An Exact and Grid-free Numerical Scheme for the Hybrid Two Phase Traffic Flow Model Based on the Lighthill-Whitham-Richards Model with Bounded Acceleration

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ABSTRACT

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In this article, we propose a new grid-free and exact solution method for computing solutions associated with an hybrid traffic flow model based on the Lighthill-Whitham-Richards (LWR) partial differential equation. In this hybrid flow model, the vehicles satisfy the LWR equation whenever possible, and have a fixed acceleration otherwise. We first present a grid-free solution method for the LWR equation based on the minimization of component functions. We then show that this solution method can be extended to compute the solutions to the hybrid model by proper modification of the component functions, for any concave fundamental diagram. We derive these functions analytically for the specific case of a triangular fundamental diagram. We also show that the proposed computational method can handle fixed or moving bottlenecks.

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

1.1 Background

One of seminal traffic flow models for highways is presented in [1] and [2], and results in the so called *Lighthill-Whitham-Richards* (LWR) model or *kinematic wave* theory. Although more sophisticated models of traffic flow are available, the LWR model is widely used to model highway traffic [3] and more recently for urban traffic [4]. The LWR *partial differential equation* (PDE) is a first order scalar hyperbolic conservation law that computes the evolution of a density function (the density of vehicles on a road section). This PDE has multiple solutions in general, among which the entropy solution [5] is recognized to be the physically meaningful solution.

While the LWR model is very robust, it fails to capture some observed features of traffic such as traffic instabilities or kinematic constraints of real vehicles. In particular, the viscosity solutions [6] to the LWR model do not model the dynamics of the vehicles [7] accurately, as the vehicle trajectories solutions to the LWR model can exhibit unrealistically large accelerations or decelerations. These modeling errors can have significant effects, for instance when computing the discharge rate of bottlenecks [8, 9, 10], or when coupling [7, 11] vehicle models (fuel consumption, noise) with traffic flow models.

These shortcomings led to a quantity of work in the field of higher order models, such as the second order models [12, 13].the two phase flow model was introduced in [9, 14], and was further developed in [7] for the case of fixed and moving bottlenecks, where it was introduced as the *field bounded acceleration* model. In the two phase flow model, the vehicles follow the LWR model in some areas of the computational domain, while satisfying kinematic constraints (bounded acceleration) otherwise. The formal definition (in terms of PDE) for such a model is particularly difficult: unlike the case of ordinary differential equations, there is no general framework for solving and even mathematically defining hybrid PDEs. As part of this article, we propose a formal mathematical definition of the solution to the two phase flow model (or equivalently field bounded acceleration model).

The LWR PDE itself can be numerically solved using a variety of computational methods, such as first order numerical schemes, for instance in [15, 16, 17]. Classical numerical methods often require a computational grid, and yield an approximate solution of the PDE. Some exceptions exist however, such as the wave tracking methods, see for instance [18], or semi-analytic algorithms [19, 20, 21]. In the present article, we propose an modification of the semi-analytical algorithm introduced in [21] for computing solutions to the hybrid two phase LWR model [9, 7]. This modification preserves the fundamental advantages of the semi-analytical algorithm: exactness and fast computational time.

The rest of this article is organized as follows. Section 2 defines the LWR and HJ PDEs investigated in this article. Section **??** introduces the concept of partial solutions to the HJ PDE. We show that these partial solutions can cause the acceleration (or deceleration) to become very large or unbounded. These partial solutions are further modified to serve as the building blocks of the solution to the two phase flow model, which is defined semi-analytically in Section 5. We further derive the expression of the partial solutions for the triangular fundamental diagram in Section 6, and present a pseud-code implementation of the resulting numerical scheme. Finally, we illustrate the performance of the algorithm on simulated traffic flow scenarios including moving and fixed bottlenecks in Section 7.

Chapter 2

Modeling

2.1 The LWR PDE

We consider a one-dimensional, homogeneous section of highway, limited by x_0 upstream and x_n downstream. For a given time $t \in [0, t_m]$ and position $x \in [x_0, x_n]$, we define the local traffic density $k(x,t)$ in vehicles per unit length, and the instantaneous flow $q(x,t)$ in vehicles per unit time. The conservation of vehicles on the highway is written as follows [1, 2, 22]:

$$\forall x, t \in [x_0, x_n] \times [0, t_m], \quad \frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \tag{2.1}$$

For first order traffic flow models, flow and density are related by the *fundamental diagram* $Q : (x, t, k(x,t)) \mapsto q(x,t)$, which is an empirically measured law [23]. Through this article, we consider the homogeneous problem [24] in which the fundamental diagram is a function of density k only, i.e. $q(x, t, k(x,t)) = Q(k(x,t))$.

The fundamental diagram is a positive function defined on $[0, \kappa]$, where κ is the maximal density (jam density). It ranges in $[0, q_{\max}]$ where $q_{\max}$ is the maximum flow (capacity). It is assumed to be differentiable at 0 and κ, with $Q'(0) = v_f > 0$ the free flow speed, and $Q'(\kappa) = w < 0$ the congested wave speed [1]. We assume that the fundamental diagram is concave and continuous. The introduction of the fundamental diagram yields the *Lighthill-Whitham-Richards (LWR) PDE*:

$$\forall x, t \in [x_0, x_n] \times [0, t_m], \quad \frac{\partial k(x,t)}{\partial t} + \frac{\partial Q(k(x,t))}{\partial x} = 0 \tag{2.2}$$

2.2 The Moskowitz function

The cumulated vehicle count $\mathbf{N}(x,t)$, also called Moskowitz function [25], represents the continuous vehicle count at location x and time t. It has been developed for instance in [3, 26, 24] in the context of transportation engineering, and goes back to [27, 25].

In the Moskowitz framework, one assumes that all vehicles are labeled by increasing integers as they pass the entry point x_0 of a highway section, and that they cannot pass each other. If the latest car that passed an observer standing at location x and time t is labeled n, then $\lfloor \mathbf{N}(x,t) \rfloor = n$. The Moskowitz function contains all traffic information that one can infer from experimental traffic measurements as long as vehicles do not pass each other. In this specific case, the isolines of $\mathbf{N}(x,t)$ correspond to vehicle trajectories.

Moreover, local density $k(x,t)$ and flow $q(x,t)$ can be inferred from vehicle counts using the formulas

$$k(x,t) = -\frac{\partial \mathbf{N}(x,t)}{\partial x} \tag{2.3}$$

$$q(x,t) = \frac{\partial \mathbf{N}(x,t)}{\partial t} \tag{2.4}$$

Introducing the Moskowitz function in (2.2) yields the *Hamilton-Jacobi PDE* [3, 26, 24, 19, 20, 21] in which the fundamental diagram Q plays the role of Hamiltonian [28]:

$$\frac{\partial \mathbf{N}(x,t)}{\partial t} - Q\left(-\frac{\partial \mathbf{N}(x,t)}{\partial x}\right) = 0 \tag{2.5}$$

2.3 Piecewise affine initial and boundary conditions

For the rest of the article, we use piecewise constant initial and boundary conditions on density and flow, which translate to piecewise-affine conditions on the Moskowitz function. Piecewise constant conditions on density and flow are a natural way to encode discrete measurements in the model, and are used for instance in the *Cell Transmission Model* (CTM) [16] and the associated Godunov scheme [15].

Let m and $n \geq 1$ be integers, $x_0 < x_1 < \cdots < x_n$ and $t_0 < t_1 < \cdots < t_m$ the space-time discretization for initial and boundary conditions where $t_0 = 0$. We assume that the initial densities $(k_{\text{ini}}^{(i)})_{0 \leq i \leq n-1} \in \mathbb{R}_+^n$, the upstream flows $(q_{\text{up}}^{(j)})_{0 \leq j \leq m-1} \in \mathbb{R}_+^m$ and the downstream flows $(q_{\text{down}}^{(j)})_{0 \leq j \leq m-1} \in \mathbb{R}_+^m$ are given. The initial densities are thus decomposed as piecewise constant in their respective measurement intervals:

$$\forall x \in [x_i, x_{i+1}], \quad k(x,0) = k_{\text{ini}}^{(i)} \tag{2.6}$$

and let the upstream and downstream flows also be prescribed as piecewise constant:

$$\forall t \in [t_j, t_{j+1}[, \quad q(x_0,t) = \quad q_{\text{up}}^{(j)} \tag{2.7}$$

$$q(x_n,t) = \quad q_{\text{down}}^{(j)} \tag{2.8}$$

Note that no assumption is made regarding the uniformity of the grid: the spacings $x_i - x_{i-1}$ and $t_i - t_{i-1}$ are not necessarily uniform over i.

The initial condition of the Moskowitz PDE is obtained by integrating the initial condition of the LWR PDE assuming that $\mathbf{N}_{\text{ini}}(x_0) = 0$ and :

$$\forall x \in [x_i, x_{i+1}], \mathbf{N}_{\text{ini}}(x) = -\int_{x_0}^{x} k(\chi,0)d\chi = -\sum_{m=0}^{i-1}(x_{m+1}-x_m)k_{\text{ini}}^{(m)} - (x-x_i)k_{\text{ini}}^{(i)} \tag{2.9}$$

Similarly, the upstream and downstream boundary conditions of the Moskowitz PDE, assuming that $\mathbf{N}_{\text{up}}(0) = 0$ and $\mathbf{N}_{\text{ini}}(x_n) = \mathbf{N}_{\text{down}}(0)$ are given by:

$$\forall t \in [t_j, t_{j+1}], \mathbf{N}_{\text{up}}(t) = \int_0^t q_{\text{up}}(\tau)d\tau = \sum_{m=0}^{j-1}(t_{m+1} - t_m)q_{\text{up}}^{(m)} + (t - t_j)q_{\text{up}}^{(j)} \tag{2.10}$$

$$\forall t \in [t_j, t_{j+1}], \mathbf{N}_{\text{down}}(t) = \mathbf{N}_{\text{ini}}(x_n) + \int_0^t q_{\text{down}}(\tau)d\tau = \mathbf{N}_{\text{ini}}(x_n) + \sum_{m=0}^{j-1}(t_{m+1} - t_m)q_{\text{down}}^{(m)} + (t - t_j)q_{\text{down}}^{(j)} \tag{2.11}$$

2.4 Affine internal boundary conditions

Similarly to the algorithm described in [21], our proposed algorithm can also integrate any number of internal boundary conditions, which can be used to model fixed or moving bottlenecks [29].

An affine internal boundary condition of the Moskowitz function is mathematically defined as :

$$c_{\text{intern}}^{(l)}(x,t) = M^{(l)} + q_{\text{intern}}^{(l)}(t - t_{min}^{(l)}) \quad : t_{min}^{(l)} \leq t \leq t_{max}^{(l)} \text{ and } x = x^{(l)} + V_{\text{intern}}^{(l)}(t - t_{min}^{(l)}) \tag{2.12}$$

In the above formula, the internal boundary condition imposes an average maximal passing rate of $q_{intern}^{(l)}$ on the domain defined by $t_{min}^{(l)} \leq t \leq t_{max}^{(l)}$ and $x = x_{min}^{(l)} + V_{\text{intern}}^{(l)}(t - t_{min}^{(l)})$. It can represent in practice a fixed ($V_{\text{intern}}^{(l)} = 0$) or a moving ($V_{\text{intern}}^{(l)} > 0$) bottleneck, restricting the relative capacity of the road to $q_{intern}^{(l)}$ on its path.

Chapter 3

Analytical Solutions to the Moskowitz HJ PDE and LWR PDE

3.1 Solutions to the Hamilton-Jacobi equation

In order to compute the analytical solution of equation (2.5) with conditions of the type (2.9), (2.10), (2.11), and (2.12), we define (based on [24, 30]) the following convex transform associated with the fundamental diagram:

$$\forall u \in [w, v_f], \quad R(u) = \sup_{k \in [0,\kappa]} \left(Q(k) - u \cdot k \right) \tag{3.1}$$

The function $-R$ is the Legendre-Fenchel transform of the function Q (fundamental diagram).

3.2 Solution components associated with affine conditions

In this section, we use the notation Q' for the derivative of the fundamental diagram (which only depends on one argument). Note that we have assumed earlier that Q has a right derivative v_f in 0 and a left derivative w in κ.

Following [21], we define below the partial components of the solutions for the four types of value conditions.

