Non-Orthogonal Opportunistic Beamforming: Performance Analysis and Implementation

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Abstract—Aiming to achieve the sum-rate capacity in multi-user multi-antenna systems where \( N_t \) antennas are implemented at the transmitter, opportunistic beamforming (OBF) generates \( N_t \) orthonormal beams and serves \( N_t \) users during each channel use, which results in high scheduling delay over the users, especially in densely populated networks. Non-orthogonal OBF with more than \( N_t \) transmit beams can be exploited to serve more users simultaneously and further decrease scheduling delay. However, the inter-beam interference will inevitably deteriorate the sum-rate. Therefore, there is a tradeoff between sum-rate and scheduling delay for non-orthogonal OBF. In this context, system performance and implementation of non-orthogonal OBF with \( N > N_t \) beams are investigated in this paper. Specifically, it is analytically shown that non-orthogonal OBF is an interference-limited system as the number of users \( K \to \infty \). When the inter-beam interference reaches its minimum for fixed \( N_t \) and \( N \), the sum-rate scales as \( \ln \left( \frac{N}{N_t - N_t} \right) \) and it degrades monotonically with the number of beams \( N \) for fixed \( N_t \). On the contrary, the average scheduling delay is shown to scale as \( \frac{1}{K} \ln K \) channel uses and it improves monotonically with \( N \). Furthermore, two practical non-orthogonal beamforming schemes are explicitly constructed and they are demonstrated to yield the minimum inter-beam interference for fixed \( N_t \) and \( N \). This study reveals that, if user traffic is light and one user can be successfully served within a single transmission, non-orthogonal OBF can be applied to obtain lower worst-case delay among the users. On the other hand, if user traffic is heavy, non-orthogonal OBF is inferior to orthogonal OBF in terms of sum-rate and packet delay.

Index Terms—Multiple-input single-output (MISO), multi-user scheduling, non-orthogonal transmission, opportunistic beamforming.

I. INTRODUCTION

For different scheduling strategies in the downlink of multi-user wireless systems, there are two conflicting goals. One is to satisfy the quality of service (QoS) requirements of different users, such as scheduling delay, and the other aims to maximize system throughput [1], [2]. Round-robin scheduling follows a strict order to serve each user once in each round such that it guarantees minimum scheduling delay over the users. On the contrary, opportunistic scheduling exploits multi-user diversity gain and achieves the sum-rate capacity as the number of users \( K \) approaches infinity [3]. When the base station (BS) is equipped with \( N_t > 1 \) transmit antennas, opportunistic scheduling can be implemented by opportunistic beamforming (OBF), which generates \( N = N_t \) beams and serves up to \( N_t \) users for each channel use [4], [5]. While it achieves the sum-rate capacity, OBF can result in significant scheduling delay over the users, especially in densely populated networks. Specifically, when the number of users \( K \) is large and all users have the same average signal-to-noise ratio (SNR), the average scheduling delay for which each user is served at least once is \( (K \ln K) / N_t \) channel uses [1]. In practice, shorter scheduling delay is much more desirable for wireless applications in general and for delay-sensitive traffic such as audio/video streaming, and is even a must for critical data such as alarm applications [6]–[10].

Conventionally, opportunistic and round-robin scheduling can be combined to keep a balance between scheduling delay and sum-rate [11]. On the other hand, for the OBF, if \( N > N_t \) beams are generated for each channel use, more users can be simultaneously served and scheduling delay will be further reduced. Unfortunately, in this case, inter-beam interference will inevitably deteriorate the sum-rate. Therefore, there is a tradeoff between sum-rate and scheduling delay when the number of transmit beams \( N \) increases. This motivates us to analytically investigate the performance and implementation of OBF with \( N > N_t \) beams, by using the same amount of system resource. Hereafter, the conventional OBF with \( N = N_t \) beams is denoted by “orthogonal OBF” while the scheme with \( N > N_t \) beams is referred to as “non-orthogonal OBF”.

In order to simultaneously transmit on more than \( N_t \) beams at the \( N_t \)-antenna BS, non-orthogonal beamforming vectors have to be designed and analyzed. In [12], [13], a Grassmannian beamforming scheme with \( N = N_t + 1 \) beams is proposed, where beamforming vectors can be constructed from the vertices of a regular simplex in the space \( \mathbb{C}^{N_t} \). [14]. However, this construction cannot be extended to the general case with \( N > N_t + 1 \). In [15], a greedy algorithm is proposed to construct non-orthogonal beamforming matrices for arbitrary \( N \). Unfortunately, the prior work [12], [13], [15] only show some simulation results and no rigorous analysis on the system performance has been made. Filling this important gap, in this work we analytically investigate system performance in terms of achievable sum-rate and scheduling delay, and design optimal beamforming matrices
achieving minimum inter-beam interference. In particular, our results include previously reported results under orthogonal transmission as special cases.

