Please fill in the name of the event you are preparing this manuscript for.	2021 SPE Middle East Oil & Gas Show and Conference	
Please fill in your 6-digit SPE manuscript number.	SPE-204906-MS	
Please fill in your manuscript title.	An Accurate Cubic Law For The Upscaling Of Discrete Natural Fractures	
Please fill in your author name(s) and company affiliation.		
Given Name	Surname	Company
Xupeng	He	King Abdullah University of Science and Technology
Marwa	Alsinan	Saudi Aramco
Hyung	Kwak	Saudi Aramco
Hussein	Hoteit	King Abdullah University of Science and Technology

This template is provided to give authors a basic shell for preparing your manuscript for submittal to an SPE meeting or event. Styles have been included (Head1, Head2, Para, FigCaption, etc) to give you an idea of how your finalized paper will look before it is published by SPE. All manuscripts submitted to SPE will be extracted from this template and tagged into an XML format; SPE's standardized styles and fonts will be used when laying out the final manuscript. Links will be added to your manuscript for references, tables, and equations. Figures and tables should be placed directly after the first paragraph they are mentioned in. The technical content of your paper WILL NOT be changed. Please start your manuscript below.

Abstract

Modeling fluid flow in fractured reservoirs requires an accurate evaluation of the hydraulic properties of discrete fractures. Full Navier-Stokes simulations provide an accurate approximation of the flow within fractures, including fracture upscaling. However, its excessive computational cost makes it impractical. The traditionally used cubic law (CL) is known to overshoot the fracture hydraulic properties significantly. In this work, we propose an alternative method based on the cubic law. We first develop geometric rules based on the fracture topography data, by which we subdivide the fracture into segments and local cells. We then modify the aperture field by incorporating the effects of flow direction, flow tortuosity, normal aperture, and local roughness. The approach is applicable for fractures in 2D and 3D spaces. This paper presented almost all existing CL-based models in the literature, which include more than twenty models. We benchmarked all these models, including our proposed model, for thousands of fracture cases. High-resolution simulations solving the full-physics Navier-Stokes (NS) equations were used to compute the reference solutions. We highlight the behavior of accuracy and limitations of all tested models as a function of fracture geometric characteristics, such as roughness. The obtained accuracy of the proposed model showed the highest for more than 2000 fracture cases with a wide range of tortuosity, roughness, and mechanical aperture variations. None of the existing methods in the literature provide this level of accuracy and applicability. The proposed model retains the simplicity and efficiency of the cubic law and can be easily implemented in workflows for reservoir characterization and modeling.

1. Introduction

Modeling fluid flow in subsurface fractured rocks is crucial for various environmental and energy applications, such as NAPLs tracking, CO₂ geological sequestration, geothermal energy, and oil recovery process in fractured reservoirs. However, modeling the complex flow behaviors in fractured rocks requires accurate quantification of the hydraulic properties of discrete rock fractures. Rock fractures in the subsurface are subject to be stress-dependent (He et al., 2020). Herein, we do not consider stress influence on the hydraulic properties of fractures.

Conventionally, discrete rock fractures have been simplified as two parallel plates with a constant aperture (Snow, 1969). With this assumption, an analytical integration of the NS equations results in the CL (Brush and Thomson, 2003; Zimmerman and Yeo, 2000), where the volumetric flow rate is a function

of the cube of the fracture's aperture. The CL is commonly used for flow predictions of fractures in various disciplines due to its efficiency and simplicity. However, natural fractures are formed of surfaces with varying apertures and anisotropic roughness with points of contact, which leads to significant deviation of the conductivity estimated by CL compared to the actual one (Konzuk and Kueper, 2004; Witherspoon et al., 1980; Zimmerman and Bodvarsson, 1996).

Many different models have been developed in the literature to increase the accuracy of CL. These models can be categorized into two groups: The first group modifies the definition of the aperture used in the CL, such as the geometric mean (Dagan, 1979; Smith and Freeze, 1979); the arithmetic mean (Brown, 1987); the volume-averaged mean (Hakami and Larsson, 1996; Schrauf and Evans, 1986); the harmonic mean (Unger and Mase, 1993); mean proposed by (Neuzil and Tracy, 1981); mean by (Tsang and Witherspoon, 1981); mean by (Brown, 1984; Deuell, 1988; Ge, 1997; Silliman, 1989; Zimmerman et al., 1991) and mean by (Zimmerman et al., 1991). The second group incorporates an additional correction factor to account for the effects of fracture roughness (Amadei and Illangasekare, 1994; Barton, 1982; Gutfraind and Hansen, 1995; Lomize, 1951; Louis, 1969; Patir and Cheng, 1978; Renshaw, 1995; Zimmerman and Bodvarsson, 1996), flow tortuosity (Brown, 1987; Waite et al., 1999), and even combined effect of roughness and tortuosity (Wang et al., 2015; Wang et al., 2018; Xiao et al., 2013). These models, however, neglect local-scale flow behavior, and therefore show inaccuracy when dealing with complex fracture cases. Oron and Berkowitz (1998) performed a local non-dimensional, 2-D order-of-magnitude analysis to the NS equations. They found that the CL approximations hold within specific segments when the geometric aspect ratio and the local roughness (non-dimensional) are less than unity. However, without treatment of non-segmented area (i.e., local cells) and approximation of local roughness effect for segments, it will have limited application to fracture flow. Wang et al. (2018) extended the work of Oron and Berkowitz (1998) by accounting for flow tortuosity and normal aperture effects for local cells following ideas from Ge (1997). They further introduced the concept of effective normal aperture for segments by including the local roughness effect. However, their assumption to estimate the flow direction within local cells is not valid when the aspect ratio of local cells exceeds one.

The Modified CL (MCL) developed herein presents an efficient and accurate approach to calculate the fractures' hydraulic properties. The remainder of this paper consists of 4 sections. Section 2 presents the development of MCL. We first propose an algorithm to subdivide the fracture into the segment and non-segment areas (i.e., local cells). We then modify the fracture aperture field by considering the effects of normal aperture, flow tortuosity, and local roughness, including 1) introducing a correction for the flow direction, which accounts for the neighboring effects; 2) deriving a new formula to calculate the normal aperture based on the corrected flow direction; 3) quantifying the local roughness effect within segments following the idea of Wang et al. (2018). In section 3, we demonstrate the model accuracy with the reference NS solutions. We also benchmark our model with other existing CL-based models in the literature. Section 4 provides the results. Finally, we conclude with a summary and propose a future extension of this work.

