Pseudo-elastic pure P-mode wave equation

Bingbing Sun* and Tariq Alkhalifah*

**Physical Sciences and Engineering,
King Abdullah University of Science and Technology,
Thuwal, 23955, Saudi Arabia*

corresponding author: Bingbing Sun (bingbing.sun@kaust.edu.sa)

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ABSTRACT

We present a pseudo-elastic wave equation describing pure pressure waves propagating in elastic media. The pure pressure-mode (P-mode) wave equation uses all the elastic parameters (such as density, P- and S-wave velocities). It produces the same amplitude variation with offsets (AVO) effects as the PP reflections computed by the conventional elastic wave equation. Because the new wave equation is free of shear waves, it does not require finer grids for simulation. This leads to a significant computational speedup when the ratio of pressure to shear wave velocities is large. We tested the performance of our method on a simple synthetic model with high-velocity contrasts. The amplitude admitted by the pseudo-elastic pure P-mode wave equation is highly consistent with that associated with the conventional elastic wave equation over a large range of incidence angles. We further verified the proposed method’s robustness and accuracy using a more complex and realistic SEAM 2D salt model. The ideal AVO behavior and computational advantage make the proposed wave equation a good candidate as a forward simulation engine for performing

elastic full-waveform inversion, especially for marine streamer datasets.

INTRODUCTION

In seismic exploration, full-waveform inversion (Tarantola, 1984; Pratt, 1999) provides us with an effective tool for velocity model building and reservoir characterization. With surface seismic data, full-waveform inversion (FWI) mainly uses the transmitted/post-critical reflected/diving waves recorded at long offsets, and as a result, the long-to-intermediate wavelengths of the subsurface Earth parameters (usually compressional wave velocities) can be constructed. FWI minimizes the difference between the observed and the simulated data, which depends on the assumption of the subsurface wave propagation behavior. If such an assumption is consistent with the observed data, FWI can invert for a realistic Earth model; otherwise, the final reconstructed Earth model may contain artifacts. From another aspect, FWI requires massive computer resources and time, especially in 3-D applications. Thus, the acoustic assumption with constant density is widely used in FWI due to its simplicity (Pratt et al., 1996; Warner et al., 2013; Sun and Alkhalifah, 2019a,b; Oh et al., 2021; Soni et al., 2021).

Acoustic FWI is usually used for marine datasets when the acquired data are mostly dominated by unconverted P-waves, and the acoustic approximation holds well for the kinematics, but not for the amplitudes, except for compressional waves with normal incidence or small incidence angle. In the literature, there are many investigations related to the validity of the acoustic assumption and the importance of incorporating elasticity in FWI. Barnes and Charara (2009) report that acoustic full-wave information is only applicable when the S-wave velocity and the density fields are smooth enough to reduce the AVO effect or when the near-offset seismograms are inverted with a good starting model. Mulder and Plessix (2008) interpret the mismatch between the data due to acoustic approximation as noise, and

such noise would increase the nonlinearity of the corresponding inversion problem. Marelli et al. (2012) investigate the validity of the acoustic approximation in full-waveform seismic cross-hole tomography. They conclude that the subsurface models comprising small low-to-medium contrast (about 30%) structures can be well resolved using the acoustic approximation. However, in the presence of extended high-contrast anomalous bodies, P-wave to S-wave conversions substantially degrade the quality of the inversion result. Cance and Capdeville (2015) studied the acoustic approximation from a different perspective, focusing on small heterogeneities. They find that in rough media, acoustic waves can only be a poor-quality approximation of elastic waves, and the small-scale heterogeneities give rise to natural acoustic effective anisotropy, which is different in nature from the effective elastic anisotropy. In summary, such an acoustic assumption in FWI tends to work when the frequency is low, and the contrast of subsurface media is small (Plessix and Pérez, 2015). When the medium contrast becomes large (such as at the sea-floor or in the presence of salt bodies) or performing inversion using high frequencies, the elastic nature of the Earth needs to be taken into consideration, and an elastic FWI should be adopted (Brossier et al., 2009; Lee et al., 2010; Vigh et al., 2014; Kamath and Tsvankin, 2016; Guitton and Alkhalifah, 2017; Mora and Wu, 2018; Oh et al., 2018; Zhang et al., 2018; He et al., 2019; Faucher et al., 2020).

One major obstacle to applying elastic FWI is the increased computational cost of elastic modeling. There are mainly two reasons for the high computational cost of the elastic modeling (Hobro et al., 2014): (1) the higher complexity of the elastic wave equation operator; (2) the finer grid spacing required by the relatively low S-wave velocities. Compared to acoustic modeling, elastic modeling needs to deal with more parameters and field variables (e.g., in the first-order formulation, elastic modeling involves six stress components

while acoustic modeling only needs one pressure field). More arithmetic operations are also required in elastic modeling in each step of the time integration. As a result, the elastic modeling computational burden is about ten times higher than its acoustic counterpart. Another more significant factor is the stricter requirements for the grid size. To avoid dispersion, a minimum number of grid points is required per wavelength. For acoustic modeling, the grid size depends on the minimum compressional velocity $\alpha_{\min}$ while for elastic modeling, it depends on the minimum shear velocity $\beta_{\min}$. To ensure temporal stability, the time step must be below the CFL number. Overall, for 3D simulation, such computation ratio is proportional to $\left(\frac{\alpha_{\min}}{\beta_{\min}}\right)^4$. For Poisson media, the ratio is about nine, while for models with low shear velocity sediments, like near the seafloor or in the reservoir, the number can be considerably higher.

In the literature, a lot of effort has been directed to reducing the high cost of elastic simulation and the development of quasi-elastic acoustic wave equations. For example, Chapman et al. (2014) and Hobro et al. (2014) propose an iterative method for correcting the acoustic wavefield for elastic effects. They approximately model the particle-velocity field in the elastic wave equation using acoustic wave solutions. Specifically, they inject errors as a secondary source using extra acoustic simulations. Their first-order correction corresponds to a Born-like wave equation. Feng and Schuster (2019) propose a similar quasi-elastic wave equation under Born approximation, except that they derive their wave equation using the pressure variable for the background wavefield propagation. In their method, the background propagated wavefield is acoustic, and with a perturbation of the elastic media using the Born approximation, they obtain the elasticity-corrected wavefield. Due to the intrinsic limitation of the Born approximation, these methods are only valid for first-order scattering with small perturbations and narrow angles, and the errors can

increase dramatically when the waves are propagating through many interfaces. Thus, although their method produces the same radiation pattern as the PP mode of the elastic wave equation, the accuracy of the amplitude produced by their method is only first order. With large contrasts in the elastic parameters (e.g., the shear modulus), the discrepancy between the elastic-corrected P-wavefield by their method and the true elastic wavefield would be large, and the errors would tend to be more severe when there is multi-scattering. Due to this reason, their initial application is limited to least-squares migration only.

Considering that the majority of the FWI applications use only the P-waves, especially for marine streamer datasets, an effective solution for mitigating the computational burden of elastic modeling, and yet, obtaining accurate elastic P-waves, is to develop a pure P-mode wave equation, which can accommodate the elasticity effects of propagation of the pressure wavefield. Thus, in this paper, we propose a pseudo-elastic pure P-mode wave equation to fulfill this need. Specifically, we project the wavefield along the polarization direction of the P-waves within the framework of the elastic wave equation. We present the resulting pseudo-elastic wave equation in both first-order and second-order forms. In addition, we show that the second-order pseudo-elastic wave equation can be decomposed into an acoustic wave equation plus a term for correcting the elastic effect. In summary, the proposed pseudo-elastic pure P-mode wave equation maintains the following features:

- It accurately describes a pure P-wave propagating in an isotropic heterogeneous elastic medium. This makes the method an ideal engine for inverting datasets in which (non-converted) P-waves are dominant, such as in the case of marine streamer acquisition. Compared to the acoustic approximation, the extra elastic correction term incorporates the elasticity effect related to the shear modulus, and thus, leads to more

accurate amplitudes (for the pre- and post- critical reflections).

- It is free of S-waves, as the resulting wave equation yields a single mode of pressure wavefield. Thus, no requirement for very finer grid spacing is needed to guarantee dispersion-free S-wave propagation. This makes the method potentially computational-efficient when the ratio between pressure velocity and shear velocity is high in the area under investigation.
- It has the same radiation pattern for all the elastic parameters as that of the PP reflections by the original elastic wave equation. In this aspect, the proposed method seemingly maintains the same level of accuracy as the method proposed by Feng and Schuster (2019). However, as the proposed method is not based on the Born approximation, the pseudo-elastic pure P-mode wave equation can produce reflections with the same amplitude variation with offset (AVO) as the PP reflections, regardless of how large the contrast in the elastic parameters. In addition, due to first-order approximation, the method by Feng and Schuster (2019) is generally limited to single scattering and small incident angles. On the contrary, the resulting pseudo-elastic pure P-mode wave equation can deal with multi-scattering and large incident angles, and thus, it can handle the inhomogeneity and nonlinearity more accurately and naturally. We further show that a Born approximation of the proposed pseudo-elastic wave equation leads to the quasi-elastic wave equation (Feng and Schuster, 2019) directly.

In the following, we first derive the pseudo-elastic wave equation, and then we analyze the radiation patterns of the resulting wave equation. In the example section, we demonstrate the features of the proposed wave equation as well as strictly verify the AVO behavior by

comparing it with the PP-mode of the conventional elastic wave equation. At last, we include some discussions and concluding remarks.