Specifically, we focus on the general non-orthogonal OBF with $N > N_t$ transmit beams. Increasing $N$ is essentially equivalent to increasing the spatial multiplexing gain. On the contrary, the inter-beam interference due to non-orthogonal beams will inevitably deteriorate the data-rate on each beam. Therefore, the key question is: how does the number of beams $N$ and inter-beam interference affect the final sum-rate? In this paper, the achievable sum-rate and the sum-rate scaling law are established. In particular, the sum-rate scaling law reveals that non-orthogonal OBF is an interference-limited technique. Moreover, when inter-beam interference is minimized for fixed $N_t$ and $N$, the sum-rate monotonically decreases with the number of beams $N$.

On the other hand, increasing $N$ allows transmitting to more users during each channel use, thus decreasing scheduling delay. Our results show that the average scheduling delay, which guarantees that each user is served at least once, improves monotonically with increasing number of beams $N$. Moreover, when practical constraints on the received SINR and outage probability are imposed, the maximum number of supported transmit beams is derived.

As aforementioned, in order to maximize the sum-rate for fixed $N_t$ and $N$, the inter-beam interference shall be kept as small as possible. But, in practice, how to explicitly construct the non-orthogonal beamforming vectors to achieve the smallest inter-beam interference remains an open problem. Thus, in the last part of this paper, two practical non-orthogonal beamforming matrices are proposed and they are shown to achieve the minimum inter-beam interference.

This study reveals that, if user traffic is light and one user can be successfully served within a single transmission, e.g., in alarm applications, non-orthogonal OBF can be applied to obtain lower worst-case delay among the users [1]. On the other hand, if multiple transmissions are needed to successfully serve a user, orthogonal transmission is preferable due to its higher achievable sum-rate and lower packet delay.

The rest of this paper is organized as follows. Section II describes the system model and the scheduling strategy. In Section III, the distribution function of the received SINR is developed. Section IV focuses on the performance analysis based on the obtained SINR distributions. Section V illustrates two practical non-orthogonal beamforming matrices. Simulation results and discussions are presented in Section VI and, finally, Section VII concludes the paper.

II. SYSTEM MODEL AND SCHEDULING STRATEGY

We consider the downlink transmission from a BS equipped with $N_t$ antennas to $K$ single-antenna users. The number of users, $K$, is assumed to be larger than the number of transmit beams, $N$. The block-fading channels of different users are mutually independent and identically distributed. The transmit power is equally allocated among different transmit antennas, and the average SNRs for all users are assumed to be identical.

At each time slot, $N$ different symbols in the vector $x = [x_1, \ldots, x_N]^H$ are simultaneously transmitted onto $N$ different beams, where $(\cdot)^H$ denotes the Hermitian transpose operator. Prior to transmission, $x$ is multiplied by a $N_t \times N$ beamforming matrix $B = [b_1, \ldots, b_N]$, where $b_n$ is the beamforming vector for beam $n$. Therefore, the received symbol of user $k$ is given by $y_k = \sqrt{\frac{\rho}{N_t}} \sum_{n=1}^{N} h_{nk}^k b_n x_n + z_k$, where $\rho$ is the average SNR at the BS, $h_{nk}^k = [h_{1,k}^k, h_{2,k}^k, \ldots, h_{N,k}^k]^H$ is the complex Gaussian channel vector between the BS and user $k$, with zero mean and unit covariance matrix $I$, and $z_k$ denotes the additive white Gaussian noise (AWGN) with zero mean and unit variance.

As to the scheduling strategy at the BS, each user calculates the received SINRs on $N$ different beams, and feeds back the maximum SINR and its corresponding beam index to the BS. Specifically, the received SINR of user $k$ on beam $n$ is given by

$$\gamma_{k,n} = \frac{|h_{nk}^k b_n|^2}{\rho + \sum_{l=1, l \neq n}^{N} |h_{lk}^k b_l|^2},$$

where $|x|$ denotes the amplitude of $x$. For user $k$, the maximum SINR among $N$ beams is determined as $\gamma_k = \max_{n=1, \ldots, N} \gamma_{k,n}$, and the corresponding beam-index is $\hat{n}_k = \arg \max_{n=1, \ldots, N} \gamma_{k,n}$. Therefore, the feedback pertaining to user $k$ is the pair $(\hat{\gamma}_k, \hat{n}_k)$.

After receiving all the feedbacks at the BS, the user with the maximum SINR among all users whose $\hat{n}_k = n$ is chosen to be served on beam $n$. More precisely, the maximum SINR achieved through beam $n$ is determined as

$$\gamma_{\max,n} = \max \{ \gamma_k \} \quad \text{s.t.} \quad \hat{n}_k = n.$$  

III. DISTRIBUTION FUNCTION OF RECEIVED SINR

In general, increasing the number of transmit beams enhances spatial multiplexing gain but also increases the total inter-beam interference, thus degrading the achievable sum-rate. Aiming at assessing this tradeoff, we analytically investigate the distribution function of the received SINR in this section. The system performance analyses are provided in the next section.