2. Proposed Model Development

This study aims to quantify the hydraulic properties of rock fractures with variable aperture and rough surfaces designed for incompressible, single-phase laminar flow. In this study, we discuss 2D fracture cases. The proposed model develops as the following steps: 1) Fracture segmentation - subdivide the fracture into segments and non-segment areas (i.e., local cells); 2) Determination of flow direction; 3) Consideration of flow tortuosity; 4) Calculation of normal aperture; 5) Quantification of local roughness.

2.1 Fracture Segmentation

Consider a cross-section of a 3D rough-walled fracture, which exhibits the characteristics of a combination of well-mated walls (i.e., the fracture walls are almost parallel) and other situations where they are not (**Figure 1a**). **Figure 1b** shows a fracture segment with the average half-aperture B_s and length L_s . The

Poiseuille equation holds when B_s is much smaller than L_s , and surface roughness of the two walls is less than B_s (Oron and Berkowitz, 1998), that is:

$$\delta = \frac{B_s}{L_s} \ll 1 \quad (1)$$

$$\varepsilon = \max\left(\frac{\sigma_u}{B_s}, \frac{\sigma_l}{B_s}\right) \ll 1 \quad (2)$$

In the above equations, δ is the geometric aspect ratio, ε represents local roughness, which is non-dimensional, σ_l and σ_u are, respectively, the standard deviations of surface roughness of the lower and upper walls.

Figure 1. Illustration of (a) a cross-section of a 3D rough fracture with combinations of well-mated and diagonal walls, (b) a fracture segment with the average half-aperture B_s and the segment length L_s , and (c) one local cell with bottom and top angles of α and β , respectively.

When the constraints (1) and (2) hold, the fracture segment with rough walls (see **Figure 1b**) can be approximated by two parallel straight lines representing the average variation of the lower and upper walls (Oron and Berkowitz, 1998). The selection of the segments of a fracture can be calculated from computing δ and ε along the fracture. However, there are some segments of the fracture that do not meet the geometric conditions. We refer to them as local cells, as shown in **Figure 1c**. Herein, we investigate the ranges of δ and ε corresponding to the applicability of the Poiseuille equation. Selecting the suitable ranges is crucial for the accuracy of our algorithm used for fracture segmentation. The geometric aspect ratio δ determines the deviation of the flow streamline from the fracture centerline. The non-dimensional local roughness ε controls the effective fracture aperture that has the highest contribution to the fluid

flux. The Poiseuille flow holds when δ is much less than 1.0, which results in a stricter requirement for forming the segments. Therefore, it is required to determine an appropriate upper limit to δ so that the segments are selected with acceptable accuracy. In this work, we choose the upper limit of δ to be 0.5, which is often utilized to generate perturbation solutions from the NS equations (Basha and El-Asmar, 2003), that is:

$$\delta \leq 0.5 \quad (3)$$

As the Poiseuille flow is assumed valid within each segment, the bulk of the flow mainly occurs within the central area, and a smaller portion of the flow occurs near-wall area (Oron and Berkowitz, 1998) (see **Figure 2**, left). **Figure 2**, right shows that 25 % of the fracture area near both walls (i.e., 12.5 % of the area near each wall) only accounts for ~10 % of the flow. The velocity's magnitude in the central section of the fracture is calculated from the aperture of open space available for the flow. The form of the disturbances at both walls is less significant when they do not extend towards the center (Oron and Berkowitz, 1998). To prevent the disturbances, we set 0.25 as the upper limit for ε , that is:

$$\varepsilon \leq 0.25 \quad (4)$$

Figure 2. **left** shows the Poiseuille flow with parabolic velocity distribution; **right** indicates a relationship between the flow rate percentage through near-wall area and ratio of near-wall to whole fracture area.

It is crucial to develop a fracture segmentation algorithm to subdivide the fracture into segments and local cells. Based on the fracture topological data, including the height of the lower and upper surfaces and their corresponding coordinates, we develop a parallel multi-linear regression (MLR) model to capture the parallel trends approximating the roughed fracture segments, as shown in **Figure 1b**. The algorithm detects the segments that honor the constraints provided in Eq. (3) and Eq. (4). The flowchart of the proposed segmentation algorithm is detailed in **Appendix A**.

2.2 Determination of Flow Direction

Consider a fracture unit M , which is not aligned with the fracture principal direction, as shown in **Figure 3**. The flow streamlines are determined by solving the high-resolution NS equations. In **Figure 3** (b) and (c), the aspect ratio δ_M is less than 1.0. In this case, the flow streamlines are aligned with the centerline of the unit M , which means that the flow direction is governed by its own geometric properties with negligible impact from neighbor units. In the other two cases, as δ_M increases, the flow streamlines deviates from the centerline of M , as observed in **Figure 3** (a) and (d). We conducted hundreds of full NS simulations to confirm this observation. Therefore, the commonly used assumption by Ge (1997), which assumes flow direction is aligned with the centerline is only valid when δ_M is sufficiently small.

Figure 3. Flow streamline distribution indicating (b) and (c) flow directions follow the centerline of M , (a) and (d) discrepancy between flow streamline and centerline. The red arrow represents the centerline orientation, the black arrow denotes the real flow direction, and the blue arrow the proposed flow direction.

In this work, we propose a new flow-direction formula, which depends on the geometric aspect ratio. Through various high-resolution NS simulations, we found when the segment is formed, the flow direction (denoted as θ) is aligned with the centerline; otherwise, the flow direction is corrected based on the geometric properties of the adjacent units, as follow:

$$\theta = \gamma_M \quad (\text{segment}) \quad (5)$$

$$\theta = \text{harmmean}(\gamma_{L \rightarrow M}^{w\_mean} + \gamma_{R \rightarrow M}^{w\_mean}) \quad (\text{local cell}) \quad (6)$$

Where,

$$\gamma_{L \rightarrow M}^{w\_mean} = \frac{\gamma_L L_L + \gamma_M L_M}{L_L + L_M} \quad \gamma_{R \rightarrow M}^{w\_mean} = \frac{\gamma_R L_R + \gamma_M L_M}{L_R + L_M} \quad (7)$$

In the above equations, L_L , L_M , and L_R are the lengths of the straight-line of left, middle, and right units, respectively; γ_L , γ_M , and γ_R are the inclination angles of the centerline within the left, middle, and right units, respectively; $\gamma_{L \rightarrow M}^{w\_mean}$ and $\gamma_{R \rightarrow M}^{w\_mean}$ are weighted arithmetic means, respectively.

2.3 Consideration of Flow Tortuosity

Flow tortuosity is defined as the actual flow-path length relative to the nominal fracture length measured in the direction of the imposed pressure gradient (Walsh and Brace, 1984). The actual flow streamlines can be described accurately by either NS simulations or experimental observation, which makes it computationally expensive or time-consuming. Herein, we apply the geometric flow tortuosity τ , defined

as the ratio of flow-direction distance d_f to the straight-line length of the fracture, d_s (see **Figure 4**).