3.2.1 Initial conditions

We want to compute the solution component induced by an affine, locally defined initial condition indexed by i:

$$\forall x \in [x_i, x_{i+1}], \quad \mathbf{c}_{\text{ini}}^{(i)}(x) = -k_i x + b_i \tag{3.2}$$

with $b_i = k_i x_i - \sum_{l=0}^{i-1}(x_{l+1} - x_l)k_{\text{ini}}^{(l)}$ allowing for the continuity of the initial conditions in $(0, x_i)$. Using the results of [20], the analytical solution to the problem associated with this sole initial condition can be written as

$$\mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(x,t) = \begin{cases} tQ(k_i) - k_i x + b_i & : x_i + tQ'(k_i) \leq x \leq x_{i+1} + tQ'(k_i) \\ tR(\frac{x-x_i}{t}) - k_i x_i + b_i & : x_i + tw \leq x \leq x_i + tQ'(k_i) \\ tR(\frac{x-x_{i+1}}{t}) - k_i x_{i+1} + b_i & : x_{i+1} + tQ'(k_i) \leq x \leq x_{i+1} + tv_f \end{cases} \tag{3.3}$$

$$k_{\mathbf{c}_{\text{ini}}^{(i)}}(x,t) = -\frac{\partial \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}}{\partial x}(x,t) = \begin{cases} k_i & : x_i + tQ'(k_i) \leq x \leq x_{i+1} + tQ'(k_i) \\ -R'(\frac{x-x_i}{t}) & : x_i + tw \leq x \leq x_i + tQ'(k_i) \\ -R'(\frac{x-x_{i+1}}{t}) & : x_{i+1} + tQ'(k_i) \leq x \leq x_{i+1} + tv_f \end{cases} \tag{3.4}$$

3.2.2 Upstream boundary conditions

We now compute the solution component induced by an affine, locally defined upstream boundary condition indexed by j.

$$\forall t \in [t_j, t_{j+1}], \mathbf{c}_{\text{up}}^{(j)}(t) = q_j t + d_j \tag{3.5}$$

with $d_j = k_j x_j + \sum_{l=0}^{i-1}(t_{l+1} - t_l) q_{\text{up}}^{(l)}$. Following [20], we define the freeflow density function K_{up}, which is the inverse of the restriction of the fundamental diagram Q to the domain $[0, k_c]$:

$$K_{\text{up}}(q) = \min\{k \in [0, \kappa] | Q(k) = q\}$$

Using the results of [20], one can prove that:

$$\mathbf{N}_{\mathbf{c}_{\text{up}}^{(j)}}(x,t) = \begin{cases} d_j + q_j t_{j+1} + (t - t_{j+1}) R(\frac{x - x_0}{t - t_{j+1}}) & : 0 \leq x - x_0 \leq Q'(K_{\text{up}}(q_j))(t - t_{j+1}) \\ d_j + q_j t - K_{\text{up}}(q_j)(x - x_0) & : Q'(K_{\text{up}}(q_j))(t - t_{j+1}) \leq x - x_0 \leq \\ & Q'(K_{\text{up}}(q_j))(t - t_j) \\ d_j + q_j t_j + (t - t_j) R(\frac{x - x_0}{t - t_j}) & : Q'(K_{\text{up}}(q_j))(t - t_j) \leq x - x_0 \leq \\ & v_f(t - t_j) \end{cases} \tag{3.6}$$

$$k_{\mathbf{c}_{\text{up}}^{(i)}}(x,t) = -\frac{\partial \mathbf{N}_{\mathbf{c}_{\text{up}}^{(i)}}}{\partial x}(x,t) = \begin{cases} -R'(\frac{x - x_0}{t - t_{j+1}}) & : 0 \leq x - x_0 \leq (t - t_{j+1}) Q'(K_{\text{up}}(q_j)) \\ K_{\text{up}}(q_j) & : Q'(K_{\text{up}}(q_j))(t - t_{j+1}) \leq x - x_0 \leq \\ & Q'(K_{\text{up}}(q_j))(t - t_j) \\ -R'(\frac{x - x_0}{t - t_j}) & : Q'(K_{\text{up}}(q_j))(t - t_j) \leq x - x_0 \leq \\ & v_f(t - t_j) \end{cases} \tag{3.7}$$

3.2.3 Downstream boundary conditions

Finally, the same process can be repeated for the downstream boundary:

$$\forall t \in [t_j, t_{j+1}], \quad \mathbf{c}_{\text{down}}^{(j)}(t) = p_j t + b_j \tag{3.8}$$

with $b_j = k_j x_j + \sum_{l=0}^{i-1}(t_{l+1} - t_l) q_{\text{down}}^{(l)}$. In a symmetric way from the upstream case, we define the congestion density function K_{down}, which is the inverse of the restriction of the fundamental diagram Q to the domain $[k_c, \kappa]$:

$$K_{\text{down}}(q) = \max\{k \in [0, \kappa] | Q(k) = q\}$$

Using the results of [20], we can similarly prove that:

$$\mathbf{N}_{\mathbf{c}_{\text{down}}^{(j)}}(x,t) = \begin{cases} b_j + p_j t + K_{\text{down}}(p_j)(x_n - x) & : Q'(K_{\text{down}}(p_j))(t - t_j) \leq x - x_n \\ & \leq Q'(K_{\text{down}}(p_j))(t - t_{j+1}) \\ b_j + p_j t_j + (t - t_j) R(\frac{x_n - x}{t_j - t}) & : w(t - t_j) \leq x - x_n \leq \\ & Q'(K_{\text{down}}(p_j))(t - t_j) \\ b_j + p_j t_{j+1} + (t - t_{j+1}) R(\frac{x_n - x}{t_{j+1} - t}) & : Q'(K_{\text{down}}(p_j))(t - t_{j+1}) \leq x - x_n \leq 0 \end{cases} \tag{3.9}$$

$$k_{\mathbf{c}_{\text{down}}^{(i)}}(x,t) = -\frac{\partial \mathbf{N}_{\mathbf{c}_{\text{down}}^{(i)}}}{\partial x}(x,t) = \begin{cases} K_{\text{down}}(p_j) & : Q'(K_{\text{down}}(p_j))(t - t_j) \leq x - x_n \\ & \leq Q'(K_{\text{down}}(p_j))(t - t_{j+1}) \\ -R'(\frac{x - x_n}{t - t_j}) & : w \leq \frac{x_n - x}{t_j - t} \leq Q'(K_{\text{down}}(p_j)) \\ -R'(\frac{x - x_n}{t - t_{j+1}}) & : Q'(K_{\text{down}}(p_j))(t - t_{j+1}) \leq x - x_n \leq 0 \end{cases} \tag{3.10}$$

3.2.4 Internal boundary conditions

In order to write the internal boundary condition component associated with (2.12) explicitly, we first have to define k_1 and k_2 such that $k_1 \leq k_2$ and:

$$Q(k_1) - k_1 V_{intern}^{(l)} = q_{intern}^{(l)} \tag{3.11}$$

$$Q(k_2) - k_2 V_{intern}^{(l)} = q_{intern}^{(l)} \tag{3.12}$$

Using the results of [20], we can similarly prove that the partial solution associated with (2.12) is:

$$\mathbf{N}_{\mathbf{c}^{(l)}_{\text{intern}}}(x,t) = \begin{cases} Q(k_1)(t-t^{(l)}_{min}) + (x^{(l)}-x)k_1 + M^{(l)} & : x^{(l)} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \leq x \text{ and} \\ & t - t^{(l)}_{max} \leq \frac{x^{(l)}+V^{(l)}_{intern}(t-tmin^{(l)})-x}{-Q'(k_1)+V^{(l)}_{intern}} \\ & \leq t - t^{(l)}_{min} \\ Q(k_2)(t-t^{(l)}_{min}) + (x^{(l)}-x)k_2 + M^{(l)} & : x^{(l)} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \geq x \text{ and} \\ & t - t^{(l)}_{max} \leq \frac{x^{(l)}+V^{(l)}_{intern}(t-tmin^{(l)})-x}{-Q'(k_2)+V^{(l)}_{intern}} \\ & \leq t - t^{(l)}_{min} \\ M^{(l)} + (t-t^{(l)}_{min})R\left(\frac{x^{(l)}-x}{t^{(l)}_{min}-t}\right) & : x^{(l)} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \leq x \text{ and} \\ & t - t^{(l)}_{min} \leq \frac{x^{(l)}+V^{(l)}_{intern}(t-tmin^{(l)})-x}{-Q'(k_1)+V^{(l)}_{intern}} \\ & \text{or} \\ & x^{(l)} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \geq x \text{ and} \\ & t - t^{(l)}_{min} \leq \frac{x^{(l)}+V^{(l)}_{intern}(t-tmin^{(l)})-x}{-Q'(k_2)+V^{(l)}_{intern}} \\ q^{(l)}_{intern}(t^{(l)}_{max}-t^{(l)}_{min}) + M^{(l)} + & : x^{(l)} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \leq x \text{ and} \\ (t-t^{(l)}_{max})R\left(\frac{x^{(l)}+V^{(l)}_{intern}(t^{(l)}_{max}-t^{(l)}_{min})-x}{t_{max}-t}\right) & \frac{x^{(l)}+V^{(l)}_{intern}(t-tmin^{(l)})-x}{-Q'(k_1)+V^{(l)}_{intern}} \leq t - t^{(l)}_{max} \\ & \text{or} \\ & x^{(l)} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \geq x \text{ and} \\ & \frac{x^{(l)}+V^{(l)}_{intern}(t-tmin^{(l)})-x}{-Q'(k_2)+V^{(l)}_{intern}} \leq t - t^{(l)}_{max} \end{cases} \tag{3.13}$$

$$k_{\mathbf{c}^{(l)}_{\text{intern}}}(x,t) = \begin{cases} k_1 & : x^{(l)} + V^{(l)}_{intern}(t - t^{(l)}_{min}) < x \text{ and } \\ & t - t^{(l)}_{max} \leq \frac{x^{(l)} + V^{(l)}_{intern}(t - tmin^{(l)}) - x}{-Q'(k_1) + V^{(l)}_{intern}} \\ & \leq t - t^{(l)}_{min} \\ k_2 & : x^{(l)} + V^{(l)}_{intern}(t - t^{(l)}_{min}) > x \text{ and } \\ & t - t^{(l)}_{max} \leq \frac{x^{(l)} + V^{(l)}_{intern}(t - tmin^{(l)}) - x}{-Q'(k_2) + V^{(l)}_{intern}} \\ & \leq t - t^{(l)}_{min} \\ R'\left(\frac{x - x^{(l)}}{t - t^{(l)}_{min}}\right) & : x^{(l)} + V^{(l)}_{intern}(t - t^{(l)}_{min}) < x \text{ and } \\ & t - t^{(l)}_{min} < \frac{x^{(l)} + V^{(l)}_{intern}(t - tmin^{(l)}) - x}{-Q'(k_1) + V^{(l)}_{intern}} \\ & \text{or} \\ & x^{(l)} + V^{(l)}_{intern}(t - t^{(l)}_{min}) > x \text{ and } \\ & t - t^{(l)}_{min} < \frac{x^{(l)} + V^{(l)}_{intern}(t - tmin^{(l)}) - x}{-Q'(k_2) + V^{(l)}_{intern}} \\ R'\left(\frac{x^{(l)} + V^{(l)}_{intern}(t^{(l)}_{max} - t^{(l)}_{min}) - x}{t^{(l)}_{max} - t}\right) & : x^{(l)} + V^{(l)}_{intern}(t - t^{(l)}_{min}) < x \text{ and } \\ & t - t^{(l)}_{max} > \frac{x^{(l)} + V^{(l)}_{intern}(t - tmin^{(l)}) - x}{-Q'(k_1) + V^{(l)}_{intern}} \\ & \text{or} \\ & x^{(l)} + V^{(l)}_{intern}(t - t^{(l)}_{min}) > x \text{ and } \\ & t - t^{(l)}_{max} > \frac{x^{(l)} + V^{(l)}_{intern}(t - tmin^{(l)}) - x}{-Q'(k_2) + V^{(l)}_{intern}} \end{cases} \tag{3.14}$$

3.3 Componentwise computation of the Moskowitz/LWR function

It is shown in [31, 30] that the solution to the HJ PDE can be computed by taking into account all contributions of initial and boundary conditions using a union property of capture basins: the inf-morphism property.

Let us consider the following mixed initial-boundary-internal conditions problem consisting of a set of initial conditions (2.9), (2.10), (2.11) and (2.12). Let **N** represent the solution (if it exists, see [32] for a formal definition of weak solutions to (2.2)) to (2.5) that satisfies at the same time all initial, boundary and internal conditions. Let The inf-morphism property, also derived by Newell in [3] states that the solution is the minimum of the corresponding partial solution components.

$$\mathbf{N}(x,t) = \min_{i,j,l} \left\{ \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(x,t), \mathbf{N}_{\mathbf{c}_{\text{up}}^{(j)}}(x,t), \mathbf{N}_{\mathbf{c}_{\text{down}}^{(j)}}(x,t), \mathbf{N}_{\mathbf{c}_{\text{intern}}^{(l)}}(x,t) \right\} \tag{3.15}$$

This last result is fundamental: it shows that in order to solve the HJ PDE, we only have to apply the formulas above (3.3,3.6,3.9) and (3.13) for each affine piece of initial, boundary and internal condition, which will give the associated solution component, and then compute the minimum of all results. .

We now show that the resulting solutions can exhibit unrealistically high accelerations or decelerations by deriving the analytical expression of vehicle trajectories.

Chapter 4

Analytical Derivation of Vehicle Trajectories

By definition of the Moskowitz function, its isolines correspond to vehicle trajectories if vehicles keep their order. Since the Moskowitz function is given by (3.15), its isolines are fragments of isolines of $\mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}$, $\mathbf{N}_{\mathbf{c}_{\text{up}}^{(j)}}$, $\mathbf{N}_{\mathbf{c}_{\text{down}}^{(j)}}$, or $\mathbf{N}_{\mathbf{c}_{\text{intern}}^{(l)}}$ which we now compute.

4.1 Initial conditions

The trajectories corresponding to an initial condition component (3.3) are defined by:

$$\mathbf{x}_{\mathbf{c}_{\text{ini}}^{(i)}}(N,t) = \begin{cases} x_i + tR^{-1}(\frac{N+K_i x_i - b_i}{t}) & : \frac{N+K_i x_i - b_i}{R(\omega)} \leq t \leq \frac{N+K_i x_i - b_i}{R(Q'(k_i))} \\ \frac{b_i + tQ(k_i) - N}{k_i} & : \frac{N+K_i x_i - b_i}{R(Q'(k_i))} \leq t \leq \frac{N+K_i x_{i+1} - b_i}{R(Q'(k_i))} \\ x_{i+1} + tR^{-1}(\frac{N+K_i x_{i+1} - b_i}{t}) & : \frac{N+K_i x_{i+1} - b_i}{R(Q'(k_i))} \leq t \leq \frac{N+K_i x_{i+1} - b_i}{R(v_f)} \end{cases} \quad (4.1)$$

Note that this trajectory exhibits kinks (or infinite accelerations) when Q' is discontinuous (for instance in the case of the triangular fundamental diagram). Even if Q' is continuous, the acceleration corresponding to this trajectory might be unbounded.

