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Abstract

Understanding the fundamental mechanism of fracture-matrix fluid exchange is crucial for the modeling of fractured reservoirs. Traditionally, high-resolution simulations for flow in fractures often neglect the matrix-fracture leakage influence on the fracture hydraulic properties, i.e., assuming impermeable fracture walls. This work introduces a micro-continuum approach to capture the matrix-fracture leakage interaction and its effect on the rock fractures' hydraulic properties.

Because of the multiscale nature of fractured media, full physics Navier-Stokes (NS) representation everywhere in the whole domain is not feasible. We thus employ NS equations to describe the flow in the fracture, and Darcy's law to model the flow in the surrounding porous rocks. Such hybrid modeling is achieved using the extended Darcy-Brinkman-Stokes (DBS) equation. With this approach, a unified conservation equation for flow in both media is applied by choosing appropriate parameters (e.g., porosity and permeability) for the corresponding domains. We apply an accurate Mixed Finite Element approach to solve the extended DBS equation. Various sensitivity analyses are conducted to explore the leakage effects on the fracture hydraulic properties by varying surrounding matrix permeability, fracture roughness, and Reynolds number (Re).

Streamline profiles show the presence of back-flow phenomena, where in-flow and out-flow are possible between the matrix and the fractures. Further, zones of stagnant (eddy) flow are observed around locations with large asperities of sharp corners under high Re conditions. Numerical results show the significant effects of roughness and inertia on flow predictions in fractures for both impermeable and leaky wall cases. Besides, the side-leakage effect can create non-uniform flow behavior within the fracture that may differ significantly from the case with impermeable wall conditions. And this matrix-fracture leakage influence on hydraulic properties of rock fractures matters especially for cases with high matrix permeability, high fracture roughness, and low Re values. In summary, we present a high-resolution micro-continuum approach to explore the flow exchange behavior between the fracture and rock matrix, and further investigate the static and dynamic effects, including variable Reynold numbers, mimicking flow near and away from the wellbore. The approach and results provide significant insights into the flow of fluids through fractures within permeable rocks and can be readily applied in field-scale reservoir simulations.

Introduction

Good knowledge of hydraulic behaviors of discrete rock fractures is crucial for modeling and further assessing various subsurface activities, such as NAPLs tracking, geothermal energy production, geological CO₂ storage, and oil recovery process in fractured reservoirs. As we know, in situ rock fractures are subjected to be stress-dependent (He et al., 2020). Herein, we investigate the hydraulic properties of rock fractures under static conditions.

The Navier-Stokes (NS) equations provide the most accurate method for calculating the hydraulic properties of rock fractures (Brush and Thomson, 2003; Zimmerman and Yeo, 2000). However, intensive computation cost makes it infeasible to apply NS equations as a practical approach. With a sufficiently small Reynolds number, we can approximate the NS equations by the Stokes equations. The computational burden of solving Stokes equations is still huge, even though cheaper than the NS equations. The cubic law, describing discrete rock fractures as two smooth parallel plates with a constant distance, has been used widely in many disciplines due to its simplicity and efficiency. Natural rock fractures, however, exhibit the characteristics of rough surface, variable aperture, and contact areas. These features of rock fractures make the validity of the cubic law questionable (Brown, 1987; Konzuk and Kueper, 2004; Witherspoon et al., 1980). Many researchers have proposed various modifications based on the cubic law to improve its performance. Examples include (1) modifying the definition of the aperture used in the cubic law, such as the arithmetic mean (Brown, 1987), geometric mean (Renshaw, 1995), harmonic mean (Unger & Mase, 1993), and among others; (2) incorporating a correction factor such as contact area (Walsh, 1981), roughness (Zimmerman and Bodvarsson, 1996), tortuosity (Waite et al., 1999), and even combined effect (Wang et al., 2015). A comprehensive given by Konzuk and Kueper (2004) shows that the cubic law using geometric mean or appropriate roughness correction factor offered reasonable flow prediction within the flow regime of Reynolds number less than 1. These models, however, show unstable performance when dealing with complex fracture cases due to the neglect of flow behavior at the local scale. As an alternative, the local cubic law, also referred to as the Reynolds equation, captures the local flow behaviors by accounting for the spatial variation in the aperture field with the assumption of the cubic law valid at each explicit location. The local cubic law, however, shows two main assumptions – flat fracture mid-surface and parabolic velocity profile, which limit its broad applicability. Many efforts have been put to improve its accuracy. Brown et al. (1995) considered the fracture mid-surface effect by introducing a correction factor. Mourzenko et al. (1995) introduced the concept of normal aperture and suggested “ball” aperture, resulting in significant improvement. Ge (1997) considered the combined effect of flow tortuosity and normal aperture. Oron and Berkowitz (1998) conducted a dimensionless, 2D order-of-magnitude analysis to the NS equations and found the cubic law may be valid within certain segments. Brush and Thomson (2003) and Nicholl et al. (1999) explored and tested various link transmissivity formulations into the Reynolds equations with the finite difference and finite volume approaches, respectively. A detailed description of these models could be found in the related references.