With the principle of orthogonal projection, the beamforming vector $b_l$ can be expressed in terms of $b_n$ via their cross-correlation coefficient $\delta_{l,n} \triangleq \left| b_l^H b_n \right|$, that is, $b_l = \delta_{l,n} b_n + \sqrt{1 - \delta_{l,n}^2} b_{l,n}^\perp$, $1 \leq l, n \leq N$, where $b_{l,n}^\perp$ is the orthonormal vector of $b_l$ to $b_n$. Substituting the previous equation into (1) and performing some algebraic manipulations, we obtain

$$\gamma_{k,n} = X \left\{ \frac{N}{\rho} + \sum_{l=1, l \neq n}^{N} \left[ \delta_{l,n}^2 X + (1 - \delta_{l,n}^2) Y_l \right] \right\}^{-1},$$

where $X \triangleq \left| h_{nk}^k b_n \right|^2$, $Y_l \triangleq \left| h_{lk}^k b_{l,n}^\perp \right|^2$, and

$$Z_l \triangleq h_{lk}^k b_n \left( h_{nk}^k b_{l,n}^\perp \right)^H + h_{lk}^k b_{l,n}^\perp \left( h_{nk}^k b_n \right)^H.$$  

Based on the theory of optimal Grassmannian line packing [17], in order to guarantee that the correlation between any
two beamforming vectors is as small as possible, beamforming vectors must be symmetric, i.e., \( \delta_l, n = \delta_0 \), where \( 1 \leq l \leq N \) and \( l \neq n \). Define \( \alpha \triangleq (N - 1)\delta_0^2 \) as the total inter-beam interference from the other \( N - 1 \) beams on beam \( n \). Also, let \( \beta \triangleq 1 - \delta_0^2 \), \( \eta \triangleq \delta_0 \sqrt{1 - \delta_0^2} \), \( Y \triangleq \sum_{l=1, l \neq n}^N Y_l \), and \( Z \triangleq \sum_{l=1, l \neq n}^N Z_l \). Then, when the inter-beam interference are symmetric, (3) can be rewritten as

\[
\gamma_{k, n} = \frac{X}{\rho + \alpha X + \beta Y + \eta Z}.
\]

Since \( h_{k, n} \) is a complex Gaussian random vector and that \( b_{n} \) and \( b_{l, n}^\perp \) are normalized constant vectors, it is clear that \( X \) is of exponential distribution with unit mean and unit variance, and \( Y \) is of chi-square distribution with \( 2(N - 1) \) degrees of freedom. Furthermore, the PDF of \( Z \) in (5) is summarized in the following lemma.

**Lemma 1:** The PDF of \( Z \) in (5) is given by

\[
f(z) = \frac{1}{2^N \Gamma(N)} z^{N-1} W_0, N - \frac{1}{2}(2z), \quad -\infty < z < +\infty,
\]

with zero mean and variance \( 2(N - 1) \), where \( W_{\kappa, \mu}(z) \) denotes the Whittaker function [19, Eq.(13.14.3)].

**Proof:** See Appendix A.

Due to the high complexity of the PDF in (6), the exact distribution function of \( \gamma_{k, n} \) in (5) is hard to obtain. However, after carefully examining the distributions of \( \alpha X \), \( \beta Y \) and \( \eta Z \) in the denominator of (5), we find that \( \eta Z \) is negligible. Specifically, \( \beta Y \) has mean \( N - 1 \) and variance \( (N - 1)(1 - \delta_0^2) \), and \( \eta Z \) has zero mean and small variance \( 2(N - 1)\delta_0^2 (1 - \delta_0^2) \), since \( \delta_0 \) is usually very small (\( \delta_0^2 \ll 1 \)) in order to avoid inter-beam interference. For example, when \( N = 5 \) and \( \delta_0 = 0.25 \), the PDF and cumulative density function (CDF) of \( \alpha X \), \( \beta Y \) and \( \eta Z \) are plotted in Fig. 1. It is observed that the PDF of \( \eta Z \) is symmetric with respect to zero and its values are generally much smaller than \( \beta Y \). Therefore, the effect of \( \eta Z \) on the received SINR \( \gamma_{k, n} \) is negligible, and (5) can be approximated using

\[
\gamma_{k, n} \approx \frac{X}{\rho + \alpha X + \beta Y}.
\]

Consequently, the CDF and PDF of \( \gamma_{k, n} \) in (7) can be, respectively, expressed as

\[
F_{\Gamma_{k, n}}(\gamma) = 1 - \exp \left( -\frac{N\gamma}{\rho(1 - \alpha \gamma)} \right) \left( 1 + \frac{\beta \gamma}{1 - \alpha \gamma} \right)^{-(N-1)}
\]

and

\[
f_{\Gamma_{k, n}}(\gamma) = \frac{\exp \left( -\frac{N\gamma}{\rho(1 - \alpha \gamma)} \right)}{(1 - \alpha \gamma) \left( 1 + \frac{\beta \gamma}{1 - \alpha \gamma} \right)} \times \frac{N}{\rho} \left( 1 + \frac{\beta \gamma}{1 - \alpha \gamma} \right) + (N - 1)\beta.
\]