4.2 Upstream and downstream boundary conditions

Similarly, the trajectories associated with the upstream and downstream boundary condition components (3.6) and (3.9) are defined by:

$$\mathbf{x}_{\mathbf{c}_{\text{up}}^{(j)}}(N,t) = \begin{cases} x_0 + (t - t_{j+1})R^{-1}(\frac{N - q_j t_{j+1} - d_j}{t - t_{j+1}}) & : \frac{N - q_j t_{j+1} - d_j}{R(0)} + t_{j+1} \leq t \leq \frac{N - q_j t_{j+1} - d_j}{R(Q'(K_{up}(q_j)))} \\ & +t_{j+1} \\ x_0 - \frac{N - q_j t - d_j}{K_{up}(q_j)} & : \frac{N - q_j t_{j+1} - d_j}{R(Q'(K_{up}(q_j)))} + t_{j+1} \leq t \leq \frac{N - q_j t_j - d_j}{R(Q'(K_{up}(q_j)))} \\ & +t_j \\ x_0 + (t - t_j)R^{-1}(\frac{N - q_j t_j - d_j}{t - t_j}) & : \frac{N - q_j t_j - d_j}{R(Q'(K_{up}(q_j)))} + t_j \leq t \leq \frac{N - q_j t_j - d_j}{R(v_f)} \\ & +t_j \end{cases} \quad (4.2)$$

$$\mathbf{x}_{\mathbf{c}_{\text{down}}^{(j)}}(N,t) = \begin{cases} x_n + (t - t_j)R^{-1}(\frac{N - p_j t_j - b_j}{t - t_j}) & : \frac{N - p_j t_j - b_j}{R(\omega)} + t_j \leq t \leq \frac{N - p_j t_j - b_j}{R(Q'(K_{down}(p_j)))} + t_j \\ x_n - \frac{N - p_j t - b_j}{K_{down}(p_j)} & : \frac{N - p_j t_j - b_j}{R(Q'(K_{down}(p_j)))} + t_j \leq t \leq \frac{N - p_j t_{j+1} - b_j}{R(Q'(K_{down}(p_j)))} \\ & +t_{j+1} \\ x_n + (t - t_{j+1})R^{-1}(\frac{N - p_j t_{j+1} - b_j}{t - t_{j+1}}) & : \frac{N - p_j t_{j+1} - b_j}{R(Q'(K_{down}(p_j)))} + t_{j+1} \leq t \leq \frac{N - p_j t_{j+1} - b_j}{R(0)} \\ & +t_{j+1} \end{cases} \quad (4.3)$$

4.3 Internal boundary conditions

The trajectories associated with an internal condition (3.13) component (moving or fixed bottleneck) are defined by the following formula:

$$\mathbf{x}_{\mathbf{c}_{\text{internal}}}(N,t) = \begin{cases} x_{min} - \frac{N-M-(t-t_{min})Q(k_1)}{k_1} & : t \geq t_{min} + \frac{N-M}{q_{int}} \\ & \text{and} \\ & t_{max} + \frac{N-M-q_{int}(t_{max}-t_{min})}{R(Q'(k_1))} \leq t \\ & \leq t_{min} + \frac{N-M}{R(Q'(k_1))} \\ x_{min} - \frac{N-M-(t-t_{min})Q(k_2)}{k_2} & : t \leq t_{min} + \frac{N-M}{q_{int}} \\ & \text{and} \\ & t_{max} + \frac{N-M-q_{int}(t_{max}-t_{min})}{R(Q'(k_2))} \geq t \\ & \geq t_{min} + \frac{N-M}{R(Q'(k_2))} \\ x_{min} + (t-t_{min})R^{-1}(\frac{N-M}{t-t_{min}}) & : t \geq t_{min} + \frac{N-M}{R(V_{int})} \\ & \text{and} \\ & t \geq t_{min} + \frac{N-M}{R(Q'(k_1))} \\ & or \\ & t \leq t_{min} + \frac{N-M}{R(V_{int})} \\ & \text{and} \\ & t \leq t_{min} + \frac{N-M}{R(Q'(k_2))} \\ x_{min} + V_{int}(t_{max}-t_{min}) & : t \geq t_{max} + \frac{N-M-q_{int}(t_{max}-t_{min})}{R(V_{int})} \\ +(t-t_{max})R^{-1}(\frac{N-M-q_{int}(t_{max}-t_{min})}{t-t_{max}}) & \text{and} \\ & t \leq t_{max} + \frac{N-M-q_{int}(t_{max}-t_{min})}{R(Q'(k_1))} \\ & or \\ & t \leq t_{max} + \frac{N-M-q_{int}(t_{max}-t_{min})}{R(V_{int})} \\ & \text{and} \\ & t \geq t_{max} + \frac{N-M-q_{int}(t_{max}-t_{min})}{R(Q'(k_2))} \end{cases} \tag{4.4}$$

As in the previous cases, the acceleration corresponding to these trajectories can be unbounded or can exceed the physical capabilities of the vehicles.

Chapter 5

Definition of the Solution to the Two Phase LWR Flow Model with Bounded Acceleration

5.1 The two phase bounded acceleration-LWR traffic flow model

The two phase bounded acceleration-LWR traffic flow model was introduced in [14], and further extended in [7] for fixed and moving bottlenecks, where it was known as the *field bounded acceleration* model. A numerical solver based on the Godunov scheme is also derived in [7].

Defining the solution to the two phase model is complex. In [14], it requires the computation of phase transition boundaries, which is very complex for general problems involving moving bottlenecks. In [7], the solution is derived from the Godunov scheme by modifying the fundamental diagram in the different areas of the computational domain.

Instead, in this article we define the solution to the two phase bounded acceleration-

LWR problem in the Moskowitz space by the following implicit formula, which does not require density-dependent modifications of the fundamental diagram nor the computation of transition zones.

We first define the vehicle speed corresponding to the initial, boundary and internal condition components $\mathbf{N}_{\mathbf{c}^{(i)}}$ as follows:

$$v_{\mathbf{c}^{(i)}}(\tilde{t},\tilde{x}) = \frac{-\frac{\partial \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t},\tilde{x})}{\partial t}}{\frac{\partial \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t},\tilde{x})}{\partial x}}$$

We then define the modified solution component $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}$ corresponding to an initial, boundary or internal condition block $\mathbf{c}^{(i)}$ as follows:

$$\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(x,t) = \inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{x},\tilde{t})$$

$$\text{such that} \left\{ \begin{array}{c} 0 \le t - \tilde{t} \le \frac{v_f - v(\tilde{t},\tilde{x})}{a} \\ \\ \text{and} \\ \\ x \ge \tilde{x} + (t-\tilde{t})v(\tilde{t},\tilde{x}) + \frac{a(t-\tilde{t})^2}{2} \end{array} \right\} \text{ or } \left\{ \begin{array}{c} t - \tilde{t} \ge \frac{v_f - v(\tilde{t},\tilde{x})}{a} \\ \\ \text{and} \\ \\ x \ge x_a + (t-t_a)v_f \end{array} \right\} \tag{5.1}$$

where

$$t_a = \tilde{t} + \frac{v_f - v(\tilde{t},\tilde{x})}{a}$$

$$x_a = \tilde{x} + (t_a - \tilde{t})v(\tilde{t},\tilde{x}) + \frac{a(t_a-\tilde{t})^2}{2}$$

Note that equation (5.1) states that for the modified solution component, the label of the vehicle at time and position (t,x) is the minimum of all labels from vehicles that can reach this point from the original component while having an acceleration equal to a (if the vehicle is accelerating) or a speed equal to v_f (the free-flow speed). Note that more complex formulations of (5.1) can be thought of, in particular if the acceleration of vehicles is a function of their speed [14]. In the remainder of this article, we will assume that the acceleration of all vehicles is upper bounded by a, irrespective of their current speed as in (5.1).

Given the definition (5.1), the solution $\mathbf{N}_{\text{modified}}(\cdot,\cdot)$ to the hybrid two phase LWR-bounded acceleration model is defined by:

$$\mathbf{N}_{\text{modified}}(x,t) = \min_{i \in I} \mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(x,t) \tag{5.2}$$

Note that the solution to the hybrid two phase LWR-bounded acceleration model (5.2) shares the same solution structure as the solution to the HJ PDE (3.15) investigated in [21]. Indeed, the only difference between both formulations is in the definition of the partial solution components (5.1).

5.2 Properties of the solutions

The solution (5.2) and the partial solution components defined by (5.1) have important properties, which we now outline.

5.2.1 Inf property

Let $\mathbf{c}^{(i)}$ represent an initial, boundary or internal condition block (3.2), (3.5), (3.8) or (2.12), $\mathbf{N}_{\mathbf{c}^{(i)}}$ be the corresponding solution component, and $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}$ be the modified solution component as in (5.1). We have that $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}} \leq \mathbf{N}_{\mathbf{c}^{(i)}}$ pointwise.

Proof — The proof of this fact follows the fact that $(\tilde{x},\tilde{t}) = (x,t)$ is always in the feasible set of (5.1). $\square$

5.2.2 Structure of the resulting solutions

The solution $\mathbf{N}_{\text{modified}}(\cdot,\cdot)$ to the hybrid two phase traffic flow model satisfies the following property:

$$
\begin{array}{ll}
(i) & \text{Either } \mathbf{N}_{\text{modified}}(x,t) = \mathbf{N}_{\mathbf{c}^{(i)}}(x,t) \\
(ii) & \text{Or } \left\{ \begin{array}{l} \exists \tilde{t}, \tilde{x} \;\; s.t. \; \tilde{t} < t \text{ and } (x \geq \tilde{x} + (t-\tilde{t})v(\tilde{t},\tilde{x}) + \frac{a(t-\tilde{t})^2}{2} \text{ or } x \geq x_a + (t-t_a)v_f) \\ \qquad\qquad \text{and } \mathbf{N}_{\text{modified}}(t,x) = \mathbf{N}_{\text{modified}}(\tilde{t},\tilde{x}) \end{array} \right.
\end{array}
\tag{5.3}
$$

Proof — Let us fix $(x,t) \in [\xi,\chi] \times \mathbb{R}^+$, and consider the feasible set $\mathcal{S}$ of (5.1). We always have that $(x,t) \in \mathcal{S}$. Two situations can arise:

1. If (x,t) minimizes the objective function of (5.1), then $\mathbf{N}_{\text{modified}}(x,t) = \mathbf{N}_{\mathbf{c}^{(i)}}(x,t)$, that is, (5.3) (i)

2. If $(\tilde{x},\tilde{t}) \neq (x,t)$ minimizes the objective function of (5.1), we have (5.3) (ii)

□

Note that (5.3) is very important: if locally $\mathbf{N}_{\text{modified}}(x,t) = \mathbf{N}_{\mathbf{c}^{(i)}}(x,t)$, we have that $\mathbf{N}_{\text{modified}}$ satisfies (2.5) locally, using the tangential properties of the capture basin (see [30]) for the construction of the solutions to the HJ PDE using capture basins). However, if (5.3) is satisfied, then $\mathbf{N}_{\text{modified}}(x,t)$ describes locally the evolution of a vehicle having a constant acceleration up to its maximal velocity v_f, and then a constant velocity.

5.2.3 Solution structure

The solution to the mixed initial-boundary-internal conditions problem (5.2) also shares the property (5.3), as it is the minimum of a finite number of functions satisfying (5.3). Though no framework for describing hybridness in partial differential equations exists, the solutions to the two phase flow model have an hybrid structure: they are either solutions to the HJ PDE (2.5) derived from the LWR PDE (2.2), or represent trajectories of vehicles with bounded acceleration.

Chapter 6

Analytical Derivation of the Modified Solution Components for the Triangular Fundamental Diagram

The solution components (5.1) are defined semi-explicitly as a minimization problem in general. However, if the fundamental diagram $Q(\cdot)$ is known explicitly, these solution components may be computed explicitly. We now derive the explicit expression of the solution components associated with affine initial, upstream and downstream boundary, or internal conditions blocks, for a triangular fundamental diagram Q defined by the following expression.

$$Q(k) = \begin{cases} v_f k & : x \in [0, k_c] \\ w(k-\kappa) & : x \in [k_c, \kappa] \end{cases} \tag{6.1}$$

We also have by continuity of $Q(k)$ k_c that $k_c = -w\frac{\kappa}{vf-w}$. The triangular fundamental diagram is widely used in the literature [16] since it is particularly simple and robust.

6.1 Initial condition

6.1.1 Free-flow initial condition

If $0 \leq k_i \leq k_c$, the modified solution component associated with the affine initial condition (3.2) is expressed by:

$$\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x) = \begin{cases} (i) & k_c(tv_f - x) + b_i + x_i(k_c - k_i) & : x_i + tw \leq x \leq x_i + tv_f \\ (ii) & k_i(tv_f - x) + b_i & : x_i + tv_f \leq x \leq x_{i+1} + tv_f \\ (iii) & -x_{i+1}k_i + b_i & : x \geq x_{i+1} + tv_f \end{cases} \tag{6.2}$$

$$k_{\mathbf{c}^{(i)}_{\text{modified}}}(x,t) = -\frac{\partial \mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}}{\partial x}(x,t) = \begin{cases} (i) & k_c & : x_i + tw \leq x \leq x_i + tv_f \\ (ii) & k_i & : x_i + tv_f \leq x \leq x_{i+1} + tv_f \\ (iii) & 0 & : x \geq x_{i+1} + tv_f \end{cases} \tag{6.3}$$

Proof — The proof is available in Appendix. □

Note that the structure of this modified component is similar to the structure of the original component (3.3), as it describes a situation corresponding to uniform velocities (though the densities are not uniform).

