Optimal Power Allocation of a Wireless Sensor Node under Different Rate Constraints

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ABSTRACT

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Wireless sensor networks consist of the placement of sensors over a broad area in order to acquire data. Depending on the application, different design criteria should be considered in the construction of the sensors but among all of them, the battery life-cycle is of crucial interest. Power minimization is a problem that has been addressed from different approaches which include an analysis from an architectural perspective and with bit error rate and/or discrete instantaneous transmission rate constraints, among others. In this work, the optimal transmit power of a sensor node while satisfying different rate constraints is derived. First, an optimization problem with an instantaneous transmission rate constraint is addressed. Next, the optimal power is analyzed, but now with an average transmission rate constraint. The optimal solution for a class of fading channels, in terms of system parameters, is presented and a suboptimal solution is also proposed for an easier, yet efficient, implementation. Insightful asymptotical analysis for both schemes, considering a Rayleigh fading channel, are shown. Furthermore, the optimal power allocation for a sensor node in a cognitive radio environment is analyzed where an optimum solution for a class of fading channels is again derived. In all cases, numerical results are...
provided for either Rayleigh or Nakagami-m fading channels. The results obtained are extended to scenarios where we have either one transmitter-multiple receivers or multiple transmitters-one receiver.
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You will teach to fly, but they will not fly your dream.

You will teach to live, but they will not live your life.

However, in every flight, in every life, in every dream it will remain the footprint of the path that you taught.

Mother Teresa of Calcutta
I thank all my friends living in all directions over the world for their support and for being with me in every step of this great adventure. Masha’Allah!

Each friend represents a world in us,
a world possibly not born until they arrive,
and it is only by this meeting
that a new world is born.

Anaïs Nin

I thank my parents and brothers for their continuous encouragement and support. It may never be possible to return everything that you have done for me but I dedicate all my achievements to you. ¡Los Amo!

When you look at your life,
the greatest happinesses are family happinesses.

Joyce Brothers

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The world needs dreamers and the world needs doers.
But above all, the world needs dreamers who do.

Sarah Ban Breathnach
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Chapter I

Introduction

Wireless sensor networks (WSNs) consist of the placement of sensors over a broad area in order to acquire data. These networks have an increasing impact in a great variety of industrial, medical, and environmental applications [1]. The energy consumption, transmission power, memory, and computational speed of the devices have a direct influence in the viability and cost of WSNs. For instance, assuming that some sensors are placed in a desertic area and each one of these devices is very distant from the others, one of the goals is to keep the sensors operational with as low maintenance as possible. Obviously, among other design criteria of sensor devices, the battery life-cycle is of crucial interest [1].

Power minimization has been addressed so far from an architectural perspective together with the design of efficient algorithms for WSNs. For instance, Chen et al. proposed in [2] a cross layer algorithm that is capable of minimizing the power consumption as well as dealing with scheduling and routing problems for WSNs with data aggregation. Also, a new routing protocol known as Pairs Energy Efficient Routing protocol that focuses on minimizing both the energy dissipated and the delay cost has been proposed as well as an optimal, computationally efficient, integer-bit allocation algorithm for discrete multi-tone modulation [3, 4]. In addition, Rhee et al. presented
in [5] a highly power-efficient sensor networking platform called the i-Bean Network. The techniques presented minimize the energy consumption at all levels of the system (node design, physical layer communications, MAC and network protocols, and system design). Also, a Lagrange multiplier power-loading algorithm has been proposed in order to solve the power-loading optimization problem under a total transmit power constraint [6]. Previous works have also considered a power minimization problem with bit error rate and/or discrete instantaneous transmission rate constraints. For instance, Berry and Gallager studied in [7] the regulation of the long-term average transmission power and the average buffer delay incurred by the traffic, assuming that the transmitter has perfect channel state information. In addition, an integer linear programming model has been proposed in order to address the problem of establishing energy-efficient state assignment to sensors under energy-efficient routing, clustering and coverage constraints [8]. In [9, 10], loading algorithms are presented to minimize transmit-power under rate and error probability constraints, using three types of channel state information at the transmitter (CSIT): deterministic (D), statistical (S) and quantized (Q). Also, the solution for the minimization of the average transmit power in a multiple-input single-output communication system subject to average rate and bit error rate requirements is proposed. It is worth pointing out that all the work in [9, 10] is presented in a discretized framework in order to address some practical challenges related to algorithms implementation.

Nowadays, more realistic scenarios are also being considered, for instance a link without CSIT or a spectrum-sharing communication system. The aim behind these scenarios is to improve the efficiency of use of the scarce spectrum because without it, wireless communications are not possible [11]. Cognitive radio (CR) is a new paradigm in this area of communications that considers the idea of sensing the environment to improve the communication without interfering with other users. In [12], a general overview of cognitive radio sensor networks (CRSNs) is given and a discussion about
the existing communication protocols and algorithms designed for cognitive radio networks and WSNs is presented along with open research avenues for the realization of CRSNs. In addition, Rezki and Slim presented in [13] the derivation of the optimum power profile and the ergodic capacity for general fading channels with respect to average and peak transmit power along with interference outage constraints.

In this work, first the problem of minimizing the transmit power of a wireless sensor node while satisfying an instantaneous or an average transmission rate constraint is addressed. Considering the challenge of efficient spectrum utilization in order to overcome the increasing demand for wireless communication systems, we also focus on the minimization of the transmit power of nodes in a WSN under an instantaneous or an average transmission rate constraint, along with an interference constraint, i.e. in a spectrum-shared / underlay CR framework. For each case, either a closed-form solution for the optimal power allocation is derived or a suboptimal power policy is proposed for a class of fading channels. In all cases, numerical results are provided for either Rayleigh or Nakagami-m fading channels. Differently from [9, 10], our framework deals with the fundamental limits of the minimum transmit power of nodes in a WSN with or without CR settings.

This work is organized as follows. Chapter II presents the mathematical formulation of the problem. In chapter III, the power minimization problem of a sensor node in a non-cognitive radio environment is addressed under different rate constraints together with asymptotic analysis and numerical results. Chapter IV shows the optimal power allocation of a sensor node in a cognitive radio environment under different rate constraints. In chapter V, the power minimization problem of a WSN with multiple receivers in a cognitive radio environment is addressed under different rate constraints. Chapter VI shows the power minimization problem of a WSN with multiple transmitters in a cognitive radio environment under different rate constraints. Chapter VII concludes the work.
Chapter II

System Model

This chapter introduces the models used throughout the rest of this work. In chapter III and IV, two scenarios are being considered. The first one is depicted in Fig. II.0.1 (solid line), where two sensors are communicating through a fading channel, denoted by $h_s$. Next, we consider a spectrum sharing communication system where a secondary sensor node (SN) is communicating with a secondary user receiver in presence of a primary SN (dashed lines in Fig. II.0.1) under certain interference constraints. The channel between the secondary SN transmitter and the primary SN receiver is denoted by $h_p$. All fading channels are assumed to be ergodic and stationary with continuous probability density functions (PDF) denoted by $f_{h_{pp}}(h_{pp})$, $f_{h_p}(h_p)$ and $f_{h_s}(h_s)$. We assume that the transmit SN is perfectly aware of the instantaneous channel gain $h_s$. However, it is only aware of the statistics, i.e. PDF, of $h_p$.

Our aim is to minimize the power $P(|h_s|) = P$ considering four different cases which are denoted as follows:
Figure II.0.1: Wireless sensor link in a spectrum sharing environment.

\[
\begin{align*}
\min_P P(|h_s|) &= P \\
C1 : & \quad R(|h_s|) \geq R_{\text{min}} \\
C2 : & \quad E_{h_s}[R(|h_s|)] \geq R_{\text{min}} \\
C3 : & \quad C1 \cup C5 \\
C4 : & \quad C2 \cup C5
\end{align*}
\] (II.0.1)

where \( R(|h_s|) = \log(1 + P(|h_s|) \cdot |h_p|^2) \), (C5) is defined by \( \Pr \{ P(|h_s|) \cdot |h_p|^2 \geq Q_{\text{peak}} \} \leq \epsilon \), \( R_{\text{min}} \) is the minimum transmission rate required, \( Q_{\text{peak}} \) is the interference limit that the primary SN can tolerate from the secondary SN, and \( \epsilon \) is the threshold for the level of interference allowed.