Note that, as a special case, the orthogonal OFB with \( N = N_1 \) implies \( \delta_0 = 0 \) and, thus, \( \alpha = 0 \) and \( \beta = 1 \). Putting \( \alpha = 0 \) and \( \beta = 1 \) into (7), (8), and (9), the expressions reduce to the exact counterparts under orthogonal case [18]. Consequently, our analysis is general and applicable to either orthogonal or non-orthogonal OFB. Also, the accuracy of the approximation in (7) will be further illustrated in Fig. 2, and is detailed in Section VI.

Since the maximum SINR in (2) is given by \( \hat{\gamma}_{\max, n} = \max_{k=1, \ldots, K} \gamma_{k, n} \), by using the results from the theory of order statistics, the CDF and PDF of \( \hat{\gamma}_{\max, n} \) in (2) is, respectively, given by

\[
F_{\Gamma_{\max, n}}(\gamma) = F_{\Gamma_{k, n}}^K(\gamma)
\]

and

\[
f_{\Gamma_{\max, n}}(\gamma) = K f_{\Gamma_{k, n}}(\gamma) F_{\Gamma_{k, n}}^{K-1}(\gamma).
\]

Note that when a user simultaneously obtains the maximum SINR on different beams, the transmitter has to choose another user who has the second largest SINR. This probability is very small when the number of users is large [4], and it is neglected herein.

**IV. PERFORMANCE ANALYSES**

Based on the distribution function of the received SINR, the sum-rate scaling law is first investigated in this section. The analysis of the average scheduling delay over the users then follows. Moreover, when practical constraints on the received SINR and outage probability are imposed, the maximum number of supported transmit beams and access probability are derived.

**A. Sum-Rate Scaling Law**

For beam \( n \) with received SINR \( \hat{\gamma}_{\max, n} \), the instantaneous data rate can be calculated by the Shannon formula \( \ln (1 + \hat{\gamma}_{\max, n}) \) in the unit of nat/s/Hz. Moreover, since there are \( N \) beams in total, the achievable sum-rate is given by

\[
R = \sum_{n=1}^{N} E \left\{ \ln (1 + \hat{\gamma}_{\max, n}) \right\}
\]

\[
= \sum_{n=1}^{N} \int_0^{+\infty} \ln (1 + \gamma) f_{\Gamma_{k, n}}(\gamma) F_{\Gamma_{k, n}}^{K-1}(\gamma) \, d\gamma
\]
where (11) was exploited to reach (13). Due to the complicated expressions of $f_{\Gamma_k,n}$ and $F_{\Gamma_k,n}$ in (8) and (9) respectively, the integration in (14) has no closed-form expression and has to be evaluated numerically. In order to gain insights into the sum-rate, we instead derive the limiting distribution of $\gamma_{\text{max},n}$ as $K \to \infty$. However, notice that the limiting distribution cannot be obtained directly applying $K \to \infty$ in (10) since, for any $F_{\Gamma_k,n}(\gamma) < 1$, (10) reduces to 0 as $K \to \infty$ and hence (10) is a degenerate distribution. In the following lemma, the asymptotic theory of extreme order statistics is exploited to attain a non-degenerate limiting distribution for $\gamma_{\text{max},n}$ such that the sum-rate scaling law is obtained.

Lemma 2: For the received SINR $\gamma_{k,n}$ of user $k$ with respect to beam $n$, with CDF $F_{\Gamma_k,n}(\gamma)$ and PDF $f_{\Gamma_k,n}(\gamma)$ given by (8) and (9) respectively, as the number of users $K \to \infty$, the limiting distribution of the maximum SINR in (2) is of the Gumbel distribution. That is,

$$
\lim_{K \to \infty} F_{\Gamma_{\text{max},n}}(\gamma) = H_{3,0}\left(\frac{\gamma - a}{b}\right),
$$

where $H_{3,0}(\gamma) = \exp(-e^{-\gamma})$, and the normalizing parameters $a$ and $b$ are given by

$$
a = \frac{\rho \ln K}{c + \rho a \ln K} + \mathcal{O}\left(\ln \ln K\right),
$$

where the notation $f(x) = \mathcal{O}(g(x))$ is defined as $\lim_{x \to \infty} |f(x)/g(x)| < \infty$, and

$$
b = \frac{c^2 (c + \rho \beta \ln K)}{(c + \rho a \ln K)^2} \left[\frac{N}{\rho} (c + \rho \beta \ln K) + c(N-1)\beta\right],
$$

respectively. In (16) and (17), the constant $c \triangleq N + \rho(N-1)\beta$.