DERIVATION OF THE PSEUDO-ELASTIC WAVE EQUATION

Describing waves propagating in a general 3D heterogeneous isotropic medium, the second-order elastic wave equation in matrix form is given by:

$$\partial_t^2 \mathbf{u} = \mathbf{E} \mathbf{u} + \frac{1}{\rho} (\mathbf{f} + \nabla \cdot \mathbf{m}), \quad (1)$$

where $\mathbf{u} = [u_x, u_y, u_z]^T$ denotes the particle displacement vector. Sources are represented by a force-density vector $\mathbf{f} = [f_x, f_y, f_z]^T$ and the moment-density tensor $\mathbf{m} = [m_{ij}]$, $i, j = x, y, z$. The operator $\mathbf{E}$, composed of medium parameters and spatial derivatives, fully describes how the wavefield interacts with the elastic medium:

$$\mathbf{E} = \frac{1}{\rho} \begin{bmatrix} \partial_x(\lambda + 2\mu)\partial_x + \partial_y\mu\partial_y + \partial_z\mu\partial_z & \partial_x\lambda\partial_y + \partial_y\mu\partial_x & \partial_x\lambda\partial_z + \partial_z\mu\partial_x \\ \partial_y\lambda\partial_x + \partial_x\mu\partial_y & \partial_x\mu\partial_x + \partial_y(\lambda + 2\mu)\partial_y + \partial_z\mu\partial_z & \partial_y\lambda\partial_z + \partial_z\mu\partial_y \\ \partial_z\lambda\partial_x + \partial_x\mu\partial_z & \partial_z\lambda\partial_y + \partial_y\mu\partial_z & \partial_x\mu\partial_x + \partial_y\mu\partial_y + \partial_z(\lambda + 2\mu)\partial_z \end{bmatrix}, \quad (2)$$

where, λ and μ are the Lamé parameters, ρ is the density of the medium.

The polarization direction for P-waves in isotropic media is along the wavefront normal direction $\mathbf{n}$, which can be represented accurately by the gradient operator $\nabla = [\partial_x, \partial_y, \partial_z]^T$. We can obtain pure P-waves by projecting the wavefield along the wavefront normal direction $\mathbf{n}$. For the resulting pressure wavefield, the displacement vector can be considered as the gradient of a scalar potential. This method is well-known as the Helmholtz decomposition and has been widely used in wavefield separation. However, such wavefield separation methods require first to simulate the elastic wave equation 1 (thus, finer grid sampling is

required to guarantee dispersion-free S-wave modeling), and the resulting separated pressure wavefield would also contain P-waves converted from S-waves (when the subsurface is complex).

Below, we introduce our method by performing a wavefield projection within the framework of the elastic wave equation. The resulting pseudo-elastic wave equation would describe pure P-waves propagating in an elastic medium.

To perform the wavefield projection, we introduce the normalized gradient operator $\tilde{\nabla}$ and the divergence operator $\tilde{\nabla} \cdot$:

$$\tilde{\nabla} = \begin{bmatrix} \frac{\partial_x}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \\ \frac{\partial_y}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \\ \frac{\partial_z}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \end{bmatrix}, \quad \tilde{\nabla} \cdot = \left[\frac{\partial_x}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \quad \frac{\partial_y}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \quad \frac{\partial_z}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \right], \quad (3)$$

in which, the following relations hold:

$$\tilde{\nabla} \cdot \tilde{\nabla} = \mathbf{I} \rightarrow [\tilde{\nabla} \cdot]^{-1} = \tilde{\nabla}. \quad (4)$$

Here, $\mathbf{I}$ represents the identity operator.

We insert the identity operator $\mathbf{I} = [\tilde{\nabla} \cdot]^{-1} \tilde{\nabla} \cdot$ after $\mathbf{E}$ in equation 1:

$$\partial_t^2 \mathbf{u} = \mathbf{E} [\tilde{\nabla} \cdot]^{-1} \tilde{\nabla} \cdot \mathbf{u} + \frac{1}{\rho} (\mathbf{f} + \nabla \cdot \mathbf{m}). \quad (5)$$

Substituting the relation $[\tilde{\nabla} \cdot]^{-1} = \tilde{\nabla}$ given by equation 4 into equation 5, we have:

$$\partial_t^2 \mathbf{u} = \mathbf{E} \tilde{\nabla} \tilde{\nabla} \cdot \mathbf{u} + \frac{1}{\rho} (\mathbf{f} + \nabla \cdot \mathbf{m}). \quad (6)$$

Applying the normalized divergence operator $\tilde{\nabla} \cdot$ on both sides of equation 6 and denoting $P = \tilde{\nabla} \cdot \mathbf{u}$, we finally obtain the pseudo-elastic pure P-mode wave equation:

$$\partial_t^2 P = \tilde{\nabla} \cdot \mathbf{E} \tilde{\nabla} P + \tilde{\nabla} \cdot \frac{1}{\rho} (\mathbf{f} + \nabla \cdot \mathbf{m}). \quad (7)$$

Ignoring the source term, we substitute a harmonic plane wave $e^{i\omega(n_x x/V + n_y y/V + n_z z/V - t)}$ defined by the direction vector (n_x, n_y, n_z) into equation 7 and obtain the phase velocity of the P-waves $\rho V^2 = \lambda + 2\mu$, indicating that the resulting pseudo-elastic wave equation does describe the propagation of a single mode of pressure wavefield.

The term $\tilde{\nabla} \cdot \frac{1}{\rho}(\mathbf{f} + \nabla \cdot \mathbf{m})$ clearly demonstrates that the divergence of the force-vector is used as the source for the pressure wavefield. Alternatively, we can easily add a scalar source to the pressure wavefield P without referring to the force-vector.

The derived equation 7 is also physically interpretable, specially the term $\tilde{\nabla} \cdot \mathbf{E} \tilde{\nabla} P$ describes how the pressure wavefield interacts with the elastic medium. Reading from right to left: the pressure wavefield P is decomposed to vertical and horizontal components by applying the normalized gradient operator $\tilde{\nabla}$. The resulting two components mimic the corresponding displacements in elastic media: at this stage, the wavefield consists of P-waves only, as such displacements are computed by the divergence of the potential P , and thus, they are irrotational. Those P-waves then interact with the elastic medium parameters through the operator $\mathbf{E}$ (see equation 1, this operator corresponds to the same one used in the conventional elastic wave equation). The resulting wavefield would contain both P- and S- waves. The final step is applying a normalized divergence operator $\tilde{\nabla}$, and the wavefield is projected along its wavefront normal direction, and the S-waves excited by the elastic operator $\mathbf{E}$ will be filtered out accordingly. Thus, as a whole, the equation given by equation 7 accurately describes a pure P-wave mode propagation, and the interaction between the P-waves and the medium is elastic. Thus, we can conclude that such pseudo-elastic pure P-mode wave equation, in each time step, performs the pressure wavefield simulation and separation simultaneously.

Solving the resulting pseudo-elastic pure P-mode wave equation 7 is similar to solving the elastic wave equation 1 when using the operator $\mathbf{E}$. However, the new wave equation requires the evaluation of the normalized gradient and divergence operators. Those operators are expressed as pseudo-differential operators in equation 7. Numerically, those pseudo-differential operators can be computed accurately and efficiently in the wavenumber domain using Fast Fourier Transform (FFT) and proper care is required to achieve stability. Specially, in our computation, in the wavenumber domain, the normalized divergence operator is implemented as $\tilde{\nabla} \cdot \sim [\frac{ik_x}{\sqrt{k_x^2+k_y^2+k_z^2}}, \frac{ik_y}{\sqrt{k_x^2+k_y^2+k_z^2}}, \frac{ik_z}{\sqrt{k_x^2+k_y^2+k_z^2}}]$, while the normalized gradient operator is implemented as $\tilde{\nabla} \sim -[\frac{ik_x}{\sqrt{k_x^2+k_y^2+k_z^2}}, \frac{ik_y}{\sqrt{k_x^2+k_y^2+k_z^2}}, \frac{ik_z}{\sqrt{k_x^2+k_y^2+k_z^2}}]^T$. We can verify that with the suggested implementation, equation 4 still holds. Although we have those extra computation of the FFTs, the elastic modeling by the pseudo-elastic pure P-mode wave equation does not require finer grids for the S-waves. When the ratio between V_p and V_s is high, the computational advantage of our method becomes significant. In the "Discussion" section, we will elaborate on the computational advantage of our proposed method in detail.

We now show the derivation of the pseudo-elastic pure P-mode wave equation for the second-order elastic wave equation expressed by displacements, leading to a pseudo-elastic wave equation expressed as an second-order wave equation using a scalar pressure wavefield P . The first-order velocity-stress formulation for elastic modeling is also popular for its robustness and accuracy (Virieux, 1986; Levander, 1988). Adopting similar strategy, we can obtain a first-order pseudo-elastic pure P-mode wave equation expressed by the pressure wavefield P and the stress $\boldsymbol{\tau} = [\tau_{ij}], i, j = x, y, z$ (Appendix A shows the detailed

derivations):

$$\begin{aligned}
\partial_t P &= \frac{\partial_x}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \left[\frac{1}{\rho} (\partial_x \tau_{xx} + \partial_y \tau_{xy} + \partial_z \tau_{xz} + f_x) \right] \\
&+ \frac{\partial_y}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \left[\frac{1}{\rho} (\partial_x \tau_{xy} + \partial_y \tau_{yy} + \partial_z \tau_{yz} + f_y) \right] \\
&+ \frac{\partial_z}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \left[\frac{1}{\rho} (\partial_x \tau_{xz} + \partial_y \tau_{yz} + \partial_z \tau_{zz} + f_z) \right], \\
\partial_t \tau_{xx} &= (\lambda + 2\mu) \frac{\partial_x^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \lambda \frac{\partial_y^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \lambda \frac{\partial_z^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \partial_t m_{xx}, \\
\partial_t \tau_{yy} &= \lambda \frac{\partial_x^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + (\lambda + 2\mu) \frac{\partial_y^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \lambda \frac{\partial_z^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \partial_t m_{yy}, \quad (8) \\
\partial_t \tau_{zz} &= \lambda \frac{\partial_x^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \lambda \frac{\partial_y^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + (\lambda + 2\mu) \frac{\partial_z^2 P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \partial_t m_{zz}, \\
\partial_t \tau_{xy} &= 2\mu \frac{\partial_x \partial_y P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \partial_t m_{xy}, \\
\partial_t \tau_{xz} &= 2\mu \frac{\partial_x \partial_z P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \partial_t m_{xz}, \\
\partial_t \tau_{yz} &= 2\mu \frac{\partial_y \partial_z P}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \partial_t m_{yz}.
\end{aligned}$$

Similar to the conventional first-order elastic wave equation, the pseudo-elastic wave equation 8 can be solved using a staggered-grid finite-difference method.