All these previous theories, including the NS, Stokes, cubic law, and even Reynolds equations, are built based on the assumption of impermeable fracture walls, i.e., neglecting the matrix-fracture leakage interaction. Few studies focus on fluid flow through discrete rock fractures in the permeable rock. Basha and El-Asmar (2003) presented the perturbation solution based on the 2D NS equations by considering the leakage effect. This perturbed solution is only applicable for fracture cases with simple geometry. It cannot capture the complex flow behaviors between matrix and fracture, such as back-flow and eddy phenomena. Crandall et al. (2010) developed a modified NS equation using finite-volume schemes to model fluid flow in fractured porous media by adding Darcy’s term. Without the treatment of choosing different parameters (i.e., porosity and permeability) in different corresponding domains, the modeling approach by Crandall et al. (2010) loses some generality.

We organize our paper as follows. Section 2 will offer the mathematical model – the extended Darcy-Brinkman-Stokes (DBS) equations. In section 3, we will provide the mixed finite formulation of the DBS equations. We will then perform sensitivity analysis to explore the matrix-fracture leakage

influence on hydraulic properties by varying fracture roughness, Reynolds number (inertial effect), and matrix permeability in section 4. Finally, we close with a summary and extension of this work.

Mathematical Model

We employ a micro-continuum approach, whereby a unified equation holds for flow in both domains: fracture and matrix. By choosing the appropriate parameters in corresponding domains, the extended Darcy-Brinkman-Stokes (DBS) equations could describe fluid flow in the fracture and surrounding matrix using the NS and Darcy equations, respectively. We assume the steady-state of incompressible, Newtonian, single-phase flow, and the extended DBS could be written as:

$$\frac{1}{\phi} \left\{ \nabla \bullet \left[\frac{\rho}{\phi} \bar{u} \otimes \bar{u} \right] \right\} = -\nabla p + \frac{\mu}{\phi} \nabla^2 \bar{u} + \bar{f} - \mu K^{-1} \bar{u} \quad (1)$$

$$\nabla \bullet \bar{u} = 0$$

In the above equations, p is pressure, $\bar{u}$ is velocity vector, ρ is fluid density, and μ is fluid viscosity; ϕ and K are porosity and permeability, respectively, which are selected based on the corresponding domains (fracture or matrix).

- In the fracture region Ω^f :

$$\begin{cases} \phi = 1 \\ K = \infty \end{cases} \quad \nabla \bullet [\rho \bar{u} \otimes \bar{u}] = -\nabla p + \mu \nabla^2 \bar{u} + \bar{f} \quad (2)$$

We can observe that **Eq. (1)** reduces to the NS equations with the selection of parameters.

- In the matrix region Ω^p :

$$\begin{cases} \phi = \phi_M \\ K = K_M \end{cases} \quad \mu K_M^{-1} \bar{u} + \nabla p + \frac{1}{\phi_M} \left\{ \nabla \bullet \left[\frac{\rho}{\phi_M} \bar{u} \otimes \bar{u} \right] \right\} - \frac{\mu}{\phi_M} \nabla^2 \bar{u} = \bar{f} \quad (3)$$

Whereas Darcy's law given as follow:

$$\mu K_M^{-1} \bar{u} + \nabla p = \bar{f} \quad (4)$$

Observe that the differences between **Eq. (3)** and **Eq. (4)** are the last two additional terms highlighted in blue. The term $\mu K_M^{-1} \bar{u}$ in **Eq. (3)** would dominate many orders of magnitude for fluid flow in porous media compared to these two additional terms, which will introduce a sufficiently small perturbation to Darcy's law.