Proof: See Appendix B.

Applying Lemma 2 to (12) yields:

$$
R \leq \sum_{n=1}^{N} \ln \left(1 + \mathcal{E}\{\gamma_{\text{max},n}\}\right)
$$

where the Jensen’s inequality was exploited to derive (18) from (12), the limiting distribution (15) was exploited to reach (19) and $\Upsilon = 0.5772\ldots$ is the Euler’s constant. Moreover, substituting (16) and (17) into (19) and examining it as $K \to \infty$, the expression (19) reduces to a simple sum-rate scaling law as given in the following theorem.

Theorem 1: When the total inter-beam interference $\alpha$ for each beam is identical, and as $K \to \infty$, the sum-rate of non-orthogonal OBF scales as

$$
R \sim N \ln \left(1 + \frac{1}{\alpha}\right),
$$

where the notation $f(K) \sim g(K)$ is defined as $\lim_{K \to \infty} f(K)/g(K) = 1$.

Special Case: The preceding analysis can be applied to orthogonal OBF without inter-beam interference, i.e., where $\delta_0 = 0$. In particular, putting $\alpha = 0$, $\beta = 1$, and $N = N_t$ into (16), (17), and (19) results in the following formula for the sum-rate as $K \to \infty$

$$
R' \sim N_t \ln \ln K,
$$

which is exactly the sum-rate scaling law under orthogonal transmission [4]. Furthermore, comparing (20) with (21), it is observed that, as $K \to \infty$, for non-orthogonal OBF, multi-user diversity gain vanishes and the sum-rate is dominated by $N$ and $\alpha$ while, for orthogonal OBF, multi-user diversity gain always benefits the sum-rate since there is no inter-beam interference.

The theorem above reveals that the sum-rate of non-orthogonal OBF increases proportionally to the number of transmit beams $N$, but it is offset by the total inter-beam interference $\alpha$. That is, for fixed $N_t$ and $N$, non-orthogonal OBF is an interference-limited system. Clearly, in order to achieve maximum $R$ with fixed $N$ and $N_t$, $\alpha$ must be kept as small as possible. Practical non-orthogonal beamforming matrices achieving the smallest $\alpha$ will be illustrated in Section V.

B. Average Scheduling Delay

For delay-sensitive traffic, the delay is usually analyzed in terms of the waiting time of packets in the input/output queues at the network layer, on average and over all users. However, in order to provide delay guarantee for all users, we have to study the delay for the most unfortunate user in the network. Therefore, in [1], the scheduling delay is defined as the minimum number of channel uses that guarantees each of the $K$ users is served at least one packet. This definition reflects the worst-case delay among the users, which roughly determines the time scale required for the system to behave fairly in a homogeneous network [1]. The guarantee of the worst-case delay for all users is increasingly important in commercial wireless systems.

Based on [1], the scheduling delay $D_{1,K}$ in this paper is defined as the minimum number of channel uses that guarantees each of the $K$ users is served at least once. For round-robin scheduling, it is clear that $D_{1,K} = K$. For opportunistic single-user scheduling, the average scheduling delay scales as $K \ln K$, while for the orthogonal OBF, it scales as $K \ln K/N_t$ [1]. Based on the probabilistic arguments similar to [22, Theorem 3], the following theorem shows the average scheduling delay of non-orthogonal OBF with $N > N_t$ transmit beams.

Theorem 2: For the non-orthogonal OBF with $N > N_t$ transmit beams, as $K \to \infty$, the average scheduling delay is given by

$$
\mathcal{E}\{D_{1,K}\} = \frac{K}{N} \ln K + \omega\left(\frac{K}{N}\right),
$$

where the notation $f(K) = \omega(g(K))$ is defined as $\lim_{K \to \infty} |f(K)/g(K)| = \infty$.

Proof: For the non-orthogonal OBF with $N$ beams, let the event $E$ denote a successful service that each of the $K$ users is served at least once after $D$ channel uses. Moreover, let $\mathcal{N}_k$, where $k = 1, \ldots, K$, refer to the actual number of service for user $k$, that is, $\sum_{k=1}^{K} \mathcal{N}_k = DN$. Then, the probability that the event $E$ occurs is given by

$$
P_r\{E\} = P_r\{\mathcal{N}_1 > 0, \mathcal{N}_2 > 0, \ldots, \mathcal{N}_K > 0\}$$
In principle, there could be infinite number of transmit beams if no constraint is imposed on the received SINR of served users. In practice, however, if one user is to be successfully served, a minimum SINR or equivalently, error probability requirement has to be satisfied at the receiver side [27]. On the other hand, an outage probability threshold has to be satisfied at the receiver successfully served, a minimum SINR or equivalently, error probability are delayed than that of the orthogonal counterpart with $N > N_{\text{orth}}$. This theorem implies that, for the non-orthogonal OBF with $K$ beams, the minimum total inter-beam interference is given by

$$
\alpha_{\min} = (N - 1) \times \left( \frac{N - N_{t}}{N_{t}(N - 1)} \right)^{2} = \frac{N}{N_{t}} - 1. 
$$

Applying (28) to Theorem 1 yields the following corollary.