Furthermore, in chapter V and VI, two extensions of the scenario proposed in Fig. II.0.1 are analyzed, where there are either multiple receivers or multiple transmitters in the secondary link, as depicted in Fig. II.0.2 and Fig. II.0.3.
Figure II.0.2: Wireless sensor link in a spectrum sharing environment with multiple secondary receivers.
Figure II.0.3: Wireless sensor link in a spectrum sharing environment with multiple secondary transmitters
Chapter III

Optimal Power Allocation for a Sensor Node in a Non-Cognitive Radio Environment

III.1 Optimal Power Allocation for a Sensor Node under Instantaneous Transmission Rate Constraint (C1)

III.1.1 Problem Formulation

Consider the minimization of the power $P(|h_s|) = P$ subject to an instantaneous transmission rate constraint given by $R(|h_s|) \geq R_{\text{min}}$. This optimization problem can be rewritten as

$$\begin{align*}
\min_P P(|h_s|) &= P \\
\text{s.t. } R(|h_s|) &\geq R_{\text{min}}
\end{align*}$$

(III.1.1)
III.1.2 General Solution

In order to solve this problem, we need to satisfy the constraint, i.e. \( R(|h_s|) = \log (1 + P(|h_s|) \cdot |h_s|^2) \geq R_{\text{min}} \). Solving for \( P(|h_s|) \) gives

\[
P(|h_s|) \geq \frac{e^{R_{\text{min}}} - 1}{|h_s|^2} \tag{III.1.2}
\]

Therefore the minimum power required under an instantaneous transmission rate constraint is given by

\[
P(|h_s|) = \begin{cases} 
\frac{e^{R_{\text{min}}} - 1}{|h_s|^2} & |h_s|^2 > 0 \\
0 & \text{otherwise}
\end{cases} \tag{III.1.3}
\]

Note that equation (III.1.3) is a function of the system parameter \( R_{\text{min}} \) as well as the channel gain \( h_s \).

III.2 Optimal Power Allocation for a Sensor Node under Average Transmission Rate Constraint (C2)

III.2.1 Problem Formulation

Consider the minimization of the power \( P(|h_s|) = P \) subject to an average transmission rate constraint given by \( E_{|h_s|^2} [R(|h_s|)] \geq R_{\text{min}} \). This optimization problem can be rewritten as

\[
\min_P P(|h_s|) = P \quad \text{s.t. } E_{|h_s|^2} [R(|h_s|)] \geq R_{\text{min}} \tag{III.2.1}
\]
III.2.2 Optimal Solution

To solve this problem, we can write the Lagrangian as follows,

\[ \mathcal{L} = P(|h_s|) - \lambda \left[ \int_0^\infty \log \left( 1 + P(|h_s|) \cdot |h_s|^2 \right) \cdot f_{|h_s|^2}(|h_s|^2) \, d|h_s|^2 - R_{\min} \right] \]  

(III.2.2)

Since both the objective function and the constraint (with the minus sign) are convex functions, the solution that we obtain is a global minimum. With this argument, we can differentiate the Lagrangian with respect to \( P(|h_s|) \) (see Appendix A) to obtain

\[ 1 - \lambda \frac{|h_s|^2}{1 + P(|h_s|) \cdot |h_s|^2} = 0, \]

which means that

\[ P(|h_s|) = \left[ \lambda - \frac{1}{|h_s|^2} \right]^+ \]  

(III.2.3)

The power profile (III.2.3) is the well-known water-filling solution which arises in information theory, in allocating power to a set of \( n \) communication channels [14].

From (III.2.3) we can see that our solution will depend on the value of the Lagrange multiplier since

\[ P(|h_s|) = \begin{cases} 
\left[ \lambda - \frac{1}{|h_s|^2} \right]^+ & \lambda > 0 \\
\text{no solution} & \lambda = 0 
\end{cases} \]  

(III.2.4)

From (III.2.4) we can see that a valid solution is obtained only when \( \lambda > 0 \), and from the first optimality conditions, this means that our constraint is achieved with equality therefore, substituting (III.2.3) into the constraint \((C2)\) gives the following implicit expression for \( \lambda \):

\[ \lambda = \exp \left( \frac{R_{\min} - \int_\chi^\infty \log \left( |h_s|^2 \right) f_{|h_s|^2} \left( |h_s|^2 \right) \, d|h_s|^2}{\int_\chi^\infty f_{|h_s|^2} \left( |h_s|^2 \right) \, d|h_s|^2} \right) \]  

(III.2.5)

which can be solved numerically. Note that the power profile (III.2.3) involves solving
the transcendental equation (III.2.5) for each channel realization \( h_s \). Now our aim is to obtain a closed-form solution that only depends on the system parameters. Let us consider a variable \( x \) such that \( x = \frac{1}{\lambda} \). The power profile in (III.2.3) can be rewritten as

\[
P( |h_s| ) = \left[ \frac{1}{x} - \frac{1}{|h_s|^2} \right]^+ \tag{III.2.6}
\]

Substituting (III.2.6) into the constraint (C2) leads to

\[
G(x) = R_{\min} \tag{III.2.7}
\]

where

\[
G(x) = -\log(x) \left[ 1 - F_{|h_s|^2}(x) \right] + \int_{x}^{\infty} \log(|h_s|^2) f_{|h_s|^2}(|h_s|^2) d|h_s|^2 \quad \tag{III.2.8}
\]

\( G(x) \) can be shown to be monotonic decreasing function (see Appendix B). Thus, the optimized power is given by

\[
P( |h_s| ) = \left[ \frac{1}{G^{-1}(R_{\min})} - \frac{1}{|h_s|^2} \right]^+ \tag{III.2.9}
\]

### III.2.3 On / Off Suboptimal Solution

The optimal solution presented in section III.2.2 would be difficult to implement because of all the numerical computations required in order to obtain the optimal power \( P \). Considering this, we propose a simple suboptimal solution based on On / Off power control, i.e.
\[ P(|h_s|) = \begin{cases} P_o & |h_s|^2 \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (\text{III.2.10}) \]

From (C2) we know that the minimum power required is the one that satisfies the constraint with equality, i.e.

\[ \int_0^\infty \log \left( 1 + P_o \cdot |h_s|^2 \right) f_{|h_s|^2} \left( |h_s|^2 \right) d|h_s|^2 = R_{\min} \quad (\text{III.2.11}) \]

Hence, (III.2.10) is a uniform power policy, i.e.

\[ P(|h_s|) = P_o \cdot |h_s|^2 \geq 0 \quad (\text{III.2.12}) \]

where \( P_o \) is obtained by solving (III.2.11). Interestingly, it is important to see that in this case, the CSIT is not needed.

### III.2.4 Application for Rayleigh Fading Channels

- Optimal Scheme

Let us consider a Rayleigh fading channel, i.e. \( f_{|h_s|^2} \left( |h_s|^2 \right) = e^{-|h_s|^2} \). The optimal power is given by (III.2.9) for \( h_s > 0 \) with

\[ G(x) = E_1(x) \quad (\text{III.2.13}) \]

where \( E_1(\cdot) \) is the exponential integral function of first-order defined by \( E_1(x) = \int_1^\infty \frac{e^{-t}}{t} dt \), [15].

It is important to notice that when \( R_{\min} \to \infty \), then \( \lambda \to \infty \) and (III.2.3) can be rewritten as (see Appendix C)

\[ P(|h_s|) \simeq \left[ e^{R_{\min}+\gamma} - \frac{1}{|h_s|^2} \right]^+ \quad (\text{III.2.14}) \]
where $\gamma \simeq 0.577216$ is Euler’s constant.

Also, when $R_{\text{min}} \to 0$, (III.2.3) can be rewritten as (see Appendix D)

$$P(|h_s|) \simeq \left( \frac{1}{W\left( \frac{1}{R_{\text{min}}} \right)} - \frac{1}{|h_s|^2} \right)^+$$  \hspace{1cm} (III.2.15)

where $W(\cdot)$ is the Lambert W-function [15].

- **On / Off Suboptimal Scheme**

Let us consider the same channel fading following the Rayleigh PDF. Thus, we can say that the suboptimal power is given by

$$P(|h_s|) = P_o \cdot |h_s|^2 \geq 0$$  \hspace{1cm} (III.2.16)

where $P_o$ is obtained by solving the equation

$$e^{P_o} E_1 \left( \frac{1}{P_o} \right) = R_{\text{min}}$$  \hspace{1cm} (III.2.17)

It is important to notice that when $R_{\text{min}} \to \infty$, then (III.2.11) can be rewritten as (see Appendix E)

$$P_o \simeq e^{R_{\text{min}} + \gamma}$$  \hspace{1cm} (III.2.18)

Also, when $R_{\text{min}} \to 0$, (III.2.11) can be rewritten as (see Appendix F)

$$P_o \simeq R_{\text{min}}$$  \hspace{1cm} (III.2.19)
III.2.5 Numerical Comparison between Optimal and On / Off Suboptimal Scheme

Figure III.2.1 depicts the average transmit power of a sensor node as a function of $R_{\text{min}}$ for a Rayleigh fading channel considering the optimal scheme, the On / Off scheme and the asymptotic approximations for high values of $R_{\text{min}}$. It can be seen that as $R_{\text{min}}$ increases, the optimal and suboptimal solutions converge, inferring that at high $R_{\text{min}}$, uniform power allocation is optimal.

Figure III.2.1: Performance comparison between the optimal and On / Off suboptimal scheme under an average rate contraint.