Figure 4. Different definitions of fracture apertures for **(a)** segment with a_{AB} representing vertical aperture; a_{EF} representing segment aperture; **(b)** local cell with a_{AB} vertical aperture; a_{CD} normal aperture based on flow direction by Ge (1997); a_{EF} normal aperture based on new proposed flow direction. α , β , $(\alpha + \beta)/2$ and θ are inclination angles of the bottom wall, top wall, centerline and the proposed flow direction, respectively.

For the 2D fracture case, the flow-direction distance D_f is the summation of flow-direction distances for all segments and local cells of the fracture. The relation between D_f and fracture length L is given by:

$$D_f = \sum d_f(x) = \sum \tau(x) d_s(x) = L \bar{\tau} \quad (9)$$

where $\bar{\tau}$ is the macroscopic flow tortuosity of the 2D fracture case.

2.4 Calculation of Normal Aperture

Based on the geometries of all segments and local cells, the vertical aperture at any location, defined as the distance between the bottom and top walls in the global xz coordinate, can be obtained. Since the aperture in the CL is defined as the length of the perpendicular to the flow direction, accordingly, the local normal aperture should be perpendicular to the flow direction (see red dashed line in **Figure 4**). We then redefine the relation between vertical aperture a_{AB} and normal aperture a_{EF} for each segment and local cell as:

$$a_{EF} = \frac{a_{AB}}{(1/\cos \alpha + 1/\cos \beta)} \left[\frac{1}{\cos(\theta - \alpha)} + \frac{1}{\cos(\theta - \beta)} \right] \quad (10)$$

A detailed derivation for this new formula for the normal aperture is given in **Appendix B**.

The link transmissivity (T) between local cells or segments can be calculated from the harmonic mean of adjacent normal apertures (Nicholl et al., 1999; Nicholl and Detwiler, 2001; Wang et al., 2018):

$$T = 2 \frac{a_{EF}(i)^3 a_{EF}(i+1)^3}{a_{EF}(i)^3 + a_{EF}(i+1)^3} \quad (11)$$

where $a_{EF}(i)$ and $a_{EF}(i+1)$ are left-hand and right-hand normal apertures, respectively.

2.5 Quantification of Local Roughness

As previously discussed, when the geometric conditions in Eq. (1) and Eq. (2) are satisfied, the rough-walled fracture segment can be approximated by two parallel straight-lines representing the average variation relative to the top and bottom walls. Here, when it comes to the local roughness effects, we only refer to the segments. The approximation error of these two parallel straight lines increases as the fracture roughness increases, as expected. Since an essential part of the pressure drop occurs at smaller apertures (Nazridoust et al., 2006), the normal aperture calculated by Eq. (10) may overestimate the hydraulic properties of the rough-walled fracture segment. Wang et al. (2018) introduced the concept of the effective normal aperture by including the local roughness effect as a correction factor.

$$a_{EF}^{eff} = a_{EF} \left(1 - \frac{\sigma_u + \sigma_l}{2B_s} \right) = a_{EF} \eta \quad (12)$$

where a_{EF}^{eff} is the effective normal aperture and other parameters are defined in Eq. (2).

2.6 Proposed Modified Cubic Law

2D fracture is subdivided into segments and local cells. The fracture aperture is modified to account for the impact of flow direction, flow tortuosity, normal aperture, and local roughness. The modified aperture of a 2D fracture is given by:

$$a_{modified}(x) = T^{1/3} \eta \quad (13)$$

where $a_{modified}(x)$ is the modified aperture field, T is the link transmissivity defined in Eq. (11), and η is the local roughness correction factor defined in Eq. (12). For the local cells, η is equal to 1.0, and Eq. (13) reduces to:

$$a_{modified}(x) = T^{1/3} \quad (14)$$

Once the modified aperture field is determined, the fracture's mechanical aperture can be obtained. Based on the relationship between mechanical and hydraulic apertures (Zimmerman and Bodvarsson, 1996), an extra factor is used to correct the classical CL to improve flow calculations. Furthermore, we modify the fracture length L to reflect the macroscopic flow tortuosity defined in Eq. (9). Therefore, our proposed physics-based modified CL becomes:

$$Q = \frac{W \langle a_{modified} \rangle^3}{12\mu\tau} \frac{\Delta p}{L} \left[1 - 1.5 \left(\frac{\sigma_{a_{modified}}}{\langle a_{modified} \rangle} \right)^2 \right] \quad (15)$$

where $a_{modified}$ represents the modified aperture and $\langle a_{modified} \rangle$ is the arithmetic mean, $\sigma_{a_{modified}}$ is the standard deviation, and W is the width of the fracture.

3. Methodology

3.1 Fracture Generation

We demonstrate the proposed MCL in terms of prediction accuracy and general applicability by applying it to 2048 2D rough fractures with tortuosity ranging from 1.16 to 1.38 and with relative roughness ranging from 4.2 to 5.6. The tortuosity is defined in Eq. (9), and the relative roughness is given as the ratio of arithmetic mean to standard deviation of aperture fields (Brown, 1987). The process of generating 2D rough fractures (see **Figure 5**) is described as follow: 3D fracture generation, then slicing it to obtain 2D fracture cases.

Figure 5. Simulation procedure including 3D fracture generation, 2D fracture sliding, mesh generation, and the NS simulations.

3.2 Full NS Simulations

Numerical simulation using the full NS equations is theoretically considered as an accurate approach for approximating the hydraulic properties of rough-walled fractures. In our study, the solutions from NS equations are considered as the “true” solutions, used to evaluate the accuracy of our model. We apply the full-physics NS equations using the mixed finite element method, by which the velocity field could be accurately approximated (Hoteit et al., 2002; Hoteit and Firoozabadi, 2008). The gravity effect is ignored,

and no-slip boundary conditions are considered at the walls of the fracture. A pressure gradient of 0.1 Pa/m is applied between the outlet and inlet. Fluid properties for water are given by $\rho = 1000 \text{ kg/m}^3$ and $\mu = 0.001 \text{ Pa s}$. Mesh sensitivity is conducted by increasing the number of grid cells so that the variation between two successive grids is within 0.1 %. Each of the generated 2048 fractures is discretized into approximately 0.2 million mesh elements with a cell size about 0.05 mm. The whole process is developed to run in parallel on a supercomputer. The algorithm automates the process to generate the 2D fracture cross-sections, create the meshes, run the steady-state NS simulations, and calculate the hydraulic apertures (see **Figure 5**). A detailed discussion of the numerical method is presented in **Appendix C**.