**Corollary 1:** For fixed $N_{t}$ and $N_{t}$, when the inter-beam interference reaches the minimum $\frac{N}{N_{t}} - 1$, the sum-rate of non-orthogonal OBF scales as

$$
R \sim N \ln \left( \frac{N}{N - N_{t}} \right). 
$$

Based on Corollary 1, it can be shown that the sum-rate monotonically decreases with increasing $N$ for fixed $N_{t}$, but it increases with $N_{t}$ for fixed $N$. More precisely, when $N_{t}$ is fixed, the first-order derivative of $R$ in (29) with respect to $N$ is given by

$$
\frac{dR}{dN} = \ln \left( \frac{N}{N - N_{t}} \right) - \frac{N_{t}}{N - N_{t}} < 0, \text{ where we}
$$
exploited the property that \( \ln x < x - 1 \) whenever \( x > 1 \) [19, Eq.(4.5.4)]. On the other hand, when \( N \) is fixed and \( N > N_t \), we have \( \frac{dR}{dN} = -\frac{N}{N - N_t} > 0 \). An intuitive explanation of the monotonicity of \( R \) with respect to \( N \) and \( N_t \) is as follows. In the complex space \( \mathbb{C}^{N_t} \), for fixed \( N_t \), increasing \( N \) yields larger cross-correlation among \( N \) vectors. Nevertheless, if \( N \) is fixed, increasing \( N_t \) reduces the cross-correlation among \( N \) vectors.

In the following, we illustrate two non-orthogonal beamforming matrices which achieve the minimum total inter-beam interference in (28).

\[ B = \frac{1}{\sqrt{N_t}} F_{N_t} \]

\[ \delta_{l,n} = \frac{1}{N_t^2} \sum_{p \in \{r_1, \ldots, r_{N_t}\}} \omega^{p(l+n)} \]

Clearly, (31) implies that the correlation coefficient \( \delta_{l,n} \) depends upon the indices of selected rows from \( F \). In total, there are \( \binom{N}{N_t} \) alternatives to select \( N_t \) rows out of \( F \). Therefore, the next question is which rows should be selected so as to obtain the minimum total inter-beam interference? The following theorem answers this question.

**Theorem 3**: For the \( N \times N \) Fourier matrix, its any \( N_t \) rows can be selected as the beamforming matrix and each beam suffers the minimum total inter-beam interference \( \frac{N_{11}}{N_t^2} - 1 \).

**Proof**: According to (31), for beam \( n \), the total inter-beam interference from the other \( N - 1 \) beams is given by

\[ \delta_n = \sum_{l=0}^{N-1} \left( \frac{1}{N_t} + \frac{1}{N_t^2} \sum_{p,q \in \{r_1, \ldots, r_{N_t}\}, p \neq q} \omega^{(p-q)(l+n)} - 1 \right) \]

where the constant 1 in the second term of the right-hand side of (32) denotes the square of the auto-correlation pertaining to the \( n \)th column of \( B_r \), and we used the fact that \( \sum_{l=0}^{N-1} \omega^{l(p-q)} = 0 \) whenever \( p \neq q \) to reach (33), since \( \omega \) is the \( N^{th} \) root of unity [26, Eq.(5.8.2)]. Comparing (33) with (28), it is evident that the truncated Fourier matrix with any \( N_t \) rows achieves the minimum total inter-beam interference, which completes the proof.

Since the Fourier scheme achieves the minimum total inter-beam interference as the Grassmannian scheme when they have the same \( N_t \) and \( N \), based on Theorem 1, both schemes achieve the same sum-rate as \( K \to \infty \). This will be further illustrated in Fig. 8 in the next section.

**VI. Simulation Results and Discussions**

In this section, numerical results based on the above analyses and Monte-Carlo simulation results are presented. All transmissions are over Rayleigh fading channels with zero mean and unit variance.

As justified from (3) to (7), the received SINR in (3) is well-approximated by (7) and then its closed-form distribution functions follow in (8) and (9). The accuracy of this approximation is also demonstrated in Fig. 2, where the simulated PDF of the received SINR in (3) is compared with
the analytical PDF in (9), based on the Fourier beamforming with $N_t = 3$ and SNR = 5 dB. From the upper panel, which corresponds to the orthogonal OBF with $N = N_t = 3$, it is seen that the numerical results of (9) coincide perfectly with the simulation results, since the PDF in (9) with $\alpha = 0$ and $\beta = 1$ is the exact PDF of the received SINR under orthogonal transmission. Furthermore, the lower panel of Fig. 2 shows that, for the non-orthogonal OBF with $N_t = 3$ and $N = 5$, the numerical results of (9) are almost indistinguishable from the simulation results.