Also shown in Fig. III.2.2, is a performance comparison of the average transmit power of a sensor node as a function of $R_{\text{min}}$ for a Rayleigh fading channel between the optimal scheme, the On / Off scheme and the asymptotical approximations for low values of $R_{\text{min}}$. Note also that in this regime of $R_{\text{min}}$, our approximation given by equation (III.2.15) matches perfectly with the optimal solution (cf. red curve in...
Figure III.2.2: Performance comparison between the optimal and the On / Off sub-optimal scheme for low $R_{\text{min}}$ values under an average rate constraint.
Chapter IV

Optimal Power Allocation for a Sensor Node in a Cognitive Radio Environment

IV.1 Optimal Power Allocation for a Sensor Node in a Cognitive Radio Environment under Instantaneous Transmission Rate Constraint (C3)

IV.1.1 Problem Formulation

Consider the minimization of the power $P(|h_s|) = P$ subject to an instantaneous transmission rate constraint given by $R(|h_s|) \geq R_{\text{min}}$, and an interference constraint given by (C5). This optimization problem can be rewritten as

$$\min_P P(|h_s|) = P$$

s.t. $R(|h_s|) \geq R_{\text{min}}$

and $\Pr \left\{ P(|h_s|) \cdot |h_p|^2 \geq Q_{\text{peak}} \right\} \leq \epsilon$ (IV.1.1)
IV.1.2 General Solution

For this case, the purpose of the interference constraint is to reduce the interference power detected at the primary receiver. Notice that the interference constraint (C5) is equivalent to

$$P (|h_s|) \leq P_{hp} (\epsilon)$$  \hspace{1cm} (IV.1.2)$$

where $P_{hp} (\epsilon) = \frac{Q_{\text{peak}}}{F_{|h_p|^2} (1-\epsilon)}$ and $F_{|h_p|^2}^{-1}$ is the inverse distribution function of $|h_p|^2$. Now, we have an optimization problem with two opponent peak constraints: (C1) and (IV.1.2).

It is easy to see that in this case, the optimal power is given by

$$P (|h_s|) = \begin{cases} \frac{e^{R_{\text{min}}-1}}{|h_s|^2} & |h_s|^2 \geq \frac{e^{R_{\text{min}}-1}}{P_{hp} (\epsilon)} \\ \text{no solution} & |h_s|^2 < \frac{e^{R_{\text{min}}-1}}{P_{hp} (\epsilon)} \end{cases}$$  \hspace{1cm} (IV.1.3)$$

From (IV.1.3), we can see that when $|h_s|^2 < \frac{e^{R_{\text{min}}-1}}{P_{hp} (\epsilon)}$ there is no solution since it is not possible to satisfy both constraints at the same time. This situation is referred as “outage”. We can then minimize the outage occurrence by setting $\text{Pr} \left\{ |h_s|^2 < \frac{e^{R_{\text{min}}-1}}{P_{hp} (\epsilon)} \right\} < \eta$, which is achieved as long as

$$P_{hp} (\epsilon) > \frac{e^{R_{\text{min}}-1}}{F_{|h_p|^2}^{-1} (\eta)}$$  \hspace{1cm} (IV.1.4)$$
IV.2 Optimal Power Allocation for a Sensor Node in a Cognitive Radio Environment under Average Transmission Rate Constraint (C4)

IV.2.1 Problem Formulation

Consider the minimization of the power $P(|h_s|) = P$, subject to an average transmission rate constraint given by $E_{|h_s|^2}[R(|h_s|)] \geq R_{min}$, and an interference constraint given by (C5). This optimization problem can be rewritten as

$$\begin{align*}
\min_{P} P(|h_s|) &= P \\
\text{s.t. } E_{|h_s|^2}[R(|h_s|)] &\geq R_{min} \\
\text{and } P(|h_s|) &\leq P_{hp}(\epsilon)
\end{align*}$$

(IV.2.1)

IV.2.2 Optimal Solution

To solve this problem, we can write the Lagrangian as,

$$L = P(|h_s|) - \lambda \left[ \int_0^\infty \log (1 + P(|h_s|) \cdot |h_s|^2) \cdot f_{|h_s|^2}(|h_s|^2) \, d|h_s|^2 - R_{min} \right] + \\
\mu \left[ P(|h_s|) - P_{hp}(\epsilon) \right]$$

(IV.2.2)

where $\lambda$ and $\mu$ are the positive Lagrange multipliers. Since both the objective function and the constraints are convex functions, then the solution that we obtain is a global minimum. Differentiating the Lagrangian with respect to $P(|h_s|)$, we get $1 + \mu - \lambda \frac{|h_s|^2}{1 + P(|h_s|) \cdot |h_s|^2} = 0$, which means that

$$P(|h_s|) = \left[ \frac{\lambda}{1 + \mu} - \frac{1}{|h_s|^2} \right]^+$$

(IV.2.3)

From the KKT conditions we know that:
• \( \mu = 0 \) if \( \lambda - \frac{1}{|h_s|^2} < P_{hp}(\epsilon) \)

• \( \mu > 0 \) if \( \frac{\lambda}{1+\mu} - \frac{1}{|h_s|^2} = P_{hp}(\epsilon) \)

Therefore, the optimum power profile can be derived as (see Appendix G):

• if \( E_{|h_s|^2} \left[ \log \left( 1 + P_{hp}(\epsilon) \cdot |h_s|^2 \right) \right] - R_{\text{min}} < 0 \), there is no solution (“outage” situation),

• if \( R_{\text{min}} \leq G \left( P_{hp}(\epsilon)^{-1} \right) \)

\[
P(|h_s|) = \min \left\{ \left[ \frac{1}{G^{-1}(R_{\text{min}})} - \frac{1}{|h_s|^2} \right]^+, \ P_{hp}(\epsilon) \right\}
\] (IV.2.4)

where \( G(x) \) is defined by (III.2.8).

• if \( R_{\text{min}} > G \left( P_{hp}(\epsilon)^{-1} \right) \)

\[
P(|h_s|) = \min \left\{ \left[ \frac{1}{H^{-1}(R_{\text{min}})} - \frac{1}{|h_s|^2} \right]^+, \ P_{hp}(\epsilon) \right\}
\] (IV.2.5)

where

\[
H(x) = G(x) + \int_{\frac{1}{1-P_{hp}(\epsilon)}}^{\infty} \log \left( \frac{1 + P_{hp}(\epsilon) |h_s|^2}{|h_s|^2} x \right) f_{|h_s|^2}(|h_s|^2) \ d|h_s|^2
\] (IV.2.6)

### IV.2.3 On / Off Suboptimal Solution

The optimal solution presented in section IV.2.2 would be difficult to implement because of all the numerical computations required in order to obtain the optimal power \( P \). Considering this, we propose a simple suboptimal solution based on the On / Off power control, i.e.

\[
P(|h_s|) = \begin{cases} P_o & |h_s|^2 \geq 0 \\ 0 & \text{otherwise.} \end{cases}
\] (IV.2.7)
The procedure to solve this problem is the following: first we consider only the average rate constraint and we compute a power $P_o$ by solving (III.2.11). Once we have this power, we compare it with the interference constraint, i.e. $P (|h_s|) \leq P_{h_p} (\epsilon)$. Therefore, we can have one of two possible cases:

- if $P_o \leq P_{h_p} (\epsilon)$ then, our uniform power policy is given by (III.2.12),
- if $P_o > P_{h_p} (\epsilon)$ then, both constraints cannot be satisfied at the same time and there is no solution.

### IV.2.4 Application for Nakagami-m and Rayleigh Fading Channels

Let us consider a Nakagami-m fading channel, i.e.

$$f_{|h|^2} (|h|^2) = \frac{m^m}{\Gamma(m)} (|h|^2)^{m-1} e^{-m|h|^2}$$

where $m \geq \frac{1}{2}$ and $\Gamma (\cdot)$ is the gamma function. The optimal power profile is given by (IV.2.4) and (IV.2.5), where

$$G(x) = \frac{1}{\Gamma(m)} G_{2,0}^{3,0} \left( m x \left| \begin{array}{cc} 1 & 1 \\ 0 & 0 & m \end{array} \right. \right)$$

(IV.2.8)

where $G_{p,q}^{i,j} \left( z \left| \begin{array}{c} a_1 \ldots a_p \\ b_1 \ldots b_q \end{array} \right. \right)$ is the Meijer G-function [15]. For the case when $m = 1$, Nakagami fading channel reduces to a Rayleigh fading channel where $G(x)$ is defined in (III.2.13). Table IV.2.1 summarizes the equations involved in a Nakagami-m and Rayleigh fading channel. In this table, the funcion $\Gamma (s, x)$ is the incomplete gamma function and it is defined as $\int_x^{\infty} t^{s-1} e^{-t} dt$. 
### Table IV.2.1: Expressions of $G(x)$ and $F_{|h_p|^2}^{-1}(x)$ for Nakagami-m and Rayleigh fading channel.