3.3 Quantification of Accuracy for MCL

To assess the proposed MCL performance, the volumetric flow rates approximated by the MCL are compared to those obtained from the NS simulations. The deviation error (D) is defined as:

$$D_{MCL} = \frac{Q_{MCL} - Q_{NSE}}{Q_{NSE}} \times 100\% \quad (16)$$

where Q_{MCL} is the volumetric flux obtained using the proposed MCL, and Q_{NSE} is the volumetric flux calculated using the NS equations.

3.4 Comparison with Other Models

To further highlight the value of the proposed MCL, we benchmark it with other 21 CL-based models (see **Table 1**). A detailed description of these models is given in related papers. These models are categorized into kinds: (1) modification of aperture definition used in the CL, and (2) introduction of correction factor that accounts for effects of roughness, tortuosity, or combined effect.

Table 1. A literature review of the CL-based models.

Model	Author(s)	Expression	Category	Method	Highlights
1	Smith and Freeze (1979)	$a_h = \text{geomean}(a)$	Aperture Definition	Theoretical Derivation	Based on perturbation analysis
2	Neuzil and Tracy (1981)	$a_h = \langle a^3 \rangle^{1/3}$	Aperture Definition	Theoretical Derivation	Assumed that fracture aperture varies only in the direction normal to the imposed pressure gradient
3	Tsang and Witherspoon (1981)	$a_h = \left[\langle a^3 \rangle^{1/3} / \langle 1/a^2 \rangle \right]^{1/3}$	Aperture Definition	Theoretical Derivation	Consider variation in aperture both parallel and perpendicular to pressure gradient
4	Brown (1984)	$a_h = \langle a^{-3} \rangle^{-1/3}$	Aperture Definition	Theoretical Derivation	Derived based on 1D Reynolds equation;
5	Schrauf and Evans (1986)	$a_h = \text{volumean}(a)$	Aperture Definition	Empirical Analysis	Remain same with vertical and normal

					aperture
6	Brown (1987)	$a_h = \langle a \rangle$	Aperture Definition	Empirical Analysis	Applicable for fractures with more-smooth surfaces and less-variable aperture
7	Zimmerman et al (1991)	$a_h = \sqrt{\langle a^3 \rangle^{1/3} \langle a^{-3} \rangle^{-1/3}}$	Aperture Definition	Semi-empirical Analysis	Geometric mean of values by Neuzil and Tracy [1981] and by Brown 1984]
8	Unger and Mase (1993)	$a_h = \text{harmmean}(a)$	Aperture Definition	Empirical Analysis	Applicable for normally distributed aperture field with large contact areas
9	Lomize (1951)	$a_h = a_m \left[\frac{1}{1 + 17.0(2a_m / \sigma)^{-1.5}} \right]^{1/3}$	Correction Factor: Roughness	Laboratory Experiment	Empirical model - fitting with experimental data
10	Louis (1969)	$a_h = a_m \left[\frac{1}{1 + 8.8(2a_m / \sigma)^{-1.5}} \right]^{1/3}$	Correction Factor: Roughness	Laboratory Experiment	Empirical model - fitting with experimental data
11	Patir and Cheng (1978)	$a_h = a_m \left[1 - 0.90e^{-0.56a_m/\sigma} \right]^{1/3}$	Correction Factor: Roughness	Numerical Simulation (FDM)	Empirical model - fitting with numerical results
12	Amadei and Illangasekare (1994)	$a_h = a_m \left[\frac{1}{1 + 0.6(a_m / \sigma)^{-1.2}} \right]^{1/3}$	Correction Factor: Roughness	Analytical Solution (ITM)	Empirical model - fitting with results solved by Reynolds equation analytically
13	Gutfraind and Hansen (1995)	$a_h = a_m \left[1 - \left(\frac{y_{max}}{a_m/2} \right)^{0.35} \right]^{1/3}$	Correction Factor: Roughness	Numerical Simulation (LGA)	Empirical model - fitting with numerical results by the NS equations
14	Renshaw (1995)	$a_h = a_m \left[1 + (\sigma/a_m)^2 \right]^{-0.5}$	Correction Factor: Roughness	Theoretical Derivation	Theoretical model; Assumed that fracture aperture is lognormally distributed
15	Zimmerman and Bodvarsson (1996)	$a_h = a_m \left[1 - 1.5(\sigma/a_m)^2 \right]^{1/3}$	Correction Factor: Roughness	Theoretical Derivation	Theoretical model; Solve NS equations via perturbation technique
16	Rasouli and Hosseini (2011)	$a_h = a_m \left[1 - 2.25(\sigma/a_m)^2 \right]^{1/3}$	Correction Factor: Roughness	Numerical Simulation (FVM)	Empirical model - fitting with numerical results by NS equations

17	Xie et al (2015)	$a_h = a_m \left[0.94 - 5.0 (\sigma / a_m)^2 \right]^{1/3}$	Correction Factor: Roughness	Numerical Simulation (FEM)	Empirical model - fitting with numerical results by NS equations
18	Brown (1987)	$a_h = a_m \left[\frac{1}{\tau^2} \right]^{1/3}$	Correction Factor: Tortuosity	Analog Analysis	Analog to fluid flow through porous media
19	Waite et al (1999)	$a_h = \left[\frac{\sum_{i=1}^n l_i}{\sum_{i=1}^n l_i / a_i^3} \right]^{1/3} \left[\frac{1}{\tau} \right]^{1/3}$	Correction Factor: Tortuosity	Empirical Analysis	τ is the macroscopic tortuosity of whole fracture
20	Xiao et al (2013)	$a_h = a_m \left[\frac{1}{\tau^2 T_s^2 (1 + 5.0 (2a_m / \sigma)^{-1.5})} \right]^{1/3}$	Correction Factor: Combined Effect	Theoretical Derivation + Laboratory Experiment	Semi-empirical Model; The model is derived theoretically, yet its coefficients are obtained by fitting with experimental data
21	Wang et al (2015)	$a_h = \left\langle \frac{T^{1/3}}{(c\tau)^{1/3}} \right\rangle \left[1 + \left(\frac{\sigma}{T^{1/3} / (c\tau)^{1/3}} \right)^2 \right]^{-0.5}$	Correction Factor: Combined Effect	Same with Renshaw [1995] model	T is harmonic mean of transmissivity; Compared with the NS equations

4. Results and Discussions

4.1 Comparison with NS Solutions

We first verify the accuracy of MCL with high-resolution NS simulations in terms of volumetric flux by applying it to 2048 fractures with a big range of tortuosity and relative roughness. For all these fractures, four combinations of δ and ε are identified: $\delta = 0.45$ and $\varepsilon = 0.20$ (denoted as C1), $\delta = 0.45$ and $\varepsilon = 0.25$ (C2), $\delta = 0.50$ and $\varepsilon = 0.20$ (C3), $\delta = 0.50$ and $\varepsilon = 0.25$ (C4), respectively. C4 provides the best results among these four combinations and, therefore, is selected for comparison with other CL-based models (see **Figure 6**).