The extended DBS equations offer the following advantages compared to the coupled Darcy-NS approach. A detailed description of the coupled Darcy-NS equations could be given in the reference (Popov et al., 2009). First, it allows a unified equation to capture flows in both domains by choosing different parameters in the corresponding domain. Second, we avoid formulating specific interface conditions, resulting in significant simplification in numerical treatment. Finally, it provides an efficient framework to simulate partially-filled fractures or particles suspension in the fluid, as this unified approach can capture the flow varying from a Darcy-dominated flow to an NS-dominated flow.

Boundary Conditions: As observed in **Figure 1**, pressure inlet and outlet values are imposed at the fracture inlet and outlet, respectively. The boundary conditions with no-slip are assigned to both the bottom and top sides. The gradient of velocity in the direction of pressure gradient is set to zero to guarantee the fully

developed flow and to avoid the inlet effect. We use fluid of water with $\rho = 1000 \text{ kg/m}^3$ and $\mu = 0.001 \text{ kg/(m}\cdot\text{s)}$. Herein, we assume matrix to be homogeneous and isotropic with constant porosity.

Figure 1. Illustration of boundary conditions imposed on the fractured rock (fracture + matrix).

Mixed Finite Element Implementation

The extended DBS equations can be arranged as follow:

$$\begin{aligned} \mu K^{-1} \bar{u} + \nabla \cdot \left[p \bar{I} - \frac{\mu}{\phi} \nabla \bar{u} \right] + \frac{1}{\phi^2} \rho (\bar{u} \cdot \nabla) \bar{u} &= \bar{f} \\ \nabla \cdot \bar{u} &= 0 \end{aligned} \quad (5)$$

The extended DBS equations are discretized in a mixed-variational form, where the pressure and the velocity are modeled simultaneously (Arbogast et al., 1997; Chavent et al., 1987; Hoteit et al., 2002; Hoteit and Firoozabadi, 2008; Mosé et al., 1994). Multiplying **Eq. (5)** by the test function $(\bar{v}, q)$ and integrating the resulting equations over the domain Ω yields:

$$\begin{aligned} \int_{\Omega} \mu K^{-1} \bar{u} \cdot \bar{v} dx + \int_{\Omega} \nabla \cdot \left[p \bar{I} - \frac{\mu}{\phi} \nabla \bar{u} \right] \cdot \bar{v} dx + \int_{\Omega} \frac{1}{\phi^2} \rho (\bar{u} \cdot \nabla) \bar{u} \cdot \bar{v} dx &= \int_{\Omega} \bar{f} \cdot \bar{v} dx \\ \int_{\Omega} (\nabla \cdot \bar{u}) q dx &= 0 \end{aligned} \quad (6)$$

Applying the integration by parts technique, we have:

$$\begin{aligned} \int_{\Omega} \nabla \cdot \left[p \bar{I} - \frac{\mu}{\phi} \nabla \bar{u} \right] \cdot \bar{v} dx &= - \int_{\Omega} \left[p \bar{I} - \frac{\mu}{\phi} \nabla \bar{u} \right] \nabla \cdot \bar{v} dx + \int_{\partial \Omega} \left[p \bar{I} - \frac{\mu}{\phi} \nabla \bar{u} \right] \bar{v} \cdot \bar{n} ds \\ &= - \int_{\Omega} p \nabla \cdot \bar{v} dx + \frac{\mu}{\phi} \int_{\Omega} \nabla \bar{u} : \nabla \bar{v} dx + \int_{\partial \Omega} \left[p \bar{I} - \frac{\mu}{\phi} \nabla \bar{u} \right] \bar{v} \cdot \bar{n} ds \end{aligned} \quad (7)$$