|          | $G(x)$       | $F_{|h_p|^2}^{-1}(x)$ |
|----------|--------------|-----------------------|
| Rayleigh | $E_1(x)$     | $-\log(1-x)$          |
| Nakagami-m | $\frac{1}{\Gamma(m)}G_2^3 \begin{pmatrix} m \bar{x} \\ 1 \ 0 \ 0 \ m \end{pmatrix} \left[ 1 - \frac{\Gamma(m, m\bar{x})}{\Gamma(m)} \right]^{-1}$ | |

**IV.2.5 Comparison between Theoretical and Numerical Solution**

The derived results in (IV.2.4) and (IV.2.5) have been used in order to display Fig. IV.2.1 and Fig. IV.2.2. Figure IV.2.1 depicts the transmit power of a sensor node as a function of $|h_s|^2$ for a Nakagami-m fading channel with a parameter $m = \frac{1}{2}, 1$ and $2$. For this case, $R_{min} \leq G \left( P_{hp} \left( \epsilon \right)^{-1} \right)$ and our numerical simulations match with our theoretical result (refer to (IV.2.4)) for the three different values of the parameter $m$.

$$R_{min} = 0.4, P_{hp} \left( \epsilon \right) = 1$$

![Figure IV.2.1: Power profile when $R_{min} > G \left( P_{hp} \left( \epsilon \right)^{-1} \right)$](image-url)
Figure IV.2.2 depicts the transmit power of a sensor node as a function of $|h_s|^2$ for a Nakagami-m fading channel. Although we computed a power profile for a parameter $m = \frac{1}{2}, 1$ and 2, there is no significant difference between them. For this case, $R_{min} > G(P_{hp}(\epsilon)^{-1})$ and our numerical simulations match with our theoretical result (refer to (IV.2.5)) for the three different values of the parameter $m$.

$R_{min} = 0.15, P_{hp}(\epsilon) = 1$

Figure IV.2.2: Power profile when $R_{min} \leq G(P_{hp}(\epsilon)^{-1})$. 

![Figure IV.2.2: Power profile when $R_{min} \leq G(P_{hp}(\epsilon)^{-1})$.](image-url)
Chapter V

Optimal Power Allocation for a Wireless Sensor Network with Multiple Secondary Receivers

V.1 Independent Peak Transmission Rate Constraint

V.1.1 Problem Formulation

In this chapter we consider an extension of what was proposed in chapter IV. Let us consider a new scenario depicted in Fig. V.1.1.

In this case, we have multiple secondary receivers and the optimization problem can be expressed as:

$$\min_{P} P (|h_{s1}|, \ldots, |h_{sn}|) = P$$

s.t. $R (|h_{si}|) \geq R_{\min} \quad \forall i \in \{1, \ldots, n\}$

and $\Pr \{ P (|h_{s1}|, |h_{s2}|, \ldots, |h_{sn}|) \cdot |h_p|^2 \geq Q_{\text{peak}} \} \leq \epsilon$

(V.1.1)

where $R (|h_{si}|) = \log (1 + P (|h_{s1}|, \ldots, |h_{sn}|) \cdot |h_{si}|^2)$. As it has been done previously, the interference constraint can be rewritten as $P (|h_{s1}|, \ldots, |h_{sn}|) \leq P_{h_p} (\epsilon)$,
Figure V.1.1: Wireless sensor link in a spectrum sharing environment with multiple secondary receivers
where \( P_{hp}(\epsilon) = \frac{Q_{\text{peak}}}{F_{|h_p|^2}(1-\epsilon)} \) and \( F^{-1}_{|h_p|^2} \) is the inverse distribution function of \( |h_p|^2 \).

\[ V.1.2 \quad \text{General Solution} \]

Making use of the solution given by (IV.1.3) and considering that all the channels are independent between them, the first constraint is satisfied as long as \( P(|h_s_1|, \ldots, |h_s_n|) = \max_{i \in \{1,n\}} \left\{ \frac{e^{R_{\text{min}} - 1}}{|h_s_i|^2} \right\} \). Taking into account the second constraint, leads to the following two cases:

- if \( \max_{i \in \{1,n\}} \left\{ \frac{e^{R_{\text{min}} - 1}}{|h_s_i|^2} \right\} \leq P_{hp}(\epsilon) \) then, \( P(|h_s_1|, \ldots, |h_s_n|) = \max_{i \in \{1,n\}} \left\{ \frac{e^{R_{\text{min}} - 1}}{|h_s_i|^2} \right\} \),
- if \( \max_{i \in \{1,n\}} \left\{ \frac{e^{R_{\text{min}} - 1}}{|h_s_i|^2} \right\} > P_{hp}(\epsilon) \) then, there is no solution.

\[ V.1.3 \quad \text{Simulation Results} \]

The general solution proposed was implemented in a numerical simulation. The results are shown in Fig. V.1.2 where we are plotting the average power per sensor node with respect to \( R_{\text{min}} \) when \( P_{hp}(\epsilon) = 2 \). As it is shown in the figure, as the number of receivers is increased, the average power per sensor node is reduced, but this also comes with the fact that it is not possible to achieve higher values of \( R_{\text{min}} \). This observation makes senses since our power depends on the most demanding sensor, and as the number of receivers increases, the probability of an outage increases as well. This fact is shown in Fig. V.1.3.
Figure V.1.2: Performance comparison between different number of receivers under an independent peak transmission rate constraint.

V.2 Independent Average Transmission Rate Constraint

V.2.1 Problem Formulation

Consider the minimization of the power $P(|h_{s1}|, \ldots, |h_{sn}|) = P$, where each of the $n$ secondary communications are subject to an average transmission rate constraint given by $E_{|h_{si}|^2}[R(|h_{si}|)] \geq R_{min} \ \forall i \in \{1, \ldots, n\}$, and an interference constraint given by $P(|h_{s1}|, \ldots, |h_{sn}|) \leq P_{hp}(\epsilon)$. This optimization problem can be rewritten as
Figure V.1.3: Frequency of outages as a function of the number of receivers under an independent peak transmission rate constraint.

\[
\begin{align*}
\min_P & \quad P(|h_{s1}|, \ldots, |h_{sn}|) = P \\
\text{s.t.} & \quad E_{|h_{si}|^2}[R(|h_{si}|)] \geq R_{\text{min}} \quad \forall i \in \{1, \ldots, n\} \\
\text{and} & \quad P(|h_{s1}|, \ldots, |h_{sn}|) \leq P_{h_p}(\epsilon)
\end{align*}
\] (V.2.1)

V.2.2 General Solution

Assuming that the distribution of the fading channels is independent and identically distributed (i.i.d.), and making use of the solution given by (IV.2.4) and (IV.2.5) then, we have the following cases:

- if \( E_{|h_{si}|^2} \left[ \log \left( 1 + P_{h_p}(\epsilon) \cdot |h_{si}|^2 \right) \right] - R_{\text{min}} < 0 \quad \forall i \in \{1, \ldots, n\}, \) there is no solution (“outage” situation),
• if \( R_{\text{min}} \leq G \left( P_{h_p} (\epsilon)^{-1} \right) \) for every channel \( i \in \{1, \ldots, n\} \) then,

\[
P (|h_{s1}|, \ldots, |h_{sn}|) = \min \left\{ \max_{i \in \{1, \ldots, n\}} \left[ \frac{1}{G^{-1} (R_{\text{min}})} - \frac{1}{|h_{si}|^2} \right]^+, P_{h_p} (\epsilon) \right\}
\]

where \( G (x) \) is defined by (III.2.8).

• if \( R_{\text{min}} > G \left( P_{h_p} (\epsilon)^{-1} \right) \) for every channel \( i \in \{1, \ldots, n\} \) then,

\[
P (|h_{s1}|, \ldots, |h_{sn}|) = \min \left\{ \max_{i \in \{1, \ldots, n\}} \left[ \frac{1}{H^{-1} (R_{\text{min}})} - \frac{1}{|h_{si}|^2} \right]^+, P_{h_p} (\epsilon) \right\}
\]

where \( H (x) \) is defined by (IV.2.6).

V.2.3 Simulation Results

The general solution proposed was implemented in a numerical simulation. The results are shown in Fig. V.2.1 where we are plotting the average power per sensor node with respect to \( R_{\text{min}} \) when \( P_{h_p} (\epsilon) = 2 \). As it is shown in the figure, as the number of receivers is increased, the average power per sensor node is reduced. For this simulation, when \( R_{\text{min}} \leq G \left( P_{h_p} (\epsilon)^{-1} \right) = 0.5597 \) the optimal power is given by (V.2.2), and when \( R_{\text{min}} > G \left( P_{h_p} (\epsilon)^{-1} \right) = 0.5597 \) then the solution is given by (V.2.3). It is important to mention that for low values of \( R_{\text{min}} \) the optimal power is equal to zero, therefore it is not plotted in the figure.

