Spacetime-Modulation in Floating Thin Elastic Plates

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We consider the propagation of flexural-gravity waves in thin elastic plates floating atop nonviscous fluids, e.g., seawater, which are governed by a partial differential equation with Laplacian and tri-Laplacian terms. We investigate the effect of time-modulation as well as spacetime-modulation on thin floating elastic plates and show the peculiarity of the phenomena of k -bandgap and rotated k -bandgap in the context of flexural-gravity waves. This makes possible floating plates with nonreciprocal features and behaving as elastodynamic analogs of luminal electromagnetic metamaterials, with exotic applications in enhanced control of ocean waves, such as filtering devices, unidirectional acoustic propagation and isolation effects and energy harvesting in maritime engineering.

I. INTRODUCTION

Metamaterials and metasurfaces consist in a spatially periodic arrangement of meta-atoms in three or two dimensions, respectively [1, 2]. These artificial materials were applied successfully in the last two decades to obtain several exciting applications, ranging from super-lensing [3], invisibility cloaks [4], plasmonic waveguides [5], to computing [6]. Such concept encompasses a broad range of disciplines, including but not limited to, acoustic [7], elastic [8], structural [9] waves, as well as engineering thermal properties [10, 11]. All these applications exploited only the space-modulation (generally subwavelength). Recently, time also emerged as a promising parameter in designing more appealing metasurfaces. In fact, the place of time in physics is very special. First, this parameter was assumed to be absolute in Newtonian mechanics [12], until Einsteinian mechanics revolutionized its very basic nature and showed that it has to be considered on an equal footing with the remaining spatial variables, i.e., it undergoes changes when moving [13]. In the same vein, there is no reason time can not be considered in the design of metamaterials. On the contrary, the use of time as a tailable variable, may open new vistas and applications not sought before in many fields of applied physics. For instance, recent works [14] have shown the importance of time-modulation in obtaining nonreciprocal effects of the so-called luminal metamaterials in the context of electromagnetism. In fact, time started recently to be considered along with geometry [15], as a design parameter for newly proposed spacetime metamaterials and metasurfaces, with many unprecedented applications [16], such as Fresnel drag [17], signal amplification [18], harmonic generation [19], and photonic circulators [20, 21], to name a few. Some earlier studies considered dispersion relations in spacetime periodic media [22, 23], and the dynamics

of light propagation in these structures [24, 25]. Interestingly, similar effects have been observed in the context of acoustic and mechanical waves in time-modulated structures [26, 27].

On the other hand, flexural-gravity waves obey a sixth order partial differential equation (PDE) and describe the flexural motion [28, 29] in thin elastic plates floating atop inviscid and incompressible fluids (e.g., water) [30]. For instance, man-made engineered offshore structures, such as airports or newly built floating islands [31, 32] are considered as thin-mat configurations where the horizontal dimensions extend for few kilometers and the thickness is around a few meters. Moreover, these structures are usually located in the offshore zone, where the depth of water is small, i.e., in the range of 20 m (shallow water approximation). Further, both the wavelength and the lateral dimensions of the plates (elastic structures) are much larger than its thickness δ [See Fig. 1(a)], so one can assume safely the thin-plates (biharmonic) approximation [33–35], or the Kirchhoff–Love plate theory [28]. Characterizing the scattering from such waves by buoyant objects is an active topic of research [36–39] with many applications, such as dispersionless weakly nonlinear flexural-gravity wave packet [40] or invisibility cloaking for such waves using both transformation optics [41] and scattering cancellation technique that was shown to possess intriguing properties from simple cylindrical thin-plates [42].

In this paper, we propose to consider time-modulation as well as spacetime-modulation in the realm of this peculiar type of waves, that are intrinsically different from acoustics and electromagnetism, due to the asymmetric role played by space and time variables (space of order 6 and time of order 2). This asymmetry will be shown to lead to some intriguing properties of the bandgap, not seen before for other wave systems. The remaining of this paper is organized as follows: We start by formulating the problem with adequate governing equation and boundary conditions. Then, we focus on investigating the time-modulation of flexural-gravity waves by imposing either a time-varying flexural rigidity, or a spacetime-modulation using an analytical model that results in closed-form expressions by coupling two modes, to get an insight into the underlying physical mechanisms. Next, we solve numeri-

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FIG. 1. Schematic of the floating thin-plate of thickness δ atop shallow seawater of depth h , in the x - z plane. The thin-plate is supposed to be infinitely extended along the y -direction. The parameters of the plate are: Young's modulus $E = 50$ GPa, thickness $\delta = 10$ m, Poisson's ratio $\nu = 0.34$, and density of 900 kg/m³, to make it floating atop water.

cally the full problem by coupling a higher number of modes and verifying the numerical convergence of the results. The results feature a peculiar behavior of the k -bandgap. We discuss also the origin of this effect such as the non-Hermiticity of the system by the spacetime-modulation, and the appearance of exceptional points (EPs) at the edges of the bandgap as well as nonreciprocal wave propagation. We also investigate the sensitivity of the EP on external parameters and fit it analytically using the Puiseux series. Finally, we summarize the obtained results and give some derivations and further discussions in the appendices.

