<< other definitions >>

With regard to the $\gamma$ factor, for no input value, it implies the no-anisotropic contribution on the horizontal plane. Therefore, $\gamma = 1$, represented in the code as

<< other definitions >>

if(NULL != thomsengamma) {
    sf_floatread(fgamma,n123,thomsengamma);
    sf_fileclose(thomsengamma);
} else { /* assume delta=0, then gamma=1*/
    for(i = 0; i < n123; i++)
        fgamma[i] = 1.;
}

<< other definitions >>

As shown for the isotropic eikonal solver, the vertical velocity field is also transformed into a slowness factor in this implementation. Among other important statements required by this code, a final declaration to the next algorithm is defined. Therefore, the following function establishes the implementation of the second code, declared as fastmarchORTHO.c. Explicitly,
/** loop over shots */
for( is = 0; is < nshot; is++) {
    sf_warning("shot %d of %d;",is+1,nshot);
    fastmarch(t,v,p, plane,
            n3,n2,n1,
            o3,o2,o1,
            d3,d2,d1,
            s[is][2],s[is][1],s[is][0],
            b3,b2,b1,
            order, fETA1, fETA2, fgamma, fvnmo, fvotra);

    sf_floatwrite (t,n123,time);
}

<< other definitions >>

where the set of $v_v$, $v_1$, $v_2$, $\eta_1$, $\eta_2$, and $\gamma$ are required as input values of the function, among other additional factors. Once the fastmarch function is declared, between other operations, an initialization process is stated by neighbors_init (in, d, n, order, time);. This definition is the actual beginning of the fast marching sequential computation. The following lines of codes represent the fields and statements defined in the fastmarch function, still within the fastmarchORTHO.c code,

<< other definitions >>

void fastmarch (float* time /* time */,
float* v /* slowness squared */,
int* in /* in/front/out flag */,
bool* plane /* if plane source */,
int n3, int n2, int n1 /* dimensions */,
float o3,float o2,float o1 /* origin */,
float d3,float d2,float d1 /* sampling */,
float s3,float s2,float s1 /* source */,
int b3, int b2, int b1 /* box around the source */,
int order /* accuracy order (1,2,3) */,
float* ETA1 /* First eta value, first plane of symmetry */,
float* ETA2 /* Second eta value, second plane of symmetry */,

/* other definitions */
float* thomsengamma /* Gamma parameter gamma=sqrt(1+2delta) */,
float* vnmo /* First nmo velocity, vnmo1 */,
float* votra /* Second nmo velocity, vnmo2*/)
/*< Run fast marching orthorhombic eikonal solver >/*
{
    float xs[3], d[3], *p;
    int n[3], b[3], npoints, i;

    n[0] = n1; xs[0] = s1-o1; b[0] = b1; d[0] = d1;

    sf_pqueue_start();
    neighbors_init (in, d, n, order, time);

    for (npoints = nearsource (xs, b, d, v, ETA1, ETA2, thomsengamma,
        vnmo, votra, plane);
        npoints > 0;
        npoints -= neighbours(i)) {
        /* Pick smallest value in the NarrowBand
         * mark as good, decrease points_left */

        p = sf_pqueue_extract();

        if (p == NULL) {
            sf_warning("%s: heap exausted!", __FILE__);
            break;
        }
        i = p - time;
        in[i] = SF_IN;
    }
}

<< other definitions >>

Now, the most important aspects of the traveltime computation are embedded in the neighborsORTH0.c code. Due to the long extension of the programming sequences involved in this section, only the updating procedures will be shown here. This section of the code implies the implementation of the derived and previously shown
orthorhombic-equation terms. Therefore, with regard to the updaten function defined in the algorithm, for the series of coefficients $A$, $B$, $C$, $D$, $E$, $F$, $G$ shown in equation (3.8),

\[
\text{static bool updaten (int m, float* res, struct Upd *v[], char ch[])} \{
\]

<< other definitions >>

\[
\text{VNMO1=vnmo1*vnmo1;}
\]

\[
\text{VNMO2=votra1*votra1;}
\]

\[
\text{vo=1/v1;}
\]

\[
\text{vnmo1=vnmo1;}
\]

\[
\text{votra1=votra1;}
\]

\[
A=vnmo1*(1+2*ETA11);
\]

\[
B=votra1*(1+2*ETA21);
\]

\[
C=vo;
\]

\[
D=(1+4*ETA11+4*ETA11*ETA11)*thomsengamma1*thomsengamma1*vnmo1*vnmo1
- (1 + 2*ETA11 + 2*ETA21 + 4*ETA11*ETA21)*vnmo1*votra1;
\]

\[
E=-2*ETA11*vnmo1*vo;
\]

\[
F=-2*ETA21*votra1*vo;
\]

\[
G=(2*(1+2*ETA11)*thomsengamma1*vnmo1*sqrt(vnmo1)*sqrt(votra1)
+ (4*ETA11*ETA21-1)*vnmo1*votra1
- (1 + 4*ETA11 + 4*ETA11*ETA11)*thomsengamma1*thomsengamma1*vnmo1*vnmo1)*vo;
\]