V.3 Sum of Peak Transmission Rate Constraint

V.3.1 Problem Formulation

Consider the minimization of the power \( P (|h_{s1}|, \ldots, |h_{sn}|) = P \), where the \( n \) secondary communications are subject to a sum of peak transmission rate constraint given by \( \sum_{i=1}^{n} R (|h_{si}|) \geq R_{\text{min}} \), and an interference constraint given by \( P (|h_{s1}|, \ldots, |h_{sn}|) \leq P_{h_p} (\epsilon) \). This optimization problem can be rewritten as
Figure V.2.1: Performance comparison between different number of receivers under an independent average transmission rate constraint.

V.3.2 General Solution

We can notice that the first constraint can be rewritten as

$$\min_P P(|h_{s1}|, \ldots, |h_{sn}|) = P$$

s.t. $\sum_{i=1}^{n} R(|h_{si}|) \geq R_{\text{min}}$ (V.3.1)

and $P(|h_{s1}|, \ldots, |h_{sn}|) \leq P_{h_p}(\epsilon)$

$$\sum_{i=1}^{n} R(|h_{si}|) = \log \left[ \prod_{i=1}^{n} \left( 1 + P(|h_{s1}|, \ldots, |h_{sn}|) \cdot |h_{si}|^2 \right) \right] \geq R_{\text{min}}$$ (V.3.2)
\[ \prod_{i=1}^{n} (1 + P(|h_{s_1}|, \ldots, |h_{s_n}|) \cdot |h_{s_i}|^2) - e^{R_{\min}} \geq 0 \quad (V.3.3) \]

is valid. Expanding (V.3.3) leads to a polynomial of order \( n \) in terms of \( P(|h_{s_1}|, \ldots, |h_{s_n}|) \) where all its coefficients \( a_i \) are larger than zero, except the independent coefficient \( a_0 = 1 - e^{R_{\min}} \) which is lower than zero. It is important to mention that the rest of the coefficients \( a_i \) follow the following rule: each \( a_i \) is equivalent to the sum of the product of all the combinations of the \( n \) channel gains taken \( i \) at a time. Making use of the Descartes’ rule of signs, from the \( n \) roots of (V.3.3) only one is real and positive therefore, we can apply Newton’s method to find it. The advantage of applying Newton’s method is that we are able to always find the root \( P^* \) that we are interested in, independently of the initial guess for the algorithm.

Taking into account the second constraint, leads to the following two cases:

- if \( P^* \leq P_{h_p}(\epsilon) \) then, \( P(|h_{s_1}|, \ldots, |h_{s_n}|) = P^* \),
- if \( P^* > P_{h_p}(\epsilon) \) then, there is no solution.

**V.3.3 Simulation Results**

The general solution proposed was implemented in a numerical simulation. The results are shown in Fig. V.3.1 where we are plotting the average power per sensor node with respect to \( R_{\min} \) when \( P_{h_p}(\epsilon) = 2 \). As it is shown in the figure, as the number of receivers is increased, the average power per sensor node is reduced. Also, it is interesting to notice that as the number of receivers increases, it is possible to achieve higher values of \( R_{\min} \). This observation makes senses since the sensor nodes are working cooperatively in order to satisfy the sum of peak transmission rate constraint, therefore the amount of power required per sensor is reduced. This also means that as the number of receivers increases, the probability of an outage is reduced (as shown in Fig. V.3.2).
V.4 Sum of Average Transmission Rate Constraint

V.4.1 Problem Formulation

Consider the minimization of the power $P (|h_{s1}|, \ldots, |h_{sn}|) = P$, where the $n$ secondary communications are subject to a sum of average transmission rate constraint given by $\sum_{i=1}^{n} E_{|h_{si}|^2} [R (|h_{si}|)] \geq R_{\text{min}}$, and an interference constraint given by $P (|h_{s1}|, \ldots, |h_{sn}|) \leq P_{h_{p}} (\epsilon)$. Assuming that the fading channels are i.i.d. then, this optimization problem can be rewritten as
Figure V.3.2: Frequency of outages as a function of the number of receivers under an independent peak transmission rate constraint.

\[
\begin{align*}
\min_p P (|h_{s1}|, \ldots, |h_{sn}|) = P \\
\text{s.t. } E_{|h_1|^2} [R (|h_s|)] & \geq \frac{R_{\min}}{n} \\
\text{and } P (|h_{s1}|, \ldots, |h_{sn}|) & \leq P_{hp} (\epsilon)
\end{align*}
\]  

(V.4.1)

V.4.2 General Solution

Making use of the solution given by (IV.2.4) and (IV.2.5) then, we have the following cases:

- if \( E_{|h_s|^2} [\log (1 + P_{hp} (\epsilon) \cdot |h_s|^2)] - \frac{R_{\min}}{n} < 0 \), there is no solution (“outage” situation),
- if \( \frac{R_{\min}}{n} \leq G (P_{hp} (\epsilon)^{-1}) \) for every channel \( i \in \{1, ..., n\} \) then,
\[ P(|h_{s1}|, \ldots, |h_{sn}|) = \min \left\{ \max_{i \in \{1, \ldots, n\}} \left[ \frac{1}{G^{-1} \left( \frac{R_{\min}}{n} \right)} - \frac{1}{|h_{s1}|^2} \right]^+, P_{hp}(\epsilon) \right\} \quad \text{(V.4.2)} \]

where \( G(x) \) is defined by (III.2.8).

- if \( \frac{R_{\min}}{n} > G \left( P_{hp}(\epsilon)^{-1} \right) \) for every channel \( i \in \{1, \ldots, n\} \) then,

\[ P(|h_{s1}|, \ldots, |h_{sn}|) = \min \left\{ \max_{i \in \{1, \ldots, n\}} \left[ \frac{1}{H^{-1} \left( \frac{R_{\min}}{n} \right)} - \frac{1}{|h_{s1}|^2} \right]^+, P_{hp}(\epsilon) \right\} \quad \text{(V.4.3)} \]

where \( H(x) \) is defined by (IV.2.6).

### V.4.3 Simulation Results

The general solution proposed was implemented in a numerical simulation. The results are shown in Fig. V.4.1 where we are plotting the average power per sensor node with respect to \( R_{\min} \) when \( P_{hp}(\epsilon) = 2 \). As it is shown in the figure, as the number of receivers is increased, the average power per sensor node is reduced. It is important to mention that for low values of \( R_{\min} \) the optimal power is equal to zero, therefore it is not plotted in the figure.

### V.5 Product of Peak Transmission Rate Constraint

#### V.5.1 Problem Formulation

Consider the minimization of the power \( P(|h_{s1}|, \ldots, |h_{sn}|) = P \), where the \( n \) secondary communications are subject to a product of peak transmission rate constraint given by \( \prod_{i=1}^{n} R(|h_{si}|) \geq R_{\min} \), and an interference constraint given by \( P(|h_{s1}|, \ldots, |h_{sn}|) \leq P_{hp}(\epsilon) \). This optimization problem can be rewritten as

\[
\begin{align*}
\min_{P} & \quad P(|h_{s1}|, \ldots, |h_{sn}|) = P \\
\text{s.t.} & \quad \prod_{i=1}^{n} R(|h_{si}|) \geq R_{\min} \\
& \quad \text{and } P(|h_{s1}|, \ldots, |h_{sn}|) \leq P_{hp}(\epsilon)
\end{align*}
\quad \text{(V.5.1)}
\]
V.5.2 General Solution

Since a closed-form solution for this problem is complicated to obtain, an algorithm to solve it is proposed. Considering only the first constraint, we can notice that the minimum power required to satisfy it is the one that solves the constraint with equality. Therefore, the power that we require is the one that solves the function

$$f(P(|h_{s_1}|, \ldots, |h_{s_n}|)) = \prod_{i=1}^{n} R(|h_{s_i}|) - R_{\text{min}} = 0,$$

i.e.

$$\prod_{i=1}^{n} \log \left(1 + P(|h_{s_1}|, \ldots, |h_{s_n}|) \cdot |h_{s_i}|^2 \right) - R_{\text{min}} = 0 \quad (V.5.2)$$

It can be shown that (V.5.2) is a monotonically increasing function of \(P(|h_{s_1}|, \ldots, |h_{s_n}|)\) (see Appendix H) therefore, the bisection method can be applied in order to find the power \(P^*\) that satisfies (V.5.2). The advantage of applying the bisection method is that it can work without problems with the function \(\log (x)\). Newton's
method fails because \( \log(x) \) is not defined for negative values of \( x \).

Taking into account the second constraint, leads to the following two cases:

- if \( P^* \leq P_{hp}(\epsilon) \) then, \( P(|h_{s1}|, \ldots, |h_{sn}|) = P^* \),

- if \( P^* > P_{hp}(\epsilon) \) then, there is no solution.