II. PROBLEM SET-UP

Flexural-gravity waves obey in the case of the approximations described above and for harmonic variation (in frequency-domain) (See Appendix A) the sixth-order PDE, i.e.,

$$D\Delta^3 W + \rho g \Delta W + \frac{\rho}{h} \omega^2 W = 0, \quad (1)$$

in terms of the water elevation (or plate vertical displacement in the plate's region) [or similarly in terms of the velocity potential ϕ , related to W through $\partial_t W = -h\Delta\phi$, as seen in Eq. (A7)] and angular frequency of the water-waves ω , with $\rho = 10^3$ kg/m³ and $h = 20$ m the density and depth of water, respectively, $g = 9.81$ ms⁻² the acceleration due to terrestrial gravity, and $D = E\delta^3/[12(1-\nu^2)]$ the flexural rigidity of the plate (See Appendix A for detailed derivation of this PDE and justification of the used approximations). The first term of Eq. (1) stands for the flexural effect (flexural rigidity is driving the surface elevation and thus the velocity potential), while the second term accounts for surface gravity effects (waves, as suggested by the presence of the Laplacian operator Δ and surface gravity g) [43].

Moreover, the dispersion relation of this kind of waves is obtained by replacing Δ by $(i\beta_1)^2$, with β_1 the flexural-gravity wavenumber, i.e., $\omega^2 = (hD/\rho)\beta_1^6 + g h \beta_1^2$. The description, above, of the interplay between surface gravity waves and flexural waves is evidenced by the presence of both the classical flexural-gravity wavenumber $\beta_1^6 = \rho \omega^2 / (hD)$ and the surface gravity wavenumber $k_0 = \omega / \sqrt{gh}$. Due to the coupling

between flexural and water-waves, the analysis on such a system including the derivation of transfer matrix requires some specific treatment [44].

III. MODULATION FOR FLEXURAL-GRAVITY WAVES

A. Time-modulation for flexural-gravity waves

Let us consider time-modulation of a floating thin-plate. This is the analog of time-modulation of the permittivity in photonic crystals [45] by allowing a periodic evolution of the flexural rigidity, i.e., $D(t+T) = D(t)$, with a temporal period T . Regarding a possible experiment, in order to change the elastic parameters directly, one can add shunted external circuits, with tunable electrical properties (resistance, capacitance, and inductance) (see SM [43] for more details on the piezo-electric potential modelling [46–52]). Equation (1) shall be re-written in time-domain and in the one-dimensional (1D) scenario

$$\gamma(t) \frac{\partial^6 W(x,t)}{\partial x^6} + \gamma_1 \frac{\partial^2 W(x,t)}{\partial x^2} - \frac{\partial^2 W(x,t)}{\partial t^2} = 0, \quad (2)$$

with $\gamma(t) = hD(t)/\rho$ and $\gamma_1 = gh$, by assuming that ρ and h are time-independent. The modulated parameter can be expressed as $\gamma(t) = \gamma_0[1 + \delta_m \cos(\Omega_m t)]$, with $\delta_m = \gamma_m/\gamma_0$ and $\gamma_0 = hD_0/\rho$ the non-modulated parameter. Using the Bloch theorem, applied to the time-periodicity [53], we can express the water elevation as

$$W(x,t) = e^{-i(\omega t - \beta_1 x)} \sum_{n=-\infty}^{\infty} \hat{W}_n e^{-in\Omega_m t}. \quad (3)$$

By inserting Eq. (3) into Eq. (2) and by assuming $\delta_m = 0$, we can get the dispersion of plane waves in free-space, shown in Fig. 2(a). The free-space eigenmodes can be thus obtained by finding the zeros of the determinant of the diagonal matrix $\text{diag}[\gamma_0 \beta_1^6 + \gamma_1 \beta_1^2 - (\omega + n\Omega_m)^2]$. It can be clearly seen from this dispersion its high-order polynomial nature, that is completely different from that of the acoustics or electromagnetism linear behavior. This peculiarity is more apparent for low wavenumbers, when the dispersion is nearly flat.