We illustrate $k_{\mathbf{c}^{(i)}_{\text{modified}}}$ as well as the isolines of $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}$ corresponding to a free-flow initial condition in Figure 6.1.

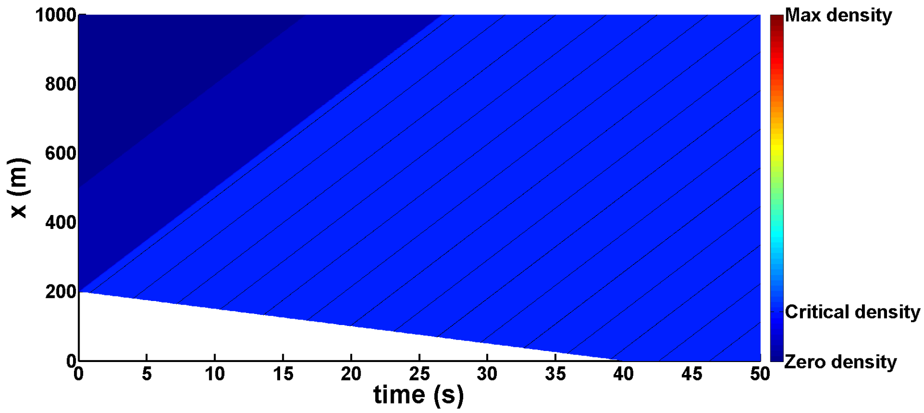

Figure 6.1: Density map $k_{\mathbf{c}^{(i)}_{\text{modified}}}$ corresponding to an uncongested initial condition, for a triangular fundamental diagram.

6.1.2 Congested initial condition

If $k_c < k_i \leq \kappa$, the modified solution component associated with the affine initial condition (3.2) is expressed by:

$$\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x) = \begin{cases} (i) & k_i(tw - x) - \kappa tw + b_i \\ & : x_i + tw \leq x \leq x_{i+1} + tw \\ (ii) & k_i(tw - x) - \kappa tw + b_i + \frac{1}{2}k_i a T_4^2 \\ & : x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \text{ when } t \geq \tau \text{ and} \\ & x_{i+1} + tw \leq x \leq x_{i+1} + tw + \frac{(at-w+v)^2-(w-v)^2}{2a} \text{ when } t \leq \tau \\ (iii) & \frac{1}{v_f - w}(x_{i+1} + \tau v + \frac{1}{2}a\tau^2 + (t-\tau)v_f - x)(k_i w - k_i v_f - \kappa w) + \\ & k_i(\tau v + \frac{1}{2}a\tau^2 + (t-\tau)v_f - x) + b_i \\ & : x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \leq x \leq x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \\ & \text{ and } \ t \geq \tau \\ (iv) & -x_{i+1}k_i + b_i \\ & : x \geq x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \text{ when } t \geq \tau \text{ or} \\ & x \geq x_{i+1} + tw + \frac{1}{2a}((at - w + v)^2 - (w - v)^2) \text{ when } t \leq \tau \end{cases} \tag{6.4}$$

$$k_{\mathbf{c}^{(i)}_{\text{modified}}}(x,t) = \begin{cases} (i) & k_i \\ & : x_i + tw \le x \le x_{i+1} + tw \\ (ii) & \frac{k_i(v-w)}{\sqrt{(v-w)^2 - 2a(x_{i+1} - x + tw)}} \\ & : x_{i+1} + tw \le x \le x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \text{ when } t \ge \tau \text{ and} \\ & x_{i+1} + tw \le x \le x_{i+1} + tw + \frac{(at-w+v)^2 - (w-v)^2}{2a} \text{ when } t \le \tau \\ (iii) & k_c \\ & : x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \le x \le x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \\ & \text{ and } \ t \ge \tau \\ (iv) & 0 \\ & : x \ge x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \text{ when } t \ge \tau \text{ or} \\ & x \ge x_{i+1} + tw + \frac{1}{2a}((at - w + v)^2 - (w-v)^2) \text{ when } t \le \tau \end{cases} \tag{6.5}$$

Proof — The proof is available in Appendix. □

Note that the structure of this modified component differs from the unmodified solution component (3.3): it now contains a transition zone in which vehicles have an uniform acceleration.

We illustrate $k_{\mathbf{c}^{(i)}_{\text{modified}}}$ as well as the isolines of $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}$ corresponding to a congested initial condition in Figure 6.2.

6.2 Upstream boundary condition

The solution component associated with a upstream boundary condition (3.5) is expressed by:

$$\mathbf{N}_{\mathbf{c}^{(j)}_{\text{modified}}}(x,t) = \begin{cases} (i) & d_j + q_j(t - \frac{x - x_0}{v_f}) & : x_0 + v_f(t - t_{j+1}) \le x \le x_0 + v_f(t - t_j) \\ (ii) & d_j + q_j t_{j+1} + & : x_0 \le x \le x_0 + v_f(t - t_{j+1}) \\ & k_c((t - t_{j+1})v_f - (x - x_0)) & \\ (iii) & d_j + q_j t_j & : x \ge x_0 + v_f(t - t_j) \end{cases} \tag{6.6}$$

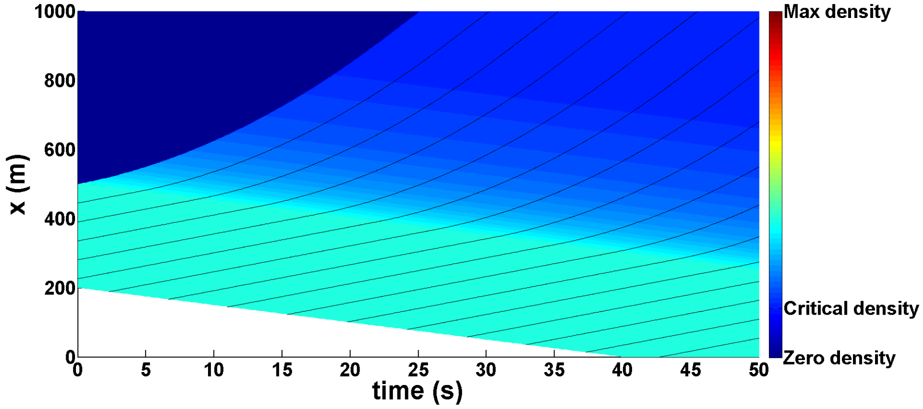

Figure 6.2: Density map $k_{\mathbf{c}^{(i)}_{\text{modified}}}$ corresponding to a congested initial condition, for a triangular fundamental diagram.

$$k_{\mathbf{c}^{(j)}_{\text{modified}}}(x,t) = -\frac{\partial \mathbf{N}_{\mathbf{c}^{(j)}_{\text{modified}}}}{\partial x}(x,t) = \begin{cases} (i) & \frac{q_j}{v_f} & : x_0 + v_f(t - t_{j+1}) \leq x \leq x_0 + v_f(t - t_j) \\ (ii) & k_c & : x_0 \leq x \leq x_0 + v_f(t - t_{j+1}) \\ (iii) & 0 & : x \geq x_0 + v_f(t - t_j) \end{cases} \tag{6.7}$$

Proof — The proof is similar to the initial condition case. □

Note that the structure of this modified component is very similar to the unmodified component (3.6), as it describes a situation corresponding to uniform velocities (though the densities are not uniform).

We illustrate $k_{\mathbf{c}^{(i)}_{\text{modified}}}$ as well as the isolines of $\mathbf{N}_{\mathbf{c}^{(j)}_{\text{modified}}}$ corresponding to an upstream boundary condition in Figure 6.3.

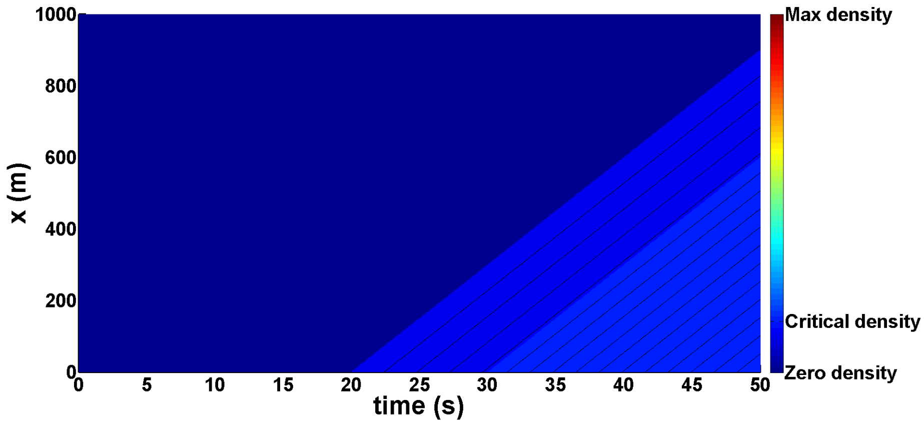

Figure 6.3: Density map $k_{\mathbf{c}^{(j)}_{\text{modified}}}$ corresponding to an upstream boundary condition, for a triangular fundamental diagram.

6.3 Downstream boundary condition

The solution component associated with a downstream boundary condition (3.8) is expressed by:

$$\mathbf{N}_{\mathbf{c}^{(j)}_{\text{modified}}}(x,t) = \begin{cases} (i) & b_j + (\frac{p_j}{w} + \kappa)(x_n - x + tv) \\ & : x_n + w(t - t_j) \leq x \leq x_n + w(t - t_{j+1}) \\ (ii) & b_j + (\frac{p_j}{w} + \kappa)(x_n - x + tv + \frac{((w-v)+\sqrt{(w-v)^2+2a(w(t_{j+1}-t)+x-x_n)})^2}{2a}) \\ & : x_n + w(t - t_{j+1}) \leq x \leq x_n + wt - wt_{j+1} + \tau(v + \frac{1}{2}a\tau - w) \text{ and} \\ & x \leq x_n + wt - wt_{j+1} + \frac{(at-w+v)^2-(w-v)^2}{2a} \\ (iii) & b_j + \frac{1}{v_f - w}(x_n - wt_{j+1} - x + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f)(p_j - (\frac{p_j}{w} + \kappa)v_f) + \\ & (\frac{p_j}{w} + \kappa)(x_n - x + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f) \\ & : x_n - wt_{j+1} + \tau v + \frac{a\tau^2}{2} + (t-\tau)w \leq x \leq x_n - wt_{j+1} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \\ & \text{ and } t \geq \tau \\ (iv) & b_j + (\frac{p_j}{w} + \kappa)wt_{j+1} \\ & : x \geq x_n - wt_{j+1} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \text{ when } t \geq \tau \text{ and} \\ & x \geq x \geq x_n + wt - wt_{j+1} + \frac{(at-w+v)^2-(w-v)^2}{2a} \text{ when } t \leq \tau \end{cases} \quad (6.8)$$

$$
k_{\mathbf{c}^{(j)}_{\text{modified}}}(x,t) = \begin{cases}
(i) & \frac{p_j}{w} + \kappa \\
& : x_n + w(t - t_j) \leq x \leq x_n + w(t - t_{j+1}) \\
(ii) & \frac{(\frac{p_j}{w}+\kappa)(v-w)}{\sqrt{(w-v)^2+2a(w(t_{j+1}-t)+x-x_n)}} \\
& : x_n + w(t - t_{j+1}) \leq x \leq x_n + wt - wt_{j+1} + \tau(v + \frac{1}{2}a\tau - w) \text{ and} \\
& x \leq x_n + wt - wt_{j+1} + \frac{(at-w+v)^2-(w-v)^2}{2a} \\
(iii) & k_c \\
& : x_n - wt_{j+1} + \tau v + \frac{a\tau^2}{2} + (t-\tau)w \leq x \leq x_n - wt_{j+1} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \\
& \text{ and } t \geq \tau \\
(iv) & 0 \\
& : x \geq x_n - wt_{j+1} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \text{ when } t \geq \tau \text{ and} \\
& x \geq x \geq x_n + wt - wt_{j+1} + \frac{(at-w+v)^2-(w-v)^2}{2a} \text{ when } t \leq \tau
\end{cases} \quad (6.9)
$$

Proof — The proof is similar to the initial condition case. □

Note that the structure of this modified component differs from the unmodified solution component (3.9): it now contains a transition zone in which vehicles have an uniform acceleration.

We illustrate $k_{\mathbf{c}^{(j)}_{\text{modified}}}$ as well as the isolines of $\mathbf{N}_{\mathbf{c}^{(j)}_{\text{modified}}}$ corresponding to a downstream boundary condition in Figure 6.4.