The problem becomes to find $(\bar{u}, p) \in W$ such that,

$$a((\bar{u}, p), (\bar{v}, q)) = L((\bar{v}, q)) \quad (8)$$

Where $(\bar{v}, q) \in W$, and

$$a((\bar{u}, p), (\bar{v}, q)) = \mu K^{-1} \int_{\Omega} \bar{u} \cdot \bar{v} dx - \int_{\Omega} p \nabla \cdot \bar{v} dx + \frac{\mu}{\phi} \int_{\Omega} \nabla \bar{u} : \nabla \bar{v} dx + \frac{\rho}{\phi^2} \int_{\Omega} (\bar{u} \cdot \nabla) \bar{u} \cdot \bar{v} dx + \int_{\Omega} (\nabla \cdot \bar{u}) q dx \quad (9)$$

$$\begin{aligned} L((\bar{v}, q)) &= \int_{\Omega} \bar{f} \cdot \bar{v} dx - \int_{\partial\Omega} \left[p \bar{I} - \frac{\mu}{\phi} \nabla \bar{u} \right] \bar{v} \cdot \bar{n} ds \\ &= \int_{\Omega} \bar{f} \cdot \bar{v} dx - \int_{\partial\Omega_{inlet}} p_{inlet} \bar{n} \cdot \bar{v} ds (inlet) - \int_{\partial\Omega_{outlet}} p_{outlet} \bar{n} \cdot \bar{v} ds (outlet) \end{aligned} \quad (10)$$

The space $W = V \times Q$ is a mixed-function space such that $q \in Q$ and $\bar{u} \in V$.

All the simulations in this study are run on the open-source platform called FEniCS (Logg et al., 2012). We automate the whole process, including mesh generation, running simulations, flow rate calculation.

Sensitivity Analysis

This section will provide high-resolution simulations to capture the matrix-fracture physics and further perform various sensitivity analyses (matrix permeability, fracture roughness, and Reynolds number) to explore the leakage influence on hydraulic properties of rock fractures.

(1) Fracture Profiles Generation

We first generate fracture profile No.1 using software SynFrac, a synthetic fracture generator. We then perform perturbation on fracture No.1 using the moving-average technique, by which we can generate fracture profiles with different roughness yet with the same mean aperture, as shown in **Table 1**. These six 2D fracture profiles used in this study, showing different roughness and tortuosity, are shown in **Figure 2**. The corresponding geometric properties are summarized in **Table 1**. We should note fractured rock (fracture + matrix) model is constructed by embedding these 2D fracture profiles into a rectangle domain (see **Figure 1**).

Figure 2. Six 2D fracture profiles used in this study (denoted as No.1-6, respectively).

Table 1. The corresponding geometric properties of fracture profiles No.1-No.6.

No.	a_m (μm)	σ_a (μm)	JRC	τ
1	500	118.5	17.0	1.031
2	500	110.5	13.6	1.021
3	500	97.3	9.7	1.013
4	500	83.6	7.0	1.009
5	500	62.6	3.1	1.004
6	500	0	0.0	1.000

Note: a represents the vertical aperture field with arithmetic mean a_m and standard deviation σ_a .

JRC refers to the joint roughness coefficient used to quantify the fracture surface roughness.

τ refers to tortuosity, the ratio of the flow-path distance to the straight-line distance of the fracture.

(2) Impermeable Walls: Inertial and Roughness Effects

We first investigate the flow behaviors in discrete rock fractures with impermeable walls, and then extend impermeable cases to fracture cases with leaky walls. Herein, we apply the high-resolution NS equations to model fluid flow inside discrete rock fractures. A detailed description of the numerical implementation of NS equations is given in **Appendix A**.