V.5.3 Simulation Results

The general solution proposed was implemented in a numerical simulation. The results are shown in Fig. V.5.1 where we are plotting the average power per sensor node with respect to \( R_{min} \) when \( P_{hp}(\epsilon) = 2 \). As it is shown in the figure, for low values of \( R_{min} \) the average power per sensor node is less when the number of receivers is reduced. As the value of \( R_{min} \) increases, this fact is reversed and the average power per sensor is less as the number of sensor nodes increases. It is important to mention that as the number of receivers increases, the probability of an outage is increased as well (as shown in Fig. V.5.2).

V.6 Product of Average Transmission Rate Constraint

V.6.1 Problem Formulation

Consider the minimization of the power \( P(|h_{s1}|, \ldots, |h_{sn}|) = P \), where the \( n \) secondary communications are subject to a product of average transmission rate constraint given by \( \prod_{i=1}^{n} \mathbb{E}[|h_{si}|^2] \geq R_{min} \), and an interference constraint given by \( P(|h_{s1}|, \ldots, |h_{sn}|) \leq P_{hp}(\epsilon) \). Assuming that the fading channels are i.i.d. then, this optimization problem can be rewritten as
Figure V.5.1: Performance comparison between different number of receivers under a product of peak transmission rate constraint.

\[
\begin{align*}
\min_{\mathbf{P}} P (|h_{s1}|, \ldots, |h_{sn}|) &= P \\
\text{s.t. } E_{|h_{si}|^2} [R (|h_{si}|)] &\geq \sqrt{R_{\min}} \\
\text{and } P (|h_{s1}|, \ldots, |h_{sn}|) &\leq P_h (\epsilon)
\end{align*}
\] (V.6.1)

V.6.2 General Solution

Making use of the solution given by (IV.2.4) and (IV.2.5) then, we have the following cases:

- if \( E_{|h_{si}|^2} [\log (1 + P_h (\epsilon) \cdot |h_{si}|^2)] - \sqrt{R_{\min}} < 0 \), there is no solution (“outage” situation),

- if \( \sqrt{R_{\min}} \leq G (P_h (\epsilon)^{-1}) \) for every channel \( i \in \{1, \ldots, n\} \) then,
Figure V.5.2: Frequency of outages as a function of the number of receivers under an independent peak transmission rate constraint.

\[ P(h_s^1, \ldots, h_s^n) = \min \left\{ \max_{i \in \{1, \ldots, n\}} \left[ \frac{1}{H^{-1} \left( \sqrt[6]{R_{\min}} \right)} - \frac{1}{|h_s^i|^2} \right]^+, P_{hp}(\epsilon) \right\} \] (V.6.2)

where \( G(x) \) is defined by (III.2.8).

- if \( \sqrt[6]{R_{\min}} > G(P_{hp}(\epsilon)^{-1}) \) for every channel \( i \in \{1, \ldots, n\} \) then,

\[ P(h_s^1, \ldots, h_s^n) = \min \left\{ \max_{i \in \{1, \ldots, n\}} \left[ \frac{1}{H^{-1} \left( \sqrt[6]{R_{\min}} \right)} - \frac{1}{|h_s^i|^2} \right]^+, P_{hp}(\epsilon) \right\} \] (V.6.3)

where \( H(x) \) is defined by (IV.2.6).

V.6.3 Simulation Results

The general solution proposed was implemented in a numerical simulation. The results are shown in Fig. V.6.1 where we are plotting the average power per sensor
node with respect to $R_{min}$ when $P_{hp}(\epsilon) = 2$. As it is shown in the figure, for low values of $R_{min}$ the average power per sensor node is less when the number of receivers is reduced. As the value of $R_{min}$ increases, this fact is reversed and the average power per sensor is less as the number of sensor nodes increases. It is important to mention that for low values of $R_{min}$ the optimal power is equal to zero, therefore it is not plotted in the figure. Also, as the value of $R_{min}$ increases, the no solution regime is achieved, and it is achieved faster as the number of receivers increases.

![Figure V.6.1: Performance comparison between different number of receivers under a product of average transmission rate constraint.](image-url)
Chapter VI

Optimal Power Allocation for a
Wireless Sensor Network with
Multiple Secondary Transmitters

VI.1 Independent Peak Transmission Rate Constraint

VI.1.1 Problem Formulation

In this chapter we consider an extension of what it was proposed in chapter IV. Let us consider a new scenario depicted in Fig. VI.1.1.

In this case, we have multiple secondary receivers and the optimization problem can be expressed as:

\[
\begin{align*}
\min_P P(|h_{s,1}|, \ldots, |h_{s,n}|) &= P \\
s.t. R(|h_{s,i}|) &\geq R_{\text{min}} \quad \forall i \in \{1, \ldots, n\} \\
\text{and } P(|h_{s,i}|) &\leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\}
\end{align*}
\]  

(VI.1.1)

where \( R(|h_{s,i}|) = \log (1 + P(|h_{s,1}|, \ldots, |h_{s,n}|) \cdot |h_{s,i}|^2) \), \( P_{hp}(\epsilon) = \frac{Q_{\text{peak}}}{F_{|h_p|^2}(1-\epsilon)} \) and \( F_{|h_p|^2}^{-1} \) is the inverse distribution function of \(|h_p|^2\).
Figure VI.1.1: Wireless sensor link in a spectrum sharing environment with multiple secondary receivers
VI.1.2 General Solution

For a specific time we select the transmitter that requires the least amount of power among the sensors that do not fall into the no-solution regime and that are not saving energy, i.e. \( P(|h_{s,i}|) = 0 \). Making use of the solution given by (IV.1.3) for the single scenario then, the optimal power is given by

\[
P(|h_{s,1}|, \ldots, |h_{s,n}|) = \min_{i} P(|h_{s,i}|)
\]  

(VI.1.2)

VI.2 Independent Average Transmission Rate Constraint

Consider the minimization of the power \( P(|h_{s,1}|, \ldots, |h_{s,n}|) = P \), where each of the \( n \) secondary communications are subject to an average transmission rate constraint given by \( E_{|h_{s,i}|^2} [R(|h_{s,i}|)] \geq R_{min} \quad \forall i \in \{1, \ldots, n\} \), and an interference constraint given by \( P(|h_{s,i}|) \leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\} \). This optimization problem can be rewritten as

\[
\begin{align*}
\min_{P} P(|h_{s,1}|, \ldots, |h_{s,n}|) &= P \\
\text{s.t.} \quad E_{|h_{s,i}|^2} [R(|h_{s,i}|)] &\geq R_{min} \quad \forall i \in \{1, \ldots, n\} \\
\text{and} \quad P(|h_{s,i}|) &\leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\}
\end{align*}
\]  

(VI.2.1)

VI.2.1 General Solution

For a specific time we select the transmitter that requires the least amount of power among the sensors that do not fall into the no-solution regime and that are not saving energy, i.e. \( P(|h_{s,i}|) = 0 \). Making use of the solution given by (IV.2.4) and (IV.2.5) for the single scenario then, the optimal power is given by
\[ P(|h_{s1}|, \ldots, |h_{sn}|) = \min_i P(|h_{si}|) \quad (VI.2.2) \]

**VI.3 Sum of Peak Transmission Rate Constraint**

**VI.3.1 Problem Formulation**

Consider the minimization of the power \( P(|h_{s1}|, \ldots, |h_{sn}|) = P \), where the \( n \) secondary communications are subject to a sum of peak transmission rate constraint given by \( \sum_{i=1}^{n} R(|h_{si}|) \geq R_{\text{min}} \), and an interference constraint given by \( P(|h_{si}|) \leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\} \). This optimization problem can be rewritten as

\[
\min_{P} P(|h_{s1}|, \ldots, |h_{sn}|) = P \\
\text{s.t.} \quad \sum_{i=1}^{n} R(|h_{si}|) \geq R_{\text{min}} \\
\text{and} \quad P(|h_{si}|) \leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\} 
\quad (VI.3.1)
\]

**VI.3.2 General Solution**

Let us consider the case when we have 2 transmitters. For this problem we need to satisfy

\[
\log [1 + P(|h_{s1}|) \cdot |h_{s1}|^2] + \log [1 + P(|h_{s2}|) \cdot |h_{s2}|^2] \geq R_{\text{min}} 
\quad (VI.3.2)
\]

and

\[
P(|h_{si}|) \leq P_{hp}(\epsilon) \quad \forall i \in \{1, 2\} 
\quad (VI.3.3)
\]

This is a multiple optimization problem where the optimal values are located along a Pareto curve that satisfy (VI.3.2) with equality as shown in Fig. VI.3.1. The
Figure VI.3.1: Representation of the feasible power values for the power minimization problem for 2 transmitters in a cognitive radio environment with a sum of rate constraint.

situation is exactly the same when we have $n$ transmitters.

Since there is no reason to give priority to a certain channel among the others, the selection of the optimal power is determined by the requirements of the implementation.