THE PSEUDO-ELASTIC ACOUSTIC WAVE EQUATION = AN ACOUSTIC WAVE EQUATION + AN ELASTIC CORRECTION TERM

In this section, we further analyze the derived equation 7 and demonstrate that the resulting pseudo-elastic pure P-mode wave equation can be expressed as an acoustic wave equation plus a term for correcting the elastic effect.

In the derivation of equation 7, we introduce the (normalized) gradient and divergence operators as shown in equation 3. Other types of (normalized) gradient and divergence operators can be used. For example, we can use:

$$\tilde{\nabla}' = \nabla = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix}, \quad \tilde{\nabla}' \cdot = \begin{bmatrix} \frac{\partial_x}{\partial_x^2 + \partial_y^2 + \partial_z^2} & \frac{\partial_y}{\partial_x^2 + \partial_y^2 + \partial_z^2} & \frac{\partial_z}{\partial_x^2 + \partial_y^2 + \partial_z^2} \end{bmatrix}. \quad (9)$$

The relation of equation 4 still holds, and we can derive the pseudo-elastic pure P-mode wave equation in the same form as in equation 7 (ignoring the source term):

$$\partial_t^2 P = \tilde{\nabla}' \cdot \mathbf{E} \tilde{\nabla}' P. \quad (10)$$

Equation 10 describes a pure pressure wave propagation in an elastic medium as well. The only difference is that the pressure wavefield P computed by equation 10 is a scaled version (the scaling factor equals $\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}$) of that by equation 7. Appendix B shows that using the parameterization given by velocity and density, and through vector and tensor operations, we show that equation 10 is equivalent to an acoustic wave equation plus a term for the elastic correction:

$$\frac{1}{\rho V_p^2} \partial_t^2 P' = \nabla' \cdot \frac{1}{\rho} \nabla' P' + \chi, \quad (11)$$

where $P' = \rho V_p^2 P$, and χ is the correction term for the elastic effect:

$$\begin{aligned} \chi = & \left[\partial_x \frac{1}{\rho} \partial_y 2\rho V_s^2 \partial_y \partial_x + \partial_y \frac{1}{\rho} \partial_x 2\rho V_s^2 \partial_x \partial_y - \partial_x \frac{1}{\rho} \partial_x 2\rho V_s^2 \partial_y^2 - \partial_y \frac{1}{\rho} \partial_y 2\rho V_s^2 \partial_x^2 \right. \\ & \partial_x \frac{1}{\rho} \partial_z 2\rho V_s^2 \partial_z \partial_x + \partial_z \frac{1}{\rho} \partial_x 2\rho V_s^2 \partial_x \partial_z - \partial_x \frac{1}{\rho} \partial_x 2\rho V_s^2 \partial_z^2 - \partial_z \frac{1}{\rho} \partial_z 2\rho V_s^2 \partial_x^2 \\ & \left. \partial_y \frac{1}{\rho} \partial_z 2\rho V_s^2 \partial_z \partial_y + \partial_z \frac{1}{\rho} \partial_y 2\rho V_s^2 \partial_y \partial_z - \partial_y \frac{1}{\rho} \partial_y 2\rho V_s^2 \partial_z^2 - \partial_z \frac{1}{\rho} \partial_z 2\rho V_s^2 \partial_y^2 \right] \frac{1}{\partial_x^2 + \partial_y^2 + \partial_z^2} \frac{P'}{\rho V_p^2}, \end{aligned} \quad (12)$$

and V_p and V_s are the P- and S-wave velocities, respectively.

From the elastic correction term χ of equation 12, we can observe that the radiation pattern for V_p as well as $\frac{1}{\rho}$ would always be zero and the compensation for the elastic scattering is only associated with the shear modulus $G = \rho V_s^2$. For constant G , the elastic correction term χ is zero. Otherwise, χ would correct for the elastic effect due to heterogeneity of the density ρ and shear-wave velocity V_s . In FWI, we usually consider constant density, and thus, the corresponding elastic correction term is simplified:

$$\begin{aligned} \chi_{\text{const}-\rho} = & [\partial_x \partial_y 4V_s^2 \partial_y \partial_x - \partial_x^2 2V_s^2 \partial_y^2 - \partial_y^2 2V_s^2 \partial_x^2 \\ & \partial_x \partial_z 4V_s^2 \partial_z \partial_x - \partial_x^2 2V_s^2 \partial_z^2 - \partial_z^2 2V_s^2 \partial_x^2 \\ & \partial_y \partial_z 4V_s^2 \partial_z \partial_y - \partial_y^2 2V_s^2 \partial_z^2 - \partial_z^2 2V_s^2 \partial_y^2] \frac{1}{\partial_x^2 + \partial_y^2 + \partial_z^2} \frac{P'}{V_p^2}. \end{aligned} \quad (13)$$

The term χ can also be used to correct for the missing elastic effect in the anisotropic acoustic wave equation (Alkhalifah, 2000). We need to replace the part of the isotropic acoustic wave equation in equation 11 with an anisotropic one. There are many options for an anisotropic acoustic wave equation. Li and Stovas (2020, 2021) propose a scalar pure P-waves acoustic VTI equation without SV-wave artifacts, which is derived by defining the SV-waves phase velocity in a form that eliminates the square root term in the P-waves dispersion relationship (not by setting the vertical S-waves velocity to be zero). However, it involves higher-order spatial derivatives and inverse of the Laplacian. We incorporate the density in the equation, such that their 2D acoustic VTI equation with the elastic correction term is given by:

$$\begin{aligned} \frac{1}{V_{p0}^2 \rho} \partial_t^2 P = & \left[(1 + 2\epsilon) \partial_x \frac{1}{\rho} \partial_x + \partial_z \frac{1}{\rho} \partial_z \right] P \\ & - 2(\epsilon - \delta) \left[\left(\frac{\partial_z^2}{\partial_x^2 + \partial_z^2} \right)^3 \partial_x \frac{1}{\rho} \partial_x + (2 - 2\delta) \left(\frac{\partial_x \partial_z}{\partial_x^2 + \partial_z^2} \right)^3 \partial_x \frac{1}{\rho} \partial_z + (1 - 2\epsilon) \left(\frac{\partial_x^2}{\partial_x^2 + \partial_z^2} \right)^3 \partial_z \frac{1}{\rho} \partial_z \right] P \\ & + (\partial_x \frac{1}{\rho} \partial_z 2\rho V_s^2 \partial_z \partial_x + \partial_z \frac{1}{\rho} \partial_x 2\rho V_s^2 \partial_x \partial_z - \partial_x \frac{1}{\rho} \partial_x 2\rho V_s^2 \partial_z^2 - \partial_z \frac{1}{\rho} \partial_z 2\rho V_s^2 \partial_x^2) \frac{1}{\partial_x^2 + \partial_z^2} \frac{P}{\rho V_{p0}^2}, \end{aligned} \quad (14)$$

where V_{p0} is the vertical P-waves velocity, ϵ and δ are Thomsen's anisotropic parameters.

We usually consider constant density, in which equation 14 reduces to:

$$\begin{aligned} \frac{1}{V_{p0}^2} \partial_t^2 P = & [(1 + 2\epsilon) \partial_x^2 + \partial_z \partial_z^2] P \\ & - 2(\epsilon - \delta) \left[\left(\frac{\partial_z^2}{\partial_x^2 + \partial_z^2} \right)^3 \partial_x^2 + (2 - 2\delta) \left(\frac{\partial_x \partial_z}{\partial_x^2 + \partial_z^2} \right)^3 \partial_x \partial_z + (1 - 2\epsilon) \left(\frac{\partial_x^2}{\partial_x^2 + \partial_z^2} \right)^3 \partial_z^2 \right] P \\ & + (\partial_x \partial_z 4V_s^2 \partial_z \partial_x - \partial_x^2 2V_s^2 \partial_z^2 - \partial_z^2 2V_s^2 \partial_x^2) \frac{1}{\partial_x^2 + \partial_z^2} \frac{P}{V_{p0}^2}. \end{aligned} \quad (15)$$

If we further set the anisotropic parameters to zero, i.e., $\epsilon = \delta = 0$, equation 15 reduces to the 2D version of equation 11 for the pure P-waves in a constant density isotropic elastic medium:

$$\frac{1}{V_{p0}^2} \partial_t^2 P = (\partial_x^2 + \partial_z^2) P + (\partial_x \partial_z 4V_s^2 \partial_z \partial_x - \partial_x^2 2V_s^2 \partial_z^2 - \partial_z^2 2V_s^2 \partial_x^2) \frac{1}{\partial_x^2 + \partial_z^2} \frac{P}{V_{p0}^2}. \quad (16)$$

We will elaborate on the anisotropy extension of our method in a separate paper.

We have presented the pseudo-elastic pure P-mode wave equation for general heterogeneous elastic media and demonstrated that it could be decomposed into an acoustic wave equation plus a term for correcting the elastic effect. We also demonstrate that such elastic correction term can also be adapted to anisotropic acoustic wave equation. In the following section, we will justify the label "pseudo-elastic": the proposed pure P-mode wave equation admits the same dynamic behavior as that of the PP mode of the original elastic wave equation. In the next section, we will demonstrate this by analyzing the radiation patterns. In the example section, the accuracy of the AVO of our pure P-mode wave equation will be further verified by numerical examples.

RADIATION PATTERN ANALYSIS

In this section, we derive the radiation patterns for the pseudo-elastic pure P-mode wave equation. In the derivation, we use the parameterization given by Lamé parameters and density and adopt the methodology of Stolt and Weglein (2012) for analyzing the radiation patterns.