In Fig. 2(a), we plot the dispersion at low frequency using two different PDEs [one using Eq. (1) and the other one omitting the second Laplacian operator]. Although the two methods give similar results, the inset clearly displays certain discrepancies and demonstrates the need for using the PDE of Eq. (1). The interesting effects due to time-modulation take place when $\delta_m \neq 0$. Solving Eq. (2) by using a Fourier expansion of $W(x,t)$ when $\delta_m \neq 0$ is generally not possible analytically, as one has to consider an infinite number of terms in the expansion, for finite coupling. So, numerically this expansion is truncated and convergence is verified, in order to obtain the dispersion diagrams. Yet, the coupling between two bands (modes) is insightful and captures the essence of the phenomena (as for the harmonic oscillator or the two-level system in quantum physics [54]). We consider the bands $n = 0$ and $n = -1$ [i.e., $\hat{W}_0$ and $\hat{W}_{-1}$, See Eq. (3)]. The characteristic

FIG. 2. (a) Dispersion curves of free plane waves (i.e., $\delta_m = 0$) in floating thin-plates for $N = 5$ (i.e., considering 11 bands) and for $\Omega_m = 10$ rad/s. The inset shows an enlarged view at low frequency using two different PDEs (See Appendix A). (b) Dispersion curves for $\delta_m = 0.4$, where the inset shows the k -bandgap around the angular frequency $\omega = \Omega_m/2$. The number of bands and modulation frequency is same as in (a). The inset shows a magnified view around the second k -bandgap. (c) Two-bands dispersion curves for the case of $\delta_m = 1.75$, i.e., considering only the bands $n = 0$ and $n = -1$. The width of the k -bandgap ($[\beta_{1-}, \beta_{1+}]$) can be calculated analytically in this case. (d) Broader domain for the dispersion similar to (b), i.e., with $\Omega_m = 10$ rad/s but for $\delta_m = 1.9$ and considering $N = 10$, i.e., 21 bands.

equation of this coupled system is

$$\begin{vmatrix} \gamma_0 \beta_1^6 + \gamma_1 \beta_1^2 - \omega^2 & \frac{\gamma_0 \delta_m}{2} \beta_1^6 \\ \frac{\gamma_0 \delta_m}{2} \beta_1^6 & \gamma_0 \beta_1^6 + \gamma_1 \beta_1^2 - (\omega - \Omega_m)^2 \end{vmatrix} = 0. \quad (4)$$

The solution of this characteristic equation gives the four eigenfrequencies, owing to the fourth order of Eq. (4) in terms of ω , i.e.,

$$\omega_j = \frac{1}{2} \left(\Omega_m \pm \sqrt{\Omega_m^2 + 4(\gamma_0 \beta_1^6 + \gamma_1 \beta_1^2)} \pm \sqrt{\Omega_m^2 (\gamma_0 \beta_1^6 + \gamma_1 \beta_1^2) + \gamma_0 \delta_m / 2 \beta_1^6} \right), \quad (5)$$

with $j = 1 \dots 4$. Figure 2(b) shows the corresponding curves (real and imaginary parts) versus the flexural-gravity wavenumber β_1 . Interestingly, for $\omega = \Omega_m/2$ one can notice the emergence of a k -bandgap, i.e. a bandgap in the wavenumber axis with analogy to ω -bandgap when space-modulation is enforced, shown in Fig. 3(a). This graph is obtained for $\delta_m = 1.75$. The emergence of this bandgap (real part of ω) is confirmed by the finite imaginary part of ω [red curve in

Fig. 2(b)] which shows that no propagation can take place in the interval denoted $[\beta_{1-}, \beta_{1+}]$. The limits of this interval in the k -space are

$$\beta_{1\pm} = \left(\frac{\Omega_m^2}{2\gamma_0 (2 \mp \delta_m)} \right)^{1/6}. \quad (6)$$

Hence, when $\delta_m \geq 2$, only one solution is positive, i.e., β_{1-} ,

FIG. 3. Two-band dispersion curves for the case of $\delta_m = 1.75$ and considering only the bands $n = 0$ and $n = -1$, for (a) $\Omega_m = 0$ rad/s and $\kappa_m = 0.06$ rad/m, i.e., classical space-modulation with frequency-bandgap, (b) $\Omega_m = 20$ rad/s and $\kappa_m = 0.06$ rad/m, i.e., spacetime-modulation with an oblique (indirect) bandgap, and (c) $\Omega_m = 20$ rad/s and $\kappa_m = 0$ rad/m, i.e., classical time-modulation with k -bandgap.

and the bandgap becomes of infinite extent, i.e., for $\beta_1 > \beta_-$ (See SM for more details [43]). Next, to get more insight, we need to consider more bands to compute the correct dispersion curves. By inserting the Fourier expansion of the water elevation field in Eq. (2), we get $\forall n \in \mathbb{Z}$,

$$\left[\gamma_0 \beta_1^6 + \gamma_1 \beta_1^2 - (\omega + n\Omega_m)^2 \right] \hat{W}_n + \frac{\gamma_0 \delta_m}{2} \beta_1^6 (\hat{W}_{n-1} + \hat{W}_{n+1}) = 0. \quad (7)$$

In Fig. 2(c) we consider $\delta_m = 0.4$ and $\Omega_m = 10$ rad/s, as well as $N = 5$ (i.e., 11 bands). At frequencies $(n + 1/2)\Omega_m$ k -bandgaps can be observed at different β_1 -intervals, of different width (the inset shows the narrow bandgap at higher β_1). Figure 2(d) shows a broader interval of both β_1 and ω , by taking into account more bands ($N = 10$), demonstrating the peculiarity of time-modulation in the framework of flexural-gravity waves. An application of this concept would be filtering and generation of ultra-narrowband water-wave signals.