The sum-rate of orthogonal and non-orthogonal transmissions is illustrated in Fig. 3, where the orthogonal OBF is implemented with $N = N_t = 3$ and the non-orthogonal scheme operates with $N = 5$ and $N_t = 3$. For the orthogonal case, it is observed that the numerical results of the integration in (14) coincides perfectly with the simulation results. Moreover, it is seen that the analytical expression (19) is very tight with (14), in which the tiny difference is due to the Jensen's loss introduced in (18). On the other hand, for the non-orthogonal transmission, the numerical results based on (14) are only slightly smaller (about 6%) than the simulation results, due to the approximation introduced in (7). The analytical expression (19) is still very tight with (14). Furthermore, as expected, Fig. 3 shows that non-orthogonal OBF underperforms the orthogonal case in terms of sum-rate, as inter-beam interference exists. However, non-orthogonal transmission benefits the scheduling delay, which is shown in Fig. 4. Figure 4 illustrates the simulation results of the average scheduling delay versus the number of users for different number of transmit beams $N = 3, 4, 5,$ and $6$ with $N_t = 3$. It is observed that, for fixed $K$, the orthogonal case with $N = N_t = 3$ has the longest scheduling delay, while the scheduling delay decreases with $N$. Moreover, as $N$ increases, the scheduling-delay improvement from each of additional beam becomes smaller and smaller. These observations coincide with Theorem 2.

Figure 5 illustrates the tradeoff between sum-rate and scheduling delay, when considering $N_t = 3$ and $K = 50, 100,$ and $200$. When $K$ is fixed, the number of transmit beams varies from $N = N_t = 3$ (corresponding to the orthogonal scenario) to $N = 4, 5, 6,$ and $7$ (corresponding to non-orthogonal scenarios). It is observed that, for fixed $K$, the orthogonal case with $N = N_t$ yields the largest sum-rate but with the longest scheduling delay. Moreover, we can see that the scheduling delay decreases with increasing $N$ at the cost of sum-rate. On the other hand, Fig. 6 shows the ratio of sum-rate to scheduling delay versus the number of beams $N$. It is seen that, for fixed number of users $K$, the ratio monotonically increases with $N$, which means that increasing $N$ reduces the scheduling delay significantly more than the sum-rate.

Figure 7 shows the maximum number of supported transmitted beams, i.e., $N_{\text{max}}$ in (26), versus the threshold of the received SINR ($\gamma_{\text{th}}$), with $N_t = 4$ and outage probability $P_{\text{th}} = 0.01$. It is observed that, for non-orthogonal OBF with fixed number of users, $N_{\text{max}}$ increases as the threshold of the received SINR decreases, but the orthogonal case always transmits $N = N_t$ beams regardless of the users’
SINR requirements. Furthermore, for a fixed threshold of the received SINR, more users allows more transmit beams. Clearly, non-orthogonal OBF can be configured with different number of transmit beams to satisfy different received SINR requirements of users, which provides a flexibility over conventional orthogonal scheme.

Finally, according to the sum-rate scaling law in Theorem 1, different non-orthogonal beamforming schemes result in the same sum-rate as $K \to \infty$, if their total inter-beam interference for each beam are identical. When $K$ is finite, this assertion is examined in Fig. 8, which depicts the simulation results of sum-rate of Grassmannian and Fourier schemes. It is observed that, when $N_t = 2$ and $N = 3$, the sum-rates of both schemes coincide perfectly in the entire range of $K$ under consideration. When $N_t = 3$ and $N = 7$, both schemes obtain almost the same sum-rates when $K > 100$. When $N_t = 4$ and $N = 13$, the sum-rate of Grassmannian beamforming is higher than that of Fourier beamforming, however, their sum-rate difference becomes smaller and smaller as $K$ increases. Therefore, Grassmannian and Fourier beamforming results in the same sum-rate as $K \to \infty$, in accordance with Theorem 1.

VII. CONCLUSIONS

The system performance of non-orthogonal OBF with $N > N_t$ transmit beams were investigated in this paper. Our results show that non-orthogonal OBF is an interference-limited system. When the inter-beam interference attains the minimum for fixed $N_t$ and $N$, the sum-rate decreases monotonically with the number of beams $N$, while the average scheduling delay improves monotonically with increasing $N$. Furthermore, two practical non-orthogonal beamforming matrices were proposed, and they were shown to achieve the minimum inter-beam interference and thus obtain the same asymptotic sum-rate. Due to the flexibility of the number of beams $N$ related to the users’ QoS requirements and the number of users
non-orthogonal OBF achieves the tradeoff between sum-rate and scheduling delay, and it is applicable to a wide range of wireless networks, especially densely populated networks with delay-sensitive traffic.