We introduce two sets of parameters: (λ', μ', ρ') and (λ, μ, ρ) . Specifically, (λ, μ, ρ) corresponds to a smooth background medium while (λ', μ', ρ') corresponds to a medium with considerable variations. The full and background propagators for the proposed pseudo-elastic pure P-mode wave equation (equation 7) are:

$$\mathcal{L}' = \omega^2 + \tilde{\nabla} \cdot \mathbf{E}' \tilde{\nabla}, \quad \text{and} \quad \mathcal{L} = \omega^2 + \tilde{\nabla} \cdot \mathbf{E} \tilde{\nabla}, \quad (17)$$

where, $\mathbf{E}'$ has the same form as $\mathbf{E}$ in equation 2, but with parameters (λ', μ', ρ') instead.

The scattering potential $\mathcal{V}$ can be computed as:

$$\mathcal{V} = \mathcal{L}' - \mathcal{L} = \tilde{\nabla} \cdot \mathbf{E}' \tilde{\nabla} - \tilde{\nabla} \cdot \mathbf{E} \tilde{\nabla}. \quad (18)$$

We define the medium perturbations as:

$$\delta\lambda = \lambda' - \lambda, \quad \delta\mu = \mu' - \mu, \quad \delta\rho = \rho' - \rho. \quad (19)$$

With a first-order Born approximation, the scattering potential in equation 18 is given by:

$$\mathcal{V} = \frac{1}{\rho} \left[-\tilde{\nabla} \cdot \delta\rho \mathbf{E} \tilde{\nabla} + \tilde{\nabla} \cdot \mathbf{D} \tilde{\nabla} \right], \quad (20)$$

where $\mathbf{D}$ corresponds to the scattering due to perturbations in the Lamé parameters:

$$\mathbf{D} = \begin{bmatrix} \partial_x(\delta\lambda + 2\delta\mu)\partial_x + \partial_y\delta\mu\partial_y + \partial_z\delta\mu\partial_z & \partial_x\delta\lambda\partial_y + \partial_y\delta\mu\partial_x & \partial_x\delta\lambda\partial_z + \partial_z\delta\mu\partial_x \\ \partial_y\delta\lambda\partial_x + \partial_x\delta\mu\partial_y & \partial_x\delta\mu\partial_x + \partial_y(\delta\lambda + 2\delta\mu)\partial_y + \partial_z\delta\mu\partial_z & \partial_y\delta\lambda\partial_z + \partial_z\delta\mu\partial_y \\ \partial_z\delta\lambda\partial_x + \partial_x\delta\mu\partial_z & \partial_z\delta\lambda\partial_y + \partial_y\delta\mu\partial_z & \partial_x\delta\mu\partial_x + \partial_y\delta\mu\partial_y + \partial_z(\delta\lambda + 2\delta\mu)\partial_z \end{bmatrix}. \quad (21)$$

Suppose that the incident and scattered wavenumbers are $\mathbf{k}_i = [k_i^x, k_i^y, k_i^z]$ and $\mathbf{k}_r = [k_r^x, k_r^y, k_r^z]$, respectively. To derive the radiation patterns, we adopt the following rules for the differential operator substitutions: for the differential operator in the right side of the parameter perturbation, we perform the substitution $\nabla \rightarrow i\mathbf{k}_i = i\omega\hat{\mathbf{k}}_i/V$; while for the differential operator in the left side of the parameter perturbation, we perform the substitution $\nabla \rightarrow i\mathbf{k}_r = i\omega\hat{\mathbf{k}}_r/V$. The normalized gradient operator $\tilde{\nabla}$ is evaluated as $\tilde{\nabla} \rightarrow \hat{\mathbf{k}}$, accordingly. Here, ω is the angular frequency and V is the P-wave velocity; $\hat{\mathbf{k}}_i$ and $\hat{\mathbf{k}}_r$ are the unit direction vectors and they are expressed as $\hat{\mathbf{k}}_i = [\hat{k}_x^i, \hat{k}_y^i, \hat{k}_z^i]$ and $\hat{\mathbf{k}}_r = [\hat{k}_x^r, \hat{k}_y^r, \hat{k}_z^r]$, for the incident and reflected waves, respectively. With the suggested operator substitution, the scattering potential $\mathcal{V}$ in equation 20 becomes:

$$\mathcal{V}(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_r) = \left[a_\lambda(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_r)\delta\lambda + a_\mu(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_r)\delta\mu + a_\rho(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_r)\delta\rho \right], \quad (22)$$

where a_λ , a_μ and a_ρ are the formulas of the radiation patterns for the Lamé parameters and density, given respectively as:

$$\begin{aligned} a_\lambda(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_r) &= -\frac{\omega^2}{\rho V^2}, \\ a_\mu(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_r) &= -\frac{2\omega^2}{\rho V^2} \left[(\hat{k}_x^r)^2 (\hat{k}_x^i)^2 + (\hat{k}_y^r)^2 (\hat{k}_y^i)^2 + (\hat{k}_z^r)^2 (\hat{k}_z^i)^2 + 2(\hat{k}_x^r \hat{k}_z^r \hat{k}_x^i \hat{k}_z^i + \hat{k}_x^r \hat{k}_y^r \hat{k}_x^i \hat{k}_y^i + \hat{k}_y^r \hat{k}_z^r \hat{k}_y^i \hat{k}_z^i) \right], \\ &= -\frac{2\omega^2}{\rho V^2} (\hat{\mathbf{k}}^r \cdot \hat{\mathbf{k}}^i)^2, \\ a_\rho(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_r) &= -\frac{\omega^2}{\rho^2 V^2} (\lambda + 2\mu) (\hat{k}_x^r \hat{k}_x^i + \hat{k}_y^r \hat{k}_y^i + \hat{k}_z^r \hat{k}_z^i) \\ &= \frac{\omega^2}{\rho^2 V^2} (\lambda + 2\mu) \hat{\mathbf{k}}^r \cdot \hat{\mathbf{k}}^i. \end{aligned} \quad (23)$$

For reflections, we have $\hat{\mathbf{k}}^r \cdot \hat{\mathbf{k}}^i = -\cos\theta$, where θ is the opening angle between the incident and reflected waves, equation 23 can be further simplified and equation 22 becomes:

$$\mathcal{V}(\theta) = -\frac{\omega^2}{\rho V^2} \left[\delta\lambda + 2\cos^2\theta\delta\mu + (\lambda + 2\mu)\cos\theta\frac{\delta\rho}{\rho} \right]. \quad (24)$$

We observe that the resulting radiation pattern is exactly that of the PP reflection of the conventional elastic wave equations (Wu and Aki, 1985). Thus, we have shown that P-waves given by the pseudo-elastic wave equation (equation 7) shares the same scattering properties as the PP mode of the original elastic wave equation.

Instead of the parameterization by (λ, μ, ρ) , in inversion, we usually use the parameterization of P- and S-waves velocities and density, i.e., (v_p, v_s, ρ) . We have following relations for the parameters perturbations between those two sets of parameterization:

$$\begin{aligned}\delta\lambda &= 2\rho v_p \delta v_p - 4\rho v_s \delta v_s + (v_p^2 - 2v_s^2) \delta\rho, \\ \delta\mu &= 2\rho v_s \delta v_s - v_s^2 \delta\rho.\end{aligned}\tag{25}$$

Substituting equation 25 into equation 24, we obtain:

$$\mathcal{V}(\theta) = -\frac{\omega^2}{\rho V^2} [2\rho v_p \delta v_p - 4\rho v_s \sin^2 \theta \delta v_s + (v_p^2 + \cos \theta v_p^2 - 2 \sin^2 \theta v_s^2) \delta\rho]\tag{26}$$

In Figure 1, we plot the resulting radiation patterns given by equations 24 and 26, respectively. From Figure 1b, we can easily observe that reflections related to the P-wave velocity has an isotropic radiation patterns while the S-wave velocity scatters P-waves at intermediate scattering angles (achieves maximum values around 90° opening angle) while the density is sensitive to near offset reflections only. We admit that the radiation pattern analysis is based on the first-order Born approximation. In the next "Example" section, using numerical examples, we show that the proposed pseudo-elastic wave equation actually produces the reflections with the same AVO effect as that by the original elastic wave equation regardless of the medium contrast.

EXAMPLES

In all the examples presented below, for pseudo-elastic pure P-waves computation, equations 8 and 7 are used as the first- and second-order wave equation, respectively, while the corresponding elastic wave equations are equations A-1 and 1. We use second-order time integration and eighth-order finite-difference approximation for evaluating the spatial derivatives for both the first- and second-order wave equations. Normalized gradient and divergence operators are implemented in the wavenumber domain using FFT. When we refer to the displacement for the pseudo-elastic wave equation, it corresponds to $\tilde{\nabla}P$, where $\tilde{\nabla}$ is the normalized gradient operator defined in equation 3. The pressure wavefield for the elastic wave equation is evaluated using the normalized divergence of the displacement vector: $\tilde{\nabla} \cdot \mathbf{u}$. The displacements for the first-order wave equation are obtained through time integration of the particle velocity. Two different types of sources are used: when we refer to a vertical source, we mean representing the source by the component f_z of the force-density vector $\mathbf{f}$ while the pressure source denotes the source applied to the diagonal components of the moment-density tensor $\mathbf{m}$, i.e., the normal stress.

Wavefields in a homogeneous model

In the first example, we demonstrate that the proposed pseudo-elastic wave equation can simulate a pure P-wave without worrying about the S-wave dispersion issue. The homogeneous model is setup with parameters: $V_p = 2500$ m/s, $V_s = 2500/\sqrt{3}$ m/s and $\rho = 2\text{g/cm}^3$. The model is discretized using 256×256 grids at a spacing of $dx = dz = 10$ m. We choose a 15 Hz Ricker wavelet and simulate for 0.4 s with time increments of 0.001 s.

Figures 2a and 2b show the snapshots (at 0.4 s) of the vertical displacement from the

first-order and second-order elastic wave equations, respectively. Figures 2c and 2d show the corresponding snapshots using the first- and second-order pseudo-elastic pure P-mode wave equation. We observe that the pseudo-elastic wave equation, either first- or second-order, can simulate pure P-waves free of S-waves. Figure 3 shows the vertical profile of the snapshots in Figure 2 in the middle of the model. The good fit of the P-wave amplitudes demonstrates the accuracy of the proposed pseudo-elastic wave equation for transmission wavefields. To test the sensitivity of the result to S-wave velocity, we reduce the S-wave velocity to 500 m/s and show the corresponding snapshots of the vertical displacement in Figure 4. Due to the lower S-wave velocity, the elastic wave equation admits numerical dispersion in the wavefield (the second-order one shows more severe dispersion). In Figure 5, we further set the S-wave velocity to zero. The second-order elastic wave equation shows strong artifacts, while the pseudo-elastic equation, either first-order or second-order, provides stable modeling of the P-waves, free of S-wave dispersion.