For instance, the picture of k -bandgap can be seen from another perspective, i.e., that of exceptional points of degeneracy (EPs) [55–57]. The edges of the bandgap correspond also to EPs as these are induced by space, time, and/or spacetime-modulation. Physically, this is different from gain-loss systems where the EPs are generated due to balanced gain and loss and parity-time (PT)-symmetry breaking. In fact, Eq. (7) (time-modulation) or Eq. (9) (spacetime-modulation) can be re-arranged in a matrix form by casting the terms with wavenumber in the left-hand side and the problem can be seen as an eigenvalue problem, that is

$$T(\beta_1) \Phi = \omega \Phi, \quad (8)$$

with $\Phi = [\hat{W}_{-N}, \dots, \hat{W}_0, \dots, \hat{W}_N]^T$ by considering $2N + 1$ har-

monics. The matrix T is thus $(2N + 1) \times (2N + 1)$ and ω is the eigenvalue (eigenfrequency) of the modulated system. This means that the results in Fig. 2(c) could be interpreted in a different but equivalent way. Explicitly, at low frequency we have two eigenvalues [both real, i.e., the exact phase of the system and imaginary part is zero as seen in Fig. 2(c)] that evolve from zero to around $\beta_1 = 0.023$ rad/m. At $\beta_1 = \beta_{1-}$, an EP takes place and the PT -symmetry is broken, so the system switches to the broken phase. This is demonstrated by the finite imaginary part of the eigenvalue [$\Im(\omega)$] (two complex conjugate eigenfrequencies). At $\beta_1 = \beta_{1+}$, a second EP occurs where the system switches back to the exact phase. For spacetime-modulation, the same reasoning can be employed with the only difference that along the broken phase (i.e., interval $[\beta_{1-}, \beta_{1+}]$) the degenerate eigenfrequency is not constant but varies linearly with β_1 and results in the oblique bandgap as explained in the SM [43]. Further analysis of the sensitivity of the EP is given in Appendix B and shows the effect of various parameters on the EP location as well as analytical derivation based on fractional Puiseux series [58, 59].

B. Spacetime-modulation and nonreciprocity

1. Spacetime-modulation

Let us move now to the modulation of the floating plate in spacetime [23], i.e., $\gamma(t) = \gamma_0[1 + \delta_m \cos(\kappa_m x - \Omega_m t)]$, with κ_m the space-modulation 'frequency'. Here, we also consider first the coupling between the bands $n = 0$ and $n = -1$. It is straightforward to show that the dispersion relation is obtained from Eq. (7) by replacing β_1 by $\beta_n = \beta_1 + n\kappa_m$, i.e.,

$$\left[\gamma_0 (\beta_1 + n\kappa_m)^6 + \gamma_1 (\beta_1 + n\kappa_m)^2 - (\omega + n\Omega_m)^2 \right] \hat{W}_n + \frac{\gamma_0 \delta_m}{2} \left[(\beta_1 + (n-1)\kappa_m)^6 \hat{W}_{n-1} + (\beta_1 + (n+1)\kappa_m)^6 \hat{W}_{n+1} \right] = 0. \quad (9)$$

When only two modes are considered, a characteristic equation similar to Eq. (4) is obtained with the proper changes, i.e.,

$$\begin{vmatrix} \gamma_0 \beta_1^6 + \gamma_1 \beta_1^2 - \omega^2 & \frac{\gamma_0 \delta_m}{2} (\beta_1 - \kappa_m)^6 \\ \frac{\gamma_0 \delta_m}{2} \beta_1^6 & \gamma_0 (\beta_1 - \kappa_m)^6 + \gamma_1 (\beta_1 - \kappa_m)^2 - (\omega - \Omega_m)^2 \end{vmatrix} = 0. \quad (10)$$

In the case of spacetime-modulation (or luminal floating structures), the eigenfrequencies are solutions to a sixth order polynomial and it is not possible to easily express them in a simple closed-form as before. Figure 3(a) shows first the space-modulation for comparison, by enforcing $\Omega_m = 0$ rad/s and $\kappa_m = 0.06$ rad/m. In this scenario, a classical frequency bandgap can be observed and is highlighted in red color. Figure 3(b) gives the plot for $\Omega_m = 20$ rad/s, $\kappa_m = 0.06$ rad/m, and $\delta_m = 0.4$. Thus, for a finite value of $\kappa_m \Omega_m$, we can observe a tilted bandgap, towards the β_1 -axis, i.e., an intermediate state between the ω -axis bandgap [Fig. 3(a)] and β_1 -axis bandgap [Fig. 3(c)]. The colored dashed regions in Figs. 3(a)-(c) give the corresponding bandgap for each modulation type.