<< other definitions >>

Now, in order to consider the directional aspect of the fast marching-calculation flow (directions in which the traveltimes are being estimated), an array declared as \text{ch} is being implemented to track the orientation of the computational process. This variable along the parameter \text{m} (with values 1, 2, and 3) leads to the tracking process that involves the grid-direction in which the traveltime is computed ($x$, $y$, or $z$) and the corresponding definition of parameters from equation (3.8), reflected on the definition of the coefficients $\beta_p$ shown in equation (3.12). After computing the series of $\beta_p$ values,
the next step is to apply the Bairstow’s Method \[59\] \[60\]. Initially for the higher order
powers of the polynomial in equation \((3.11)\), where

\[
\text{<< Computation of beta coefficients}
\]
\[
\text{defined initially as } a_6, a_5, a_4, a_3, a_2, a_1, a_0
\]

\[
\text{if}(a_6!=0 \&\& a_5!=0)\{
\]
\[
c = 1; \quad d = 1; \quad u_0 = a_5/a_6; \quad v_0 = a_4/a_6;
\]
\[
b_6 = 0; \quad b_5 = 0; \quad f_6 = 0; \quad f_5 = 0;
\]
\[
\text{while } (|c|>0.00001 \&\& |d|>0.00001)\{
\]
\[
b_4 = a_6-u_0*b_5-v_0*b_6;
\]
\[
b_3 = a_5-u_0*b_4-v_0*b_5;
\]
\[
b_2 = a_4-u_0*b_3-v_0*b_4;
\]
\[
b_1 = a_3-u_0*b_2-v_0*b_3;
\]
\[
b_0 = a_2-u_0*b_1-v_0*b_2;
\]
\[
c = a_1-u_0*b_0-v_0*b_1;
\]
\[
d = a_0-v_0*b_0;
\]
\[
f_4 = b_6-u_0*f_5-v_0*f_6;
\]
\[
f_3 = b_5-u_0*f_4-v_0*f_5;
\]
\[
f_2 = b_4-u_0*f_3-v_0*f_4;
\]
\[
f_1 = b_3-u_0*f_2-v_0*f_3;
\]
\[
f_0 = b_2-u_0*f_1-v_0*f_2;
\]
\[
g = b_1-u_0*f_0-v_0*f_1;
\]
\[
h = b_0-v_0*f_0;
\]
\[
u_{00} = u_0 - (-h*c+g*d)/(v_0*g*g+h*(h-u_0*g));
\]
\[
v_{00} = v_0 - (-g*v_0*c+(g*u_0-h)*d)/(v_0*g*g+h*(h-u_0*g));
\]
\[
u_0 = u_{00}; \quad v_0 = v_{00};\}
\]
\[
\text{else}
\]
\[
b_4=a_4; \quad b_3=a_3; \quad b_2=a_2; \quad b_1=a_1; \quad b_0=a_0;
\]
\[
}\]
\[
\text{if}(a_6!=0 \&\& a_5!=0)\{
\]
\[
tf_5 = (-u_0+csqrt(u_0*u_0-4*v_0))/(2.0);
\]
\[
tf_6 = (-u_0-csqrt(u_0*u_0-4*v_0))/(2.0);
\]
\[
}\]
\[
\text{else}
\]
\[
tf_5=0.; \quad tf_6=0.;
\]
\[
}\]
\[
\text{<< other definitions >>}
In this sequence, once the values $c$ and $d$ are reduced to approximately zero, the iterative process implied in the Bairstow’s Method computes the optimized coefficients $b_4$, $b_3$, $b_2$, $b_1$, and $b_0$. Then, the computed $u_0$ and $v_0$ values are used to estimate the first two polynomial roots. Now, equivalently for the remainder roots,