Reflections at an interface

In the second example, we investigate the reflections from one interface and demonstrate that the proposed pseudo-elastic pure P-mode wave equation produces reflections with the same AVO as that of the PP-reflections by the elastic wave equation.

We consider one interface at a depth of 2 km. We set the upper layer with the parameters: $V_p = 2000$ m/s, $V_s = 1000$ m/s and $\rho = 2$ g/cm³. To represent a relatively strong elastic contrast, we set the bottom layer with the parameters 50% higher than the upper layer given by: $V_p = 3000$ m/s, $V_s = 1500$ m/s and $\rho = 3$ g/cm³. We test the cases in which one of the parameters are higher in the bottom layer or all of them. In the simulation, we

use a pressure source with a 15 Hz Ricker wavelet, and the pressure wavefields are recorded for both the second-order elastic wave equation 1 and the pseudo-elastic pure P-mode wave equation 7. For both methods, we use the same time and space sampling intervals of $dt = 0.001$ s and $dx = dz = 10$ m, respectively. In the computation, a maximum offset of 30 km is recorded. Thus, the offset to depth ratio extends to 15:1, which is reasonably large and covers the range for any realistic acquisition. We apply a gain proportional to 0.5 power of the time to the record and flatten the first arrivals to help in visualizing and analyzing the results.

Figure 6 corresponds to the result for which only V_p has contrast, i.e., we set the V_p to 3000 m/s in the bottom layer, while V_s and ρ are the same as the upper layer. Accordingly, Figures 7 and 8 are the results corresponding to the models that have similar contrasts in V_s and ρ , respectively.

For reflections due to the contrast in V_p (Figure 6), we observe that the pseudo-elastic pure P-mode wave equation (red curves) accurately produces the P-waves reflections with the same amplitude as that by the elastic wave equation (dotted blue curves). When only V_s has a contrast, Figure 7a shows that the pseudo-elastic pure P-mode wave equation can produce the reflections with the same AVO features as that by the elastic wave equation (The amplitude becomes larger with increased offset, and this is consistent with our radiation pattern analysis (see Figure 1b)). When we take a closer look at the traces in Figure 7b, we see small amplitude differences at the rather large offsets (e.g., 20 km or 30 km). We attribute such small errors to numerical roundoff in the approximation of the elastic correction operator (equation 12). Similarly, Figure 8 shows the result for the density ρ with a contrast, and the pseudo-elastic wave equation and elastic wave equation produces the reflections with the same radiation patterns (the larger the offset, the smaller the amplitude

and this also agrees with our radiation pattern analysis in Figure 1b) and their amplitudes match with high accuracy. In this case, it seems that at far offsets, the errors between the elastic wave equation and the pseudo-elastic one is large (Figure 8b). However, due to the nature of the radiation pattern for density, the amplitudes at larger offsets are rather small (note we apply gain here), and the errors between these two methods are mainly numerical errors and can be considered negligible in many applications.

Figure 9 shows the shot gathers when we set all the parameters of the bottom layer to be 150% of the upper layer. To demonstrate the effect of elasticity, we also show the result from acoustic modeling (setting V_s to be zero in both layers). For the current setup, the critical angle corresponds to an offset of about 3.5 km. In Figure 9a, we observe amplitude differences between acoustic and elastic (including the pseudo-elastic) modeling results for the pre-critical reflections and phase difference (phase rotation) for the post-critical reflections. This observation is consistent with the analytic AVO response obtained from solving the Zoeppritz equations (Figure 10). In the selected traces shown in Figure 9b, such amplitude and phase differences between the acoustic and elastic modeling are more obvious, while the records by the elastic and pseudo-elastic modeling show a high level of agreement in both amplitude and phase. Considering that we perform the comparison with relatively strong elasticity contrasts and large offset to depth ratios, we conclude that the reflections produced by the pseudo-elastic wave equation exhibit the same AVO features as the PP reflections by the elastic wave equation. As demonstrated, the elasticity provides by the elastic correction term of equation 12 can compensate for the amplitude and phase differences for the pre-and post-critical reflections. Such a remedy is essential for performing FWI in areas where elastic effects dominates.

Wave propagation in multi-layered model

In this example, we increase the complexity of the model and consider the multi-layer model shown in Figure 11. We want to evaluate the behavior of the wavefields propagating through multi-interface for the pseudo-elastic pure P-mode wave equation. A 25 Hz peak frequency Ricker wavelet is used as a vertical source and placed in the middle on the surface. The space and time sampling values are set to be relatively small, i.e., $dx = dz = 2.5$ m and $dt = 0.00025$ s to avoid S-wave dispersion for the elastic wave equation. Figures 12 and 13 show snapshots of the vertical displacement for the pseudo-elastic pure P-mode wave equation and the elastic wave equation, respectively. From the snapshots, we can easily see the complicated wavefield conversions simulated by the elastic wave equation, while the pseudo-elastic pure P-mode wave equation maintains a single wave-mode propagation for both transmissions and reflections. Figure 14a shows the resulting common shot record of the vertical displacement with receivers placed at the surface, and it further demonstrates the complexity of the wavefield simulated by the elastic wave equations (the dotted blue curves) and the simplicity of the wavefield by the pseudo-elastic wave equations (red curves). Note the area marked with blue rectangles, the P-waves produced by the elastic and pseudo-elastic wave equations fit each other well. Figure 14b shows one trace at offset 0.5 km, and we can see that both reflections (P1 corresponds to the first interface and P2 for the second interface) are well simulated by the pseudo-elastic wave equation and fit the one corresponding to the elastic wave equation. Note that the P2 reflection encountered the first interface twice, and yet the pseudo-elastic wave equation still maintains the simulation accuracy and produces reflections with the same phase and amplitude as those by the elastic wave equation.

SEAM 2D salt model

In the last example, we evaluate our method on a realistic SEAM 2D salt model. This model represents a realistic earth model of a salt canopy region of the Gulf of Mexico, complete with fine-scale stratigraphy. As shown in Figure 15, the model extends 15 km in depth and 35 km laterally. The grid size for simulation is $10 \text{ m} \times 10 \text{ m}$ (x, z). The time interval is 0.001 s, and we simulate for 12 s. A 15 Hz Ricker wavelet is used as a pressure source and placed at 20 m depth. We record the pressure wavefields with receivers placed at a depth of 10 m. Considering that the first-order equation is more robust in dealing with large contrast (such as the salt body in the model), we test and evaluate the elastic wave equation and the pseudo-elastic wave equation based on the first-order schemes (i.e., equations 8 and A-1). We also perform modeling using acoustic modeling for comparison (the acoustic wave equation is obtained by setting the shear-wave velocity in the pseudo-elastic wave equation to zero.)

Figures 16-18 show common shot gathers produced by the elastic wave equation, pseudo-elastic pure P-mode wave equation, and the acoustic wave equation, respectively. We can observe that all gathers share similar features for the first arrivals and salt body reflections. While considering the varying assumptions used in the simulation, there are significant differences in reflections for later arrivals. Figure 19 shows the difference between elastic and acoustic modeling, and Figure 20 shows the difference between elastic and pseudo-elastic modeling. Those two figures are plotted at the same scale. Overall, we observe that the difference between the elastic and pseudo-elastic is smaller than that between the elastic and acoustic (for example, the area marked by the ovals). To perform a more qualitative analysis of the discrepancy between different modeling methods, we compute

the L2 norm of the difference of each trace and show the result in Figure 21a. The pseudo-elastic modeling admits relatively smaller errors across all the offset. We can also observe that in the ± 5 km, the performance of the pseudo-elastic is relatively good and this is mainly due to the strong elasticity effect introduced by the salt body in the shallow area. We also use another metric based on the normalized cross-correlation (see Figure 21b). The same conclusion holds: the pseudo-elastic modeling produces reflections closer to the elastic dataset than acoustic modeling. We can also notice that in both Figures 21a and 21b, there is a spike around 2.4 km offset, indicating a large disagreement between the elastic and acoustic modeling at this position. We plot the trace at that position in Figure 22, and we can note that pseudo-elastic modeling produces first arrivals closer to those from elastic modeling, while the acoustic modeling is less accurate with larger amplitude. For the latter arrivals around 2.5 s, though the pseudo-elastic wave equation produces reflections with large amplitudes, its phase is consistent with that of the elastic modeling. In contrast, the acoustic modeling shows the opposite phase. This explains the spike in Figure 21a and further demonstrates the importance of including elasticity. Though the pseudo-elastic wave equation only produces accurate PP reflections, it shows considerable improvements in fitting the elastic dataset compared to the acoustic modeling. Such improvements will be manifested in elastic full-waveform inversion, leading to fast convergence and a more accurate model with fewer artifacts.

DISCUSSION

In the discussion section, we present the AVO formula for the P-waves reflections to further demonstrate the effect of the shear modulus, and show that the quasi-elastic wave equation proposed by Feng and Schuster (2019) is the Born equation of the pseudo-elastic pure P-

mode wave equation proposed here. We will comment on the computational advantage of our method and include a short remark on the numerical stability and dispersion associated with modeling using our method. At last, we will discuss the potential application of the proposed method.