For the complete treatment, we consider $N = 5$ (e.g., $2N + 1 = 11$ bands) and plot in Fig. 4 the dispersion curves of this luminal floating structure. In comparison with Fig. 2(c), we can see the rotation of the bandgap due to the additional space-modulation. In the limit where $\kappa_m / \Omega_m \rightarrow \infty$ we can recover classical bandgap in the frequency-domain as demonstrated in Appendix C: Figs. C.4 and C.5. However, these tilted bandgaps are a unique feature of spacetime-modulated thin-plates and due to the fact that in electromagnetic or acoustic systems, time and space have some sort of “duality”, i.e., they can exchange their positions in the equation (up to the speed of wave) while still get the same solution. But in thin-plates, time and space are not interchangeable, as exemplified by the governing PDE of Eq. (1) and the corresponding dispersion relation. In SM, we show an investigation of the role of this asymmetry between space and time in different wave systems [e.g., acoustics (order 2), flexural (order 4), and flexural-gravity (order 6)] as well as hypothetical PDEs. In the plot of Fig. 3 of the SM [43], only the acoustic waves (equal order between space and time) do not present oblique bandgap, due to the symmetry between space and time. The other PDEs have different space and time orders, and thus possess tilted bandgaps, confirming thus the origin of the tilted bandgap. The same effect is also seen in Fig. 3(b) for the simple case of coupling between two bands (See Appendix C for the progression of the bandgap with varying space and time-modulation frequencies, showing a nontrivial dependence).

2. Nonreciprocity and unidirectionality

Figure 4 showcases another intriguing property directly originating from spacetime-modulation, i.e., nonreciprocity. For instance, to grasp these nonreciprocal features we plot the eigenfrequencies (or dispersion relation) for a flexural-gravity wavenumber β_1 spanning the range $[-0.045, 0.045]$ rad/s. Due to the tilting of the bandgaps (owing to spacetime-modulation) and the overall rotation of the band diagram, we clearly observe a strong asymmetrical behavior for positive and negative opposite values of β_1 . For example, the operation point marked with a green dot ($\beta_1 = 0.034$) and the one marked with a blue dot ($\beta_1 = -0.034$ and opposite to the green one) have markedly different properties. The blue point lies

FIG. 4. Upper panel: Dispersion curves for $N = 5$, i.e., considering 11 bands for $\delta_m = 0.8$, $\Omega_m = 30$ rad/s, and $\kappa_m = 0.0065$ rad/m, where the inset (red curves) is a magnified view of the oblique (rotated) k -bandgap. Lower panel: Imaginary part of the first eigenfrequency $\Im(\omega)$ in the same range of flexural-gravity wavenumbers. The blue (green) dots correspond to negative (positive) $\beta_1 = \mp 0.034$ rad/m.

in a bandgap region where no propagation is possible while the green point corresponds to propagating regime (this is confirmed by the observation of the imaginary part of the eigenfrequencies: finite or zero, respectively). Hence, our spacetime metamaterial may be used for unidirectional water-wave propagation (only right direction propagation is possible in this specific case). In a sense, our spacetime-modulation plays the role of topologically nontrivial metal-metal interfaces, where these structures were shown to result in unidirectional propagation of plasmons due to the topology of the band structure [60, 61]. Here, we propose the first of its kind unidirectional device for flexural-gravity waves without the use of complex strategies (such as presence of a magnetic field in plasmonics or the use of gain and loss systems).

On the application side, this idea may open new avenues for unprecedented control and harvesting of water-waves in floating thin elastic systems. In fact, cities and critical infrastructures located nearby coastlines are subject to flooding and thus floating systems might play a role in the near future to either replace some of the existing infrastructures (airports, artificial islands) or to help protect and isolate land from some ocean waves. Our nonreciprocal floating metamaterial can be

of use as it will selectively stop water-waves incoming from a preferred direction.

IV. CONCLUDING REMARKS

To conclude, flexural-gravity waves are investigated and their scattering properties quantified in the context of spacetime-modulation of floating plates. In this context, we must handle the coupling between water-waves and flexural waves of thin floating plates. It is shown that these waves possess one propagating and two evanescent solutions, responsible for the novel observed effects. As a potential application, we consider a time-modulation of the Young's modulus both in time and spacetime, and we show the emergence of k -bandgap with peculiar properties as well as tilted bandgaps. This results in particular in nonreciprocal features with evident use for unidirectional propagation of flexural-gravity waves. The presented research may open new vistas in manipulation and harvesting of water-waves in floating structures, with promising applications in maritime engineering.

For instance, one intriguing application may be the rainbow trapping of water-waves by leveraging of spacetime-modulation in order to collect some of the ocean energy and then convert it to useful electricity via piezo-electric materials for example. Thus, it can be clearly seen that spacetime-modulation in the realm of floating thin-plates can have a plethora of intriguing applications and novel properties not found with other classical waves (i.e., obeying the Helmholtz equation).