Figure 6. Violin plot shows the deviation error (D) between ‘true’ volumetric flux by full NS simulations and proposed MCL with four different combinations of δ and ε . Negative values of D indicate under-estimation of flow rate, while positive values over-estimation the flow rate. For quartiles, the central mark (white circle) refers to the median (50%), and the edges of the black box correspond to 25th and 75th percentiles, respectively.

4.2 Accuracy of MCL Relative to Other CL-based Models

Figure 7 shows the violin plot of the deviation error (D) between the NSE simulations and other CL-based models using (a) vertical apertures and (b) normal apertures in the literature. The proposed MCL shows better performance in terms of volumetric flux, with the derivation error D ranging from -13.64 % to 19.64 %. The arithmetic mean of $|D|$ is 5.29 %, which shows the most accurate results (see **Table 2**).

Comparing **Figure 7a** and **Figure 7b** shows that the flow rates calculated from vertical apertures are larger and give less accurate results compared to the ones calculated from normal apertures (except for Xiao et al. [2013]’s method). This result agrees with the findings of other authors (Ge (1997), Mourzenko et al. (1995), Oron and Berkowitz (1998)) and also agrees that the rate should be computed based on Darcy’s Law from the cross-sectional area, which is perpendicular to the local flow direction. Therefore, the results calculated using normal apertures will be presented herein. The mean by Tsang and Witherspoon (1981) (Model 3) is found to be the only approach that offers a good match with simulations, which is within 15% for these 2D fracture cases. Of the various correction factors, volumetric flux calculated using roughness factor by Louis (1969) (Model 10) and Xie et al. (2015) (Model 17) are within $\pm 10\%$ of that by NS simulations, with $|D|$ of 9.98 % and 9.52 %, respectively (see **Table 2**).

Figure 7. Violin plot shows the deviation error (D) between ‘true’ volumetric flux by full NS simulations and different CL-based models using (a) vertical apertures and (b) normal apertures. The proposed model with C4 shows the best accuracy.

We also investigate how $|D|$ changes with increasing relative roughness for different CL-based models and proposed MCL (**Figure 8**). As observed, except for roughness correction factor by Gutfraind and Hansen (1995), the deviation error of $|D|$ increases essentially as the fractures become rougher. We should note that, in this study, the smaller value of relative roughness, the rougher the fracture is. This finding further confirms the limitations of other CL-based models with unstable performance when

dealing with complex fracture cases. Our proposed model (C4) shows good consistency in the behavior of its accuracy regardless of relative roughness. This critical feature enhances its applicability for fracture cases with a wide range of roughness.

Figure 8. Relationship between the deviation error (D) and relative roughness (d_m/σ) for CL-based models and proposed MCL (a) for vertical apertures and (b) for normal apertures. One point represents 200 simulations (2048 simulations in total).

Table 2. Accuracy of proposed MCL relative to other 21 CL-based models from the literature. The top three models are highlighted in dark gray.

Model	Vertical Aperture			Model	Normal Aperture		
	Range of D	$ \overline{D} $	$\sigma_{ D }$		Range of D	$ \overline{D} $	$\sigma_{ D }$
2	(125.00 %, 271.04 %)	189.73 %	23.03 %	2	(52.54 %, 120.37 %)	77.44 %	11.30 %
6	(106.05 %, 233.77 %)	159.53 %	19.28 %	5	(36.92 %, 97.03 %)	59.05 %	9.16 %
5	(106.01 %, 233.93 %)	159.53 %	19.29 %	6	(38.30 %, 94.61 %)	58.46 %	9.06 %
11	(97.17 %, 215.00 %)	144.30 %	17.28 %	14	(24.70 %, 74.99 %)	42.49 %	7.61 %
14	(95.98 %, 214.34 %)	143.97 %	17.49 %	15	(23.48 %, 73.18 %)	41.00 %	7.44 %
15	(95.56 %, 213.34 %)	143.14 %	17.37 %	12	(23.13 %, 72.96 %)	40.84 %	7.69 %
1	(95.24 %, 211.62 %)	142.55 %	17.15 %	11	(22.98 %, 72.44 %)	40.42 %	7.37 %
12	(91.00 %, 206.77 %)	138.19 %	17.19 %	1	(23.26 %, 72.08 %)	40.45 %	7.27 %
21	(91.39 %, 207.18 %)	137.79 %	16.93 %	8	(8.47 %, 48.11 %)	22.30 %	5.55 %
7	(93.33 %, 194.65 %)	136.69 %	15.07 %	21	(2.69 %, 48.69 %)	18.87 %	6.38 %
3	(91.16 %, 196.52 %)	134.58 %	15.40 %	7	(-2.92 %, 35.95 %)	15.74 %	5.44 %
8	(84.25 %, 187.12 %)	124.97 %	14.83 %	18	(-5.30 %, 48.04 %)	14.48 %	6.92 %
10	(65.36 %, 160.27 %)	101.39 %	13.65 %	3	(-4.64 %, 37.19 %)	14.00 %	5.71 %
4	(65.77 %, 133.99 %)	93.47 %	10.62 %	10	(-3.29 %, 35.27 %)	9.98 %	5.45 %
17	(58.71 %, 145.66 %)	89.34 %	12.24 %	17	(-25.63 %, 11.84 %)	9.52 %	4.29 %
18	(59.98 %, 126.19 %)	87.21 %	11.25 %	9	(-24.45 %, 5.40 %)	14.18 %	4.14 %
9	(39.66 %, 115.95 %)	66.64 %	10.84 %	4	(-43.60 %, -10.64 %)	24.36 %	5.08 %
19	(45.90 %, 89.86 %)	64.30 %	7.05 %	19	(-54.25 %, -21.63 %)	35.67 %	5.09 %
16	(18.29 %, 82.09 %)	39.83 %	8.78 %	16	(-50.87 %, -23.71 %)	38.10 %	3.62 %
20	(-31.86 %, 6.99 %)	16.60 %	5.09 %	20	(-67.54 %, -33.81 %)	52.34 %	4.71 %
13	(-82.63 %, -44.33 %)	62.92 %	7.63 %	13	(-91.17 %, -75.14 %)	82.83 %	2.74 %
C4	(-13.94 %, 19.64 %)	5.29 %	3.88 %	C4	(-13.94 %, 19.64 %)	5.29 %	3.88 %