6.4 Internal boundary condition

The modified solution component associated with an internal condition (2.12) is expressed by:

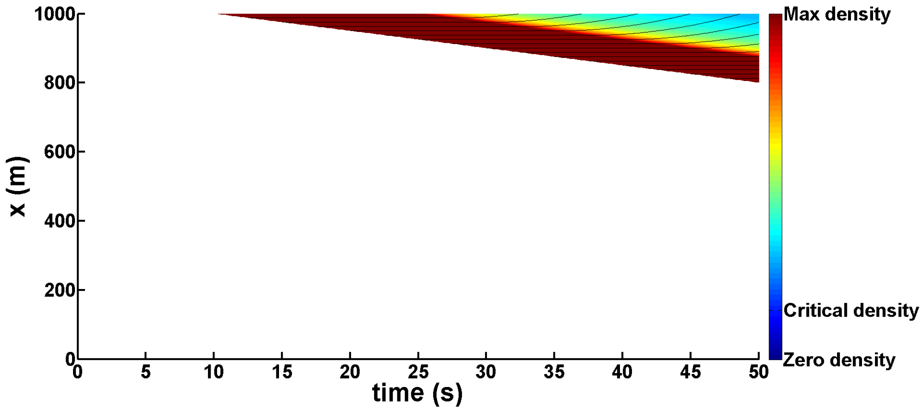

Figure 6.4: Density map $k_{\mathbf{c}^{(j)}_{\text{modified}}}$ corresponding to a downstream boundary condition, for a triangular fundamental diagram.

$$\mathbf{N}_{\mathbf{c}^{(l)}_{\text{modified}}}(x,t) = \begin{cases} (i) & M^{(l)} - w(k_2-\kappa)t^{(l)}_{min} + (x^{(l)}_{min} - x) + tv)k_2 \\ & : x \leqslant x^{(l)}_{min} + V^{(l)}_{intern}(t - t^{(l)}_{min}) \text{ and} \\ & x \geqslant x^{(l)}_{min} + w(t - t^{(l)}_{min}) \text{ and} \\ & x \leqslant x^{(l)}_{max} + w(t - t^{(l)}_{max}) \\ (ii) & M^{(l)} - w(k_2-\kappa)t^{(l)}_{min} + (x^{(l)}_{min} - x + tv)k_2 + \frac{k_2 a}{2}T_4^2 \\ & : x \geqslant x^{(l)}_{max} + w(t - t^{(l)}_{max}) \text{ and} \\ & x \leqslant \frac{(a\tau - w + v)^2 - (w-v)^2}{2a} + x^{(l)}_{max} + w(t - t^{(l)}_{max}) \text{ and} \\ & x \leqslant \frac{a(t - t^{(l)}_{max})^2}{2} + v(t - t^{(l)}_{max}) + x^{(l)}_{max} \text{ and} \\ & t \geqslant t^{(l)}_{max} + \frac{V^{(l)}_{intern} - v}{a} \\ (iii) & M^{(l)} - w(k_2-\kappa)t^{(l)}_{min} + (x^{(l)}_{min} - x + tv)k_2 + \frac{k_2 a}{2}T_2^2 \\ & : x \geqslant -\frac{(V^{(l)}_{intern} - v)^2}{2a} + x^{(l)}_{min} + V^{(l)}_{intern}(t - t^{(l)}_{min}) \text{ and} \\ & x \leqslant \frac{(a\tau - w + v)^2 - (w-v)^2}{2a} + x^{(l)}_{max} + w(t - t^{(l)}_{max}) \text{ and} \\ & x \leqslant \frac{a(t - t^{(l)}_{min})^2}{2} + v(t - t^{(l)}_{min}) + x^{(l)}_{min} \text{ and} \\ & t \geqslant t^{(l)}_{min} + \frac{V^{(l)}_{intern} - v}{a} \\ & x \leqslant \frac{(a\tau - V^{(l)}_{intern} + v)^2 - (V^{(l)}_{intern} - v)^2}{2a} + x^{(l)}_{min} + \\ & V^{(l)}_{intern}(t - t^{(l)}_{min}) \text{ and} \\ & x \geqslant x^{(l)}_{min} + V^{(l)}_{intern}(t - t^{(l)}_{min}) \\ & x \geqslant \frac{a(t - t^{(l)}_{max})^2}{2} + v(t - t^{(l)}_{max}) + x^{(l)}_{max}, \\ & t \geqslant t^{(l)}_{max} + \frac{V^{(l)}_{intern} - v}{a} \text{ or } t \leqslant t^{(l)}_{max} + \frac{V^{(l)}_{intern} - v}{a} \end{cases} \tag{6.10}$$

$$\mathbf{N}_{\mathbf{c}^{(l)}_{\text{modified}}}(x,t) = \begin{cases} (iv) & w(k_2-\kappa)(t_1-t^{(l)}_{min}) - v_f k_2 t_1 + (x^{(l)}_{min} - x + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f)k_2 + M^{(l)} \\ & : x \geqslant x^{(l)}_{min} - V^{(l)}_{intern} t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)V^{(l)}_{intern} \text{ and} \\ & x \geqslant \frac{1}{w-V^{(l)}_{intern}}((w-V^{(l)}_{intern})(\tau v + \frac{a\tau^2}{2} + (t-\tau)v_f) + \\ & (v_f - V^{(l)}_{intern})(x_{max} - w t^{(l)}_{max}) - (v_f - w)(x_{min} - V^{(l)}_{intern} t^{(l)}_{min}) \text{ and} \\ & x \leqslant x^{(l)}_{min} - v_f t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \text{ and} \\ & t \geqslant \tau \\ (v) & w(k_2-\kappa)(t_2-t^{(l)}_{min}) - v_f k_2 t_2 + (x^{(l)}_{min} - x + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f)k_2 + M^{(l)} \\ & : x \geqslant x_{max} - w t_{max} + \tau v + \frac{a\tau^2}{2} + (t-\tau)w \text{ and} \\ & x \leqslant x_{max} - w t_{max} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \text{ and} \\ & x \leqslant \frac{1}{w-V^{(l)}_{intern}}((w-V^{(l)}_{intern})(\tau v + \frac{a\tau^2}{2} + (t-\tau)v_f) + \\ & (v_f - V^{(l)}_{intern})(x_{max} - w t^{(l)}_{max}) - (v_f - w)(x_{min} - V^{(l)}_{intern} t^{(l)}_{min}) \text{ and} \\ & t \geqslant \tau \\ (vi) & M^{(l)} \\ & : x \geqslant \frac{a(t-t^{(l)}_{min})^2}{2} + v(t-t^{(l)}_{min}) + x^{(l)}_{min} \text{ and} \\ & x \leqslant x^{(l)}_{min} - V^{(l)}_{intern} t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)V^{(l)}_{intern} \text{ and} \\ & t \geqslant t^{(l)}_{min} \\ & \text{ or} \\ & x \geqslant x^{(l)}_{min} - v_f t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \text{ and} \\ & x \geqslant x^{(l)}_{min} - V^{(l)}_{intern} t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)V^{(l)}_{intern} \text{ and} \\ & t \geqslant t^{(}_{min} l) \end{cases} \quad (6.11)$$

$$k_{\mathbf{c}^{(l)}_{\text{modified}}}(x,t) = \begin{cases} (i) & k_2 \\ & : x \leqslant x^{(l)}_{min} + V^{(l)}_{intern}(t - t^{(l)}_{min}) \text{ and} \\ & x \geqslant x^{(l)}_{min} + w(t - t^{(l)}_{min}) \text{ and} \\ & x \leqslant x^{(l)}_{max} + w(t - t^{(l)}_{max}) \\ (ii) & \frac{k_2(v-w)}{\sqrt{(v-w)^2 + 2a(x - x^{(l)}_{max} - w(t - t^{(l)}_{max}))}} \\ & : x \geqslant x^{(l)}_{max} + w(t - t^{(l)}_{max}) \text{ and} \\ & x \leqslant \frac{(a\tau - w + v)^2 - (w-v)^2}{2a} + x^{(l)}_{max} + w(t - t^{(l)}_{max}) \text{ and} \\ & x \leqslant \frac{a(t-t^{(l)}_{max})^2}{2} + v(t - t^{(l)}_{max}) + x^{(l)}_{max} \text{ and} \\ & t \geqslant t^{(l)}_{max} + \frac{V^{(l)}_{intern} - v}{a} \end{cases} \quad (6.12)$$

$$k_{\mathbf{c}^{(l)}_{\text{modified}}}(x,t) = \begin{cases} (iii) & \frac{k_2(V^{(l)}_{intern}-w)}{\sqrt{(V^{(l)}_{intern}-w)^2+2a(x-x^{(l)}_{min}-V^{(l)}_{intern}(t-t^{(l)}_{min}))}} \\ & : x \geqslant -\frac{(V^{(l)}_{intern}-v)^2}{2a} + x^{(l)}_{min} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \text{ and} \\ & x \leqslant \frac{(a\tau-w+v)^2-(w-v)^2}{2a} + x^{(l)}_{max} + w(t-t^{(l)}_{max}) \text{ and} \\ & x \leqslant \frac{a(t-t^{(l)}_{min})^2}{2} + v(t-t^{(l)}_{min}) + x^{(l)}_{min} \text{ and} \\ & t \geqslant t^{(l)}_{min} + \frac{V^{(l)}_{intern}-v}{a} \\ & x \leqslant \frac{(a\tau-V^{(l)}_{intern}+v)^2-(V^{(l)}_{intern}-v)^2}{2a} + x^{(l)}_{min} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \text{ and} \\ & \\ & x \geqslant x^{(l)}_{min} + V^{(l)}_{intern}(t-t^{(l)}_{min}) \\ & x \geqslant \frac{a(t-t^{(l)}_{max})^2}{2} + v(t-t^{(l)}_{max}) + x^{(l)}_{max}, \\ & t \geqslant t^{(l)}_{max} + \frac{V^{(l)}_{intern}-v}{a} \text{ or } t \leqslant t^{(l)}_{max} + \frac{V^{(l)}_{intern}-v}{a} \\ (iv) & k_2 + \frac{w(k_2-\kappa)-v_f k_2}{v_f-V^{(l)}_{intern}} \\ & : x \geqslant x^{(l)}_{min} - V^{(l)}_{intern}t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)V^{(l)}_{intern} \text{ and} \\ & x \geqslant \frac{1}{w-V^{(l)}_{intern}}((w-V^{(l)}_{intern})(\tau v + \frac{a\tau^2}{2} + (t-\tau)v_f) + \\ & (v_f - V^{(l)}_{intern})(x_{max} - wt^{(l)}_{max}) - (v_f-w)(x_{min} - V^{(l)}_{intern}t^{(l)}_{min}) \text{ and} \\ & x \leqslant x^{(l)}_{min} - v_f t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \text{ and} \\ & t \geqslant \tau \\ (v) & k_2 + \frac{w(k_2-\kappa)-v_f k_2}{v_f-w} \\ & : x \geqslant x_{max} - wt_{max} + \tau v + \frac{a\tau^2}{2} + (t-\tau)w \text{ and} \\ & x \leqslant x_{max} - wt_{max} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \text{ and} \\ & x \leqslant \frac{1}{w-V^{(l)}_{intern}}((w-V^{(l)}_{intern})(\tau v + \frac{a\tau^2}{2} + (t-\tau)v_f) + \\ & (v_f - V^{(l)}_{intern})(x_{max} - wt^{(l)}_{max}) - (v_f-w)(x_{min} - V^{(l)}_{intern}t^{(l)}_{min}) \text{ and} \\ & t \geqslant \tau \\ (vi) & 0 \\ & : x \geqslant \frac{a(t-t^{(l)}_{min})^2}{2} + v(t-t^{(l)}_{min}) + x^{(l)}_{min} \text{ and} \\ & x \leqslant x^{(l)}_{min} - V^{(l)}_{intern}t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)V^{(l)}_{intern} \text{ and} \\ & t \geqslant t^{(l)}_{min} \\ & \text{ or} \\ & x \geqslant x^{(l)}_{min} - v_f t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)v_f \text{ and} \\ & x \geqslant x^{(l)}_{min} - V^{(l)}_{intern}t^{(l)}_{min} + \tau v + \frac{a\tau^2}{2} + (t-\tau)V^{(l)}_{intern} \text{ and} \\ & t \geqslant t^{(l)}_{min} \end{cases} \tag{6.13}$$

where T_2, T_4, t_1 and t_2 are defined as follows:

$$T_2 = \frac{(V_{intern}^{(l)} - v) + \sqrt{(V_{intern}^{(l)} - v)^2 + 2a(x - x_{min}^{(l)} - V_{intern}^{(l)}(t - t_{min}^{(l)}))}}{a}$$

$$T_4 = \frac{(w - v) + \sqrt{(w - v)^2 + 2a(x - x_{max}^{(l)} - w(t - t_{max}^{(l)}))}}{a}$$

$$t_1 = \frac{1}{v_f - V_{intern}^{(l)}}(x_{min}^{(l)} - V_{intern}^{(l)} t_{min}^{(l)} - x + \tau v + \frac{a\tau^2}{2} + (t - \tau)v_f)$$

$$t_2 = \frac{1}{v_f - w}(x_{max}^{(l)} - w(l) t_{max}^{(l)} - x + \tau v + \frac{a\tau^2}{2} + (t - \tau)v_f)$$

Proof — The proof is similar to the initial condition case. □

Note that the structure of this modified component differs from the unmodified solution component (3.13): it now contains a transition zone in which vehicles have an uniform acceleration.

We illustrate $k_{\mathbf{c}^{(l)}_{\text{modified}}}$ as well as the isolines of $\mathbf{N}_{\mathbf{c}^{(l)}_{\text{modified}}}$ corresponding to an internal condition in Figure 6.5.