**VI.4 Sum of Average Transmission Rate Constraint**

**VI.4.1 Problem Formulation**

Consider the minimization of the power $P (|h_{s1}|, \ldots, |h_{sn}|) = P$, where the $n$ secondary communications are subject to a sum of average transmission rate constraint given by $\sum_{i=1}^{n} E_{|h_{si}|^2} [R (|h_{si}|)] \geq R_{min}$, and an interference constraint given by $P (|h_{s1}|)$
\[ \leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\} \]. Assuming that the fading channels are i.i.d. then, this optimization problem can be rewritten as

\[
\min_P P(|h_{s1}|, \ldots, |h_{sn}|) = P \\
\text{s.t. } \sum_{i=1}^{n} E|h_{si}|^2 [R(|h_{si}|)] \geq R_{min} \\
\text{and } P(|h_{si}|) \leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\}
\]

(VI.4.1)

VI.4.2 General Solution

As stated in section VI.3.2, this is a multiple optimization problem where the optimal values are located along a Pareto curve that satisfy \[ \sum_{i=1}^{n} E|h_{si}|^2 [R(|h_{si}|)] = R_{min} \]. Since there is no reason to give priority to a certain channel among the others, the selection of the optimal power is determined by the requirements of the implementation.

VI.5 Product of Peak Transmission Rate Constraint

VI.5.1 Problem Formulation

Consider the minimization of the power \[ P(|h_{s1}|, \ldots, |h_{sn}|) = P \], where the \( n \) secondary communications are subject to a product of peak transmission rate constraint given by \[ \prod_{i=1}^{n} R(|h_{si}|) \geq R_{min} \], and an interference constraint given by \[ P(|h_{si}|) \leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\} \]. This optimization problem can be rewritten as

\[
\min_P P(|h_{s1}|, \ldots, |h_{sn}|) = P \\
\text{s.t. } \prod_{i=1}^{n} R(|h_{si}|) \geq R_{min} \\
\text{and } P(|h_{si}|) \leq P_{hp}(\epsilon) \quad \forall i \in \{1, \ldots, n\}
\]

(VI.5.1)
VI.5.2 General Solution

As stated in section VI.3.2, this is a multiple optimization problem where the optimal values are located along a Pareto curve that satisfy \( \prod_{i=1}^{n} R (|h_s i|) = R_{\text{min}} \). Since there is no reason to give priority to a certain channel among the others, the selection of the optimal power is determined by the requirements of the implementation.

VI.6 Product of Average Transmission Rate Constraint

VI.6.1 Problem Formulation

Consider the minimization of the power \( P (|h_s 1|, \ldots, |h_s n|) = P \), where the \( n \) secondary communications are subject to a product of average transmission rate constraint given by \( \prod_{i=1}^{n} E_{|h_s i|^2} [R (|h_s i|)] \geq R_{\text{min}} \), and an interference constraint given by \( P (|h_s i|) \leq P_{\text{hp}} (\epsilon) \quad \forall i \in \{1, \ldots, n\} \). Assuming that the fading channels are i.i.d. then, this optimization problem can be rewritten as

\[
\min_{P} P (|h_s 1|, \ldots, |h_s n|) = P \\
\text{s.t.} \quad \prod_{i=1}^{n} E_{|h_s i|^2} [R (|h_s i|)] \geq R_{\text{min}} \\
\text{and} \quad P (|h_s i|) \leq P_{\text{hp}} (\epsilon) \quad \forall i \in \{1, \ldots, n\}
\]  

(VI.6.1)

VI.6.2 General Solution

As stated in section VI.3.2, this is a multiple optimization problem where the optimal values are located along a Pareto curve that satisfy \( \prod_{i=1}^{n} E_{|h_s i|^2} [R (|h_s i|)] = R_{\text{min}} \). Since there is no reason to give priority to a certain channel among the others, the selection of the optimal power is determined by the requirements of the implementation.
Chapter VII

Conclusions

The problem of minimizing the transmit power of a sensor node while satisfying different rate constraints is analyzed. First, an optimization problem with instantaneous transmission rate constraint is covered. Next, a power minimization problem with an average transmission rate constraint is considered and two approaches are proposed to solve it. The first one provides the optimal solution, where we obtained the minimum power required as a function of the channel gain. The implementation of the optimal scheme implies a perhaps costly processor at the sensor node level which is not a desirable choice. In order to overcome this issue, a suboptimal On/Off scheme is proposed which turns out to be a uniform power policy as a function of $R_{\text{min}}$, that operates independently of the CSI gains at the sensor node level. Both schemes were analyzed asymptotically and it has been found that:

- At high $R_{\text{min}}$, the On/Off scheme is optimal,

- At low $R_{\text{min}}$, the On/Off scheme is strictly suboptimal. Hence, an accurate, yet easy to implement, approximation of the optimal solution is derived.

Furthermore, the optimal power policy for a sensor in a CR environment under either an instantaneous or an average rate constraint, is derived. Our analysis reveals that there might be “outage” events, where it is not possible to satisfy all the constraints at
the same time. These “outage” events have been identified and characterized. Simulations have been provided to confirm our claims. Our results are extended to scenarios where we have either one transmitter-multiple receivers or multiple transmitters-one receiver in a CR environment under different rate constraints. For each case, either a closed-form solution for a class of fading channels, or an algorithm for the optimal power allocation is derived depending on the complexity of the problem. For the multiple receivers scenario, different simulations are done considering a Rayleigh fading channel. These simulations show the average power per sensor node required as a function of $R_{min}$. For the problems with a peak constraint, it is plotted the frequency of outages with respect to the number of receivers in order to highlight the no solution regime. For the problems with an average constraint, the simulations highlight the solution of the two regimes derived. Finally, for the multiple transmitters scenario, it is defined a multiple optimal solution given by a Pareto curve. An extension for this work will be considered where we add a minimum sum of powers constraint to identify a unique solution.
Appendix A

Proof for Derivative of (III.2.2)

We can rewrite the Lagrangian (III.2.2) as follows:

\[
\mathcal{L} \approx \frac{1}{N} \sum_{k=1}^{N} \left[ P(|h_{s,k}|) - \lambda \cdot \log \left( 1 + P(|h_{s,k}|) \cdot |h_{s,k}|^2 \right) \right] - \lambda R_{\text{min}}
\]

The solution to this optimization problem is given by \( \partial \mathcal{L} / \partial P(|h_s|) = 0 \), therefore,

\[
\frac{\partial \mathcal{L}}{\partial P(|h_s|)} = 1 - \lambda \frac{|h_s|^2}{1 + P(|h_s|) \cdot |h_s|^2} = 0
\]
Appendix B

Proof for Decremental Monotonicity
of Equation (III.2.8)

Consider the function

\[ G(x) = -\log(x) \left[ 1 - F_{|h|^2}(x) \right] + \int_{x}^{\infty} \log(|h|^2) f_{|h|^2}(|h|^2) \, d|h|^2 \]

Let us get the first derivative of \( G(x) \). By applying Leibniz integral rule, we get

\[ G'(x) = -\frac{1}{x} \int_{x}^{\infty} f_{|h|^2}(|h|^2) \, d|h|^2 = -\frac{1}{x} \left[ 1 - F_{|h|^2}(x) \right] \]

which means that,

\[ G'(x) < 0 \quad \forall \, x > 0 \]

Thus, we can say that \( G(x) \) is a monotonic decreasing function.
Appendix C

Proof for Equation (III.2.14)

In order to prove that when $R_{\text{min}} \to \infty$, then $\lambda \to \infty$ we need to work with (III.2.5), considering that $f_{|h_s|^2}(t) = e^{-t}$ for $t \geq 0$, i.e.

$$
\lambda = \exp \left( \frac{R_{\text{min}} - \int_{\frac{\lambda}{X}}^{\infty} \log \left( |h_s|^2 \right) e^{-|h_s|^2} d|h_s|^2}{\int_{\frac{1}{X}}^{\infty} e^{-|h_s|^2} d|h_s|^2} \right)
$$

This equation can be rewritten as

$$
\exp \left\{ \exp \left( \frac{1}{\lambda} \right) \left[ R_{\text{min}} - E_1 \left( \frac{1}{\lambda} \right) \right] \right\} = 1
$$

which means that

$$
R_{\text{min}} = E_1 \left( \frac{1}{\lambda} \right)
$$

Let us establish an upper bound for the function $E_1 \left( \frac{1}{\lambda} \right)$

$$
E_1 \left( \frac{1}{\lambda} \right) = \int_{\frac{1}{X}}^{\infty} \frac{e^{-t}}{t} \, dt \leq \int_{\frac{1}{X}}^{\infty} \frac{e^{-t}}{\lambda} \, dt = \lambda e^{-\frac{1}{X}}
$$

Therefore
Now, let us make use of the inequality \( \log x \leq x - 1 \) which is valid for every value \( x \). Using this inequality, we can write that

\[
\log \frac{1}{\lambda} \leq \frac{1}{\lambda} - 1
\]

which by algebraic manipulation can be transformed into the following expression

\[
\lambda e^{-\frac{1}{\lambda}} \leq \frac{\lambda^2}{e}
\]