Summary

The introduced MCL provides more accurate flow prediction than all other commonly used CL-based models in the literature. It also keeps the simplicity of cubic law but improves its accuracy in

approximating the hydraulic properties for fractures under incompressible, single-phase laminar flow. Future work will focus on 3D rock fracture cases. We summarize the main findings as follow:

- The proposed MCL model aims to calculate the hydraulic properties of 2D rock fractures by accounting for the effects of flow direction, flow tortuosity, normal aperture, and local roughness.
- We introduce a new algorithm for subdividing the 2D fractures into a series of segments and local cells, based on the geometric rules reflecting the aspect ratio and non-dimensional local roughness.
- We develop a new formula to determine the flow direction within local cells, which depends on the geometric properties of adjacent units. This new finding challenges the commonly used flow direction, which assumes flow direction always aligns with the centerline orientation of local cells.
- We derive a more general formula to calculate the normal aperture based on the newly corrected flow direction. The formula proposed by Ge (1997) is a special case of ours.
- We benchmark the proposed MCL with the full NS solutions solved from 2048 2D fractures with different roughness and tortuosity, which shows the generality of our proposed model.
- A comprehensive review of almost (21) all CL-based models is presented and further demonstrated the superiority of MCL.

Appendix A – Fracture Segmentation Algorithm

The proposed fracture segmentation algorithm (see **Figure A1**):

Figure A1. Flowchart of the proposed algorithm to detect parallel segments in a 2D roughed fracture.

- Input Parameters: include optimal values δ and ε (denoted as $\delta_{optimal}$ and $\varepsilon_{optimal}$, respectively), check steps for the outer loop and inner loop (denoted as ς_{outer} and ς_{inner} , respectively), maximum steps that each segment contains, and the initial start point $i_{start} = 1$.
- Outer Loop: aims to check the availability of potential segments. Based on 2D fracture topological data, the MLR model is applied to the potential segment for calculating the values of δ and ε . The segment is formed if both the calculated δ and ε fall into the range of $\delta_{optimal}$ and $\varepsilon_{optimal}$. Then we update the i_{start} with i_{end} and resume the checking process for a new potential segment. Otherwise, the process switches into the inner loop.
- Inner Loop: extends the range of the potential segment by adding the inner check steps to the ending points. The newly formed potential segment is offered to the outer loop for further checking. When the steps that a potential segment contains exceed ς_{max} , we update the start point i_{start} with $i_{start} + 1$ to resume the whole process.

Appendix B – Normal Aperture Calculation

The flow direction proposed by Ge (1997) follows the orientation of the centerline of units (segment or local cell), which has been widely used by many researchers. The relationship between the normal aperture a_{CD} and the apparent aperture a_{AB} (see **Figure B1**) can be given by:

$$a_{CD} = a_{OC} + a_{OD} = 2a_{OD} = \frac{2a_{AB}}{\cos[(\beta - \alpha)/2](1/\cos \alpha + 1/\cos \beta)} \quad (B1)$$

Based on Eq. (B1), we have:

$$a_{OC} = a_{OD} = \frac{a_{AB}}{\cos[(\beta - \alpha)/2](1/\cos \alpha + 1/\cos \beta)} \quad (B2)$$

The step-by-step procedure for the calculation of normal aperture a_{EF} (see **Figure B1**) is as follow:

Since the normal aperture is perpendicular to the corresponding flow direction, in $\triangle ODF$:

$$\begin{aligned} \angle ODF &= 90^\circ - \angle OGD = 90^\circ - (\beta - \alpha)/2 \\ \angle DOF &= \angle BOD - \angle BOF = (\beta + \alpha)/2 - \theta \end{aligned} \quad (B3)$$

Interior angles sum up to 180° . Thus, we can get:

$$\angle OFD = 180^\circ - \angle ODF - \angle DOF = 90^\circ + \theta - \alpha \quad (B4)$$

According to the Law of Sines:

$$\frac{a_{OF}}{\sin(\angle ODF)} = \frac{a_{OD}}{\sin(\angle OFD)} \quad (B5)$$

which can be arranged to:

$$\begin{aligned}\rho(\bar{\mathbf{u}} \bullet \nabla \bar{\mathbf{u}}) &= -\nabla p + \mu \nabla^2 \bar{\mathbf{u}} \\ \nabla \bullet \bar{\mathbf{u}} &= 0\end{aligned}\tag{C1}$$

In the above equations, ρ is fluid density, p is pressure, $\bar{\mathbf{u}}$ is velocity vector, and μ is fluid viscosity. No-slip boundary conditions are set at the walls of the fracture, and pressure values are imposed at the outlet and inlet of the fracture (see **Figure C1**). The gradient of velocity in the direction of the pressure gradient is set to zero to guarantee the fully developed flow and to avoid the inlet effect.

Figure C1. Illustration of boundary conditions imposed on a rough-walled fracture.

The NSEs can be formulated in a mixed variational form. Multiplying Eq. (C1) by the test function $(\bar{\mathbf{v}}, q)$ and integrating the resulting equations over the domain Ω yields:

$$\begin{aligned}\int_{\Omega} \rho(\bar{\mathbf{u}} \bullet \nabla \bar{\mathbf{u}}) \bullet \bar{\mathbf{v}} dx &= \int_{\Omega} (-\nabla p) \bullet \bar{\mathbf{v}} dx + \int_{\Omega} (\mu \nabla^2 \bar{\mathbf{u}}) \bullet \bar{\mathbf{v}} dx \\ \int_{\Omega} (\nabla \bullet \bar{\mathbf{u}}) q dx &= 0\end{aligned}\tag{C2}$$

Applying the integration by parts technique, we have:

$$\begin{aligned}\int_{\Omega} (-\nabla p) \bullet \bar{\mathbf{v}} dx &= \int_{\Omega} p \nabla \bullet \bar{\mathbf{v}} dx - \int_{\partial\Omega} p \bar{\mathbf{v}} \bullet \bar{\mathbf{n}} ds \\ \int_{\Omega} (\mu \nabla^2 \bar{\mathbf{u}}) \bullet \bar{\mathbf{v}} dx &= -\mu \int_{\Omega} \nabla \bar{\mathbf{u}} : \nabla \bar{\mathbf{v}} dx + \mu \int_{\partial\Omega} \nabla \bar{\mathbf{u}} \bullet \bar{\mathbf{n}} \bar{\mathbf{v}} ds\end{aligned}\tag{C3}$$