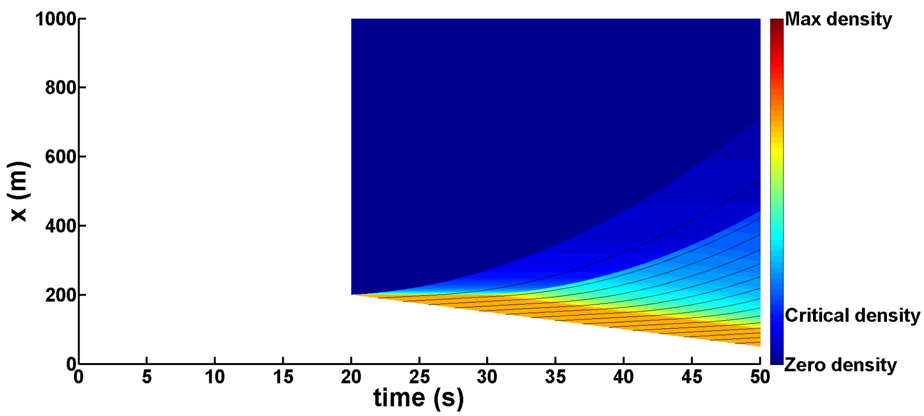

Figure 6.5: Density map $k_{\mathbf{c}^{(l)}_{\text{modified}}}$ corresponding to an internal condition, for a triangular fundamental diagram.

Chapter 7

Implementation

7.1 Algorithm structure

Given the derivations of the modified component solutions associated with the initial, boundary and internal conditions, one can build a numerical scheme for solving the hybrid two phase LWR-bounded acceleration model semi-analytically with a low computational cost, as in [21].

The proposed numerical scheme is based on the inf-morphism property (3.15). It is based on the minimization of analytic formulas, and is thus guaranteed to be exact.

Algorithm 1 Pseudo-code implementation for the Lax-Hopf based computation of the Moskowitz function and the associated density at a single point (x,t) prescribed by the user.

Input: $x \in [x_0, x_n], t \in [0, t_m]$, {input space domain, time domain}

$\mathbf{N} \leftarrow +\infty$ {initialization of the Moskowitz function to infinity}
for $j = j_{\min}$ to $j_{\max}$ **do** {iteration on initial conditions}
  compute $\mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(x,t)$ using (6.2) or (6.4) {component induced by the initial condition $\mathbf{c}_{\text{ini}}^{(i)}$}
  if $\mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(x,t) < \mathbf{N}$ **then** {if the current component contributes to the solution}
    $\mathbf{N} \leftarrow \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(x,t)$ {update Moskowitz function}
    $k \leftarrow k_{\mathbf{c}_{\text{ini}}^{(i)}}(x,t)$, computed using (6.3) or (6.5) {compute density}
  end if
end for
for $j = 0$ to j_{up} **do** {iteration on upstream boundary conditions}
  compute $\mathbf{N}_{\mathbf{c}_{\text{up}}^{(j)}}(x,t)$ using (6.6) {component induced by the upstream boundary condition $\mathbf{c}_{\text{up}}^{(j)}$}
  if $\mathbf{N}_{\mathbf{c}_{\text{up}}^{(j)}}(x,t) < \mathbf{N}$ **then** {if the current component contributes to the solution}
    $\mathbf{N} \leftarrow \mathbf{N}_{\mathbf{c}_{\text{up}}^{(j)}}(x,t)$ {update Moskowitz function}
    $k \leftarrow k_{\mathbf{c}_{\text{up}}^{(j)}}(x,t)$, computed using (6.7) {compute density}
  end if
end for
for $j = 0$ to j_{down} **do** {iteration on downstream boundary conditions}
  compute $\mathbf{N}_{\mathbf{c}_{\text{down}}^{(j)}}(x,t)$ using (6.8 {component induced by the downstream boundary condition $\mathbf{c}_{\text{down}}^{(j)}$}
  if $\mathbf{N}_{\mathbf{c}_{\text{down}}^{(j)}}(x,t) < \mathbf{N}$ **then** {if the current component contributes to the solution}
    $\mathbf{N} \leftarrow \mathbf{N}_{\mathbf{c}_{\text{down}}^{(j)}}(x,t)$ {update Moskowitz function}
    $k \leftarrow k_{\mathbf{c}_{\text{down}}^{(j)}}(x,t)$, computed using (6.9) {compute density}
  end if
end for
for $l = 0$ to l_{intern} **do** {iteration on internal conditions}
  compute $\mathbf{N}_{\mathbf{c}_{\text{intern}}^{(l)}}(x,t)$ using (6.11) {component induced by the internal condition $\mathbf{c}_{\text{intern}}^{(l)}$}
  if $\mathbf{N}_{\mathbf{c}_{\text{intern}}^{(l)}}(x,t) < \mathbf{N}$ **then** {if the current component contributes to the solution}
    $\mathbf{N} \leftarrow \mathbf{N}_{\mathbf{c}_{\text{intern}}^{(l)}}(x,t)$ {update Moskowitz function}
    $k \leftarrow k_{\mathbf{c}_{\text{intern}}^{(l)}}(x,t)$, computed using (6.13) {compute density}
  end if
end for
Output: $\mathbf{N}, k$

7.2 Numerical examples

We implemented 1 as a `Matlab` Toolbox, freely downloadable from XXX. All numerical computations were performed using an `Intel` i3 CPU running at 3.0 GHz, operated by `Windows 7` (32 bit), with 4 GB of RAM.

We consider a triangular fundamental diagram defined by (6.1), where the parameters are defined as $vf = 30\ m/s$, $w = -5\ m/s$, $\kappa = 0.1\ veh/m$ and $k_c = -w\frac{\kappa}{vf-w} = 0.014\ veh/m$. The acceleration of the vehicles during the bounded acceleration phases is set to $a = 2m/s^2$, consistent with the capabilities of modern vehicles.

The computational domain consists in a 1000 m section, for a total duration of 50 s.

We first compute the density and trajectories (isolines of N) corresponding to a given set of initial and boundary conditions, defined as follows:

initial condition:

$x_{ini} = [0, 250, 500, 750, 1000]$;

$k_{ini} = [10E-3, 40E-3, 5E-3, 50E-3]$;

upstream boundary condition

$t_{up} = [0, 20, 30, 50]$;

$q_{up} = [1, 0.3, 0.1]$;

Though the algorithm is gridless, we compute the exact solutions on a rectangular grid (for visualization purposes) of $500,000$ points (1000×500). By construction, the numerical solutions are exact up to machine accuracy, *i.e.* the numerical errors are on the order of machine zeros.

The solution corresponding to the set of initial and boundary conditions outlined above was computed in $1.14s$, and is illustrated in Figure 7.1.

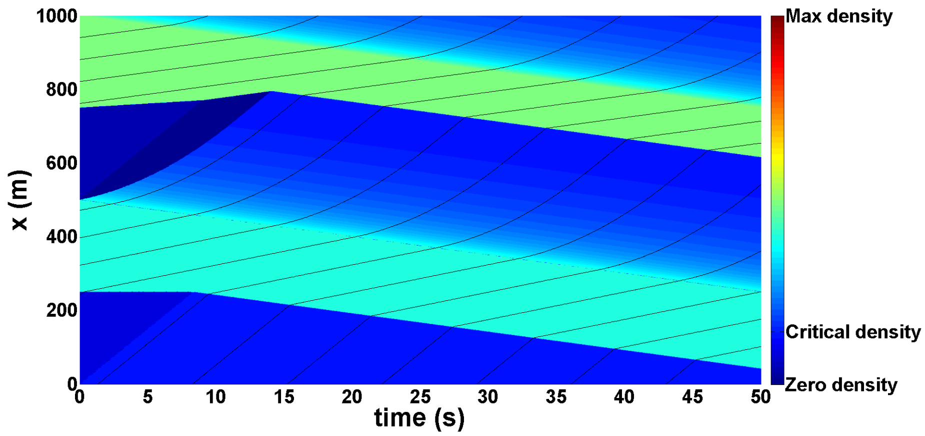

Figure 7.1: Density map and vehicle trajectories corresponding to a set of initial and upstream boundary conditions.

We now consider the same initial and boundary conditions, while also assuming that a fixed ($V_{intern} = 0$) obstruction (for instance a red traffic light, or a car accident) prevents traffic propagation ($q_{intern} = 0$) at location $x = 400m$, between times 20 s and 30 s. The solution was computed in 1.44 s, and is illustrated in Figure 7.2.

While internal conditions can model fixed obstructions, they can also model more complex traffic flow scenarios, such as moving bottlenecks. To illustrate this, we simulate the solution to the same initial and boundary conditions, but we now assume that a slow bus moving at $V_{intern} = 5\ m/s$ is restricting the road capacity along its path, allowing a maximal passing flow of $q_{intern} = 0.025\ veh/s$. The solution was computed in 1.38 s, and is illustrated in Figure 7.3.

While we only show numerical computations involving one internal condition for simplicity, an arbitrary number of fixed and moving bottlenecks can be integrated in our proposed algorithm. As illustrated in the three above examples, incorporating fixed or moving bottlenecks does not dramatically increase the computational time much.

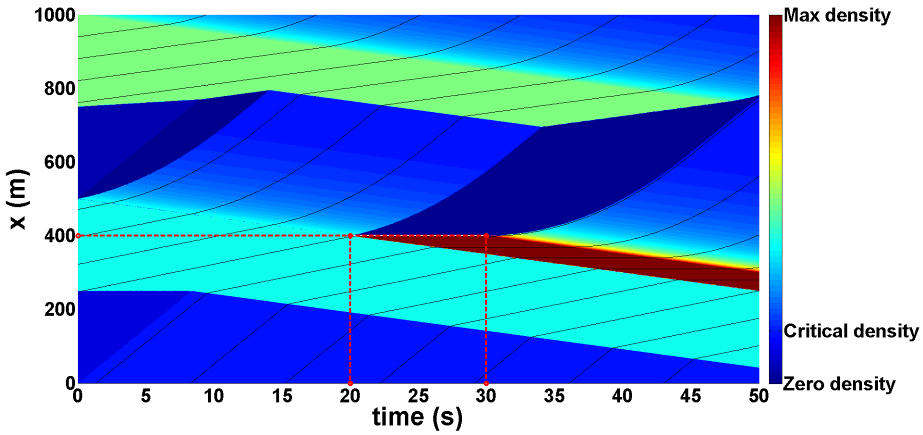

Figure 7.2: Density map and vehicle trajectories corresponding to a set of initial and upstream boundary conditions, with a fix bottleneck preventing traffic propagation. The fixed bottleneck is highlighted by a red dash.

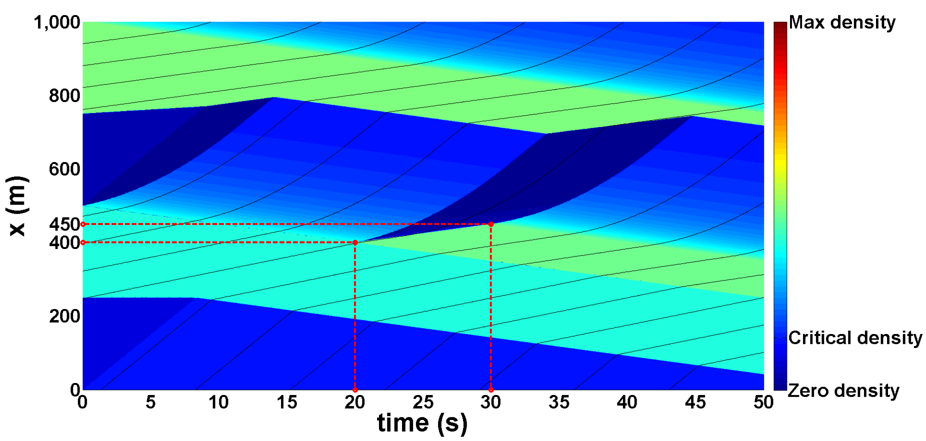

Figure 7.3: Density map and vehicle trajectories corresponding to a set of initial and upstream boundary conditions, with a moving bottleneck. The moving bottleneck, represents a bus restricting the road capacity and allowing a passing flow of $q_{intern} = 0.025\ veh/s$. We highlight the trajectory of the bus by a red dash.

7.3 Benefits of a grid-free and exact method

The fact that the proposed numerical scheme is both exact and gridless is very important for solving practical problems involving the kinematic effects of vehicles on traffic flow propagation. For instance, dealing with the effects of traffic flow (pollution, energy consumption, noise...) often requires the coupling between traffic flow propagation models and vehicle property models (for instance the energy consumption of the vehicle as a function of its target path). Optimizing these effects (for instance minimizing pollution of a bus driving through traffic) thus requires the computation of an optimal single vehicle trajectory, which is itself a function of the surrounding traffic. Most algorithms compute these trajectories iteratively, and thus any error of the numerical traffic solver is propagated through the iterations, leading to poor overall accuracy. Since the algorithm described in this article is exact, it does not add any uncertainty to the results.

The fact that the algorithm is grid-free also allows for a faster search over all possible vehicle trajectories for single vehicle trajectory optimization problems.

Chapter 8

Conclusion

This article presents a new semi analytical expression for the solutions to the Lighthill-Whitham-Richards traffic flow model with bounded vehicle acceleration. Based on this semi-analytical expression, we compute the analytical solution component blocks associated with the triangular fundamental diagram. These analytical solution blocks allow us to construct the solution to the modified traffic flow model as a finite minimization of functions. The resulting numerical scheme is both grid-free and exact, which are very favorable characteristics when dealing with optimization problems involving the coupling between the modified LWR model and vehicle models. We then use the algorithm to compute the solutions to various problems involving fixed and moving bottlenecks of increased complexity.