The influence of shear modulus G on Amplitude Versus offset (AVO)

For small contrasts in the elastic properties, the approximated fully linearized AVO for the PP reflections can be written as (Rüger, 2002):

$$R_P(i) = \frac{1}{2} \frac{\Delta Z}{\bar{Z}} + \frac{1}{2} \left\{ \frac{\Delta V_p}{\bar{V}_p} - \left(\frac{2\bar{V}_s}{\bar{V}_p} \right)^2 \frac{\Delta G}{\bar{G}} \right\} \sin^2 i + \frac{1}{2} \frac{\Delta V_p}{\bar{V}_p} \sin^2 \theta \tan^2 i, \quad (27)$$

where $Z = \rho V_p$ is the impedance, $G = \rho \mu^2$ is the shear modulus and i denotes incidence angle. Over-line denotes average values of the upper and lower layer parameters, while Δ denotes the difference (perturbation). We can use other parameterizations, but the one used here can help us to find the effective range of incident angle for the shear modulus. Also note that the AVO formula is originally obtained from the elastic wave equation, however, as we have shown in the example section, the pseudo-elastic wave and elastic wave equation shares the same AVO, and thus, the AVO formula given by equation 27 applies to the pseudo-elastic wave equation as well.

By equation 27, we can identify that the influence of the shear modulus G is mainly on the incidence angles i of moderate values (middle offsets). For small i , $\sin i \approx 0$, the first term dominates while for large i , $\sin^2 i \tan^2 i \gg \sin^2 i$, and thus, the third term would be the leading term. As demonstrated in equation 11, the pseudo-elastic wave equation introduces an elasticity-correction term for the acoustic wave equation. Such a correction term would include the elastic effect related to the shear modulus G . As demonstrated by equation

27, the influence of the shear modulus G is limited to moderate incidence angles (middle offsets). This region covers the post- and pre-critical reflection zone. These findings further support our results in the example section (see Figure 9).

The Born formula of the pseudo-elastic pure P-mode wave equation

The pseudo-elastic pure P-mode wave equation given in equation B-9 can be simplified:

$$\begin{aligned}
 \partial_t^2 P &= \nabla \cdot \left\{ \frac{1}{\rho} [\nabla(\lambda + 2\mu)P] \right\} + \nabla \cdot \left\{ \frac{1}{\rho} [2(\nabla\mu \cdot \nabla)\nabla - 2\nabla\mu\Delta] \frac{P}{\Delta} \right\} \\
 &= \nabla \cdot \left[\frac{1}{\rho} \nabla(\lambda P) \right] + 2\nabla \cdot \left[\frac{1}{\rho} \nabla(\mu P) \right] + 2\nabla \cdot \left[\frac{1}{\rho} (\nabla\mu \cdot \nabla) \frac{\nabla}{\Delta} P \right] - 2\nabla \cdot \left[\frac{1}{\rho} (\nabla\mu)P \right] \\
 &= \nabla \cdot \left[\frac{1}{\rho} \nabla(\lambda P) \right] + 2\nabla \cdot \left[\frac{1}{\rho} (\nabla\mu \cdot \nabla) \frac{\nabla}{\Delta} P \right] + 2\nabla \cdot \left[\frac{1}{\rho} \mu \nabla P \right] \\
 &= \nabla \cdot \left[\frac{1}{\rho} \nabla(\lambda P) \right] + 2\nabla \cdot \left[\frac{1}{\rho} \nabla\mu \cdot \frac{\nabla\nabla}{\Delta} P \right] + 2\nabla \cdot \left[\frac{1}{\rho} \mu \nabla \cdot \frac{\nabla\nabla}{\Delta} P \right] \\
 &= \nabla \cdot \left[\frac{1}{\rho} \nabla(\lambda P) \right] + 2\nabla \cdot \left[\frac{1}{\rho} \nabla \cdot \left(\mu \frac{\nabla\nabla}{\Delta} P \right) \right]
 \end{aligned} \tag{28}$$

where ∇ is the gradient operator $\nabla = [\partial_x, \partial_y, \partial_z]^T$ and Δ is the Laplacian operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. Considering perturbations of the Lamé parameters and density in equation 28, we can obtain the corresponding Born equation for the pseudo-elastic pure P-mode wave equation:

$$\begin{aligned}
 &\partial_t^2 \delta P - \nabla \cdot \left[\frac{1}{\rho_0} \nabla(\lambda_0 \delta P) \right] - 2\nabla \cdot \left[\frac{1}{\rho_0} \nabla \cdot \left(\mu_0 \frac{\nabla\nabla}{\Delta} \delta P \right) \right] \\
 &= -\nabla \cdot \left[\frac{\delta\rho}{\rho_0^2} \nabla(\lambda_0 P_0) \right] - 2\nabla \cdot \left[\frac{\delta\rho}{\rho_0^2} \nabla \cdot \left(\mu_0 \frac{\nabla\nabla}{\Delta} P_0 \right) \right] + \nabla \cdot \left[\frac{1}{\rho_0} \nabla(\delta\lambda P_0) \right] + 2\nabla \cdot \left[\frac{1}{\rho_0} \nabla \cdot \left(\delta\mu_0 \frac{\nabla\nabla}{\Delta} P_0 \right) \right],
 \end{aligned} \tag{29}$$

where P_0 is the background wavefield computed with the background model $(\lambda_0, \mu_0, \rho_0)$ and $(\delta\lambda, \delta\mu, \delta\rho)$ are the model perturbations. We set the background model to be acoustic, i.e., $\lambda_0 = \lambda + 2\mu = \rho V_p^2$, $\mu_0 = 0$, $\rho_0 = \rho$ and further perform the substitution $\delta P' = (\lambda + 2\mu)\delta P$

and $P'_0 = (\lambda + 2\mu)P'_0$, equation 29 can be written as:

$$\frac{1}{\lambda + 2\mu} \partial_t^2 \delta P' - \nabla \cdot \left[\frac{1}{\rho} \nabla \delta P' \right] = -\nabla \cdot \left(\frac{\delta \rho}{\rho^2} \nabla P'_0 \right) + \nabla \cdot \left[\frac{1}{\rho} \nabla \left(\frac{\delta \lambda P'_0}{\lambda + 2\mu} \right) \right] + 2\nabla \cdot \left[\frac{1}{\rho} \nabla \cdot \left(\delta \mu \frac{\nabla \nabla}{\Delta} \frac{P'_0}{\lambda + 2\mu} \right) \right]. \quad (30)$$

Considering that the background model (λ, μ, ρ) is smoothly varying, we can move the parameters outside of the differential operator, and thus, equation 30 can be further simplified as

$$\frac{1}{V_p^2} \partial_t^2 \delta P' - \Delta \delta P' = -\frac{1}{\rho} \nabla \cdot (\delta \rho \nabla P'_0) + \frac{1}{\lambda + 2\mu} \Delta (\delta \lambda P'_0) + \frac{2}{\lambda + 2\mu} \nabla \cdot \nabla \cdot \left(\delta \mu \frac{\nabla \nabla}{\Delta} P'_0 \right) \quad (31)$$

In this simplification, we use the relation $(\lambda + 2\mu) = \rho V_p^2$ and cancel out $1/\rho$ on both sides.

The relationship between the inverse Laplacian of the wavefield and the time integration is given by

$$\frac{1}{\Delta} = \frac{1}{\partial_x^2 + \partial_y^2 + \partial_z^2} = \frac{V_p^2}{\partial_t^2} = V_P^2 \iint d^2t, \quad (32)$$

where $\iint d^2t$ denotes twice integrations over time. Substituting equation 26 into equation 25, we obtain the Born form for the pseudo-elastic wave equation:

$$\frac{1}{V_p^2} \partial_t^2 \delta P' - \Delta \delta P' = -\frac{1}{\rho} \nabla \cdot (\delta \rho \nabla P'_0) + \frac{1}{\lambda + 2\mu} \Delta (\delta \lambda P'_0) + \frac{2}{\rho} \nabla \cdot \nabla \cdot \left(\delta \mu \nabla \nabla \iint d^2t P'_0 \right). \quad (33)$$

The resulting equation 33 is the same as the quasi-elastic wave equation 14 in Feng and Schuster (2019). Note that the Born formula of equation 33 has an intrinsic limitation on model perturbation while the original pseudo-elastic wave equation does not.

Computation complexity analysis

In 3D elastic wavefield modeling, suppose the time and space grid size using $n_x = n_y = n_z = n_t = m$ is sufficient for modeling the P-waves without dispersion, then the computation cost is about m^4 . If the ratio between P- and S- wave velocities is $a = V_p/V_s$, to properly

simulate S-waves without dispersion, the time and space grids size should be modified to $n'_x = n'_y = n'_z = n'_t = am$, and the corresponding computation cost for elastic modeling is a^4m^4 . Due to the elimination of S-waves in the pseudo-elastic modeling, we do not need the finer grid for simulation, and thus, in this aspect, the computational cost is proportional to m^4 . Another component of the cost related to pseudo-elastic modeling is the evaluating of the FFTs. For example, in solving equation 7, we need to evaluate the normalized gradient and divergence operators, which, in the wavenumber domain, accounts for two forward FFTs and three inverse FFTs. The overall computational cost is about $5 \log(m)m^4$. Thus, the computation cost ratio between the elastic and pseudo-elastic modeling is:

$$r = \frac{a^4}{1 + \log(m)}. \quad (34)$$

We note that the resulting ratio is positively proportional to the fourth power of the Vp-Vs ratio a , while negatively proportional to the logarithm of the grid size m , i.e., $\log(m)$. For a typical grid size of 512 and a Vp-Vs ratio, $a = 3$, the ratio $r = 1.76$, and thus, the pseudo-elastic modeling is approximately two times faster. If we raise a to 5, the ratio $r = 13.5$, and thus, the pseudo-elastic modeling becomes ten times faster. Alternatively, if the simulation is performed with a relatively small offset, i.e., $m = 256$, the acceleration would be doubled. Thus, we conclude that the pseudo-elastic wave equation is effective for modeling and inversion when the maximum offset is small (e.g., in reflection waveform inversion) while the Vp/Vs ratio is high.

Stability and dispersion analysis

In finite-difference simulation of the elastic wave equation, the stability limits and dispersion errors of the scheme should be taken into consideration. The stability and dispersion of

the pseudo-elastic wave equation follow the same guidelines as the original elastic wave equation. From our experience, the stability condition for the pseudo-elastic wave equation is the same as the original elastic wave equation, and it depends on the maximum P-wave velocity and the FD schemes used. The numerical dispersion relation of the P-waves for the pseudo-elastic wave equation is also the same as the original elastic wave equation, and because there are no S-waves, we can avoid the dispersion limitations of the S-waves when performing simulation using the pseudo-elastic wave equation.