The problem becomes $(\bar{\mathbf{u}}, p) \in W$,

$$a((\bar{\mathbf{u}}, p), (\bar{\mathbf{v}}, q)) = L((\bar{\mathbf{v}}, q))\tag{C4}$$

where $(\bar{\mathbf{v}}, q) \in W$, and

$$a((\bar{\mathbf{u}}, p), (\bar{\mathbf{v}}, q)) = \rho \int_{\Omega} (\bar{\mathbf{u}} \bullet \nabla \bar{\mathbf{u}}) \bullet \bar{\mathbf{v}} dx - \int_{\Omega} p \nabla \bullet \bar{\mathbf{v}} dx + \mu \int_{\Omega} \nabla \bar{\mathbf{u}} : \nabla \bar{\mathbf{v}} dx + \int_{\Omega} (\nabla \bullet \bar{\mathbf{u}}) q dx\tag{C5}$$

$$L((\bar{\mathbf{v}}, q)) = - \int_{\partial\Omega_N} p_{inlet} \bar{\mathbf{v}} \bullet \bar{\mathbf{n}} ds - \int_{\partial\Omega_N} p_{outlet} \bar{\mathbf{v}} \bullet \bar{\mathbf{n}} ds\tag{C6}$$

In the above equation, $W = V \times Q$ is a mixed-function space with $q \in Q$, $\bar{\mathbf{u}} \in V$.

Acknowledgments

We would like to thank Saudi Aramco for funding this research. We would also like to thank King Abdullah University of Science and Technology (KAUST) for providing a license of MATLAB.

References

- Amadei, B., & Illangasekare, T. (1994). A mathematical model for flow and solute transport in non-homogeneous rock fractures. *International Journal of Rock Mechanics and Mining Sciences*, 31(6), 719–731. [https://doi.org/10.1016/0148-9062\(94\)90011-6](https://doi.org/10.1016/0148-9062(94)90011-6)
- Barton, N. (1982). Modelling Rock Joint Behaviour from In Situ Block Tests: Implications for Nuclear Waste Repository Design. *Office of Nuclear Waste Isolation, ONWI-308*(March), 114. Retrieved from http://inis.iaea.org/search/search.aspx?orig_q=RN:14736771
- Basha, H. A., & El-Asmar, W. (2003). The fracture flow equation and its perturbation solution. *Water Resources Research*, 39(12), 1–15. <https://doi.org/10.1029/2003WR002472>
- Brown, D. M. (1984). *Stochastic analysis of flow and solute transport in a variable-aperture rock fracture*.
- Brown, S., Caprihan, A., & Hardy, R. (1998). Experimental observation of fluid flow channels in a single fracture. *Journal of Geophysical Research: Solid Earth*, 103(3), 5125–5132. <https://doi.org/10.1029/97jb03542>
- Brown, S. R. (1987). Fluid flow through rock joints: the effect of surface roughness. *Journal of Geophysical Research*, 92(B2), 1337–1347. <https://doi.org/10.1029/JB092iB02p01337>
- Brown, Stephen R., Stockman, H. W., & Reeves, S. J. (1995). Applicability of the Reynolds Equation for modeling fluid flow between rough surfaces. *Geophysical Research Letters*, 22(18), 2537–2540. <https://doi.org/10.1029/95GL02666>
- Brown, Stephen R. (1987). Fluid Flow Through Rock Joints: The Effect of Surface Roughness. *Journal of Geophysical Research*, 92(B2), 1337–1347. <https://doi.org/10.1029/JB092iB02p01337>
- Brush, D. J., & Thomson, N. R. (2003). Fluid flow in synthetic rough-walled fractures: Navier-Stokes, Stokes, and local cubic law simulations. *Water Resources Research*, 39(4). <https://doi.org/10.1029/2002WR001346>
- Dagan, G. (1979). Models of groundwater flow in statistically homogeneous porous formations. *Water Resources Research*, 15(1), 47–63. <https://doi.org/10.1029/WR015i001p00047>
- Deuell, R. J. (1988). *An analysis of the microscopic hydraulic characteristics of a discrete rock fracture using finite elements*. Elsevier Ltd. <https://doi.org/10.1016/j.compgeo.2008.02.010>
- Ge, S. (1997). A governing equation for fluid flow in rough fractures. *Water Resources Research*, 33(1), 53–61. <https://doi.org/10.1029/96WR02588>
- Gutfraind, R., & Hansen, A. (1995). Study of fracture permeability using Lattice Gas Automata. *Transport in Porous Media*, 18(2), 131–149. <https://doi.org/10.1007/BF01064675>
- Hakami, E. V. A., & Larsson, E. (1996). Aperture measurements and flow experiments on a single natural fracture. *International Journal of Rock Mechanics and Mining Sciences and Geomechanics*, 33(4), 395–404. [https://doi.org/10.1016/0148-9062\(95\)00070-4](https://doi.org/10.1016/0148-9062(95)00070-4)
- He, X., Hoteit, H., AlSinan, M. M., & Kwak, H. T. (2020). Modeling hydraulic response of rock fractures under effective normal stress. *ARMA/DGS/SEG International Geomechanics Symposium 2020, IGS 2020*.
- Hoteit, H., Erhel, J., Mosé, R., Philippe, B., & Ackerer, P. (2002). Numerical reliability for mixed methods applied to flow problems in porous media. *Computational Geosciences*, 6(2), 161–194. <https://doi.org/10.1023/A:1019988901420>
- Hoteit, Hussein, & Firoozabadi, A. (2008). Numerical modeling of two-phase flow in heterogeneous permeable media with different capillarity pressures. *Advances in Water Resources*, 31(1), 56–73. <https://doi.org/10.1016/j.advwatres.2007.06.006>
- Konzuk, J. S., & Kueper, B. H. (2004). Evaluation of cubic law based models describing single-phase