**APPENDIX A**

**Proof of Lemma 1**

According to (4), $Z_l$ can be rewritten as

$$Z_l = h^H_k (b_{1, l, n} b^H_{1, l, n} + b^H_{l, n} b_{l, n}) h_k.$$  

where $W = b_{1, l, n} b^H_{1, l, n} + b^H_{l, n} b_{l, n}$, it is straightforward that $W$ is a Hermitian matrix with rank two. Since $\text{trace}(W) = \text{trace}(b_{1, l, n} b^H_{1, l, n}) + \text{trace}(b^H_{l, n} b_{l, n}) = 0$, we have $\text{trace}(W) = \lambda_1 + \lambda_2 = 0$, where $\lambda_1$ and $\lambda_2$ denote the two real eigenvalues of $W$. Thus, we obtain $\lambda_1 = -\lambda_2$. Furthermore, in the following, it is shown that $|\lambda_1| = |\lambda_2| = 1$.

Assuming that $x_i$ is an eigen-vector of $W$ with respect to $\lambda_i$, where $i = 1, 2$, we have

$$b_{1, l, n} b^H_{1, l, n} x_i = \lambda_i x_i.$$  

Multiplying both sides of (34) with $b_{l, n}^H$ and in view of the facts that $b_{l, n}^H b_{l, n} = 1$ and $b_{l, n}^H b_{1, l, n} = 0$, we get $b_{l, n}^H x_i = \lambda_i^{b_{l, n}} x_i$. Similarly, multiplying both sides of (34) with $b_{1, l, n}^H$ yields $b_{1, l, n}^H x_i = \lambda_1^{b_{1, l, n}} x_i$. Substituting the former into the latter results in $b_{l, n}^H x_i = \lambda_i^{b_{l, n}} x_i$ and, finally, taking account of the fact that $\lambda_i$ is a real number, we obtain that $\lambda_i \in \{ \pm 1 \}$, where $i = 1, 2$.

Applying the spectral decomposition theorem over $W$, $Z_l$ can be rewritten as

$$Z_l = h^H_k U D U^H h_k = \sum_{i=1}^{2} \lambda_i \left| h_i^H u_{k,i} \right|^2 E_i$$

where $U = [u_1, \ldots, u_N]$ is a unitary matrix and $D$ is a diagonal matrix with the first two diagonal elements as $\lambda_1$ and $\lambda_2$ and other elements being zero. Since $h_k$ is a complex Gaussian random vector, $E_i$, $i = 1, 2$ in (35), are two independent exponentially-distributed variables. Therefore, no matter (8) or (9) in (36).

Furthermore, it is clear that the Laplace transform of $Z = \sum_{i=1}^{N} Z_l$ is given by $1/(1 - s^2)^{N-1}$. Accordingly, the inverse transform yields $n = \text{trace}(h_k^H u_k^H) = \lambda_1^{b_{1, l, n}} x_i$ and, finally, taking account of the fact that $\lambda_i$ is a real number, we obtain that $\lambda_i \in \{ \pm 1 \}$, where $i = 1, 2$.

In the following, it is shown that $|\lambda_1| = |\lambda_2| = 1$.

**APPENDIX B**

**Proof of Lemma 2**

We exploit the von Mises criteria [21, Theorem 2.7.2] to determine the limiting distribution of $\tilde{\gamma}_{\text{max}, n}$. Accordingly, two conditions need to be satisfied. The first one requires that the PDF $f_{\tilde{F}_{k, n}}(\gamma)$ in (8) is differentiable for all $\gamma \in (0, 1/\alpha)$, where $1/\alpha$ is the supremum of $\gamma_k, n$ in (7). This is obviously true. Furthermore, substituting (8) and (9) into the function $g(\gamma) = (1 - F_{k, n}(\gamma))/f_{\tilde{F}_{k, n}}(\gamma)$ and performing some algebraic manipulations, it can be shown that $\lim_{\gamma \rightarrow 1/\alpha} g'(\gamma) = 0$, which is the second condition. Consequently, the limiting

CDF of $\tilde{\gamma}_{\text{max}, n}$ is of the Gumbel distribution [21, p.54], as shown in (15). Moreover, the normalizing parameter $\alpha$ is the solution to the equation $1 - F_{k, n}(\alpha) = 1/K$ [21, Theorem 2.1.3]. Substituting (9) into the preceding equation yields a transcendental equation:

$$\exp \left( -\frac{N a}{\rho(1 - \alpha)} \right) \left( 1 + \alpha \left( 1 - \gamma_0^2 \right)^{-1} \right)^{-(N-1)} = \frac{1}{K}$$

Now, taking the logarithm on both sides of (36), applying the first-order Taylor approximation of $\ln(1 + x)$ with small $x$, and performing further derivations, $\alpha$ can be given by (16).

Once $\alpha$ is determined, the value of $b$ is given by $b = g(\alpha)$ [21, Remark 2.7.1], which results in the desired (17).
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