Potential application

Our proposed pseudo-elastic pure P-mode wave equation can simulate a wavefield with a more realistic AVO effect for the P-mode than the oversimplified acoustic approximation. The shear waves, as well as the converted waves, are mitigated yielding clean pressure wavefields, which is in line with our processing of marine datasets focussed on enhancing P-waves. Although the full-elastic wave equation can intrinsically incorporate the elasticity effect in its wavefield, there is considerable nonlinearity introduced (between the model and the wavefield) due to the shear and converted waves. The pseudo-elastic equation injects the elasticity into the pressure wavefield with limited addition in nonlinearity. Thus, the introduction of elasticity with reduced nonlinearity by our method can be potentially beneficial to elastic FWI. We are actively investigating this aspect and will report the result shortly.

CONCLUSION

We derived a pseudo-elastic pure P-mode wave equation. The resulting wave equation describes a pure P- wave propagating in elastic media. It has the same AVO behavior for the reflections as that of the PP reflections of the original elastic wave equation. Due to the pure P-waves nature of the new wave equation, its simulation does not require fine grids to avoid the numerical dispersion of the S-waves. We demonstrated that the proposed wave equation could be formulated as an acoustic wave equation plus an elastic correction term. The elastic correction term can be potentially used for compensating the elastic effects missing in the anisotropic acoustic wave equation. The proposed pseudo-elastic pure P-waves equation can help us in multi-parameter elastic full-waveform inversion, especially for marine streamer datasets.

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APPENDIX A

DERIVATION OF THE FIRST-ORDER PSEUDO-ELASTIC PURE
P-WAVES WAVE EQUATION

Describing waves propagating in a general 3D heterogeneous isotropic medium, the first-order velocity-stress elastic wave equation in a matrix form is expressed as:

$$\partial_t \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\tau} \end{bmatrix} = \boldsymbol{\Phi} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\tau} \end{bmatrix} + \begin{bmatrix} \mathbf{f} \\ \partial_t \mathbf{m} \end{bmatrix}, \quad (\text{A-1})$$

where $\mathbf{v} = [v_x, v_y, v_z]^T$ denotes the particle velocity and $\boldsymbol{\tau} = [\tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}]^T$ denotes stress components. $\mathbf{f} = [f_x, f_y, f_z]^T$ and $\mathbf{m} = [m_{ij}]$, $i, j = x, y, z$ are the force-density vector and the moment-density vector, respectively. The matrix operator $\boldsymbol{\Phi}$ fully describes the interaction between the wavefield and the elastic parameters, is given by:

$$\boldsymbol{\Phi} = \begin{bmatrix} & & & \frac{1}{\rho} \partial_x & 0 & 0 & \frac{1}{\rho} \partial_y & \frac{1}{\rho} \partial_z & 0 \\ & \mathbf{O}_{3 \times 3} & & 0 & \frac{1}{\rho} \partial_y & 0 & \frac{1}{\rho} \partial_x & 0 & \frac{1}{\rho} \partial_z \\ & & & 0 & 0 & \frac{1}{\rho} \partial_z & 0 & \frac{1}{\rho} \partial_x & \frac{1}{\rho} \partial_y \\ (\lambda + 2\mu) \partial_x & \lambda \partial_y & \lambda \partial_z & & & & & & \\ \lambda \partial_x & (\lambda + 2\mu) \partial_y & \lambda \partial_z & & & & & & \\ \lambda \partial_x & \lambda \partial_y & (\lambda + 2\mu) \partial_z & & & & \mathbf{O}_{6 \times 6} & & \\ \mu \partial_y & \mu \partial_x & 0 & & & & & & \\ \mu \partial_z & 0 & \mu \partial_x & & & & & & \\ 0 & \mu \partial_z & \mu \partial_y & & & & & & \end{bmatrix}, \quad (\text{A-2})$$

where $\mathbf{O}$ denotes a matrix with zero value components, ρ is the density, and (λ, μ) are the Lamé parameters.

To derive the first-order pure P-wave equation, we project the particle velocity wavefield

along the wavefront normal direction. Thus, we introduce the following operator $\mathbf{\Pi}$ and its adjoint $\mathbf{\Pi}^T$:

$$\mathbf{\Pi} = \begin{bmatrix} \frac{\partial_x}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} & & & \\ \frac{\partial_y}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} & \mathbf{O}_{3 \times 6} & & \\ \frac{\partial_z}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} & & & \\ \mathbf{O}_{6 \times 1} & & \mathbf{I}_{6 \times 6} & \end{bmatrix}, \quad \mathbf{\Pi}^T = \begin{bmatrix} \frac{\partial_x}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} & \frac{\partial_y}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} & \frac{\partial_z}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} & \mathbf{O}_{1 \times 6} \\ & \mathbf{O}_{6 \times 3} & & \mathbf{I}_{6 \times 6} \end{bmatrix}. \quad (\text{A-3})$$

The relation $\mathbf{\Pi}^T \mathbf{\Pi} = \mathbf{I}$ holds, where $\mathbf{I}$ denotes the identity operator. Thus, we have $(\mathbf{\Pi}^T)^{-1} = \mathbf{\Pi}$. We insert $\mathbf{I} = (\mathbf{\Pi}^T)^{-1} \mathbf{\Pi}^T$ after Φ in equation A-1:

$$\partial_t \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\tau} \end{bmatrix} = \Phi (\mathbf{\Pi}^T)^{-1} \mathbf{\Pi}^T \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\tau} \end{bmatrix} + \begin{bmatrix} \mathbf{f} \\ \partial_t \mathbf{m} \end{bmatrix}. \quad (\text{A-4})$$

We then apply operator $\mathbf{\Pi}^T$ on both sides, and substitute $(\mathbf{\Pi}^T)^{-1} = \mathbf{\Pi}$, we obtain:

$$\partial_t \mathbf{\Pi}^T \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\tau} \end{bmatrix} = \mathbf{\Pi}^T \Phi \mathbf{\Pi} \mathbf{\Pi}^T \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\tau} \end{bmatrix} + \mathbf{\Pi}^T \begin{bmatrix} \mathbf{f} \\ \partial_t \mathbf{m} \end{bmatrix}. \quad (\text{A-5})$$

In both sides of equation A-5, we have the term $\mathbf{\Pi}^T [\mathbf{v}, \boldsymbol{\tau}]^T$, which is given by:

$$\mathbf{\Pi}^T \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\tau} \end{bmatrix} = \begin{bmatrix} \frac{\partial_x v_x}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \frac{\partial_y v_y}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \frac{\partial_z v_z}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} \\ \boldsymbol{\tau} \end{bmatrix} \quad (\text{A-6})$$

Thus, we introduce the pressure wavefield $P = \frac{\partial_x v_x}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \frac{\partial_y v_y}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}} + \frac{\partial_z v_z}{\sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2}}$ and equation A-5 becomes:

$$\partial_t \begin{bmatrix} P \\ \boldsymbol{\tau} \end{bmatrix} = \mathbf{\Pi}^T \Phi \mathbf{\Pi} \begin{bmatrix} P \\ \boldsymbol{\tau} \end{bmatrix} + \mathbf{\Pi}^T \begin{bmatrix} \mathbf{f} \\ \partial_t \mathbf{m} \end{bmatrix}. \quad (\text{A-7})$$

Equation A-7 is the resulting first-order pseudo-elastic wave equation, and it shares a similar form as the second-order equation 7. It is expressed by the pressure and stress

tensor, and if we expand equation A-7 and write out each component, we obtain equation 8. Staggered-grid finite-difference can be used to solve equation A-7. FFT can be used to evaluate the normalized gradient and divergence operators in the matrix operator $\mathbf{II}$ and $\mathbf{II}^T$. In staggered-grid FD applications, attention should be taken when variables are not in the same grid, shifting properties of Fourier transform will be used, and it can be performed efficiently together with the evaluation of the normalized gradient and divergence operators.

APPENDIX B

DERIVATION OF THE ELASTIC CORRECTION TERM χ

Ignoring the source term, the elastic wave equation 2 can be rewritten as:

$$\rho \partial_t^2 \mathbf{u} = (\nabla \lambda)(\nabla \cdot \mathbf{u}) + \nabla \mu \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + (\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u}. \quad (\text{B-1})$$

It is equivalent to

$$\rho \partial_t^2 \mathbf{u} = [\nabla(\lambda + 2\mu)](\nabla \cdot \mathbf{u}) + \nabla \mu \cdot [(\nabla \mathbf{u})^T - \nabla \mathbf{u}] + (\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} - 2(\nabla \mu)(\nabla \cdot \mathbf{u}) + 2\nabla \mu \cdot \nabla \mathbf{u}. \quad (\text{B-2})$$

Using the following relations,

$$\nabla[(\lambda + 2\mu) \nabla \cdot \mathbf{u}] = [\nabla(\lambda + 2\mu)](\nabla \cdot \mathbf{u}) + (\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) \quad (\text{B-3})$$

$$\nabla \mu \times (\nabla \times \mathbf{u}) = \nabla \mu \cdot [\nabla \mathbf{u} - (\nabla \mathbf{u})^T] \quad (\text{B-4})$$

equation B-2 can be simplified as:

$$\rho \partial_t^2 \mathbf{u} = \nabla[(\lambda + 2\mu) \nabla \cdot \mathbf{u}] - \nabla \mu \times (\nabla \times \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} + 2\nabla \mu \cdot \nabla \mathbf{u} - 2(\nabla \mu)(\nabla \cdot \mathbf{u}). \quad (\text{B-5})$$