- flow through a rough-walled fracture. *Water Resources Research*, 40(2), 1–17.
<https://doi.org/10.1029/2003WR002356>
- Kucza, J., & Ilek, A. (2016). The effect of the shape parameters of a sample on the hydraulic conductivity. *Journal of Hydrology*, 534, 230–236. <https://doi.org/10.1016/j.jhydrol.2016.01.010>
- Lomize, G. M. (1951). Water Flow Through Jointed Rock (in Russian). *Gosenergoizdat*, 127.
<https://doi.org/10.1016/j.enggeo.2017.11.022>
- Louis, C. (1969). A Study of Groundwater flow in Jointed Rock and Its Influence on the Stability of Rock Masses. *Rock. Mech. Res. Rep.* <https://doi.org/10.1016/j.geoderma.2012.02.014>
- Mourzenko, V. V., Thovert, J.-F., & Adler, P. M. (1995). Permeability of a Single Fracture; Validity of the Reynolds Equation. *Journal de Physique II*, 5(3), 465–482. <https://doi.org/10.1051/jp2:1995133>
- Nazridoust, K., Ahmadi, G., & Smith, D. H. (2006). A new friction factor correlation for laminar, single-phase flows through rock fractures. *Journal of Hydrology*, 329(1–2), 315–328.
<https://doi.org/10.1016/j.jhydrol.2006.02.032>
- Neuzil, C. E., & Tracy, J. V. (1981). Flow through fractures. *Water Resources Research*, 17(1), 191–199. <https://doi.org/10.1029/WR017i001p00191>
- Nicholl, M. J., Rajaram, H., Glass, R. J., & Detwiler, R. (1999). Saturated flow in a single fracture: Evaluation of the Reynolds equation in measured aperture fields. *Water Resources Research*, 35(11), 3361–3373. <https://doi.org/10.1029/1999WR900241>
- Nicholl, Michael J., & Detwiler, R. L. (2001). Simulation of flow and transport in a single fracture: Macroscopic effects of underestimating local head loss. *Geophysical Research Letters*, 28(23), 4355–4358. <https://doi.org/10.1029/2001GL013647>
- Oron, A. P., & Berkowitz, B. (1998). Flow in rock fractures: The local cubic law assumption reexamined. *Water Resources Research*, 34(11), 2811–2825. <https://doi.org/10.1029/98WR02285>
- Piggott, A. R., & Elsworth, D. (1993). Laboratory assessment of the equivalent apertures of a rock fracture. *Geophysical Research Letters*, 20(13), 1387–1390. <https://doi.org/10.1029/93GL01384>
- Renshaw, C. E. (1995). On the relationship between mechanical and hydraulic apertures in rough-walled fractures. *Journal of Geophysical Research*, 629–636.
- Schrauf, T. W., & Evans, D. D. (1986). Laboratory Studies of Gas Flow Through a Single Natural Fracture. *Water Resources Research*, 22(7), 1038–1050.
<https://doi.org/10.1029/WR022i007p01038>
- Silliman, S. E. (1989). An interpretation of the difference between aperture estimates derived from hydraulic and tracer tests in a single fracture. *Water Resources Research*, 25(10), 2275–2283.
<https://doi.org/10.1029/WR025i010p02275>
- Smith, L., & Freeze, R. A. (1979). Stochastic analysis of steady state groundwater flow in a bounded domain: 2. Two-dimensional simulations. *Water Resources Research*, 15(6), 1543–1559.
<https://doi.org/10.1029/WR015i006p01543>
- Snow, D. (1969). Anisotropic Permeability of Fractured Media. *Water Resources Research*, 5(6), 1273–1289.
- Tsang, Y. W. (1984). The Effect of Tortuosity on Fluid Flow Through a Single Fracture. *Water Resources Research*, 20(9), 1209–1215. <https://doi.org/10.1029/WR020i009p01209>
- Tsang, Y. W., & Tsang, C. F. (1990). Hydrological characterization of variable-aperture fractures. *Rock Joints*, (July), 35–43.
- Tsang, Y. W., & Witherspoon, P. A. (1981). Hydromechanical Behavior of a Deformable Rock Fracture Subject to Normal Stress. *JOURNAL OF GEOPHYSICAL RESEARCH*, 86, 9287–9298.
- Unger, A. J. A., & Mase, C. W. (1993). Numerical study of the hydromechanical behavior of two rough fracture surfaces in contact. *Water Resources Research*, 29(7), 2101–2114.
<https://doi.org/10.1029/93WR00516>
- Waite, M. E., Ge, S., & Spetzler, H. (1999). A new conceptual model for fluid flow in discrete fractures: An experimental and numerical study. *Journal of Geophysical Research: Solid Earth*, 104(B6), 13049–13059. <https://doi.org/10.1029/1998jb900035>

- Walsh, J. B., & Brace, W. F. (1984). The effect of pressure on porosity and the transport properties of rock. *Journal of Geophysical Research*, 89(B11), 9425–9431. <https://doi.org/10.1029/JB089iB11p09425>
- Wang, L., Cardenas, M. B., Slottke, D. T., Ketcham, R. A., & Sharp, J. M. (2015). Modification of the Local Cubic Law of fracture flow for weak inertia, tortuosity, and roughness. *Water Resources Research*, 51(4), 2064–2080. <https://doi.org/10.1002/2014WR015815>
- Wang, Z., Xu, C., & Dowd, P. (2018). A Modified Cubic Law for single-phase saturated laminar flow in rough rock fractures. *International Journal of Rock Mechanics and Mining Sciences*, 103(March 2017), 107–115. <https://doi.org/10.1016/j.ijrmms.2017.12.002>
- Witherspoon, P. A., Wang, J. S. Y., Iwai, K., & Gale, J. E. (1980). Validity of Cubic Law for fluid flow in a deformable rock fracture. *Water Resources Research*, 16(6), 1016–1024. <https://doi.org/10.1029/WR016i006p01016>
- Xiao, W., Xia, C., Wei, W., & Bian, Y. (2013). Combined effect of tortuosity and surface roughness on estimation of flow rate through a single rough joint. *Journal of Geophysics and Engineering*, 10(4). <https://doi.org/10.1088/1742-2132/10/4/045015>
- Xie, L. Z., Gao, C., Ren, L., & Li, C. B. (2015). Numerical investigation of geometrical and hydraulic properties in a single rock fracture during shear displacement with the Navier–Stokes equations. *Environmental Earth Sciences*, 73(11), 7061–7074. <https://doi.org/10.1007/s12665-015-4256-3>
- Zimmerman, R. W., Kumar, S., & Bodvarsson, G. S. (1991). Lubrication theory analysis of the permeability of rough-walled fractures. *International Journal of Rock Mechanics and Mining Sciences And*, 28(4), 325–331. [https://doi.org/10.1016/0148-9062\(91\)90597-F](https://doi.org/10.1016/0148-9062(91)90597-F)
- Zimmerman, Robert W., & Bodvarsson, G. S. (1996). Hydraulic conductivity of rock fractures. *Transport in Porous Media*, 23(1), 1–30. <https://doi.org/10.1007/BF00145263>
- Zimmerman, Robert W., & Yeo, I. W. (2000). Fluid flow in rock fractures: From the navier-stokes equations to the cubic law. *Geophysical Monograph Series*, 122(January 2000), 213–224. <https://doi.org/10.1029/GM122p0213>
- Zimmerman, Robert W., Chen, D. W., & Cook, N. G. W. (1992). The effect of contact area on the permeability of fractures. *Journal of Hydrology*, 139(1–4), 79–96. [https://doi.org/10.1016/0022-1694\(92\)90196-3](https://doi.org/10.1016/0022-1694(92)90196-3)