Regarding the issue of practical realization, if the goal is to model and design space-time metamaterials for floating mega-structures such as floating airports or artificial islands, the size of mega-structures ranging from a few hundreds of meters to kilometers has to be studied. Of course, this is a long-term experiment that requires great accuracy in the modeling. For instance, to model such mega-structures, one cannot consider a fixed height (or depth of water). Also, the thickness of the plate and its material elastic properties (Young's modulus and Poisson's ratio) may vary from one region to another depending on the structure and the tackled application. Thus, a more detailed numerical modeling needs to be developed and performed, for example by using finite element methods devoted to fluid dynamics that would account for various non linear effects beyond the scope of our study. Yet, it should be emphasized that our concept is scalable and that it can work efficiently for a much smaller scale that may be used for example in other applications such as energy harvesting (see SM for some results [43]).

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Appendix A: Flexural-Gravity Governing Equation

1. Derivation

The structures considered in this work are thin elastic plates floating atop water. The thickness δ of a given plate is assumed to be very small in comparison to its lateral dimensions L_p and the wavelength of the water-waves, i.e. $0 < \delta \ll L$ and $0 < \delta \ll \lambda$. Also, we assume that the depth of water h is small in comparison to wavelength, i.e. the shallow water approximation that is $0 < h \ll \lambda$. Further, it is assumed that the flow of water is irrotational, thus the velocity field can be expressed as $\mathbf{v} = \nabla\phi$, with ϕ the velocity potential. The water elevation in the plate's region is denoted W . Thanks to the shallow water equation, one has

$$\frac{\partial W}{\partial t} + h\Delta\phi = 0. \quad (\text{A1})$$

In the plate's region, one can express the relationship between the liquid elevation and pressure exerted by the thin-plate using the linearized Bernoulli equation, i.e.,

$$p = -\rho gW - \rho \frac{\partial \phi}{\partial t}, \quad (\text{A2})$$

where $g = 9.81 \text{ m/s}^2$ is the surface gravity of Earth and ρ the mass density of water. Last, the pressure exerted by the plate can also be expressed by the dynamic condition as

$$p = D\Delta^2 W + M \frac{\partial^2 W}{\partial t^2}, \quad (\text{A3})$$

where the Laplacian Δ is understood as operating in the two dimensional space, i.e., in the x - y -plane. Moreover, M is the surface density of the plate (i.e., unit of mass per unit of surface) and D its flexural rigidity. By combining Eqs. (A1)-(A3), and taking time derivative of both Eqs. (A2)-(A3), one can derive the equation obeyed by the velocity potential in its domain of validity, in case of isotropic and homogeneous physical parameters, i.e., D , h , M , and ρ , that is

$$D\Delta^3\phi + M \frac{\partial^2}{\partial t^2}\Delta\phi + \rho g\Delta\phi - \frac{\rho}{h} \frac{\partial^2 \phi}{\partial t^2} = 0. \quad (\text{A4})$$

Yet, in the case the flexural rigidity D depends on time, it is more convenient to write the PDE in terms of W by taking the Laplacian of both Eqs. (A1)-(A3). This leads to

$$D\Delta^3 W + M \frac{\partial^2}{\partial t^2}\Delta W + \rho g\Delta W - \frac{\rho}{h} \frac{\partial^2 W}{\partial t^2} = 0. \quad (\text{A5})$$

2. Approximations

Let us denote the terms of Eq. (A4) by

$$T_1 = D\Delta^2\phi, \quad T_2 = M\partial_t^2\Delta\phi, \quad \text{and} \quad T_3 = \rho g\Delta\phi \approx \rho/h\partial_t^2\phi.$$

FIG. A.1. (a) Comparison of the strength of the terms T_2 and T_3 , using dimensional analysis, normalized by the first term T_1 , versus the water wavelength $\lambda_0 = 2\pi/k_0$. (b) Dispersion relation (flexural wavelength versus water wavelength) of flexural-gravity waves, using the PDE of Eq. (A7) and the other PDE (same as in Eq. (A7) without the term T_2). (c) Two-band dispersion curves in free-space, i.e., no coupling $\delta_m = 0$. (d) Two-band dispersion considering only the bands $n = 0$ and $n = -1$, for $\Omega_m = 20$ rad/s, $\kappa_m = 0.06$ rad/m (spacetime-modulation with an oblique (indirect) bandgap), and for $\delta_m = 1.75$.