If we choose the normalized gradient and divergence operators defined in equation 9, similar to equation B-5, the pseudo-elastic pure P-mode wave equation becomes:

$$\begin{aligned} \partial_t^2 P = & \nabla \cdot \left\{ \frac{1}{\rho} \nabla \left[(\lambda + 2\mu) \frac{\nabla \cdot \nabla}{\Delta} P \right] \right\} \\ & - \nabla \cdot \left\{ \frac{1}{\rho} \left[\nabla \mu \times \frac{\nabla \times \nabla P}{\Delta} + \mu \nabla \times \frac{\nabla \times \nabla P}{\Delta} \right] \right\} \\ & + \nabla \cdot \left\{ \frac{1}{\rho} \left[2 \frac{\nabla \mu \cdot \nabla}{\Delta} \nabla P - 2(\nabla \mu) \left(\frac{\nabla \cdot \nabla}{\Delta} P \right) \right] \right\}, \end{aligned} \quad (\text{B-6})$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplacian operator and we have

$$\nabla \cdot \nabla = \Delta. \quad (\text{B-7})$$

Because ∇P is curl-free, we also have

$$\nabla \times \nabla P = 0. \quad (\text{B-8})$$

Using equations B-7 and B-8, equation B-6 can be simplified as:

$$\partial_t^2 P = \nabla \cdot \left\{ \frac{1}{\rho} \nabla [(\lambda + 2\mu) P] \right\} + \nabla \cdot \left\{ \frac{1}{\rho} [2(\nabla \mu \cdot \nabla) \nabla - 2\nabla \mu \Delta] \frac{P}{\Delta} \right\}. \quad (\text{B-9})$$

If we choose to use the parameterization of velocity and density, and perform the substitution of $P' = \rho V_p^2 P$, equation B-9 becomes:

$$\frac{1}{\rho V_p^2} \partial_t^2 P' = \nabla \cdot \frac{1}{\rho} \nabla P' + \chi, \quad (\text{B-10})$$

where the elastic correction term χ is :

$$\chi = \nabla \cdot \left\{ \frac{1}{\rho} [2(\nabla(\rho V_s^2) \cdot \nabla) \nabla - 2\nabla(\rho V_s^2) \Delta] \frac{1}{\Delta} \frac{P'}{\rho V_p^2} \right\}. \quad (\text{B-11})$$

Equation B-11 can be further expanded into equation 12.

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Figure 1a: Radiation patterns for the pseudo-elastic pure P-mode wave equation: using parameterization of a) Lamé paramters and density, i.e., (λ , μ , ρ); b) P- and S-waves velocity and density, i.e. (V_p , V_s , ρ). Note the pseudo-elastic wave equation shares the exact radiation patterns as that of the PP reflections for the vector elastic wave equation.

317x337mm (118 x 118 DPI)

Figure 1b: Radiation patterns for the pseudo-elastic pure P-mode wave equation: using parameterization of a) Lamé parameters and density, i.e., (λ, μ, ρ) ; b) P- and S-waves velocity and density, i.e. (V_p, V_s, ρ) . Note the pseudo-elastic wave equation shares the exact radiation patterns as that of the PP reflections for the vector elastic wave equation.

310x338mm (118 x 118 DPI)

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356x356mm (236 x 236 DPI)

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356x356mm (236 x 236 DPI)

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356x356mm (236 x 236 DPI)

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356x356mm (236 x 236 DPI)

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1476x1209mm (38 x 38 DPI)

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356x356mm (236 x 236 DPI)

Figure 4c: Snapshots of the wavefields using a) first-order elastic modeling; b) second-order elastic modeling; c) first-order pseudo-elastic modeling; d) second-order pseudo-elastic modeling. Compared to the results in Figure 2, a smaller S-waves velocity is used.

356x356mm (236 x 236 DPI)

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356x356mm (236 x 236 DPI)

Figure 5a: Snapshots of the wavefields using a) first-order elastic modeling; b) second-order elastic modeling; c) first-order pseudo-elastic modeling; d) second-order pseudo-elastic modeling. In the modeling, the shear wave velocity is set to zero.

356x356mm (236 x 236 DPI)

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356x356mm (236 x 236 DPI)

Figure 5c: Snapshots of the wavefields using a) first-order elastic modeling; b) second-order elastic modeling; c) first-order pseudo-elastic modeling; d) second-order pseudo-elastic modeling. In the modeling, the shear wave velocity is set to zero.

356x356mm (236 x 236 DPI)

Figure 5d: Snapshots of the wavefields using a) first-order elastic modeling; b) second-order elastic modeling; c) first-order pseudo-elastic modeling; d) second-order pseudo-elastic modeling. In the modeling, the shear wave velocity is set to zero.

356x356mm (236 x 236 DPI)

Figure 6a: a) Common shot gathers and b) traces at different offset for the single interface elastic model. The simulation is performed when V_p has contrast.

604x306mm (118 x 118 DPI)

Figure 6b: a) Common shot gathers and b) traces at different offset for the single interface elastic model. The simulation is performed when V_p has contrast.

633x292mm (118 x 118 DPI)

Figure 7a:a) Common shot gathers and b) traces at different offset for the single interface elastic model. The simulation is performed when V_s has contrast.

604x306mm (118 x 118 DPI)

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633x292mm (118 x 118 DPI)

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604x306mm (118 x 118 DPI)

Figure 8b: a) Common shot gathers and b) traces at different offset for the single interface elastic model. The simulation is performed when ρ has contrast.

633x292mm (118 x 118 DPI)

Figure 9a: a) Common shot gathers and b) traces at different offset for the single interface elastic model. The simulation is performed when V_p , V_s and ρ all have contrast.

658x281mm (118 x 118 DPI)

Figure 9b: a) Common shot gathers and b) traces at different offset for the single interface elastic model. The simulation is performed when V_p , V_s and ρ all have contrast.

633x292mm (118 x 118 DPI)

Figure 10a: AVO responses of the PP reflections by solving the Zoeppritz equations: a) the amplitude; b) the phase angle.

563x328mm (118 x 118 DPI)

Figure 10b: AVO responses of the PP reflections by solving the Zoeppritz equations: a) the amplitude; b) the phase angle.

571x324mm (118 x 118 DPI)

Figure 11: A multi-layer elastic model.

429x430mm (118 x 118 DPI)

Figure 12a: Snapshots of the vertical displacement wavefield simulated using pseudo-elastic pure P-mode wave equation.

297x623mm (118 x 118 DPI)

Figure 12b: Snapshots of the vertical displacement wavefield simulated using pseudo-elastic pure P-mode wave equation.

297x623mm (118 x 118 DPI)

Figure 12c: Snapshots of the vertical displacement wavefield simulated using pseudo-elastic pure P-mode wave equation.

297x623mm (118 x 118 DPI)

Figure 12d: Snapshots of the vertical displacement wavefield simulated using pseudo-elastic pure P-mode wave equation.

297x623mm (118 x 118 DPI)

Figure 12e: Snapshots of the vertical displacement wavefield simulated using pseudo-elastic pure P-mode wave equation.

297x623mm (118 x 118 DPI)

Figure 12f: Snapshots of the vertical displacement wavefield simulated using pseudo-elastic pure P-mode wave equation.

297x623mm (118 x 118 DPI)

Figure 13a: Snapshots of the vertical displacement wavefield simulated using elastic wave equation.
297x623mm (118 x 118 DPI)

Figure 13b: Snapshots of the vertical displacement wavefield simulated using elastic wave equation.

297x623mm (118 x 118 DPI)

Figure 13c: Snapshots of the vertical displacement wavefield simulated using elastic wave equation.

297x623mm (118 x 118 DPI)

Figure 13d: Snapshots of the vertical displacement wavefield simulated using elastic wave equation.

297x623mm (118 x 118 DPI)

Figure 13e: Snapshots of the vertical displacement wavefield simulated using elastic wave equation.

297x623mm (118 x 118 DPI)

Figure 13f: Snapshots of the vertical displacement wavefield simulated using elastic wave equation.

297x623mm (118 x 118 DPI)

Figure 14a: a) Common shot gathers for the multi-layer model; b) the trace at offset 0.5 m.

586x315mm (118 x 118 DPI)

Figure 14b: a) Common shot gathers for the multi-layer model; b) the trace at offset 0.5 m.

779x237mm (118 x 118 DPI)

Figure 15a: The SEAM 2D Salt model, a) V_p ; b) V_s ; and c) density. The star symbol denotes the source.

620x298mm (118 x 118 DPI)

Figure 15b: The SEAM 2D Salt model, a) V_p ; b) V_s ; and c) density. The star symbol denotes the source.

620x298mm (118 x 118 DPI)

Figure 15c: The SEAM 2D Salt model, a) V_p ; b) V_s ; and c) density. The star symbol denotes the source.

620x298mm (118 x 118 DPI)

Figure 16: A common shot gather for the SEAM 2D Salt model simulated using the first-order elastic wave equation.

525x352mm (118 x 118 DPI)

Figure 17:A common shot gather for the SEAM 2D Salt model simulated using the first-order pseudo-elastic wave equation.

525x352mm (118 x 118 DPI)

Figure 18: A common shot gather for the SEAM 2D Salt model simulated using the acoustic wave equation.

525x352mm (118 x 118 DPI)

Figure 19: The difference between the common shot gathers simulated by elastic wave equation (Figure 16) and the acoustic wave equation (Figure 18).

525x352mm (118 x 118 DPI)

Figure 20: The difference between the common shot gathers simulated by elastic wave equation (Figure 16) and the pseudo-elastic wave equation (Figure 17).

525x352mm (118 x 118 DPI)

Figure 21a: Measuring the similarity between records by the pseudo-elastic modeling (as well as the acoustic modeling) and the elastic modeling using a) the L2 norm of the record difference; b) the normalized cross-correlation between the records.

649x285mm (118 x 118 DPI)

Figure 21b: Measuring the similarity between records by the pseudo-elastic modeling (as well as the acoustic modeling) and the elastic modeling using a) the L2 norm of the record difference; b) the normalized cross-correlation between the records.

654x282mm (118 x 118 DPI)

Figure 22:One trace from the common shot gathers at offset 2.4 km where the acoustic modeling (black curve) produces the signal with considerable phase error as compared with the one corresponding to the elastic modeling (dashed blue curve).

572x323mm (118 x 118 DPI)

DATA AND MATERIALS AVAILABILITY

Data associated with this research are available and can be obtained by contacting the corresponding author.