Future work will deal with the extension of this algorithm to networks, as bounded accelerations frequently occur in junctions, whenever congestion occurs because of capacity restriction. Another important avenue is the development of an hybrid model based on the Lighthill-Whitham-Richards traffic flow model with both bounded acceleration and deceleration, which is to date an open problem.

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APPENDICES

A Derivation of the modified initial condition component for an uncongested initial condition

If $0 \leq k_i \leq k_c$, the original solution component associated with the affine initial condition (3.2) is expressed by:

$$\mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x},\tilde{t}) = \begin{cases} (i) \quad k_i(\tilde{t}v_f - \tilde{x}) + b_i & : x_i + \tilde{t}v_f \leq \tilde{x} \leq x_{i+1} + \tilde{t}v_f \\ (ii) \quad k_c(\tilde{t}v_f - \tilde{x}) + b_i + x_i(k_c - k_i) & : x_i + \tilde{t}w \leq \tilde{x} \leq x_i + \tilde{t}v_f \end{cases} \tag{A.1}$$

Applying the first condition of (5.1) to equation $(A.1)(i)$ yields: $v(\tilde{t},\tilde{x}) = v_f$, $t = \tilde{t}$ and $x \geq \tilde{x}$. Thus,

$$\mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t},\tilde{x}) = k_i(\tilde{t}v_f - \tilde{x}) + b_i \geq k_i(tv_f - x) + b_i$$

with equality when $x = \tilde{x}$. When $x = \tilde{x}$, the condition $x_i + \tilde{t}v_f \leq \tilde{x} \leq x_{i+1} + \tilde{t}v_f$

yields $x_i + tv_f \leq x \leq x_{i+1} + tv_f$, which implies:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t}, \tilde{x}) = k_i(tv_f - x) + b_i \quad \text{for } x_i + tv_f \leq x \leq x_{i+1} + tv_f \tag{A.2}$$

For $x \geq x_{i+1} + tv_f$, we plug $\tilde{x} = x_{i+1} + tv_f$ into equation $(A.1)(i)$ to compute $\inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t}, \tilde{x})$:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t}, \tilde{x}) = -k_i x_{i+1} + b_i \quad \text{for } x \geq x_{i+1} + tv_f \tag{A.3}$$

Applying the second condition of (5.1) to equation $(A.1)(i)$ yields the following expression:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t}, \tilde{x}) = \begin{cases} (i) \quad k_i(tv_f - x) + b_i & : x_i + tv_f \leq x \leq x_{i+1} + tv_f \\ (ii) \quad -x_{i+1}k_i + b_i & : x \geq x_{i+1} + tv_f \end{cases} \tag{A.4}$$

Similarly, applying (5.1) to equation $(A.1)(ii)$ yields the following candidate values for $\inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t}, \tilde{x})$. Applying the first condition of (5.1) yields:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t}, \tilde{x}) = \begin{cases} (i) \quad k_c(tv_f - x) + b_i + x_i(k_c - k_i) & : x_i + tw \leq x \leq x_i + tv_f \\ (ii) \quad -x_i k_i + b_i & : x \geq x_i + tv_f \end{cases} \tag{A.5}$$

Applying the second condition of (5.1) yields:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t}, \tilde{x}) = \begin{cases} (i) \quad k_c(tv_f - x) + b_i + x_i(k_c - k_i) & : x_i + \tilde{t}(w - v_f) + tv_f \leq x \leq x_i + tv_f \\ (ii) \quad -x_i k_i + b_i & : x \geq x_i + tv_f \end{cases} \tag{A.6}$$

Since $t \geq \tilde{t}$ and $w \leq 0$, we have $x_i + \tilde{t}(w - v_f) + tv_f = x_i + \tilde{t}w + (t - \tilde{t})v_f \geq x_i + tw$. Therefore, the domain of equation $(A.6)(i)$ can be extended to $x_i + tw \leq x \leq x_{i+1} + tv_f$

and (A.6) becomes:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}}(\tilde{t},\tilde{x}) = \begin{cases} (i) & k_c(tv_f - x) + b_i + x_i(k_c - k_i) & : x_i + tw \leq x \leq x_i + tv_f \\ (ii) & -x_i k_i + b_i & : x \geq x_i + tv_f \end{cases} \tag{A.7}$$

Using the definition of (5.1), we the minimum value among (A.2),(A.3),(A.4),(A.5) and (**??**) in their corresponding domains, which yeilds the following expression for $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}$ when $0 \leq k_i \leq k_c$:

$$\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x) = \begin{cases} (i) & k_c(tv_f - x) + b_i + x_i(k_c - k_i) & : x_i + tw \leq x \leq x_i + tv_f \\ (ii) & k_i(tv_f - x) + b_i & : x_i + tv_f \leq x \leq x_{i+1} + tv_f \\ (iii) & -x_{i+1}k_i + b_i & : x \geq x_{i+1} + tv_f \end{cases} \tag{A.8}$$

$$k_{\mathbf{c}^{(i)}_{\text{modified}}}(x,t) = -\frac{\partial \mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}}{\partial x}(x,t) = \begin{cases} (i) & k_c & : x_i + tw \leq x \leq x_i + tv_f \\ (ii) & k_i & : x_i + tv_f \leq x \leq x_{i+1} + tv_f \\ (iii) & 0 & : x \geq x_{i+1} + tv_f \end{cases} \tag{A.9}$$

B Derivation of the modified initial condition component for an congested initial condition

If $k_c < k_i \leq \kappa$, the initial condition imposes a congested state:

$$\mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x},\tilde{t}) = \begin{cases} (i) \quad k_i(\tilde{t}w - \tilde{x}) - \kappa\tilde{t}w + b_i & : x_i + \tilde{t}w \leq \tilde{x} \leq x_{i+1} + \tilde{t}w \\ (ii) \quad k_c(\tilde{t}w - \tilde{x}) - \kappa\tilde{t}w + x_{i+1}(k_c - k_i) + b_i & : x_{i+1} + \tilde{t}w \leq \tilde{x} \leq x_{i+1} + \tilde{t}v_f \end{cases} \tag{B.1}$$

Applying the first condition of (5.1) to equation $(B.1)(i)$ yields $v(\tilde{t},\tilde{x}) = v = w(1-\frac{\kappa}{k_i})$, $x \geq \tilde{x} + (t-\tilde{t})v + \frac{a(t-\tilde{t})^2}{2}$ and $0 \leq t - \tilde{t} \leq \tau$, where $\tau = \frac{v_f - v(\tilde{t},\tilde{x})}{a}$. Thus,

$$N_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x},\tilde{t}) = k_i(\tilde{t}w - \tilde{x}) - \kappa\tilde{t}w + b_i \geq k_i(tw - x) - \kappa tw + b_i + \frac{k_i a(t-\tilde{t})^2}{2}$$

with equality when $x = \tilde{x} + (t-\tilde{t})v + \frac{a(t-\tilde{t})^2}{2}$. Thus:

$$\inf \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x},\tilde{t}) = \inf(k_i(tw - x) - \kappa tw + b_i + \frac{k_i a(t-\tilde{t})^2}{2}), \text{ for } (\tilde{t},\tilde{x}) \text{ satisfying:}$$

$$\begin{cases} (i) & 0 \leq t - \tilde{t} \leq \tau \\ (ii) & x = \tilde{x} + (t - \tilde{t})v + \frac{a(t-\tilde{t})^2}{2} \\ (iii) & x + \tilde{t}w \leq \tilde{x} \leq x_{i+1} + \tilde{t}w \\ (iv) & \tilde{t} \geq 0 \end{cases} \tag{B.2}$$

To compute the minimum value of $N_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t})$, we first have to determine the range of $t - \tilde{t}$ under the constraint (B.2). Let $T = t - \tilde{t}$. By plugging $(B.2)(ii)$ into $(B.2)(iii)$, we can rewrite (B.2) as:

$$\begin{cases} (i) & 0 \leq T \leq \tau \\ (ii) & -\frac{aT^2}{2} + (w - v)T - (x_i + tw - x) \geq 0 \\ (iii) & -\frac{aT^2}{2} + (w - v)T - (x_{i+1} + tw - x) \leq 0 \\ (iv) & T \leq t \end{cases} \tag{B.3}$$

Let us define the following auxiliary variables:

$$\Delta_1 = (w - v)^2 - 2a(x_i + tw - x)$$

$$T_1 = \frac{(w - v) - \sqrt{\Delta_1}}{a}$$

$$T_2 = \frac{(w - v) + \sqrt{\Delta_1}}{a}$$

$$\Delta_2 = (w - v)^2 - 2a(x_{i+1} + tw - x)$$

$$T_3 = \frac{(w - v) - \sqrt{\Delta_2}}{a}$$

$$T_4 = \frac{(w - v) + \sqrt{\Delta_2}}{a}$$

where:

$T = T_1$ and $T = T_2$ are solutions to $-\dfrac{aT^2}{2} + (w - v)T - (x_i + tw - x) = 0$ $(T_1 \leq T_2)$

$T = T_3$ and $T = T_4$ are solutions of $-\frac{aT^2}{2} + (w-v)T - (x_{i+1} + tw - x) = 0$ $(T_3 \leq T_4)$

We have the following cases:

1. If $\Delta_1 < 0$, then $(B.3)(ii)$ has no real solution.

2. If $\Delta_1 \geq 0$ and $\Delta_2 \leq 0$, we have $x_i + tw - \frac{(w-v)^2}{2a} \leq x \leq x_{i+1} + tw - \frac{(w-v)^2}{2a}$. The solution of $(B.3)(ii)$ is $T_1 \leq T \leq T_2$, while the solution of (B.3)(iii) is $T \in \mathbb{R}$. Hence, (B.3) can be rewritten as:

$$\begin{cases} (i) & 0 \leq T \leq \tau \\ (ii) & T_1 \leq T \leq T_2 \\ (iii) & T \leq t \end{cases}$$

From its definition, we have that $T_1 \leq 0$. Thus, $N_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t})$ has its minimum value at $T = t - \tilde{t} = 0$ if and only if $T_2 \geq 0$, that is, if $x \geq x_i + tw$. Therefore:

$$\begin{aligned} &\inf \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t}) = k_i(tw - x) - \kappa tw + b_i \\ &\text{for: } x_i + tw \leq x \leq x_{i+1} + tw - \frac{(w-v)^2}{2a} \end{aligned} \tag{B.4}$$

3. If $\Delta_1 \geq 0$ and $\Delta_2 \geq 0$, (B.3) can be simplified to:

$$\begin{cases} (i) & 0 \leq T \leq \tau \\ (ii) & T_1 \leq T \leq T_2 \\ (iii) & T \leq T_3 \text{ or } \ T \geq T_4 \\ (iv) & T \leq t \end{cases} \tag{B.5}$$

From their definitions, we have that $T_1 \leq 0$, $T_2 \leq 0$ and $T_4 \leq T_2$. If $T_2 < 0$, the (B.5) has no solution. So we need $T_2 \geq 0$, which implies $x \geq x_i + tw$. Using this last inequality, (B.5) can be rewritten as:

$$\begin{cases} (i) & 0 \leq T \leq \tau \\ (ii) & T_4 \leq T \leq T_2 \\ (iii) & T \leq t \end{cases} \tag{B.6}$$

There are two cases for which (B.6) has a solution in T:

- $T_4 \leq 0$ and $T_2 \geq 0$. In this case, $x_i + tw \leq x \leq x_{i+1} + tw$ and $\mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t})$ has its minimum value at $T = t - \tilde{t} = 0$. Therefore:

$$\inf \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t}) = k_i(tw - x) - \kappa tw + b_i \quad \text{for: } x_i + tw \leq x \leq x_{i+1} + tw \tag{B.7}$$

  We can see that the case (B.4) is included in (B.7).