The second term, i.e., T_2 of this equation can be shown to be less significant than the remaining ones, so it can be ignored, as with our set of approximations, this term is much smaller than the remaining terms of the left-hand side of Eq. (A4). Figure A.1(a) compares the terms T_2 and T_3 versus the wavelength of water-waves. It shows that for most of the spectrum, the term T_2 can be safely ignored. The exception occurs only for small wavelengths. But since we are working in the shallow water approximation, we do not consider these wavelengths, and hence, T_2 can be neglected. Thus, the equation satisfied in the plate's region is

$$D\Delta^3\varphi + \rho g\Delta\varphi - \frac{\rho}{h}\frac{\partial^2\varphi}{\partial t^2} = 0. \quad (\text{A6})$$

In the frequency-domain, when we assume a time-harmonic dependence, i.e., $\partial/\partial t = -i\omega$, we obtain (by uniformly and interchangeably denoting the velocity potential and its harmonic component by φ)

$$\Delta^3\varphi + \frac{\rho g}{D}\Delta\varphi + \frac{\rho}{hD}\omega^2\varphi = 0. \quad (\text{A7})$$

Evidently, the same approximations apply to Eq. (A5), i.e.,

$$\Delta^3 W + \frac{\rho g}{D}\Delta W + \frac{\rho}{hD}\omega^2 W = 0. \quad (\text{A8})$$

Figure A.1(b) plots the dispersion relation of Eq. (A7) and that of the other PDE, where the term $T_3 = \rho g\Delta$ is neglected. The two PDEs coincide at small wavelengths, but as the wavelength increases, the mismatch increases considerably.

3. Boundary Conditions

In order to solve a scattering problem involving Eq. (A7) or Eq. (A8) we need to supply the boundary conditions, which in the case of a plate-plate boundary, consist in ensuring the continuity of the six parameters $\partial_t\varphi$, $\partial_n\varphi$, $\Delta\varphi$, $\partial_n(\Delta\varphi)$, $M_n(\Delta\varphi)$, and $V_n(\Delta\varphi)$, corresponding to the six unknowns (See Fig. 1 of the manuscript [32], with n the normal to the boundary and s the tangential coordinate, i.e.,

$$M_n(\Delta\varphi) = -D \left(\frac{\partial^2\Delta\varphi}{\partial n^2} + \nu \frac{\partial^2\Delta\varphi}{\partial s^2} \right),$$

$$V_n(\Delta\varphi) = \frac{\partial M_n(\Delta\varphi)}{\partial n} - 2 \frac{\partial M_{ns}(\Delta\varphi)}{\partial s},$$

where the operator $M_{ns} = D(1 - \nu)\partial^2/\partial n\partial s$ and where ∂_t , ∂_n , and ∂_s denote the partial derivative with respect to time, normal component, and tangent component, respectively. For a

plate-water boundary, we have instead four boundary conditions, that are continuity of $\partial_t \phi$ as well as $\partial_n \phi$ and $M_n = V_n = 0$ [32]. In the case of a layered-structure as the one shown in Fig. 1 of the manuscript, these conditions simplify greatly as can be shown easily.

4. Evanescent Waves

We wish to emphasize here that flexural-gravity waves propagating within a floating thin-plate are shown to obey the order six PDE [Eq. (1) of the manuscript] in frequency-domain; in addition to propagating flexural-gravity waves, i.e., $e^{\pm i\beta_1 x}$, there exist evanescent (inhomogeneous) flexural-gravity wave solutions, differentiating the floating plate from its acoustic counterpart (See SM [43]), in which only propagating waves are considered. In free propagating domain, only the propagating component survives, as the evanescent wave is proportional to $e^{\pm \beta_1 x}$ and since these evanescent waves decay exponentially as they travel away from their corresponding interfaces, they do not contribute to the scattering coefficients, which are measured in the far-field. But importantly, in order to fully characterize the scattering of flexural-gravity waves, we have to take into account the contribution of all waves at the inner interfaces of the metamaterial. And this is exactly what we do in our treatment, where these waves (coupled to the other propagating ones) implicitly influence the physics of the spacetime-modulation. Hence, it should be understood that these waves are present at all steps of the calculation on an equal footing with the propagating ones; in a sense, we solve the full problem without approximation.

Appendix B: Sensitivity of EP and Analytical Modeling via Puiseux Fractional-Power Series

Based on the result of Eq. (8) that shows that our problem can be thought of as eigenvalue problem and that the k -bandgap is reminiscent of EP and parity-time symmetry breaking as discussed in Section III. In the following we analyze the sensitivity of the EP and bandgap with respect to small perturbation on the physical parameters of the structure and show the extreme sensitivity of the spacetime-modulation scheme that further demonstrates its inner relation with EPs. Let us consider any parameter that we want to perturb and denote its perturbed value by Λ_p and its value at the EP by Λ_{EP} . Hence, the relative perturbation can be given by $\Delta = (\Lambda_p - \Lambda_{EP})/\Lambda_{EP}$ or equivalently $\Lambda_p = \Lambda_{EP}(1 + \Delta)$. Δ can thus assume positive or negative values and in Fig. B.2 we plot the effect of this perturbation on the eigenfrequencies for different scenarios, i.e., $\Lambda = (\Omega_m, \delta_m, \gamma_0, \gamma_1)$. We notice that for $\Lambda = \Omega_m$ the splitting in the eigenfrequencies occurs for $\Delta > 0$, whereas for the other scenarios this occurs for $\Delta < 0$, showing the peculiar role of perturbing the modulation frequency. Note also that the lower panel corresponding to $\Lambda = \gamma_1$ has a different scale in the x -axis, as it is two orders of magnitude less sensitive to perturbation than the remaining parameters. We go one step further and we use the Puiseux series that