Inertial Effect: We investigate the inertial impact by varying the Reynolds number on the same fracture profile. **Figure 3** shows the eddy evolution (size and shape) with the increasing Reynolds number in part of fracture profile No.1. Eddy generally occurs around the locations with local-large asperity of sharp corner under high Reynold number conditions (see **Figure 3**). Eddy exerts a significant influence on fluid flow through rock fractures by reducing the effective flow channel. Given that eddies, an enclosed and separated region, show no flux interaction with the flow channel, the expansion in Eddy's zones further reduces the effective flow channel, and thus reduces the fracture transmissivity (see **Figure 3**). The inertial influence on fracture transmissivity is quantitatively shown in **Figure 5**, left, which agrees with previous simulation observations in **Figure 3**. The existence of Eddy implies the existence of dynamic trapping mechanisms that may impact the relative permeabilities and residual saturations within the fractures.

Figure 3. Flow streamlines in part of the fracture profile No.1 for different Reynolds numbers varying from 1 to 150, showing the size and shape of eddy increases as Reynolds number increases (effective flow channel indicating a reverse trend).

Roughness Effect: We investigate the roughness influence by varying the pressure values at fracture inlet under the same Reynolds number of 1. We observe, in **Figure 4**, more pressure drop is required in rougher fractures than smooth fractures under the same Re . As roughness introduces resistance to flow, resulting in reducing fracture hydraulic properties. We quantitatively provide the relation between normalized transmissivity and JRC in **Figure 5**, right, for fracture profiles No.1-6 under $Re=1$.

Figure 4. Pressure distribution for different fracture profiles under the same $Re=1$, showing the roughness influence on fluid flow through rock fractures.

Figure 5. Normalized transmissivity (T/T_o) versus Reynolds number for fracture No.1 (left); Normalized transmissivity ($T/T_{JRC=0}$) versus JRC for fracture No.1 – 6 under flow regime of $Re=1$.

(3) Leaky Walls: Matrix-Fracture Interaction

We solve the extended DBS equations using mixed finite formulation on the FEniCS platform. We perform mesh refinement until the difference of successive flow solutions is within 0.1 %. We set the matrix porosity to be 0.4 for all simulations below and the formation height of 40 mm.

We can observe, in **Figure 6**, left, the pressure tends to decrease linearly along the direction of fracture length in far-fracture areas. This observation agrees with Darcy's law. However, the pressure exhibits a significant deviation from the essential linear trend near the fracture profile due to the matrix-fracture interaction. These conclusions are further illustrated in **Figure 6**, right.

Figure 6. Left: Pressure distributions within fracture profile No.1 with matrix permeability of 50 mD and pressure gradient of 10 Pa/m (corresponding Re 0.05). Right: Pressure across the lines of $y=-0.016$, -0.010 , -0.004 , and 0 .

We conduct the high-resolution extended DBS simulations on different fracture profiles (No. 1, 3, 5) with varying matrix permeability (50, 500, 5000 mD). We also add impermeable cases (denoted as “no-slip”) to highlight the matrix-fracture influence. As observed in **Figure 7** left, the matrix-fracture leakage shows an increasing impact on flux through **fracture outlet** with the increase of matrix permeability. It increases 13.7 % flux compared to flow in fracture with impermeable walls (denoted as “no-slip”) for fracture profile No.1. The leakage influence would decrease as the fracture roughness decreases under the same matrix permeability, as expected.

We then perform the simulation on fracture profile No.1 with varying matrix permeability (50, 500, 5000 mD) under different pressure gradients (10, 100, 1000 Pa/m). Impermeable cases also are included for comparison. We observe, in **Figure 8**, back-flow phenomena occur, with in-flow and out-flow between matrix-fracture. **Figure 7** right shows the matrix-fracture interaction exerts decreasing influence as Reynolds number increases for the same fracture profile due to the existence of eddies. This observation implies the existence of eddies, to some extent, prevents the occurrence of matrix-fracture interaction (leakage).

Figure 7. Left: Flux through fracture outlet versus matrix permeability and fracture roughness under the pressure gradient of 10 Pa/s. Right: Flux through fracture outlet versus matrix permeability and pressure gradient for fracture profile 1.

Figure 8. Top: Velocity profiles for fracture profile No.1 with the pressure gradient of 1000 Pa/m and matrix permeability of 500 mD. Bottom: streamline distribution in the exaggerated area of the white box.