<< other definitions >>

```c
if(b4!=0 && b3!=0){
    cc=1; dd=1; uu0=b3/b4; vv0=b2/b4;
    bb4=0; bb3=0; ff4=0; ff3=0;
    while(fabs(cc)>0.00001 && fabs(dd)>0.00001){
        bb2 = b4-uu0*bb3-vv0*bb4;
        bb1 = b3-uu0*bb2-vv0*bb3;
        bb0 = b2-uu0*bb1-vv0*bb2;
        cc = b1-uu0*bb0-vv0*bb1;
        dd = b0-vv0*bb0;
        ff2 = bb4-uu0*ff3-vv0*ff4;
        ff1 = bb3-uu0*ff2-vv0*ff3;
        ff0 = bb2-uu0*ff1-vv0*ff2;
        gg = bb1-uu0*ff0-vv0*ff1;
        hh = bb0-vv0*ff0;
        uu00 = uu0 - (-hh*cc+gg*dd)/(vv0*gg*gg+hh*(hh-uu0*gg));
        vv00 = vv0 - (-gg*vv0*cc+(gg*uu0-hh)*dd)/(vv0*gg*gg+hh*(hh-uu0*gg));
        uu0 = uu00; vv0 = vv00;
    }
    else{
        bb2=b2; bb1=b1; bb0=b0;
    }
    // Solution from remaining polynomial:
    tf1 = (-bb1+csqrt(bb1*bb1-4*bb2*bb0))/(2.0*bb2);
    tf2 = (-bb1-csqrt(bb1*bb1-4*bb2*bb0))/(2.0*bb2);
    if(b4!=0 && b3!=0){
        tf3 = (-uu0+csqrt(uu0*uu0-4*vv0))/(2.0);
        tf4 = (-uu0-csqrt(uu0*uu0-4*vv0))/(2.0);
    }
    else
    {
```
In this case, the parameters to be optimized are $cc$ and $dd$. Once they reach a lower value than $0.00001$, $tf4$ and $tf3$ are computed and the remainder coefficients $bb1$, $bb2$ and $bb0$, are used to calculate the last pair of roots, $tf2$ and $tf1$. Since complex conjugate roots may be found in the set of solutions, only the real part of the roots are considered for this particular implementation. Once the set of roots are computed, the corresponding values are sorted in the array $Ta$, from highest $Ta[0]$ to lowest $Ta[6]$ ($Ta[6]$, for the case when all six roots exist).

```
if (a4==0 && a3==0 && a6==0 && a5==0){t=Ta[0];}
if (a6==0 && a5==0 && b4!=0 && b3!=0){t=Ta[1];}
if (a6!=0 && a5!=0){t=Ta[2];}
```

Thus, for the quadratic case (isotropic symmetry), $a6=a5=a4=a3=0$ or equivalently, $\beta_6 = \beta_5 = \beta_4 = \beta_3 = 0$, the fastest P-wave arrival is given by the first (highest value) element of the array $Ta$. Subsequently, the quartic solution (VTI symmetry) is given by $Ta[1]$, where $\beta_6 = \beta_5 = 0$, $\beta_4 \neq 0$, and $\beta_3 \neq 0$. Lastly, for a sixth order contribution (orthorhombic symmetry) that implies $\beta_6 \neq 0$ and $\beta_5 \neq 0$, the fastest P-wave arrival traveltime is given by $Ta[2]$. The ordering-selection of solutions taken from the array $Ta$ (with respect to the possible cases; second, fourth, and sixth order) is based on the comparisons made with the corresponding full solutions given by the wavefield extrapolation.

Once a traveltime solution is chosen, it is then assigned to a specific grid point. Thus, at each grid point of the model, the set of values $v_v$, $v_1$, $v_2$, $\eta_1$, $\eta_2$, and $\gamma$, are used to compute and solve the resultant polynomial equation that provides the fastest P-wave solution.
B  Code Development. Effective Velocity Inversion

The algorithm used to compute the set of spatial derivatives shown in equation (4.2) is presented in this section. Due to the extension of the codes, only the most relevant parts of the implemented sequences are presented in this document. The objective of this code consists in the inversion of $v_{eff}$ at each grid point. To do so, a finite difference scheme is used to approximate the corresponding spatial derivatives of the traveltime field at a particular location $x, y, z$. Depending on the number of values or grid points available on each side on the spatial domain, different finite difference schemes are implemented. First, to define the main output function,

\[
\text{static void veff_phase(float *veff, const float *dtdz, const float *dtdy, const float *dtdx)}
\]

\[
\text{for (int i=0; i < nxyz; i++)} 
\text{veff[i] = 1. / hypotf(hypotf(dtdy[i],dtdz[i]),dtdx[i]);}} 
\]