FIG. B.2. Variation of the eigenfrequencies ω versus the normalized perturbation parameter Δ compute numerically for $\Lambda = \Omega_m$, $\Lambda = \gamma_0$, $\Lambda = \delta_m$, and $\Lambda = \gamma_1$ reading from top to bottom, respectively. The parameter of the spacetime-modulation are $\Omega_m = 10$ rad/s and $\delta_m = 1.75$, while the other parameters are same as in the manuscript.

are fractional-power series used to fit the observed behavior of the eigenfrequencies variation versus the perturbation Δ [58, 59, 62]. As in our case the eigenvalue is ω as given by Eq. (8) it can be shown that the eigenvalues are (here we have a doubly degenerate EP, so we expect to have two eigenvalues)

$$\omega_{\pm}(\Delta) = \omega_{EP} \pm \tau_1 \sqrt{\Delta}, \quad (\text{B1})$$

FIG. B.3. Analytical (blue dashed line) and numerical (black squares) calculation of the variation of the eigenvalues versus the normalized perturbation parameter Δ , with remaining properties similar as in Fig. B.2.

where

$$\tau_1 = i \sqrt{\frac{\frac{\partial \mathcal{L}}{\partial \Delta}(\Delta, \omega)}{\frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial \omega^2}(\Delta, \omega)}} \bigg|_{\Delta=0, \omega=\omega_{EP}}, \quad (\text{B2})$$

with $\mathcal{L} = |T(\Delta) - \omega \mathbb{1}|$ and $\mathbb{1}$ denotes the identity matrix of same dimension as T . The sensitivity of the device is demonstrated by the fact that the eigenfrequencies vary as the square root of the perturbation Δ , meaning a small value of Δ can still result in a dramatic effect on $\omega_{\pm}$. Figure B.3 plots the comparison between the analytical and numerical calculation of the eigenvalues when $\Lambda = \delta_m$, i.e., we perturb the modulation amplitude. In this case, we compute $\tau_1 = 1.87i$ from Eq. (B2). An excellent agreement is observed for small values of Δ demonstrating thus the validity of the analytical method as well as sensitivity of the spacetime-modulation effect.

Appendix C: Tilted bandgap

Here, we wish to understand the origin and variation of the tilted bandgap due to spacetime-modulation of flexural-gravity waves. Figure A.1(c)-(d) shows the comparison between the two-band in free-space and in case of spacetime-modulation, showing how the bandgap appears due to coupling.

1. Bandgap Variation with Modulation

Figures C.4 and C.5 show the dependence of the oblique (tilted) bandgap for a time-modulation frequency $\Omega_m = 20$ rad/s and for a varying space-modulation frequency κ_m ranging from 0 to 0.25 rad/m. These two figures show how the bandgap is gradually transformed from horizontal to vertical, via the oblique bandgap of various angles with increasing values of κ_m . These figures show for instance a nontrivial behavior (i.e., a nonlinear variation of the bandgap rotation) as the oblique bandgap first disappears for $\kappa_m = 0.03, 0.04$ rad/m and then reappears before transforming into a classical (frequency) bandgap when the spatial-modulation becomes strong enough (i.e., for $\kappa_m \geq 0.1$ rad/m). Hence, this progression shows that we can get all bandgap types by controlling the spatial-modulation frequency and keeping the time-modulation constant or equivalently by sweeping the time-modulation for a constant spatial-modulation: k -bandgap for $\kappa_m = 0$ for the first plot in Fig. C.4, oblique bandgap, and frequency bandgap for the remaining plots in Figs C.5.

FIG. C.4. Two-band dispersion curves for the case of $\delta_m = 1$ and considering only the bands $n = 0$ and $n = -1$, for $\Omega_m = 20$ rad/s and for varying κ_m from 0 to 0.05 rad/m (spacetime-modulation with an oblique, i.e., indirect bandgap). The black curves show the real part of the eigenfrequencies while the red curves show their imaginary part. The remaining physical parameters of the elastic plate and liquid are same to those defined in Section II of the main manuscript.

FIG. C.5. Two-band dispersion curves for the case of $\delta_m = 1$ and considering only the bands $n = 0$ and $n = -1$, for $\Omega_m = 20$ rad/s and for varying κ_m from 0.06 to 0.25 rad/m (spacetime-modulation with an oblique (indirect) bandgap). The black curves show the real parts of the eigenfrequencies while the red curves show their imaginary parts. The remaining physical parameters of the elastic plate and liquid are same to those defined in Section II of the main manuscript.

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