With this, we can write that

\[
R_{\text{min}} \leq \lambda e^{-\frac{1}{\lambda}} \Rightarrow \lambda^2 \geq e \cdot R_{\text{min}}
\]

and finally we can conclude that as \( R_{\text{min}} \to \infty \), then \( \lambda \to \infty \). With this fact, it is possible to simplify (III.2.5) since \( \lim_{\lambda \to \infty} \int_{\frac{1}{\lambda}}^{\infty} \log \left( \left| h_s \right|^2 \right) e^{-\left| h_s \right|^2} d \left| h_s \right|^2 = -\gamma \), where \( \gamma \) is the Euler’s constant, and \( \lim_{\lambda \to \infty} \int_{\frac{1}{\lambda}}^{\infty} e^{-\left| h_s \right|^2} d \left| h_s \right| = 1 \). Thus, we get that \( \lambda = \exp \left( R_{\text{min}} + \gamma \right) \), which yields to the result shown in (III.2.14).
Appendix D

Proof for Equation (III.2.15)

We need to work with (III.2.5), considering that $f_{|h_s|^2}(t) = e^{-t}$ for $t \geq 0$, i.e. From (III.2.5), we know that

$$
\lambda = \exp \left( \frac{R_{\text{min}} - \int_{\frac{1}{\lambda}}^{\infty} \log (|h_s|^2) f_{|h|^2} (|h_s|^2) d |h_s|^2) \right)
$$

and by assuming that $R_{\text{min}} \to 0$, this means that $\lambda \to 0$, therefore the equation above can be simplified by using the Taylor series expansion

$$
\int_{\frac{1}{\lambda}}^{\infty} \log (|h_s|^2) e^{-|h_s|^2} d |h_s|^2 = \exp \left( -\frac{1}{\lambda} \right) \left[ - \log (\lambda) + \lambda + O [\lambda^2] \right]
$$

which yields the following result,

$$
\lambda = \exp \left( \frac{R_{\text{min}} - e^{-\frac{1}{\lambda}} (- \log \lambda + \lambda)}{e^{-\frac{1}{\lambda}}} \right)
$$

Thus, $R_{\text{min}} = \lambda \exp \left( -\frac{1}{\lambda} \right)$. Solving this equation for $\lambda$ gives the following equality

$$
\lambda = \frac{1}{W \left( \frac{1}{R_{\text{min}}} \right)}
$$

which yields to (III.2.15).
Appendix E

Proof for Equation (III.2.18)

Let us consider (III.2.11) written as follows

\[ e^{R_{\text{min}}} = \exp \left[ e^{\frac{1}{P_o}} E_1 \left( \frac{1}{P_o} \right) \right] \]

It is known that as \( R_{\text{min}} \to \infty \), then \( P_o \to \infty \) thus, by taking a Taylor series expansion of the exponent in the right hand side (RHS) of the equation when \( \frac{1}{P_o} \to 0 \), we get that

\[ e^{\frac{1}{P_o}} E_1 \left( \frac{1}{P_o} \right) \simeq \log (P_o) - \gamma + \frac{1}{P_o} [1 - \gamma + \log (P_o)] + O \left[ \left( \frac{1}{P_o} \right)^2 \right] \]

where \( \gamma \) is the Euler’s constant. With this result, (III.2.11) can be written as

\[ e^{R_{\text{min}}} \simeq P_o e^{-\gamma} + O \left[ \left( \frac{1}{P_o} \right)^2 \right] \]

Therefore,

\[ P_o \simeq e^{R_{\text{min}}+\gamma} \]

which is the result given by (III.2.18).
Proof for Equation (III.2.19)

Let us consider (III.2.11) written as

\[ e^{R_{\text{min}}} = \exp \left[ e^{\frac{1}{P_o}} E_1 \left( \frac{1}{P_o} \right) \right] \]

It is known that as \( R_{\text{min}} \to 0 \), then \( P_o \to 0 \) thus, by taking a Taylor series expansion of the exponent in the RHS of the equation when \( P_o \to 0 \), we get that

\[ e^{\frac{1}{P_o}} E_1 \left( \frac{1}{P_o} \right) \simeq P_o + O \left[ P_o^2 \right] \]

With this result, (III.2.11) can be written as

\[ e^{R_{\text{min}}} \simeq \exp \left( P_o + O \left[ P_o^2 \right] \right) \]

Therefore,

\[ P_o \simeq R_{\text{min}} \]

which is the result given by (III.2.19).
Appendix G

Proof for Equations (IV.2.4) and (IV.2.5)

The derivative of (IV.2.2) with respect to $\lambda$ is given by

$$\frac{\partial L}{\partial \lambda} = R_{\text{min}} - E_{h_s} \left[ \log \left( 1 + P (|h_s|) \cdot |h_s|^2 \right) \right]$$

Considering that $P (|h_s|) = \min \left\{ \left[ \lambda - \frac{1}{|h_s|^2} \right]^+, P_{h_p} (\epsilon) \right\}$ then, $\frac{\partial L}{\partial \lambda}$ evaluated at $\lambda = 0$ is equal to $R_{\text{min}}$, which means that the Lagrangian is increasing. In case that

$$\lim_{\lambda \to \infty} \frac{\partial L}{\partial \lambda} = R_{\text{min}} - E_{h_s} \left[ \log \left( 1 + P_{h_p} (\epsilon) \cdot |h_s|^2 \right) \right] > 0,$$

then there is no solution (this is an “outage” situation).

If a solution exists, we need to find the value of $\lambda$ where $\frac{\partial L}{\partial \lambda} = 0$, i.e. $E_{h_s} \left[ \log \left( 1 + P_{h_p} (\epsilon) \cdot |h_s|^2 \right) \right] = R_{\text{min}}$. To make the analysis simpler, we make the change $\lambda = \frac{1}{x}$, therefore

$$\int_{\frac{1}{x}}^{\infty} \min \left\{ \log \left( \frac{h}{x} \right), \log \left( 1 + P_{h_p} (\epsilon) \cdot h \right) \right\} f_{|h_s|^2} \left( |h_s|^2 \right) d |h_s|^2 = R_{\text{min}}$$

From this point, we have two cases:

- if $x > P_{h_p} (\epsilon)^{-1}$ then, $P (|h_s|) = \min \left\{ \left[ \frac{1}{x} - \frac{1}{|h_s|^2} \right]^+, P_{h_p} (\epsilon) \right\}$, where $x$ is ob-
\[ G(x) = \int_x^\infty \log \left( \frac{h}{x} \right) f_{|h_s|^2} (|h_s|^2) \, d|h_s|^2 = R_{\text{min}} \]

- if \( x < P_{h_p}(\epsilon)^{-1} \) then, \( P(|h_s|) = \min \left\{ \left[ \frac{1}{x} - \frac{1}{|h_s|^2} \right]^+, P_{h_p}(\epsilon) \right\} \), where \( x \) is obtained from

\[ H(x) = G(x) + \int_x^{\infty} \log \left( 1 + P_{h_p}(\epsilon) \frac{|h_s|^2}{|h_s|^2} \left( \frac{1}{x} - \frac{1}{|h_s|^2} \right) \right) f_{|h_s|^2} (|h_s|^2) \, d|h_s|^2 = R_{\text{min}} \]

-
Appendix H

Proof for Incremental Monotonicity of Equation (V.5.2)

If we differentiate the function

\[ f[P(\mid h_{s1}\mid, \mid h_{s2}\mid, \ldots, \mid h_{sn}\mid)] = \prod_{i=1}^{n} \log(1 + P(\mid h_{s1}\mid, \mid h_{s2}\mid, \ldots, \mid h_{sn}\mid) \cdot |h_{si}|^2) - R_{\min} \]

with respect to \( P(\mid h_{s1}\mid, \mid h_{s2}\mid, \ldots, \mid h_{sn}\mid) \), we get

\[
\frac{\partial f}{\partial P(\mid h_{s1}\mid, \mid h_{s2}\mid, \ldots, \mid h_{sn}\mid)} = \sum_{i=1}^{n} \left[ \frac{|h_{si}|^2}{1 + P(\mid h_{s1}\mid, \mid h_{s2}\mid, \ldots, \mid h_{sn}\mid) \cdot |h_{si}|^2} \right]
\]

\[
\prod_{j=1, j \neq i}^{n} \log(1 + P(\mid h_{s1}\mid, \mid h_{s2}\mid, \ldots, \mid h_{sn}\mid) \cdot |h_{si}|^2)
\]

Since \( \frac{\partial f}{\partial P(\mid h_{s1}\mid, \mid h_{s2}\mid, \ldots, \mid h_{sn}\mid)} > 0 \) then, we can say that (V.5.2) is a monotonic increasing function.
Bibliography