This expression represented in this function defines the effective velocity $v_{eff}$ at the index $i$, where the spatial components $x, y, z$ are embedded. On the other hand, computing the corresponding derivative components involves different finite difference schemes with different order of accuracy. As shown in the following segment of code, when at least two grid points are available on both sides of the domain, a central difference scheme with a second order truncation-error is implemented. Explicitly,

\[
\text{<< other definitions >>} 
\]
int beg = ( 0 == i1);
int end = (n1-1 == i1);
int beg1 = ( 1 == i1);
int end1 = (n1-2 == i1);

switch ( (beg << 1) + end ) {
    case 0 :
        switch ( (beg1 << 1) + end1 ) {
            case 0 :
                der = FACTOR * id * (f[JO(i1+1,i2,i3)] - f[JO(i1-1,i2,i3)]); break;
            case 1 :
                der = FACTOR * id * (f[JO(i1+1,i2,i3)] - f[JO(i1-1,i2,i3)]); break;
            case 2 :
                der = FACTOR * id * (f[JO(i1+1,i2,i3)] - f[JO(i1-1,i2,i3)]); break;
            default:
                der = 0.;
        } break;
    case 1 :
        der = id * (f[JO(i1 ,i2,i3)] - f[JO(i1-1,i2,i3)]); break;
    case 2 :
        der = id * (f[JO(i1+1,i2,i3)] - f[JO(i1 ,i2,i3)]); break;
    default:
        der = 0.;
}
return der;

<< other definitions >>

The value of FACTOR corresponds in this case to 0.5, as required by the central finite difference representation. On the other hand, for the boundary values, forward and backward finite difference schemes are being implemented. These operations are applied when the outer switch gives the options case 1 or case 2, depending on which side of the model the derivative is being computed. Only one component (with i1 index) is presented in this examples.

Due to the symmetric aspect of the central finite difference scheme and the geometrically equivalent first traveltime values around the source, when the finite operation reaches the center of the traveltime field, it yields a zero value for the corresponding derivative. Therefore, when the three derivative components are estimated at the
source location, a zero value is then given as a result for the set of derivatives. This automatically leads to a singular value at the source location in the effective velocity model. Replacing this singular value with an average term allows to solve the problem.
C Seismic Wave Extrapolation Using Lowrank Symbol Approximation

The theory and the corresponding methodology used in this work for the lowrank wavefield extrapolations is a novel algorithm presented by Sergey Fomel, Lexing Ying, and Xiaolei Song [8, 16] at the SEG 2010 annual meeting, Denver, Colorado, USA.

The construction of the wavefield operator implies Fourier transforms in space domain, incorporated with a lowrank approximation for the corresponding space-wavenumber wave-propagator matrix. As mentioned earlier, extrapolation of wavefields in time represents a critical role in a wide set of seismic imaging processes, such as modeling, reverse-time migration, and full waveform inversion. Most common extrapolations in time are performed by finite difference-methods [7]. However, alternative approaches, such as spectral methods, provide more accurate results with the important advantage of suppressing dispersion artifacts [69]. As a critical characteristic of this method, the lowrank approximation only involves a small set of representative wavenumbers and representative spatial locations.

C.1 Wavefield Extrapolation

For a seismic wavefield evaluated at the location $x$ and time $t$, the next time step $t + \Delta t$ of the wavefield can be approximated using the mixed-domain operator [67]

$$ P(x, t + \Delta t) = \int \hat{P}(k, t) \exp \{i\phi(x, k, \Delta t)\} dk, \quad (C.1) $$

where the field $\hat{P}(k, t)$ corresponds to the spatial Fourier transform of $P(x, t)$ represented by

$$ \hat{P}(k, t) = \frac{1}{(2\pi)^3} \int P(x, t) \exp \{-i k \cdot x\} dx. $$

In order to define the phase term $\phi(x, k, t)$, equation (C.1) can be substituted into the isotropic wave equation to obtain the high frequency asymptotic approximation of it. The given result yields the following eikonal equation

$$ \frac{\partial \phi}{\partial t} = \pm v(x, k) |\nabla \phi|, \quad (C.2) $$
where \( v(x, k) \) represents the phase velocity. Since the directional aspect of \( v(x, k) \) is defined by \( k \), for the isotropic case, \( v \) does not depend on \( k \) and the initial condition for equation (C.2) is represented as \( \phi(x, k, 0) = k \cdot x \), which transforms equation (C.1) into an inverse Fourier transform. Expanding the phase term \( \phi \) (for small steps \( \Delta t \)) into a Taylor series,

\[
\phi(x, k, t) \approx k \cdot x + \phi_1(x, k)t + \phi_2(x, k)t^2 + \cdots \quad (C.3)
\]

Then,

\[
|\nabla \phi| \approx |k| + \frac{\nabla \phi_1 \cdot k}{|k|}t + \cdots \quad (C.4)
\]