- $0 \leq T_4 \leq \tau$ and $T_4 \leq t$. In this case, $x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w)$ when $t \geq \tau$ and $x_{i+1} + tw \leq x \leq x_{i+1} + tw + \frac{(at-w+v)^2-(w-v)^2}{2a}$ when $t \leq \tau$. $\mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t})$ has its minimum value for $T = t - \tilde{t} = T_4$:

$$\begin{gathered} \inf \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t}) = k_i(tw - x) - \kappa tw + b_i + \tfrac{k_i a}{2} T_4^2 \\ \text{for: } x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \tfrac{1}{2}a\tau - w) \text{ when } t \geq \tau \text{ or} \\ x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tfrac{(at-v+v)^2-(w-v)^2}{2a} \text{ when } t \leq \tau \end{gathered} \tag{B.8}$$

For $x \geq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w)$ when $t \geq \tau$, we should plug $x = x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w)$ into (B.8) to compute $\inf \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t})$:

$$\inf \mathbf{N}_{\mathbf{c}_{\text{ini}}^{(i)}}(\tilde{x}, \tilde{t}) = \kappa(\tau - t)w - k_i x_{i+1} + b_i \text{ for: } x \geq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \text{ and } t \geq \tau \tag{B.9}$$

For $x \geq x_{i+1} + tw + \frac{(at-w+v)^2-(w-v)^2}{2a}$ when $t \leq \tau$, we should plug $x = x_{i+1} + tw + \frac{(at-v+v)^2-(w-v)^2}{2a}$ into (B.8) to compute $\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t})$:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t}) = -k_i x_{i+1} + b_i \text{ for: } x \geq x_{i+1} + tw + \frac{(at-v+v)^2-(w-v)^2}{2a} \text{ and } t \leq \tau \tag{B.10}$$

Applying the second condition yields: $v(\tilde{t}, \tilde{x}) = v = w(1 - \frac{\kappa}{k_i})$, $t - \tilde{t} \geq \tau$ and $x \geq \tilde{x} + \tau v + \frac{1}{2}a\tau^2 + (t - \tilde{t} - \tau)v_f$.

$$\mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t}) = k_i(\tilde{t}w - \tilde{x}) - \kappa\tilde{t}w + b_i \geq \tilde{t}(k_i w - k_i v_f - \kappa w) + k_i(-x + \tau v + \frac{1}{2}\tau^2 + (t-\tau)v_f) + b_i$$

with equality when $x = \tilde{x} + \tau v + \frac{1}{2}\tau^2$. Hence:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t}) = \inf(\tilde{t}(k_i w - k_i v_f - \kappa w) + k_i(-x + \tau v + \frac{1}{2}\tau^2 + (t-\tau)v_f) + b_i), \text{ for } (\tilde{t}, \tilde{x}) \text{ satisfy:}$$

$$\begin{cases} (i) & t - \tilde{t} \geq \tau \\ (ii) & x = \tilde{x} + \tau v + \frac{1}{2}\tau^2 \\ (iii) & x_i + \tilde{t}w \leq \tilde{x} \leq x_{i+1} + \tilde{t}w \\ (iv) & \tilde{t} \geq 0 \end{cases} \tag{B.11}$$

Because of the negative coefficient in $\tilde{t}$, $(k_i w - k_i v_f - \kappa w)$, $N_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t})$ has its minimum value when $\tilde{t}$ is maximal. Plugging $(B.11)(ii)$ into $(B.11)(iii)$, we can rewrite $(B.11)$ as:

$$\begin{cases} (i) & \tilde{t}_1 \leq \tilde{t} \leq \tilde{t}_2 \\ (ii) & 0 \leq \tilde{t} \leq t - \tau \end{cases} \tag{B.12}$$

where $\tilde{t}_1 = \frac{1}{v_f - w}(x_i + \tau v + \frac{1}{2}a\tau^2 + (t-\tau)v_f - x)$ and $\tilde{t}_2 = \frac{1}{v_f - w}(x_{i+1} + \tau v + \frac{1}{2}a\tau^2 + (t-\tau)v_f - x)$. There are two cases to solve this minimum value problem:

- $0 \leq \tilde{t}_2 \leq t - \tau$. In this case, $x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \leq x \leq x_{i+1} + tv_f - \frac{1}{2}a\tau^2$

and $t \geq \tau$. $\tilde{t}_{max} = \tilde{t}_2$, therefore:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t}) = \frac{1}{v_f - w}(x_{i+1} + \tau v + \tfrac{1}{2}a\tau^2 + (t-\tau)v_f - x)(k_i w - k_i v_f - \kappa w) + k_i(\tau v + \tfrac{1}{2}a\tau^2 + (t-\tau)v_f - x) + b_i$$
$$\text{for: } x_{i+1} + tw + \tau(v + \tfrac{1}{2}a\tau - w) \leq x \leq x_{i+1} + tv_f - \tfrac{1}{2}a\tau^2 \text{ and } \ t \geq \tau \tag{B.13}$$

- $\tilde{t}_1 \leq t - \tau \leq \tilde{t}_2$ and $t \geq \tau$. In this case, $x_i + tw + \tau(v + \frac{1}{2}\tau - w) \leq x < x_{i+1} + tw + \tau(v + \frac{1}{2}\tau - w)$ and $\tilde{t}_{max} = t - \tau$. Therefore:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t}) = k_i(tw - x) - \kappa tw + b_i + \tfrac{1}{2}k_i\tau^2$$
$$\text{for: } x_i + tw + \tau(v + \tfrac{1}{2}a\tau - w) \leq x < x_{i+1} + tw + \tau(v + \tfrac{1}{2}a\tau - w) \text{ and } \ t \geq \tau \tag{B.14}$$

For $x \geq x_{i+1} + tv_f - \frac{1}{2}a\tau^2$ and $t \geq \tau$, we can plug $x = x_{i+1} + tv_f - \frac{1}{2}a\tau^2$ into (B.13) to compute $\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t})$:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t}) = -k_i x_{i+1} + b_i \text{ for } x \geq x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \text{ and } t \geq \tau \tag{B.15}$$

For the case $(B.1)(ii)$, it is easy to compute $\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t})$ by using a similar procedure:

$$\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t}) = \begin{cases} (i) \quad k_c(tw - x) - \kappa tw + x_{i+1}(k_c - k_i) + b_i & : x_{i+1} + tw \leq x \leq x_{i+1} + tv_f \\ (ii) \quad -x_{i+1}k_i + b_i & : x \geq x_{i+1} + tv_f \end{cases} \tag{B.16}$$

After obtaining $\inf \mathbf{N}_{\mathbf{c}^{(i)}_{\text{ini}}}(\tilde{x}, \tilde{t})$ in different domains, it is needed to determine the minimum value of $(B.7) - (B.10)$ and $(B.13) - (B.16)$ in their corresponding domains to compute $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t, x)$. For the domain $x_i + tw \leq x \leq x_{i+1} + tw$, it is obvious that $(B.14) \geq (B.7)$. Thus:

$$\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t, x) = k_i(tw - x) - \kappa wt + b_i \text{ for: } x_i + tw \leq x \leq x_{i+1} + tw \tag{B.17}$$

For the domain $x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w)$ and $t \geq \tau$, we need to compare the values of (B.8),(B.14) and (B.16)(i). $T_4 \leq \tau$ is needed to have (B.8), which implies that $(B.8) \leq (B.14)$. We now show that $(B.8) \leq (B.16)(i)$. Let:

$$\begin{gathered} F(x) = (B.8) - (B.16)(i) = (k_i - k_c)(tw - x + x_{i+1}) + \\ \tfrac{1}{2a}k_i(w - v + \sqrt{(w-v)^2 - 2a(x_{i+1} + tw - x)})^2 \\ \text{for:} x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \tfrac{1}{2}a\tau - w) \end{gathered} \tag{B.18}$$

Let us define the change of variables $z = \sqrt{(w-v)^2 - 2a(tw - x + x_{i+1})}$. With this change of variables, (B.18) becomes:

$$F(z) = \frac{1}{2a}z^2 + \frac{1}{a}(w-v)z + \frac{1}{2a}(w-v)^2(2k_i - k_c) \tag{B.19}$$

$z = z_1 = v - w$ and $z = z_2 = \frac{1}{k_c}(2k_i - k_c)(v - w)$ are the roots of $F(z) = 0$. Hence $F(z) \leq 0$ when $z_1 \leq z \leq z_2$. $z = z_1 = v - w$ implies $x = x_{i+1} + tw$ and $z = z_2 = \frac{1}{k_c}(2k_i - k_c)(v - w)$ implies $x = x_{i+1} + tw + \frac{1}{a}(v_f - v)(v_f - w)$, therefore:

$$F(x) \leq 0 \text{ for } x_{i+1} + tw \leq x \leq x_{i+1} + tw + \frac{1}{a}(v_f - v)(v_f - w) \tag{B.20}$$

Note that we can write $x_{i+1} + tw + \frac{1}{a}(v_f - v)(v_f - w) - (x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w)) = \frac{1}{2a}(v - v_f)^2 \geq 0$, which enables us to rewrite (B.20) as:

$$F(x) \leq 0 \text{ for } x_{i+1} + tw \leq x \leq x_{i+1} + tw +_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \tag{B.21}$$

Therefore, $(B.8) \leq (B.16)(i)$ and (B.8) minimizes our objective function in the domain $x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w)$ when $t \geq \tau$ (it also includes the domain $x_{i+1} + tw \le x \le x_{i+1} + tw + \frac{(at-w+v)^2-(w-v)^2}{2a}$ when $t \le \tau$):

$$\begin{gathered}\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x) = k_i(tw-x) - \kappa tw + b_i + \tfrac{1}{2}k_i a T_4^2 \\ \text{for: } x_{i+1} + tw \le x \le x_{i+1} + tw + \tau(v + \tfrac{1}{2}a\tau - w) \text{ when } t \ge \tau \text{ or} \\ x_{i+1} + tw \le x \le x_{i+1} + tw + \tfrac{(at-w+v)^2-(w-v)^2}{2a} \text{ when } t \le \tau\end{gathered} \tag{B.22}$$

For the domain of $x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \le x \le x_{i+1} + tv_f - \frac{1}{2}a\tau^2$ and $t \ge \tau$, we need to compare the values of (B.9),(B.13) and (B.16)(i). In this domain, (B.9) is independent of x, while (B.13) and (B.16)(i) are linearly decreasing functions of x. Thus, we just need to evaluate the values at the lower boundary of the spatial domain to find the minimizer. At this lower boundary, $x = x_{i+1}+tw+\tau(v+\frac{1}{2}a\tau-w)$, $(B.13) = (B.9) = \kappa(\tau - t)w - x_{i+1}k_i + b_i$, so $(B.13) \le (B.9)$ in this domain. At the upper boundary, we have $x = x_{i+1}+tw-\frac{1}{2}a\tau^2$, $(B.9)-(B.16)(i) = \kappa\tau w - \frac{1}{2}ak_c\tau^2 \le 0$, which implies that $(B.9) \le (B.16)(i)$ in this domain. Therefore,(B.13) minimizes our objective function in the domain of $x_{i+1}+tw+\tau(v+\frac{1}{2}a\tau-w) \le x \le x_{i+1}+tv_f-\frac{1}{2}a\tau^2$ and $t \ge \tau$:

$$\begin{gathered}\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x) = \tfrac{1}{v_f-w}(x_{i+1} + \tau v + \tfrac{1}{2}a\tau^2 + (t-\tau)v_f - x)(k_i w - k_i v_f - \kappa w) + \\ k_i(\tau v + \tfrac{1}{2}a\tau^2 + (t-\tau)v_f - x) + b_i \\ \text{for: } x_{i+1} + tw + \tau(v + \tfrac{1}{2}a\tau - w) \le x \le x_{i+1} + tv_f - \tfrac{1}{2}a\tau^2 \text{ and } \ t \le \tau\end{gathered} \tag{B.23}$$

Similarly, it is easy to determine the smallest value in the rest of the domain by using the linearity property. For the rest of the domain, $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x)$ is:

$$\begin{gathered}\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x) = -x_{i+1}k_i + b_i \\ \text{for: } x \ge x_{i+1} + tv_f - \tfrac{1}{2}a\tau^2 \text{ if } t \ge \tau \text{ or } x \ge x_{i+1} + tw + \tfrac{1}{2a}((at-w+v)^2 - (w-v)^2) \text{ if } t \le \tau\end{gathered} \tag{B.24}$$

Finally, using (B.17), (B.22), (B.23) and (B.24), we can write $\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x)$ on its different domains for a congested initial condition ($k_c \leq k_i \leq \kappa$) as:

$$\mathbf{N}_{\mathbf{c}^{(i)}_{\text{modified}}}(t,x) = \begin{cases} (i) & k_i(tw-x) - \kappa tw + b_i \\ & : x_i + tw \leq x \leq x_{i+1} + tw \\ (ii) & k_i(tw-x) - \kappa tw + bi + \frac{1}{2}k_i a T_4^2 \\ & : x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \text{ when } t \geq \tau \text{ and} \\ & x_{i+1} + tw \leq x \leq x_{i+1} + tw + \frac{(at-w+v)^2-(w-v)^2}{2a} \text{ when } t \leq \tau \\ (iii) & \frac{1}{v_f - w}(x_{i+1} + \tau v + \frac{1}{2}a\tau^2 + (t-\tau)v_f - x)(k_i w - k_i v_f - \kappa w) + \\ & k_i(\tau v + \frac{1}{2}a\tau^2 + (t-\tau)v_f - x) + b_i \\ & : x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \leq x \leq x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \\ & \text{and} \quad t \geq \tau \\ (iv) & -x_{i+1}k_i + b_i \\ & : x \geq x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \text{ if } t \geq \tau \text{ or} \\ & x \geq x_{i+1} + tw + \frac{1}{2a}((at-w+v)^2 - (w-v)^2) \text{ when } t \leq \tau \end{cases} \quad \text{(B.25)}$$

$$k_{\mathbf{c}^{(i)}_{\text{modified}}}(x,t) == \begin{cases} (i) & k_i \\ & : x_i + tw \leq x \leq x_{i+1} + tw \\ (ii) & \frac{k_i(v-w)}{\sqrt{(v-w)^2 - 2a(x_{i+1} - x + tw)}} \\ & : x_{i+1} + tw \leq x \leq x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \text{ when } t \geq \tau \text{ and} \\ & x_{i+1} + tw \leq x \leq x_{i+1} + tw + \frac{(at-w+v)^2-(w-v)^2}{2a} \text{ when } t \leq \tau \\ (iii) & k_c \\ & : x_{i+1} + tw + \tau(v + \frac{1}{2}a\tau - w) \leq x \leq x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \\ & \text{and} \quad t \geq \tau \\ (iv) & 0 \\ & : x \geq x_{i+1} + tv_f - \frac{1}{2}a\tau^2 \text{ if } t \geq \tau \text{ or} \\ & x \geq x_{i+1} + tw + \frac{1}{2a}((at-w+v)^2 - (w-v)^2) \text{ when } t \leq \tau \end{cases} \quad \text{(B.26)}$$

C Papers Submitted and Under Preparation

• Shanwen Qiu, Mohannad Abdelaziz, Fadl Abdellatif,and Christian G. Claudel "An exact and grid-free numerical scheme for the hybrid two phase traffic flow model based on the Lighthill-Whitham-Richards model with bounded acceleration", *Submitted to Transportation Research Part B.*