Conclusions

This work provides a high-resolution, micro-continuum approach to explore the matrix-fracture interaction and further quantify the leakage influence on the rock fractures' hydraulic properties. We summarize the main finding as follows:

- Fracture roughness and inertia show a significant effect on flow predictions for both fracture cases with impermeable and leaky walls.
- Eddies generally occur around locations with large asperities of sharp corners under high Re conditions. The existence of eddies reduces fracture hydraulic properties by narrowing the effective flow channel.
- The matrix-fracture interaction makes the pressure contours showing significant fluctuation from the linear trend near fracture profile areas.
- The matrix-fracture leakage easily takes place for cases with high matrix permeability, high fracture roughness, and low Re values.
- The leakage phenomena take place bidirectionally, where in-flow and out-flow are possible between matrix and fracture.

Future work will focus on developing an analytical model for estimating the permeability of fractured rocks (matrix+fracture) to avoid expensive computation cost via high-resolution extended DBS equations.

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Appendix A: Mixed Finite Element Formulation for NS Equations

Consider the steady-state of incompressible, Newtonian flow with no gravity effects, and the full-physics NS equations can be given as:

$$\begin{aligned}\rho(\bar{\mathbf{u}} \bullet \nabla \bar{\mathbf{u}}) &= -\nabla p + \mu \nabla^2 \bar{\mathbf{u}} \\ \nabla \bullet \bar{\mathbf{u}} &= 0\end{aligned}\tag{A1}$$

In the above equations, p is pressure, $\bar{\mathbf{u}}$ is velocity vector, ρ is fluid density, and μ is fluid viscosity. No-slip boundary conditions are imposed at the walls of the fracture, and pressure values are set at the outlet and inlet of fracture (see **Figure A1**). The gradient of velocity in the direction of pressure gradient is set to zero to guarantee the fully developed flow and to avoid the inlet effect.

Figure A1. Illustration of boundary conditions imposed on a rough-walled fracture.

The NS equations are approximated in a mixed variational form. Multiplying Eq. (A1) by the test function $(\bar{\mathbf{v}}, q)$ and integrating the resulting equations over the domain Ω yields:

$$\begin{aligned} \int_{\Omega} \rho(\bar{u} \cdot \nabla \bar{u}) \cdot \bar{v} dx &= \int_{\Omega} (-\nabla p) \cdot \bar{v} dx + \int_{\Omega} (\mu \nabla^2 \bar{u}) \cdot \bar{v} dx \\ \int_{\Omega} (\nabla \cdot \bar{u}) q dx &= 0 \end{aligned} \quad (A2)$$

Applying the integration by parts technique, we have:

$$\begin{aligned} \int_{\Omega} (-\nabla p) \cdot \bar{v} dx &= \int_{\Omega} p \nabla \cdot \bar{v} dx - \int_{\partial\Omega} p \bar{v} \cdot \bar{n} ds \\ \int_{\Omega} (\mu \nabla^2 \bar{u}) \cdot \bar{v} dx &= -\mu \int_{\Omega} \nabla \bar{u} : \nabla \bar{v} dx + \mu \int_{\partial\Omega} \nabla \bar{u} \cdot \bar{n} \bar{v} ds \end{aligned} \quad (A3)$$

the problem becomes,

$$a((\bar{u}, p), (\bar{v}, q)) = L((\bar{v}, q)) \quad (A4)$$

For $(\bar{v}, q) \in W$, and

$$a((\bar{u}, p), (\bar{v}, q)) = \rho \int_{\Omega} (\bar{u} \cdot \nabla \bar{u}) \cdot \bar{v} dx - \int_{\Omega} p \nabla \cdot \bar{v} dx + \mu \int_{\Omega} \nabla \bar{u} : \nabla \bar{v} dx + \int_{\Omega} (\nabla \cdot \bar{u}) q dx \quad (A5)$$

$$L((\bar{v}, q)) = - \int_{\partial\Omega_N} p_{inlet} \bar{v} \cdot \bar{n} ds - \int_{\partial\Omega_N} p_{outlet} \bar{v} \cdot \bar{n} ds \quad (A6)$$

The space $W = V \times Q$ is a mixed-function space, where $q \in Q$ and $\bar{u} \in V$.

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