Replacing the results of equations (C.3) and (C.4) into equation (C.2), and taking the common terms with different powers of \( t \), gives \( \phi_1(x, k) = v(x, k)|k| \) and \( \phi_2(x, k) = v(x, k)\nabla v \cdot k \). Assuming small values of \( \nabla v \) or \( \Delta t \), the Taylor series shown in equation (C.3) can be simplified to only two terms. This approximation reduces equation (C.1) into the well known expression [70]

\[
P(x, t + \Delta t) \approx \int \hat{P}(k, t) \exp \{i(k \cdot x + v(x, k)|k|\Delta t)\} dk, \quad (C.5)
\]

or equivalently

\[
P(x, t + \Delta t) + P(x, t - \Delta t) \approx 2 \int \hat{P}(k, t) \exp \{i(k \cdot x)\} \cos [v(x, k)|k|\Delta t] dk. \quad (C.6)
\]

The inherent computational cost for the implementation of equation (C.1) is \( \mathcal{O}(N_x^2) \), where \( N_x \) represents the complete three-dimensional contribution of the model. However, the algorithm presented in the next section represents an alternative approach that reduces the computational cost of seismic wavefield extrapolations.

### C.2 Reduction of Computational Cost with Lowrank Approximation

The important aspect considered under the lowrank approximation approach is given by the decomposition of the wave extrapolation matrix

\[
W(x, k) = \exp \{i[\phi(x, k, \Delta t) - k \cdot x]\},
\]

into a different representation that allows to separate the contribution from spatial and wavenumber domains at a certain time \( \Delta t \). Therefore, under the lowrank representation approach, this alternative representation can be expressed as
\[ W(x, k) \approx \sum_{m=1}^{M} \sum_{n=1}^{N} W(x, k_m) a_{mn} W(x, k). \quad (C.7) \]

The term in equation (C.7) allows to speed up the computational process of \( P(x, t + \Delta t) \), due to the previous reduced selection of the \( N \) and \( M \) components from the full wave extrapolation matrix. Thus,

\[ P(x, t + \Delta t) = \int \exp \{i x \cdot k\} W(x, k) \hat{P}(k, t) dk \quad (C.8) \]

\[ \approx \sum_{m=1}^{M} W(x, k_m) \left( \sum_{n=1}^{N} a_{mn} \left( \int \exp \{i x \cdot k\} W(x, k) \hat{P}(k, t) dk \right) \right) \]

The application of equation (C.8) is equivalent to perform \( N \) inverse FFT. Thus, the separable lowrank approximation approach allows to select a set of \( M \) representative wavenumbers and \( N \) representative spatial locations. On the other hand, the approximation approach presented in equation (C.7) may be interpreted as a matrix decomposition problem, where

\[ W \approx W_1 \cdot A \cdot W_2. \]

Therefore, matrix \( W \) is a \( N_x \times N_x \) matrix with the input domain \( W(x, k) \), \( W_1 \) corresponds to a submatrix of \( W \) defined in terms of the columns associated with \( \{k_m\} \), \( W_2 \) represents the submatrix defined in terms of the rows associated with the \( \{x_n\} \), and \( A = \{a_{mn}\} \). An important condition to have a lowrank decomposition for matrix \( W \) is to assume adequately small values of \( \Delta t \). The validity of the separation method used for the lowrank approximation is based on fact that the columns of \( W_1 \) and the rows of \( W_2 \) should span the corresponding column space and row space of initial matrix \( W \) [71]. Thus, the lowrank algorithm does not need to perform any computation on the initial matrix \( W \), it only access the selected columns and rows. Therefore, compared to the approach that requires the implementation of the full wave extrapolation matrix shown in equation (C.1), the lowrank approximation shown in equation (C.8), reduces the cost from \( O(N_x^2) \) to \( O(N N_x \log N_x) \), where \( N \) represents the number of inverse FFT applied in equation (C.8).

### C.3 Lowrank Approximation of Pseudo-acoustic P-waves in Orthorhombic Media

The approach presented in this section is an extension of the lowrank approximation [8] introduced by Xiaolei Song and Tariq Alkhalifah [19]. In this case, due to the anisotropic aspect of the wavefield, the lowrank approximation [8] is adopted to
handle the given mixed-domain operator. Therefore, for acoustic orthorhombic media, Song and Alkhalifah [19] derived a phase operator \( \phi(x, k) \) to substitute the term \(|k|v(x)\) of the isotropic model shown in equation (C.5). This mixed-domain operator is then obtained after solving the sixth order polynomial equation in \( \phi \). For additional information, see references [8, 16, 66, 52, 19].